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Equivalent Conditions of the Reverse Hardy-Type Integral Inequalities

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Abstract: Hardy-type integral inequalities play a prominent role in the study of analytic inequalities, which are essential in mathematical analysis and its various applications, such as in the study of symmetry and asymmetry phenomena. In this paper, employing methods of real analysis and using weight functions, we investigate some equivalent conditions of two kinds of reverse Hardy-type integral inequalities with a particular non-homogeneous kernel. A few equivalent conditions of two kinds of reverse Hardy-type integral inequalities with a particular homogeneous kernel are deduced in the form of applications.

Keywords: hardy-type integral inequality; weight function; equivalent form; reverse; gamma function

MSC: 26D15; 47A05

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1. Introduction

In 1925, by introducing one pair of conjugate exponents (p,q) $\left(p>1, \frac{1}{p}+\frac{1}{q}=1\right)$, Hardy [1] established the following extension of Hilbert's integral inequality:
For $f(x), g(y) \geq 0$,

$$0<\int_0^\infty f^p(x)dx<\infty \quad \text{and} \quad 0<\int_0^\infty g^q(y)dy<\infty,$$

we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}, \tag{1}$$

with the best possible constant factor

$$\frac{\pi}{\sin(\pi/p)}.$$

Inequality (1) as well as Hilbert's integral inequality (for $p=q=2$ in (1), cf. [2]) have proved to be essential in analysis and its various applications (cf. [3,4]). In 1934, Hardy et al. established an extension of (1) with the kernel $k_1(x,y)$, where $k_1(x,y)$ is a non-negative homogeneous function of degree -1 (cf. [3], Theorem 319). The following Hilbert-type

integral inequality with the non-homogeneous kernel is proved:
If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, h(u) > 0,$

$$\phi(\sigma) = \int_0^\infty h(u)u^{\sigma-1}du \in \mathbf{R}_+,$$

then

$$\begin{aligned} &\int_0^\infty \int_0^\infty h(xy)f(x)g(y)dxdy \\ < \quad & \phi\left(\frac{1}{p}\right)\left(\int_0^\infty x^{p-2}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}, \end{aligned} \tag{2}$$

with the best possible constant factor $\phi(\frac{1}{p})$ (cf. [3], Theorem 350).

In 1998, by introducing an independent parameter $\lambda > 0$, Yang presented an extension of (1) for $p = q = 2$ with the kernel $\frac{1}{(x+y)^\lambda}$ (cf. [5,6]). In 2004, by introducing another pair of conjugate exponents (r, s) $(r > 1, \frac{1}{r} + \frac{1}{s} = 1)$, Yang [7] proved an extension of (1) with the kernel $\frac{1}{x^\lambda + y^\lambda}$ $(\lambda > 0)$. In 2005, Yang et al. [8] also established an extension of (1) and the result of [5]. Krnic et al. in [9–14] presented as well some extensions of (1).

In 2009, Yang proved the following extension of (3) (cf. [15,16]):

If $\lambda_1 + \lambda_2 = \lambda \in \mathbf{R} = (-\infty, \infty)$, $k_\lambda(x, y)$ is a non-negative homogeneous function of degree $-\lambda$, satisfying

$$k_\lambda(ux, uy) = u^{-\lambda}k_\lambda(x, y)(u, x, y > 0),$$

$$k(\lambda_1) = \int_0^\infty k_\lambda(u, 1)u^{\lambda_1-1}du \in \mathbf{R}_+ = (0, \infty),$$

then for $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, we have

$$\begin{aligned} &\int_0^\infty \int_0^\infty k_\lambda(x, y)f(x)g(y)dxdy \\ < \quad & k(\lambda_1)\left(\int_0^\infty x^{p(1-\lambda_1)-1}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{q(1-\lambda_2)-1}g^q(y)dy\right)^{\frac{1}{q}}, \end{aligned} \tag{3}$$

with the best possible constant factor $k(\lambda_1)$. For $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1$, we derive the reverse of (3). The following extension of (2) has been proved:
For $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, we have

$$\begin{aligned} &\int_0^\infty \int_0^\infty h(xy)f(x)g(y)dxdy \\ < \quad & \phi(\sigma)\left(\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{q(1-\sigma)-1}g^q(y)dy\right)^{\frac{1}{q}}, \end{aligned} \tag{4}$$

where the constant factor $\phi(\sigma)$ is the best possible. For $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1$, we obtain the reverse of (4) (cf. [17]).

Some equivalent inequalities of (3) and (4) were considered in [16]. In 2013, Yang [17] also studied the equivalency between (3) and (4). In 2017, Hong [18] presented an equivalent condition between (3) and some parameters. Other similar works are provided in [19–27].

Remark 1 (cf. [17])**.** *If $h(xy) = 0$, for $xy > 1$, then*

$$\phi(\sigma) = \int_0^1 h(u)u^{\sigma-1}du = \phi_1(\sigma) \in \mathbf{R}_+,$$

and the reverse of (4) reduces to the following reverse Hardy-type integral inequality with the non-homogeneous kernel:

$$\int_0^\infty g(y)\left(\int_0^{\frac{1}{y}} h(xy)f(x)dx\right)dy$$
$$> \quad \phi_1(\sigma)\left(\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{q(1-\sigma)-1}g^q(y)dy\right)^{\frac{1}{q}}; \tag{5}$$

if $h(xy) = 0$, *for* $xy < 1$, *then*

$$\phi(\sigma) = \int_1^\infty h(u)u^{\sigma-1}du = \phi_2(\sigma) \in \mathbf{R}_+,$$

and the reverse of (4) reduces to the following reverse Hardy-type integral inequality with non-homogeneous kernel:

$$\int_0^\infty g(y)\left(\int_{\frac{1}{y}}^\infty h(xy)f(x)dx\right)dy$$
$$> \quad \phi_2(\sigma)\left(\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{q(1-\sigma)-1}g^q(y)dy\right)^{\frac{1}{q}}. \tag{6}$$

Hardy-type integral inequalities play a prominent role in the study of analytic inequalities, which are essential in mathematical analysis and its various applications in Physics and Engineering, such as in the study of symmetry and asymmetry phenomena (cf. [23,28]).

In the present work, employing methods of real analysis as well as using weight functions, we obtain a few equivalent conditions of (5) (resp. (6)) with a particular non-homogeneous kernel

$$\frac{(\min\{xy,1\})^\alpha|\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}}\ (\beta > -1).$$

Some equivalent conditions of two kinds of reverse Hardy-type integral inequalities with a particular homogeneous kernel are deduced in the form of applications. We also consider some interesting corollaries.

