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Relativistic Corrections to the Higgs Boson Decay into a Pair of Vector Quarkonia †

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Abstract: We studied different mechanisms in the Higgs boson decay into a pair of J/Ψ and Υ particles within the perturbative Standard Model and relativistic quark model. The relativistic corrections to the decay amplitude and decay width were calculated. We obtained numerical values of the decay widths of the Higgs boson, which can be used for comparison with experimental data.

Keywords: Higgs boson decay; pair quarkonium production; relativistic quark model

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Citation: Martynenko, A.P.; Martynenko, F.A. Relativistic Corrections to the Higgs Boson Decay into a Pair of Vector Quarkonia. *Symmetry* **2023**, *15*, 448. https://doi.org/10.3390/sym15020448

Academic Editor: Maxim Y. Khlopov

Received: 29 December 2022
Revised: 15 January 2023
Accepted: 1 February 2023
Published: 8 February 2023

1. Introduction

After the discovery of the Higgs boson [1,2], the study of the Higgs sector has become one of the most-important areas in particle physics. For the investigation of the nature of the Higgs boson, it is necessary to create particle colliders, by which Higgs bosons can be produced in significant quantities. It is possible that the reactions with the Higgs boson can provide a transition to a New Physics that lies beyond the Standard Model.

Due to the Higgs boson's large mass and the presence of coupling constants with different particles, the Higgs boson has numerous decay channels. The decay channel of the Higgs boson into a pair of heavy quarks is interesting because it creates the possibility of the production of bound states of heavy quarks. Thus, the rare exclusive decay processes of the Higgs boson decay into a pair of charmoniums or bottomoniums are of obvious interest both for studying the decay mechanisms and for studying the properties of bound states of quarks.

The CMS collaboration began the search for rare Higgs decays into a pair of heavy vector quarkonia in 2019 [3]. The results of new upper limits on the branching fractions were obtained in [4].

The theoretical calculation of the pair production of heavy quarkonia in the decays of the Higgs boson began about 40 years ago in [5,6]. In these papers, basic theoretical expressions were obtained for estimating the decay rates in the nonrelativistic approximation for some decay mechanisms. Recent theoretical studies of these processes were carried out in [7–11].

In this work, we continue the study of relativistic effects in the exclusive paired charmonium and bottomonium production in the Higgs boson decay, which was begun in [8,9] for B_c mesons. Our calculation of the decay widths was performed on the basis of the relativistic quark model used previously for the investigation of relativistic corrections in different reactions in [12,13]. Despite the rare nature of the Higgs boson decays being studied, one can hope that such processes can be investigated at the new Higgs boson factories. The aim of the present work was to give a new calculation of quarkonia pair production in the Higgs boson decays in the Standard Model. We considered different production mechanisms (quark–gluon, quark–photon, photon–photon, and Z-boson mechanisms) of the charmonium and bottomonium pair production and accounted for relativistic corrections connected with the relative motion of heavy quarks both in the production amplitude itself and in the wave functions of heavy quarkonia. As a result, new numerical estimations for the Higgs boson decay rates were obtained.

2. General Formalism

From the very beginning, it should be emphasized that there are several groups of amplitudes for the decay of the Higgs boson with the formation of a pair of charmoniums J/Ψ (bottomoniums Υ), which contribute the same order of magnitude to the decay width. We studied the amplitudes that are shown in Figures 1 and 2 (see also [11]). They represent different decay mechanisms with pair vector meson production: quark–gluon, quark–photon, quark loop, W-boson loop and Z-boson. The first group includes the quark–gluon amplitudes shown in Figure 1 (quark–gluon mechanism). The factor determining the order of contribution can be represented as α_s/M_H^4, where M_H is the mass of the Higgs boson. Such a factor can be distinguished from the very beginning due to the structure of the interaction vertices and the denominators of the particle propagators. The second group is formed by the quark–photon amplitudes shown in Figure 2 (quark–photon mechanism). The order of contribution is determined here by the factor $\alpha/M_H^2 M_{Q\bar{Q}}^2$. The third group is formed by the amplitudes that contain the quark or W-boson loop with two photons, which create a pair of J/Ψ (or Υ) mesons (quark loop and W-boson loop mechanisms) [11]. The primary common factor in this case takes the form $\alpha^2/M_{Q\bar{Q}}^4$. Finally, the last group of amplitudes is determined by the interaction of the Higgs boson with a pair of Z-bosons (Z-boson mechanism) [11]. To achieve good calculation accuracy, it is necessary to take into account the contribution of all amplitudes from these groups.