2. An Example and Two Lemmas

Example 1. *Setting*

$$h(u) = \frac{(\min\{u,1\})^\alpha|\ln u|^\beta}{(\max\{u,1\})^{\lambda+\alpha}}\ (u > 0),$$

we then obtain that

$$h(xy) = \frac{(\min\{xy,1\})^\alpha|\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}},$$

and for $\beta > -1, \sigma, \mu > -\alpha, \sigma + \mu = \lambda \in \mathbf{R}$,

$$\begin{aligned} k_\lambda^{(1)}(\sigma) \quad : &= \int_0^1 h(u)u^{\sigma-1}du = \int_0^1 \frac{(\min\{u,1\})^\alpha(-\ln u)^\beta}{(\max\{u,1\})^{\lambda+\alpha}}u^{\sigma-1}du \\ &= \int_0^1 u^{\alpha+\sigma-1}(-\ln u)^\beta du = \frac{\Gamma(\beta+1)}{(\sigma+\alpha)^{\beta+1}} \in \mathbf{R}_+, \end{aligned}$$

$$\begin{aligned} k_\lambda^{(2)}(\sigma) \quad : &= \int_1^\infty h(u)u^{\sigma-1}du = \int_1^\infty \frac{(\min\{u,1\})^\alpha(\ln u)^\beta}{(\max\{u,1\})^{\lambda+\alpha}}u^{\sigma-1}du \\ &= \int_0^1 v^{\alpha+\mu-1}(-\ln v)^\beta dv = \frac{\Gamma(\beta+1)}{(\mu+\alpha)^{\beta+1}} = k_\lambda^{(1)}(\mu) \in \mathbf{R}_+, \end{aligned}$$

where

$$\Gamma(\eta) := \int_0^\infty v^{\eta-1} e^{-v} dv \ (\eta > 0)$$

stands for the gamma function (cf. [29]).

In the following, we assume that $0 < p < 1 (q < 0)$, $\frac{1}{p} + \frac{1}{q} = 1$, $\beta > -1$, $\lambda, \sigma_1 \in \mathbf{R}$.

Lemma 1. *If $\sigma > -\alpha$ and there exists a constant $M_1 > 0$ such that for any non-negative measurable functions $f(x)$, $g(y)$ in $(0, \infty)$ the following inequality*

$$\begin{aligned} &\int_0^\infty g(y) \left[\int_0^{\frac{1}{y}} \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} f(x) dx \right] dy \\ \geq \ & M_1 \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{q}} \end{aligned} \tag{7}$$

holds true, then we have

$$\sigma_1 = \sigma \ \textit{and} \ k_\lambda^{(1)}(\sigma) \geq M_1.$$

Proof. If $\sigma_1 < \sigma$, then for $n \in \mathbf{N}$, we consider the following two functions

$$f_n(x) := \begin{cases} x^{\sigma + \frac{1}{pn} - 1}, 0 < x \leq 1 \\ 0, x > 1 \end{cases}, \quad g_n(y) := \begin{cases} 0, 0 < y < 1 \\ y^{\sigma_1 - \frac{1}{qn} - 1}, y \geq 1 \end{cases},$$

and obtain that

$$\begin{aligned} J_1 \ :&= \left[\int_0^\infty x^{p(1-\sigma)-1} f_n^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} g_n^q(y) dy \right]^{\frac{1}{q}} \\ &= \left(\int_0^1 x^{\frac{1}{n}-1} dx \right)^{\frac{1}{p}} \left(\int_1^\infty y^{-\frac{1}{n}-1} dy \right)^{\frac{1}{q}} = n. \end{aligned}$$

Setting $u = xy$, for $0 < p < 1$, we derive that

$$\begin{aligned} I_1 \ :&= \int_0^\infty g_n(y) \left[\int_0^{\frac{1}{y}} \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} f_n(x) dx \right] dy \\ &= \int_1^\infty \left[\int_0^{\frac{1}{y}} \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} x^{\sigma + \frac{1}{pn} - 1} dx \right] y^{\sigma_1 - \frac{1}{qn} - 1} dy \\ &= \int_1^\infty y^{(\sigma_1 - \sigma) - \frac{1}{n} - 1} dy \int_0^1 \frac{(\min\{u,1\})^\alpha |\ln u|^\beta}{(\max\{u,1\})^{\lambda+\alpha}} u^{\sigma + \frac{1}{pn} - 1} du \\ &\leq \frac{1}{\sigma - \sigma_1 + \frac{1}{n}} \int_0^1 \frac{(\min\{u,1\})^\alpha |\ln u|^\beta}{(\max\{u,1\})^{\lambda+\alpha}} u^{\sigma-1} du \leq \frac{k_\lambda^{(1)}(\sigma)}{\sigma - \sigma_1}, \end{aligned}$$

and then by (7), it follows that

$$\frac{k_\lambda^{(1)}(\sigma)}{\sigma - \sigma_1} \geq I_1 \geq M_1 J_1 = M_1 n. \tag{8}$$

By (8), letting $n \to \infty$, in view of $k_\lambda^{(1)}(\sigma) < \infty$, $\sigma > \sigma_1$ and $M_1 > 0$, we get that

$$\infty > \frac{k_\lambda^{(1)}(\sigma)}{\sigma - \sigma_1} \geq \infty,$$

which is a contradiction.