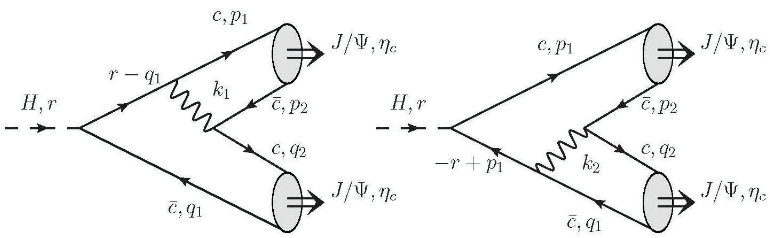

Figure 1. Quark–gluon mechanism of the charmonium pair production. The Higgs boson is presented by a dashed line. The gluon is indicated by a wavy line.

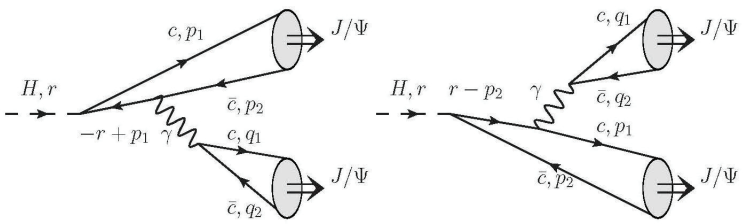

Figure 2. Quark–photon mechanism of the charmonium pair production. The Higgs boson is presented by a dashed line. The gluon is indicated by a wavy line.

Let us consider firstly the Higgs boson decay amplitudes shown in Figures 1 and 2. Two other diagrams can be obtained by the corresponding permutations. For the production of quarkonium pairs in the leading order of perturbation theory, it is necessary to obtain at the first stage two free quarks and two free antiquarks. Then, they can form bound states with some probability at the next stage. In the quasipotential approach, the decay amplitude can be presented as a convolution of a perturbative production amplitude of two c-quarks and two $\bar{c}$-antiquarks and the quasipotential wave functions of the vector mesons [8,9]:

$$\mathcal{M}(P,Q) = -i(\sqrt{2}G_F)^{\frac{1}{2}} \frac{2\pi}{3} M_{Q\bar{Q}} \int \frac{d\mathbf{p}}{(2\pi)^3} \int \frac{d\mathbf{q}}{(2\pi)^3} \times \quad (1)$$

$$Tr\Big\{\Psi^{\mathcal{V}}(p,P)\Gamma_1^{\nu}(p,q,P,Q)\Psi^{\mathcal{V}}(q,Q)\gamma_\nu+\Psi^{\mathcal{V}}(q,Q)\Gamma_2^{\nu}(p,q,P,Q)\Psi^{\mathcal{V}}(p,P)\gamma_\nu\Big\},$$

where $M_{Q\bar{Q}}$ is quarkonium mass. Four-momenta p_1 and p_2 of c-quarks and $\bar{c}$-antiquarks in the pair forming the first $(\mathcal{Q}\bar{\mathcal{Q}})$ meson and four-momenta q_1 and q_2 for quarks and antiquarks in the second meson are expressed in terms of relative and total four-momenta as follows:

$$p_{1,2}=\frac{1}{2}P\pm p,\quad (pP)=0;\qquad q_{1,2}=\frac{1}{2}Q\pm q,\quad (qQ)=0. \tag{2}$$

A superscript $\mathcal{V}$ indicates a vector meson $(\mathcal{Q}\bar{\mathcal{Q}})$. The vertex functions $\Gamma_{1,2}$ are presented below in leading order and in Figure 1. Heavy quarks c, b and antiquarks $\bar{c}$, $\bar{b}$ do not lie on the mass shell: $p_{1,2}^2\neq m^2$, so that $p_1^2-m^2=p_2^2-m^2$, which means that there is a symmetrical exit of particles from the mass shell.