If $\sigma_1 > \sigma$, then for

$$n \geq \frac{1}{|q|(\sigma_1 - \sigma)} \ (n \in \mathbf{N}),$$

we consider the following two functions:

$$\widetilde{f}_n(x) := \begin{cases} 0, 0 < x < 1 \\ x^{\sigma - \frac{1}{pn} - 1}, x \geq 1 \end{cases}, \quad \widetilde{g}_n(y) := \begin{cases} y^{\sigma_1 + \frac{1}{qn} - 1}, 0 < y \leq 1 \\ 0, y > 1 \end{cases},$$

and deduce that

$$\begin{aligned} \widetilde{J}_1 \quad : &= \left[\int_0^\infty x^{p(1-\sigma)-1} \widetilde{f}_n^p(x) dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} \widetilde{g}_n^q(y) dy\right]^{\frac{1}{q}} \\ &= \left(\int_1^\infty x^{-\frac{1}{n}-1} dx\right)^{\frac{1}{p}} \left(\int_0^1 y^{\frac{1}{n}-1} dy\right)^{\frac{1}{q}} = n. \end{aligned}$$

Setting $u = xy$, in view of $\sigma_1 + \frac{1}{qn} \geq \sigma$ $(q < 0)$, we obtain

$$\begin{aligned} \widetilde{I}_1 &:= \int_0^\infty \widetilde{f}_n(x) \left[\int_0^{\frac{1}{x}} \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} \widetilde{g}_n(y) dy\right] dx \\ &= \int_1^\infty \left[\int_0^{\frac{1}{x}} \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} y^{\sigma_1 + \frac{1}{qn} - 1} dy\right] x^{\sigma - \frac{1}{pn} - 1} dx \\ &= \int_1^\infty x^{(\sigma - \sigma_1) - \frac{1}{n} - 1} dx \int_0^1 \frac{(\min\{u,1\})^\alpha |\ln u|^\beta}{(\max\{u,1\})^{\lambda+\alpha}} u^{\sigma_1 + \frac{1}{qn} - 1} du \\ &\leq \frac{1}{\sigma_1 - \sigma + \frac{1}{n}} \int_0^1 \frac{(\min\{u,1\})^\alpha |\ln u|^\beta}{(\max\{u,1\})^{\lambda+\alpha}} u^{\sigma - 1} du \leq \frac{k_\lambda^{(1)}(\sigma)}{\sigma_1 - \sigma}, \end{aligned}$$

and then by Fubini's theorem (cf. [30]) and (7), we derive that

$$\begin{aligned} \frac{k_1(\sigma)}{\sigma_1 - \sigma} &\geq \widetilde{I}_1 = \int_0^\infty \widetilde{g}_n(y) \left[\int_0^{\frac{1}{y}} \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} \widetilde{f}_n(x) dx\right] dy \\ &\geq M_1 \widetilde{J}_1 = M_1 n. \end{aligned} \tag{9}$$

By (9), letting $n \to \infty$, we obtain that

$$\infty > \frac{k_\lambda^{(1)}(\sigma)}{\sigma_1 - \sigma} \geq \infty,$$

which is a contradiction.

Hence, we conclude that $\sigma_1 = \sigma$.

For $\sigma_1 = \sigma$, we deduce that $I_1 \geq M_1 J_1$ and then

$$\begin{aligned} k_\lambda^{(1)}(\sigma) &= \int_0^1 \frac{(\min\{u,1\})^\alpha |\ln u|^\beta}{(\max\{u,1\})^{\lambda+\alpha}} u^{\sigma-1} du \\ &\geq \int_0^1 \frac{(\min\{u,1\})^\alpha |\ln u|^\beta}{(\max\{u,1\})^{\lambda+\alpha}} u^{\sigma + \frac{1}{pn} - 1} du \geq M_1. \end{aligned}$$

This completes the proof of the lemma. □

Lemma 2. *If $\mu > -\alpha, \sigma = \lambda - \mu$ and there exists a constant $M_2 > 0$ such that for any non-negative measurable functions $f(x)$, $g(y)$ in $(0, \infty)$, the following inequality*

$$\begin{aligned} & \int_0^\infty g(y)\left[\int_{\frac{1}{y}}^\infty \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} f(x)dx\right]dy \\ \geq\ & M_2\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y)dy\right]^{\frac{1}{q}} \end{aligned} \quad (10)$$

holds true, then we have

$$\sigma_1 = \sigma \quad and \quad k_\lambda^{(2)}(\sigma) \geq M_2.$$

Proof. If $\sigma_1 > \sigma$, then for $n \in \mathbf{N}$, we consider two functions $\widetilde{f}_n(x)$ and $\widetilde{g}_n(y)$ as in Lemma 1 and derive that

$$\widetilde{J}_1 = \left[\int_0^\infty x^{p(1-\sigma)-1} \widetilde{f}_n^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(1-\sigma_1)-1} \widetilde{g}_n^q(y)dy\right]^{\frac{1}{q}} = n.$$

Setting $u = xy$, we obtain

$$\begin{aligned} \widetilde{I}_2 \ :&= \int_0^\infty \widetilde{g}_n(y)\left[\int_{\frac{1}{y}}^\infty \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} \widetilde{f}_n(x)dx\right]dy \\ &= \int_0^1 \left[\int_{\frac{1}{y}}^\infty \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} x^{\sigma-\frac{1}{pn}-1}dx\right] y^{\sigma_1+\frac{1}{qn}-1}dy \\ &= \int_0^1 y^{(\sigma_1-\sigma)+\frac{1}{n}-1}dy \int_1^\infty \frac{(\min\{u,1\})^\alpha |\ln u|^\beta}{(\max\{u,1\})^{\lambda+\alpha}} u^{\sigma-\frac{1}{pn}-1}du \\ &\leq \frac{k_\lambda^{(2)}(\sigma)}{\sigma_1-\sigma}, \end{aligned}$$

and then by (10), it follows that

$$\frac{k_\lambda^{(2)}(\sigma)}{\sigma_1-\sigma} \geq \widetilde{I}_2 \geq M_2 \widetilde{J}_1 = M_2 n. \quad (11)$$

By (11), letting $n \to \infty$, we get that

$$\infty > \frac{k_\lambda^{(2)}(\sigma)}{\sigma_1-\sigma} \geq \infty,$$

which is a contradiction.

If $\sigma_1 < \sigma$, then for

$$n \geq \frac{1}{|q|(\sigma-\sigma_1)} \ (n \in \mathbf{N}),$$

we consider two functions $f_n(x)$ and $g_n(y)$ as in Lemma 1 and get that

$$J_1 = \left[\int_0^\infty x^{p(1-\sigma)-1} f_n^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(1-\sigma_1)-1} g_n^q(y)dy\right]^{\frac{1}{q}} = n.$$

Setting $u = xy$, we obtain

$$\begin{aligned} I_2 \quad : \quad &= \int_0^\infty f_n(x)\left[\int_{\frac{1}{x}}^\infty \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} g_n(y)dy\right]dx \\ &= \int_0^1 \left[\int_{\frac{1}{x}}^\infty \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} y^{\sigma_1-\frac{1}{qn}-1}dy\right] x^{\sigma+\frac{1}{pn}-1}dx \\ &= \int_0^1 x^{(\sigma-\sigma_1)+\frac{1}{n}-1}dx \int_1^\infty \frac{(\min\{u,1\})^\alpha |\ln u|^\beta}{(\max\{u,1\})^{\lambda+\alpha}} u^{\sigma_1-\frac{1}{qn}-1}du \\ &\leq \frac{1}{\sigma-\sigma_1}\int_1^\infty \frac{(\min\{u,1\})^\alpha |\ln u|^\beta}{(\max\{u,1\})^{\lambda+\alpha}} u^{\sigma-1}du = \frac{k_\lambda^{(2)}(\sigma)}{\sigma-\sigma_1}, \end{aligned}$$

and then by Fubini's theorem (cf. [30]) and (10), it follows that

$$\begin{aligned} \frac{k_\lambda^{(2)}(\sigma)}{\sigma-\sigma_1} &\geq I_2 = \int_0^\infty g_n(y)\left[\int_{\frac{1}{y}}^\infty \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} f_n(x)dx\right]dy \\ &\geq M_2 J_1 = M_2 n. \end{aligned} \tag{12}$$

By (12), letting $n \to \infty$, we derive that

$$\infty > \frac{k_\lambda^{(2)}(\sigma)}{\sigma-\sigma_1} \geq \infty,$$

which is a contradiction.

Hence, we conclude that $\sigma_1 = \sigma$.

For $\sigma_1 = \sigma$, we deduce $\widetilde{I}_2 \geq M_2 \widetilde{J}_2$ and then it follows that

$$\begin{aligned} k_\lambda^{(2)}(\sigma) &= \int_1^\infty \frac{(\min\{u,1\})^\alpha |\ln u|^\beta}{(\max\{u,1\})^{\lambda+\alpha}} u^{\sigma-1}du \\ &\geq \int_1^\infty \frac{(\min\{u,1\})^\alpha |\ln u|^\beta}{(\max\{u,1\})^{\lambda+\alpha}} u^{\sigma-\frac{1}{pn}-1} \geq M_2. \end{aligned}$$

This completes the proof of the lemma. □

3. Reverse Hardy-Type Inequalities of the First Kind

Theorem 1. *If $\sigma > -\alpha$, then the following conditions are equivalent:*

(i) There exists a constant $M_1 > 0$, such that for any $f(x) \geq 0$ satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx < \infty,$$

we have the following reverse Hardy-type integral inequality of the first kind with the non-homogeneous kernel:

$$\begin{aligned} J \quad : \quad &= \left\{\int_0^\infty y^{p\sigma_1-1}\left[\int_0^{\frac{1}{y}} \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} f(x)dx\right]^p dy\right\}^{\frac{1}{p}} \\ &> M_1\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}. \end{aligned} \tag{13}$$

(ii) There exists a constant $M_1 > 0$, such that for any $g(y) \geq 0$ satisfying

$$0 < \int_0^\infty y^{q(1-\sigma_1)-1} g^q(y)dy < \infty,$$

we have the following reverse Hardy-type integral inequality of the first kind with the non-homogeneous kernel:

$$\begin{aligned}&\left\{\int_0^\infty x^{q\sigma-1}\left[\int_0^{\frac{1}{x}}\frac{(\min\{xy,1\})^\alpha|\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}}g(y)dy\right]^q dx\right\}^{\frac{1}{q}}\\ &> M_1\left[\int_0^\infty y^{q(1-\sigma_1)-1}g^q(y)dy\right]^{\frac{1}{q}}. \end{aligned} \tag{14}$$

(iii) There exists a constant $M_1 > 0$, such that for any $f(x), g(y) \geq 0$ satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx < \infty \text{ and } 0 < \int_0^\infty y^{q(1-\sigma_1)-1}g^q(y)dy < \infty,$$

we have the following inequality:

$$\begin{aligned} I &:= \int_0^\infty g(y)\left[\int_0^{\frac{1}{y}}\frac{(\min\{xy,1\})^\alpha|\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}}f(x)dx\right]dy\\ &> M_1\left[\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(1-\sigma_1)-1}g^q(y)dy\right]^{\frac{1}{q}}. \end{aligned} \tag{15}$$

(iv) $\sigma_1 = \sigma$.

If Condition (iv) holds, then the constant $M_1 = k_\lambda^{(1)}(\sigma)$ in (13)–(15) is the best possible.

Proof. $(i) \Rightarrow (ii)$. By the reverse Hölder inequality (cf. [31]), we have

$$\begin{aligned} I &= \int_0^\infty\left[y^{\sigma_1-\frac{1}{p}}\int_0^{\frac{1}{y}}\frac{(\min\{xy,1\})^\alpha|\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}}f(x)dx\right]\left(y^{\frac{1}{p}-\sigma_1}g(y)\right)dy\\ &\geq J\left[\int_0^\infty y^{q(1-\sigma_1)-1}g^q(y)dy\right]^{\frac{1}{q}}. \end{aligned} \tag{16}$$

Then by (13), we obtain (14).

$(ii) \Rightarrow (iv)$. By Lemma 1, we have $\sigma_1 = \sigma$.

$(iv) \Rightarrow (i)$. Setting $u = xy$, we obtain the following weight function:

$$\begin{aligned} \omega_1(\sigma,y) &:= y^\sigma\int_0^{\frac{1}{y}}\frac{(\min\{xy,1\})^\alpha|\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}}x^{\sigma-1}dx\\ &= \int_0^1\frac{(\min\{u,1\})^\alpha|\ln u|^\beta}{(\max\{u,1\})^{\lambda+\alpha}}u^{\sigma-1}du = k_\lambda^{(1)}(\sigma)(y>0). \end{aligned} \tag{17}$$