Equation (1) contains the integration over the relative three-momenta of quarks and antiquarks. The accounting of relativistic corrections that are determined by the relative quark momenta p and q in the decay amplitude is important for making reliable predictions of observables. Relative four-momenta $p=L_P(0,\mathbf{p})$ and $q=L_Q(0,\mathbf{q})$ are obtained by the Lorentz transformation of four-vectors $(0,\mathbf{p})$ and $(0,\mathbf{q})$ to the reference frames moving with four-momenta P and Q.

The amplitude (1) contains the relativistic wave functions of vector mesons, which are obtained taking into account the transformation law from the rest frame to the moving one with four-momenta P, and Q:

$$\Psi(p,P)=\frac{\Psi_0(\mathbf{p})}{\left[\frac{\epsilon(p)}{m}\frac{\epsilon(p)+m}{2m}\right]}\left[\frac{\hat{v}_1-1}{2}+\hat{v}_1\frac{\mathbf{p}^2}{2m(\epsilon(p)+m)}-\frac{\hat{p}}{2m}\right]\times\hat{\varepsilon}(P,S_z) \tag{3}$$

$$(1+\hat{v}_1)\left[\frac{\hat{v}_1+1}{2}+\hat{v}_1\frac{\mathbf{p}^2}{2m(\epsilon(p)+m)}+\frac{\hat{p}}{2m}\right],$$

$$\Psi(q,Q)=\frac{\Psi_0(\mathbf{q})}{\left[\frac{\epsilon(q)}{m}\frac{\epsilon(q)+m}{2m}\right]}\left[\frac{\hat{v}_2-1}{2}+\hat{v}_2\frac{\mathbf{q}^2}{2m(\epsilon(q)+m)}+\frac{\hat{q}}{2m}\right]\times\hat{\varepsilon}(Q,S_z) \tag{4}$$

$$(1+\hat{v}_2)\left[\frac{\hat{v}_2+1}{2}+\hat{v}_2\frac{\mathbf{q}^2}{2m(\epsilon(q)+m)}-\frac{\hat{q}}{2m}\right],$$

where $v_1=P/M_{Q\bar{Q}}$, $v_2=Q/M_{Q\bar{Q}}$; $\epsilon(p)=\sqrt{m^2+\mathbf{p}^2}$, m is the $c(b)$-quark mass. $\varepsilon^\lambda(P,S_z)$ is the polarization vector of the $J/\Psi(\Upsilon)$ meson.

Expressions (3) and (4) represent complicated functions depending on relative momenta $\mathbf{p}$, $\mathbf{q}$, including the bound state wave function in the rest frame $\Psi_0(\mathbf{p})$. The color part of the meson wave function in the amplitudes (3) and (4) is taken as $\delta_{ij}/\sqrt{3}$ (color indexes $i,j,k=1,2,3$).

The general structure of Expressions (3) and (4) allows us to say that they are the product of the wave functions of mesons in the rest frame and special projection operators resulting from the transformation from the moving reference frame to the reference frame in which the meson is at rest. Expressions (3) and (4) make it possible to correctly take into account the relativistic corrections connected with the relative momenta of quarks in the final states. It is useful to note that the expression for the projection operators was obtained in the framework of nonrelativistic quantum chromodynamics in [14] in a slightly different form for the case when the quark momenta lie on the mass shell. The transformation law of the bound state wave functions of quarks, which is used in the derivation of Equations (3) and (4), was obtained in the Bethe–Salpeter approach in [15] and in quasipotential method in [16].