By the reverse Hölder inequality with weight and (17), for $y \in (0, \infty)$, we have

$$\begin{aligned}
&\left[\int_0^{\frac{1}{y}} \frac{(\min\{xy,1\})^{\alpha}|\ln xy|^{\beta}}{(\max\{xy,1\})^{\lambda+\alpha}} f(x)dx\right]^p \\
= &\left\{\int_0^{\frac{1}{y}} \frac{(\min\{xy,1\})^{\alpha}|\ln xy|^{\beta}}{(\max\{xy,1\})^{\lambda+\alpha}} \left[\frac{y^{(\sigma-1)/p}}{x^{(\sigma-1)/q}} f(x)\right]\left[\frac{x^{(\sigma-1)/q}}{y^{(\sigma-1)/p}}\right] dx\right\}^p \\
\geq &\int_0^{\frac{1}{y}} \frac{(\min\{xy,1\})^{\alpha}|\ln xy|^{\beta}}{(\max\{xy,1\})^{\lambda+\alpha}} \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x)dx \\
&\times\left[\int_0^{\frac{1}{y}} \frac{(\min\{xy,1\})^{\alpha}|\ln xy|^{\beta}}{(\max\{xy,1\})^{\lambda+\alpha}} \frac{x^{\sigma-1}}{y^{(\sigma-1)q/p}} dx\right]^{p-1} \\
= &\left[\frac{\omega_1(\sigma,y)}{y^{q(\sigma-1)+1}}\right]^{p-1} \int_0^{\frac{1}{y}} \frac{(\min\{xy,1\})^{\alpha}|\ln xy|^{\beta}}{(\max\{xy,1\})^{\lambda+\alpha}} \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x)dx \\
= &(k_\lambda^{(1)}(\sigma))^{p-1} y^{-p\sigma+1} \int_0^{\frac{1}{y}} \frac{(\min\{xy,1\})^{\alpha}|\ln xy|^{\beta}}{(\max\{xy,1\})^{\lambda+\alpha}} \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x)dx. \qquad (18)
\end{aligned}$$

If (18) obtains the form of equality for some $y \in (0, \infty)$, then (cf. [31]) there exist constants A and B, such that they are not both zero, and

$$A\frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) = B\frac{x^{\sigma-1}}{y^{(\sigma-1)q/p}} \ a.e. \text{ in } \mathbf{R}_+.$$

We suppose that $A \neq 0$ (otherwise $B = A = 0$). It follows that

$$x^{p(1-\sigma)-1} f^p(x) = y^{q(1-\sigma)} \frac{B}{Ax} \ a.e. \text{ in } \mathbf{R}_+,$$

which contradicts the fact that $0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx < \infty$. Hence, (18) becomes a strict inequality.

For $\sigma_1 = \sigma$, by Fubini's theorem (cf. [30]) and the above result, we have

$$\begin{aligned}
J &> (k_\lambda^{(1)}(\sigma))^{\frac{1}{q}} \left\{\int_0^\infty \left[\int_0^{\frac{1}{y}} \frac{(\min\{xy,1\})^{\alpha}|\ln xy|^{\beta}}{(\max\{xy,1\})^{\lambda+\alpha}} \frac{y^{\sigma-1} f^p(x)}{x^{(\sigma-1)p/q}} dx\right] dy\right\}^{\frac{1}{p}} \\
&= (k_\lambda^{(1)}(\sigma))^{\frac{1}{q}} \left\{\int_0^\infty \left[\int_0^{\frac{1}{x}} \frac{(\min\{xy,1\})^{\alpha}|\ln xy|^{\beta}}{(\max\{xy,1\})^{\lambda+\alpha}} \frac{y^{\sigma-1} dy}{x^{(\sigma-1)(p-1)}}\right] f^p(x)dx\right\}^{\frac{1}{p}} \\
&= (k_\lambda^{(1)}(\sigma))^{\frac{1}{q}} \left[\int_0^\infty \omega_1(\sigma,x) x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}} \\
&= k_\lambda^{(1)}(\sigma) \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}.
\end{aligned}$$

Setting $0 < M_1 \leq k_\lambda^{(1)}(\sigma)(< \infty)$, (13) follows.

Therefore, Conditions (i), (iii), and (iv) are equivalent. Since the Conditions (i) and (iii) are equivalent, similarly, by Fubini's theorem, we have

$$I = \int_0^\infty f(x)\left[\int_0^{\frac{1}{x}} \frac{(\min\{xy,1\})^{\alpha}|\ln xy|^{\beta}}{(\max\{xy,1\})^{\lambda+\alpha}} g(y)dy\right]dx,$$

and we deduce that Conditions (ii) and (iii) are equivalent. Hence, the conditions (i), (ii), (iii), and (iv) are equivalent.

When Condition (iv) is satisfied, if there exists a constant $M_1 \geq k_\lambda^{(1)}(\sigma)$, such that (14) is true, then by Lemma 1 we have $k_\lambda^{(1)}(\sigma) \geq M_1$. Hence, the constant factor $M_1 = k_\lambda^{(1)}(\sigma)$ in (14) is the best possible.

The constant factor $M_1 = k_\lambda^{(1)}(\sigma)$ in (13) is still the best possible. Otherwise, by (16) (for $\sigma_1 = \sigma$), we would conclude that the constant factor $M_1 = k_\lambda^{(1)}(\sigma)$ in (15) is not the best possible. Similarly, we can prove that the constant factor $M_1 = k_\lambda^{(1)}(\sigma)$ in (14) is the best possible.

This completes the proof of the theorem. □

In particular, for $\sigma = \sigma_1 = \frac{1}{p}$ in Theorem 1, we derive the following corollary.

Corollary 1. *If $\alpha > -\frac{1}{p}$, then the following conditions are equivalent:*

(i) There exists a constant $M_1 > 0$, such that for any $f(x) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p-2} f^p(x) dx < \infty,$$

we have the following inequality:

$$\left\{ \int_0^\infty \left[\int_0^{\frac{1}{y}} \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} f(x) dx \right]^p dy \right\}^{\frac{1}{p}} > M_1 \left(\int_0^\infty x^{p-2} f^p(x) dx \right)^{\frac{1}{p}}. \quad (19)$$

(ii) There exists a constant $M_1 > 0$, such that for any $g(y) \geq 0$, satisfying

$$0 < \int_0^\infty g^q(y) dy < \infty,$$

we have the following inequality:

$$\left\{ \int_0^\infty x^{q-2} \left[\int_0^{\frac{1}{x}} \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} g(y) dy \right]^q dx \right\}^{\frac{1}{q}} > M_1 \left(\int_0^\infty g^q(y) dy \right)^{\frac{1}{q}}. \quad (20)$$

(iii) There exists a constant $M_1 > 0$, such that for any $f(x), g(y) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p-2} f^p(x) dx < \infty \text{ and } 0 < \int_0^\infty g^q(y) dy < \infty,$$

we have the following inequality:

$$\int_0^\infty g(y) \left[\int_0^{\frac{1}{y}} \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} f(x) dx \right] dy > M_1 \left(\int_0^\infty x^{p-2} f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y) dy \right)^{\frac{1}{q}}. \quad (21)$$

The constant $M_1 = k_\lambda^{(1)}(\frac{1}{p})$ in (19)–(21) is the best possible.

Setting $y = \frac{1}{Y}$, $G(Y) = g(\frac{1}{Y}) \frac{1}{Y^{2-\lambda}}$ in Theorem 1, and then replacing Y by y, we deduce the following corollary.

Corollary 2. *If $\sigma > -\alpha$, then the following conditions are equivalent:*

(i) There exists a constant M_1, such that for any $f(x) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx < \infty,$$

we have the following inequality:

$$\left\{\int_0^\infty y^{p(\lambda-\sigma_1)-1}\left[\int_0^y \frac{(\min\{x,y\})^\alpha |\ln(x/y)|^\beta}{(\max\{x,y\})^{\lambda+\alpha}} f(x)dx\right]^p dy\right\}^{\frac{1}{p}} > M_1\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}. \tag{22}$$

(ii) There exists a constant $M_1 > 0$, such that for any $G(y) \geq 0$, satisfying

$$0 < \int_0^\infty y^{q[1-(\lambda-\sigma_1)]-1} G^q(y)dy < \infty,$$

we have the following reverse Hardy-type integral inequality:

$$\left\{\int_0^\infty x^{q\sigma-1}\left[\int_x^\infty \frac{(\min\{x,y\})^\alpha |\ln(x/y)|^\beta}{(\max\{x,y\})^{\lambda+\alpha}} G(y)dy\right]^q dx\right\}^{\frac{1}{q}} > M_1\left[\int_0^\infty y^{q[1-(\lambda-\sigma_1)]-1} G^q(y)dy\right]^{\frac{1}{q}}. \tag{23}$$

(iii) There exists a constant $M_1 > 0$, such that for any $f(x), G(y) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx < \infty \ \text{ and } \ 0 < \int_0^\infty y^{q[1-(\lambda-\sigma_1)]-1} G^q(y)dy < \infty,$$

we have the following inequality:

$$\int_0^\infty G(y)\left[\int_0^y \frac{(\min\{x,y\})^\alpha |\ln(x/y)|^\beta}{(\max\{x,y\})^{\lambda+\alpha}} f(x)dx\right]dy > M_1\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q[1-(\lambda-\sigma_1)]-1} G^q(y)dy\right]^{\frac{1}{q}}. \tag{24}$$

(iv) $\sigma_1 = \sigma$.

If Condition (iv) holds true, then the constant $M_1 = k_\lambda^{(1)}(\sigma)$ in (22)–(24) is the best possible.

For $g(y) = G(y)$ and $\mu = \lambda - \sigma_1$ in Corollary 2, we deduce the corollary below.

Theorem 2. *If $\sigma > -\alpha$, then the following conditions are equivalent:*

(i) There exists a constant $M_1 > 0$, such that for any $f(x) \geq 0$ satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx < \infty,$$

we have the following reverse Hardy-type inequality of the first kind with the homogeneous kernel:

$$\left\{\int_0^\infty y^{p\mu-1}\left[\int_0^y \frac{(\min\{x,y\})^\alpha |\ln(x/y)|^\beta}{(\max\{x,y\})^{\lambda+\alpha}} f(x)dx\right]^p dy\right\}^{\frac{1}{p}} > M_1\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}. \tag{25}$$

(ii) There exists a constant $M_1 > 0$*, such that for any* $g(y) \geq 0$ *satisfying*

$$0 < \int_0^\infty y^{q(1-\mu)-1} g^q(y) dy < \infty,$$

we have the following reverse Hardy-type inequality of the first kind with the homogeneous kernel:

$$\left\{ \int_0^\infty x^{q\sigma-1} \left[\int_x^\infty \frac{(\min\{x,y\})^\alpha |\ln(x/y)|^\beta}{(\max\{x,y\})^{\lambda+\alpha}} g(y) dy \right]^q dx \right\}^{\frac{1}{q}} > M_1 \left[\int_0^\infty y^{q(1-\mu)-1} g^q(y) dy \right]^{\frac{1}{p}}. \quad (26)$$

(iii) There exists a constant $M_1 > 0$*, such that for any* $f(x), g(y) \geq 0$ *satisfying*

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty \text{ and } 0 < \int_0^\infty y^{q(1-\mu)-1} g^q(y) dy < \infty,$$

we have the following inequality:

$$\int_0^\infty g(y) \left[\int_0^y \frac{(\min\{x,y\})^\alpha |\ln(x/y)|^\beta}{(\max\{x,y\})^{\lambda+\alpha}} f(x) dx \right] dy > M_1 \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\mu)-1} g^q(y) dy \right]^{\frac{1}{q}}. \quad (27)$$

(iv) $\mu + \sigma = \lambda$.
If Condition (iv) holds, then the constant $M_1 = k_\lambda^{(1)}(\sigma)$ *in (25)–(27) is the best possible.*

In particular, for $\lambda = 1, \sigma = \frac{1}{q}, \mu = \frac{1}{p}$ in Theorem 2, we deduce the corollary below.

Corollary 3. *If* $\alpha > -\frac{1}{q}$*, then the following conditions are equivalent:*
(i) There exists a constant $M_1 > 0$*, such that for any* $f(x) \geq 0$ *satisfying*

$$0 < \int_0^\infty f^p(x) dx < \infty,$$

we have the following inequality:

$$\left\{ \int_0^\infty \left[\int_0^y \frac{(\min\{x,y\})^\alpha |\ln(x/y)|^\beta}{(\max\{x,y\})^{1+\alpha}} f(x) dx \right]^p dy \right\}^{\frac{1}{p}} > M_1 \left(\int_0^\infty f^p(x) dx \right)^{\frac{1}{p}}. \quad (28)$$

(ii) There exists a constant $M_1 > 0$*, such that for any* $g(y) \geq 0$ *satisfying*

$$0 < \int_0^\infty g^q(y) dy < \infty,$$

we have the following inequality:

$$\left\{ \int_0^\infty \left[\int_0^x \frac{(\min\{x,y\})^\alpha |\ln(x/y)|^\beta}{(\max\{x,y\})^{1+\alpha}} g(y) dy \right]^q dx \right\}^{\frac{1}{q}} > M_1 \left(\int_0^\infty g^q(y) dy \right)^{\frac{1}{p}}. \quad (29)$$

(iii) There exists a constant $M_1 > 0$*, such that for any* $f(x), g(y) \geq 0$ *satisfying*

$$0 < \int_0^\infty f^p(x) dx < \infty \text{ and } 0 < \int_0^\infty g^q(y) dy < \infty,$$

we have the following inequality:

$$\begin{aligned} I &= \int_0^\infty g(y)\left[\int_0^y \frac{(\min\{x,y\})^\alpha |\ln(x/y)|^\beta}{(\max\{x,y\})^{\lambda+\alpha}} f(x)dx\right]dy \\ &> M_1\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}. \end{aligned} \tag{30}$$

The constant $M_1 = k_1^{(1)}(\frac{1}{q})$ in (28)–(30) is the best possible.