The total Higgs boson decay amplitude for the case of the quark–gluon mechanism in the leading order over α_s can be presented in the form:

$$\mathcal{M} = \frac{4\pi}{3} M_{Q\bar{Q}} \alpha_s \Gamma_Q \int \frac{d\mathbf{p}}{(2\pi)^3} \int \frac{d\mathbf{q}}{(2\pi)^3} \mathrm{Tr}\{\mathcal{T}_{12} + \mathcal{T}_{34}\}, \tag{5}$$

$$\mathcal{T}_{12} = \Psi^{\mathcal{V}}(p,P)\left[\frac{\hat{p}_1 - \hat{r} + m}{(r-p_1)^2 - m^2}\gamma_\mu + \gamma_\mu \frac{\hat{r} - \hat{q}_1 + m}{(r-q_1)^2 - m^2}\right] D^{\mu\nu}(k_2)\Psi^{\mathcal{V}}(q,Q)\gamma_\nu, \tag{6}$$

$$\mathcal{T}_{34} = \Psi^{\mathcal{V}}(q,Q)\left[\frac{\hat{p}_2 - \hat{r} + m}{(r-p_2)^2 - m^2}\gamma_\mu + \gamma_\mu \frac{\hat{r} - \hat{q}_2 + m}{(r-q_2)^2 - m^2}\right] D^{\mu\nu}(k_1)\Psi^{\mathcal{V}}(p,P)\gamma_\nu, \tag{7}$$

where $\alpha_s = \alpha_s\left(\frac{M_H^2}{4\Lambda^2}\right)$, $\Gamma_Q = m(\sqrt{2}G_F)^{\frac{1}{2}}$.

The four-momentum of the squared Higgs boson $r^2 = M_H^2 = (P+Q)^2 = 2M_{Q\bar{Q}}^2 + 2PQ$. The gluon four-momenta are $k_1 = p_1 + q_1$, $k_2 = p_2 + q_2$. Relative momenta p, q of heavy quarks enter in the gluon propagators $D_{\mu\nu}(k_{1,2})$ and quark propagators, as well as in relativistic wave functions (3) and (4). To simplify the denominators of quark and gluon propagators, one can use the smallness of the values of the characteristic relative momenta of quarks p and q compared to the mass of the Higgs boson M_H:

$$\frac{1}{(p_1+q_1)^2} \approx \frac{1}{(p_2+q_2)^2} = \frac{4}{M_H^2}, \quad \frac{1}{(r-q_1)^2 - m_1^2} = \frac{2}{M_H^2}. \tag{8}$$

In (8), we completely neglected the corrections of the form $|\mathbf{p}|/M_H$, $|\mathbf{q}|/M_H$. At the same time, we kept in the amplitudes (6), (7) the second-order relativistic corrections $|\mathbf{p}|/m$, $|\mathbf{q}|/m$ relative to the leading order result. Calculating the trace in the obtained expression, we found relativistic amplitudes of the paired vector meson production in the form:

$$\mathcal{M}_{\mathcal{V}\mathcal{V}}{}^{(1)} = \frac{256\pi}{3M_H^4}(\sqrt{2}G_F)^{\frac{1}{2}} m M_{Q\bar{Q}} \alpha_s \varepsilon_1^\lambda \varepsilon_2^\sigma F_{1,\mathcal{V}\mathcal{V}}^{\lambda\sigma} |\tilde{\Psi}_{\mathcal{V}}(0)|^2, \tag{9}$$

where ε_1^λ, ε_2^σ are the polarization vectors of spin 1 mesons. The superscript in the amplitude and the subscript in the tensor function $F_{\mathcal{V}\mathcal{V}}$ denote the contribution of the quark–gluon mechanism.

The contribution of the amplitudes in Figure 2 must also be taken into account, because these amplitudes have the same order as the previous ones, despite the replacement $\alpha_s \to \alpha$. This is due to the presence in the denominator of the mass of the meson instead of the mass of the Higgs boson [17]. The expression for the production amplitude of the pair J/Ψ has a similar structure with slight changes in the common factors:

$$\mathcal{M}_{\mathcal{V}\mathcal{V}}{}^{(2)} = \frac{288\pi}{M_H^2 M_{Q\bar{Q}}}(\sqrt{2}G_F)^{\frac{1}{2}} m e_Q^2 \alpha \varepsilon_1^\lambda \varepsilon_2^\sigma F_{2,\mathcal{V}\mathcal{V}}^{\lambda\sigma} |\tilde{\Psi}_{\mathcal{V}}(0)|^2. \tag{10}$$

The tensor corresponding to the quark or W-boson loops in decay amplitudes [11] has the structure (the subscript denotes the contribution of the quark or bosonic loop):

$$T_{Q,W}^{\mu\nu} = A_{Q,W}(t)(g^{\mu\nu}(v_1v_2) - v_1^\nu v_2^\mu) + B_{Q,W}(t)[v_2^\mu - v_1^\mu(v_1v_2)][v_1^\nu - v_2^\nu(v_1v_2)], \tag{11}$$

$t = \frac{M_h^2}{4m_Q^2}$ or $t = \frac{M_h^2}{4m_W^2}$. The structure functions $A_{Q,W}(t)$, $B_{Q,W}(t)$ can be obtained using an explicit expression for loop integrals (see Appendices A and B in [11]).

The amplitudes with quarks, W-boson loops, and ZZ in an intermediate state contain the contributions of direct and crossed diagrams. The direct diagrams, in which virtual photons or Z-bosons give vector quarkonia in the final state, are dominant in terms of the mass factor. However, the structure of the numerators of the direct and cross-amplitudes is different, which can eventually lead to numerically close contributions. We present the expressions for these amplitudes only in the leading order:

$$\mathcal{M}_{\mathcal{V}\mathcal{V}}{}^{(3)} = \frac{2052\pi^2}{m_Q M_{\mathcal{Q}\bar{\mathcal{Q}}}}(\sqrt{2}G_F)^{\frac{1}{2}} e_q^2 e_Q^2 \alpha^2 \varepsilon_1^\lambda \varepsilon_2^\sigma F_{3,\mathcal{V}\mathcal{V}}^{\lambda\sigma} |\tilde{\Psi}_{\mathcal{V}}(0)|^2, \tag{12}$$

$$\mathcal{M}_{\mathcal{V}\mathcal{V}}{}^{(4)} = \frac{48\pi^2 M_Z M_W}{M_{\mathcal{Q}\bar{\mathcal{Q}}}^4}(\sqrt{2}G_F)^{\frac{1}{2}} e_q^2 \alpha^2 \cos\theta_W \varepsilon_1^\lambda \varepsilon_2^\sigma F_{4,\mathcal{V}\mathcal{V}}^{\lambda\sigma} |\tilde{\Psi}_{\mathcal{V}}(0)|^2, \tag{13}$$

$$\mathcal{M}_{\mathcal{V}\mathcal{V}}{}^{(5)} = \frac{48\pi\alpha}{M_Z^2 \sin^2 2\theta_W}(\sqrt{2}G_F)^{\frac{1}{2}} \varepsilon_1^\lambda \varepsilon_2^\sigma F_{5,\mathcal{V}\mathcal{V}}^{\lambda\sigma} |\tilde{\Psi}_{\mathcal{V}}(0)|^2, \tag{14}$$

where m is the mass of the heavy quarks c, b, produced in the vertex of Higgs boson decay, m_Q is the heavy quark mass in the quark loop, e_q is the charge (in units e) of heavy quarks (c or b) entering in the final mesons, and e_Q is the charge (in units e) of quarks (c or b) in the quark loop.

The tensor functions in the amplitudes (9)–(14) have the following form:

$$F_{i,\mathcal{V}\mathcal{V}}^{\alpha\beta} = g_1^{(i)} v_1^\alpha v_2^\beta + g_2^{(i)} g^{\alpha\beta}, \quad g_1^{(1)} = -2 + \frac{2}{9}\omega_1^2, \tag{15}$$

$$g_2^{(1)} = -1 - 2r_2 + r_1^2 + \frac{4}{3}r_2\omega_1 + \frac{1}{9}\omega_1^2 + \frac{2}{3}r_2\omega_1^2 - \frac{1}{9}r_1^2\omega_1^2,$$