4. Reverse Hardy-Type Inequalities of the Second Kind

Similarly, we obtain the following weight function:

$$\begin{aligned} \omega_2(\sigma,y) \quad : &= y^\sigma \int_{\frac{1}{y}}^\infty \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} x^{\sigma-1}dx \\ &= \int_1^\infty \frac{(\min\{u,1\})^\alpha |\ln u|^\beta}{(\max\{u,1\})^{\lambda+\alpha}} u^{\sigma-1}du = k_\lambda^{(2)}(\sigma)\ (y>0). \end{aligned}$$

Given Lemma 2, we similarly derive the following theorem.

Theorem 3. *If $\lambda-\sigma>-\alpha$, then the following conditions are equivalent:*

(i) There exists a constant $M_2>0$, such that for any $f(x)\geq 0$, satisfying

$$0<\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx<\infty,$$

we have the following reverse Hardy-type inequality of the second kind with the non-homogeneous kernel:

$$\begin{aligned} &\left\{\int_0^\infty y^{p\sigma_1-1}\left[\int_{\frac{1}{y}}^\infty \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} f(x)dx\right]^p dy\right\}^{\frac{1}{p}} \\ &> M_2\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}. \end{aligned} \tag{31}$$

(ii) There exists a constant $M_2>0$, such that for any $g(y)\geq 0$ satisfying

$$0<\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y)dy<\infty,$$

we have the following reverse Hardy-type integral inequality of the second kind with the non-homogeneous kernel:

$$\begin{aligned} &\left\{\int_0^\infty x^{q\sigma-1}\left[\int_{\frac{1}{x}}^\infty \frac{(\min\{xy,1\})^\alpha |\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} g(y)dy\right]^q dx\right\}^{\frac{1}{q}} \\ &> M_2\left[\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y)dy\right]^{\frac{1}{q}}. \end{aligned} \tag{32}$$

(iii) There exists a constant $M_2>0$, such that for any $f(x), g(y)\geq 0$ satisfying

$$0<\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx<\infty \text{ and } 0<\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y)dy<\infty,$$

we have the following inequality:

$$\int_0^\infty g(y)\left[\int_{\frac{1}{y}}^\infty \frac{(\min\{xy,1\})^\alpha|\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} f(x)dx\right]dy$$
$$> \quad M_2\left[\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(1-\sigma_1)-1}g^q(y)dy\right]^{\frac{1}{q}}. \quad (33)$$

(iv) $\sigma_1 = \sigma$.

If Condition (iv) holds, then the constant $M_2 = k_\lambda^{(2)}(\sigma)$ in (31)–(33) is the best possible.

In particular, for $\sigma = \sigma_1 = \frac{1}{p}$ in Theorem 3, we have

Corollary 4. *If $\lambda > \frac{1}{p} - \alpha$, then the following conditions are equivalent:*

(i) There exists a constant $M_2 > 0$, such that for any $f(x) \geq 0$ satisfying

$$0 < \int_0^\infty x^{p-2}f^p(x)dx < \infty,$$

we have the following inequality:

$$\left\{\int_0^\infty \left[\int_{\frac{1}{y}}^\infty \frac{(\min\{xy,1\})^\alpha|\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} f(x)dx\right]^p dy\right\}^{\frac{1}{p}}$$
$$> \quad M_2\left(\int_0^\infty x^{p-2}f^p(x)dx\right)^{\frac{1}{p}}. \quad (34)$$

(ii) There exists a constant $M_2 > 0$, such that for any $g(y) \geq 0$ satisfying

$$0 < \int_0^\infty g^q(y)dy < \infty,$$

we have the following inequality:

$$\left\{\int_0^\infty x^{q-2}\left[\int_{\frac{1}{x}}^\infty \frac{(\min\{xy,1\})^\alpha|\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} g(y)dy\right]^q dx\right\}^{\frac{1}{q}}$$
$$> \quad M_2\left[\int_0^\infty g^q(y)dy\right]^{\frac{1}{q}}. \quad (35)$$

(iii) There exists a constant $M_2 > 0$, such that for any $f(x), g(y) \geq 0$ satisfying

$$0 < \int_0^\infty x^{p-2}f^p(x)dx < \infty, \text{ and } 0 < \int_0^\infty g^q(y)dy < \infty,$$

we have the following inequality:

$$\int_0^\infty g(y)\left[\int_{\frac{1}{y}}^\infty \frac{(\min\{xy,1\})^\alpha|\ln xy|^\beta}{(\max\{xy,1\})^{\lambda+\alpha}} f(x)dx\right]dy$$
$$> \quad M_2\left(\int_0^\infty x^{p-2}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}. \quad (36)$$

The constant $M_2 = k_\lambda^{(2)}(\frac{1}{p})$ in (34)–(36) is the best possible.

Setting $y = \frac{1}{Y}$, $G(Y) = g(\frac{1}{Y})\frac{1}{Y^2}$ in Theorem 3, and then replacing Y by y, we deduce the following corollary.

Corollary 5. *If $\lambda - \sigma > -\alpha$, then the following conditions are equivalent:*
(i) There exists a constant $M_2 > 0$, such that for any $f(x) \geq 0$ satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

we have the following inequality:

$$\left\{ \int_0^\infty y^{-p\sigma_1 - 1} \left[\int_y^\infty \frac{(\min\{x/y,1\})^\alpha |\ln(x/y)|^\beta}{(\max\{x/y,1\})^{\lambda+\alpha}} f(x) dx \right]^p dy \right\}^{\frac{1}{p}} > M_2 \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}. \tag{37}$$

(ii) There exists a constant $M_2 > 0$, such that for any $G(y) \geq 0$ satisfying

$$0 < \int_0^\infty y^{q(1+\sigma_1)-1} G^q(y) dy < \infty,$$

we have the following reverse Hardy-type integral inequality:

$$\left\{ \int_0^\infty x^{q\sigma - 1} \left[\int_0^x \frac{(\min\{x/y,1\})^\alpha |\ln(x/y)|^\beta}{(\max\{x/y,1\})^{\lambda+\alpha}} G(y) dy \right]^q dx \right\}^{\frac{1}{q}} > M_2 \left[\int_0^\infty y^{q(1+\sigma_1)-1} G^q(y) dy \right]^{\frac{1}{q}}. \tag{38}$$

(iii) There exists a constant $M_2 > 0$, such that for any $f(x), G(y) \geq 0$ satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty, \text{ and } 0 < \int_0^\infty y^{q(1+\sigma_1)-1} G^q(y) dy < \infty,$$

we have the following inequality:

$$\int_0^\infty G(y) \left[\int_y^\infty \frac{(\min\{x/y,1\})^\alpha |\ln(x/y)|^\beta}{(\max\{x/y,1\})^{\lambda+\alpha}} f(x) dx \right] dy > M_2 \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1+\sigma_1)-1} G^q(y) dy \right]^{\frac{1}{q}}. \tag{39}$$

(iv) $\sigma_1 = \sigma$.
If Condition (iv) is satisfied, then the constant $M_2 = k_\lambda^{(2)}(\sigma)$ in (37)–(39) is the best possible.

For $g(y) = y^\lambda G(y)$ and $\mu = \lambda - \sigma_1$ in Corollary 5, we obtain the following theorem.

Theorem 4. *If $\lambda - \sigma > -\alpha$, then the following conditions are equivalent:*
(i) There exists a constant $M_2 > 0$, such that for any $f(x) \geq 0$ satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

we have the following reverse Hardy-type integral inequality of the second kind with the homogeneous kernel:

$$\left\{\int_0^\infty y^{p\mu-1}\left[\int_y^\infty \frac{(\min\{x,y\})^\alpha|\ln(x/y)|^\beta}{(\max\{x,y\})^{\lambda+\alpha}}f(x)dx\right]^p dy\right\}^{\frac{1}{p}}$$
$$> \quad M_2\left[\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}. \tag{40}$$

(ii) There exists a constant $M_2 > 0$, such that for any $g(y) \geq 0$ satisfying

$$0 < \int_0^\infty y^{q(1-\mu)-1}g^q(y)dy < \infty,$$

we have the following reverse Hardy-type inequality of the second kind with the homogeneous kernel:

$$\left\{\int_0^\infty x^{q\sigma-1}\left[\int_0^x \frac{(\min\{x,y\})^\alpha|\ln(x/y)|^\beta}{(\max\{x,y\})^{\lambda+\alpha}}g(y)dy\right]^q dx\right\}^{\frac{1}{q}}$$
$$> \quad M_2\left[\int_0^\infty y^{q(1-\mu)-1}g^q(y)dy\right]^{\frac{1}{p}}. \tag{41}$$

(iii) There exists a constant $M_2 > 0$, such that for any $f(x), g(y) \geq 0$ satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx < \infty, \ \ \textit{and} \ \ 0 < \int_0^\infty y^{q(1-\mu)-1}g^q(y)dy < \infty,$$

we have the following inequality:

$$\int_0^\infty g(y)\left[\int_y^\infty \frac{(\min\{x,y\})^\alpha|\ln(x/y)|^\beta}{(\max\{x,y\})^{\lambda+\alpha}}f(x)dx\right]dy$$
$$> \quad M_2\left[\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(1-\mu)-1}g^q(y)dy\right]^{\frac{1}{q}}. \tag{42}$$

(iv) $\mu + \sigma = \lambda$.
If Condition (iv) is satisfied, then the constant $M_2 = k_\lambda^{(1)}(\mu)$ in (40)–(42) is the best possible.

In particular, for $\lambda = 1, \sigma = \frac{1}{q}, \mu = \frac{1}{p}$ in Theorem 4, we derive the corollary below.

Corollary 6. *If $\alpha > -\frac{1}{p}$, then the following conditions are equivalent:*
(i) There exists a constant $M_2 > 0$, such that for any $f(x) \geq 0$ satisfying

$$0 < \int_0^\infty f^p(x)dx < \infty,$$

we have the following inequality:

$$\left\{\int_0^\infty\left[\int_y^\infty \frac{(\min\{x,y\})^\alpha|\ln(x/y)|^\beta}{(\max\{x,y\})^{1+\alpha}}f(x)dx\right]^p dy\right\}^{\frac{1}{p}} > M_2\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}. \tag{43}$$

(ii) There exists a constant $M_2 > 0$, such that for any $g(y) \geq 0$ satisfying

$$0 < \int_0^\infty g^q(y)dy < \infty,$$

we have the following inequality:

$$\left\{\int_0^\infty \left[\int_0^x \frac{(\min\{x,y\})^\alpha |\ln(x/y)|^\beta}{(\max\{x,y\})^{1+\alpha}} g(y)dy\right]^q dx\right\}^{\frac{1}{q}} > M_2\left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{p}}. \tag{44}$$

(iii) There exists a constant $M_2 > 0$, such that for any $f(x), g(y) \geq 0$ satisfying

$$0 < \int_0^\infty f^p(x)dx < \infty, \text{ and } 0 < \int_0^\infty g^q(y)dy < \infty,$$

we have the following inequality:

$$\begin{aligned} &\int_0^\infty g(y)\left[\int_y^\infty \frac{(\min\{x,y\})^\alpha |\ln(x/y)|^\beta}{(\max\{x,y\})^{1+\alpha}} f(x)dx\right]dy \\ > \; & M_2\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}. \end{aligned} \tag{45}$$

The constant $M_2 = k_1^{(1)}(\frac{1}{p})$ in (43)–(45) is the best possible.

5. Conclusions

Hardy-type integral inequalities play a prominent role in the study of analytic inequalities, which are essential in mathematical analysis and its various applications, such as in the study of symmetry and asymmetry phenomena. In the present work, in Theorem 1 and Theorem 3, employing methods of real analysis as well as using weight functions, we obtain a few equivalent conditions of (5) (resp. (6)) with a particular non-homogeneous kernel. Some equivalent conditions of two kinds of reverse Hardy-type integral inequalities with a particular homogeneous kernel are deduced in the form of applications in Theorem 2 and Theorem 4. We also consider some interesting corollaries. In further studies, some Hardy-type integral inequalities involving the Riemann zeta function are obtained. The lemmas and theorems proved within this work provide an extensive account of this type of inequalities.

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