$$g_1^{(2)} = 4 - \frac{4}{9}\omega_1^2,\ g_2^{(2)} = 2 + 4r_2 - 2r_1^2 - \frac{8}{3}r_2\omega_1 - \frac{2}{9}\omega_1^2 - \frac{4}{3}r\omega_1^2 + \frac{2}{9}r_1^2\omega_1^2, \tag{16}$$

$$g_{1,Q}^{(3)} = -A_Q(t)(1 + \frac{2}{3}\omega_1 + \frac{1}{9}\omega_1^2) + B_Q(t)(1 + \frac{2}{3}\omega_1 + \frac{1}{9}\omega_1^2), \tag{17}$$

$$g_{2,Q}^{(3)} = A_Q(t)\left(-1 - \frac{2}{3}\omega_1 - \frac{1}{9}\omega_1^2 + \frac{1}{2}r_1^2 + \frac{1}{3}\omega_1 r_1^2 + \frac{1}{18}\omega_1^2 r_1^2\right),$$

$$g_1^{(4)} = -A_W(t)\left(1 + \frac{2}{3}\omega_1 + \frac{1}{9}\omega_1^2\right) + B_W(t)\left(1 + \frac{2}{3}\omega_1 + \frac{1}{9}\omega_1^2\right), \tag{18}$$

$$g_2^{(4)} = A_W(t)\left(-1 - \frac{2}{3}\omega_1 - \frac{1}{9}\omega_1^2 + \frac{1}{2}r_1^2 + \frac{1}{3}\omega_1 r_1^2 + \frac{1}{18}\omega_1^2 r_1^2\right), \tag{19}$$

$$g_2^{(5)} = \left(1 + \frac{1}{3}\omega_1\right)^2 \left(\frac{1}{2} - a_z\right)^2 - \frac{M_z^4}{3\left(\frac{M_h^2}{4} - M_Z^2\right)^2}\Big(-\frac{1}{4} - \frac{1}{6}\omega_1 - \frac{1}{36}\omega_1^2 + \tag{20}$$

$$\frac{1}{2}a_z + \frac{1}{3}\omega_1 a_z + \frac{1}{18}\omega_1^2 a_z - \frac{1}{2}a_z^2 - \frac{1}{3}\omega_1 a_z^2 - \frac{1}{18}\omega_1^2 a_z^2\Big), \quad g_1^{(5)} = 0,$$

where $r_1 = \frac{M_H}{M_{\mathcal{Q}\bar{\mathcal{Q}}}}$, $r_2 = \frac{m}{M_{\mathcal{Q}\bar{\mathcal{Q}}}}$, $a_z = 2|e_Q|\sin^2\theta_W$.

The decay rates of the Higgs boson into a pair of vector quarkonia states are determined by the following analytical expressions (see also [8,9]):

$$\Gamma_{\mathcal{V}\mathcal{V}} = \frac{2^{14}\sqrt{2}\pi\alpha_s^2 m^2 G_F |\tilde{\Psi}_{\mathcal{V}}(0)|^4 \sqrt{\frac{r_1^2}{4} - 1}}{9M_H^5 r_1^5} \sum_{\lambda,\sigma} |\varepsilon_1^\lambda \varepsilon_2^\sigma F_{\mathcal{V}\mathcal{V}}^{\lambda\sigma}|^2, \tag{21}$$

$$F_{\mathcal{V}\mathcal{V}}^{\lambda\sigma} = \Big[g_1^{(1)} + \frac{9}{16}r_1^2\frac{e_q^2\alpha}{\alpha_s}g_1^{(2)} + \sum_Q \frac{27\pi}{8}r_1^4\frac{e_Q^2 e_q^2\alpha^2 m_Q^2}{\alpha_s m M_{\mathcal{Q}\bar{\mathcal{Q}}}}g_{1,Q}^{(3)} + \frac{9\pi e_q^2\alpha^2 r_1^4 M_Z M_W}{64\alpha_s m M_{\mathcal{Q}\bar{\mathcal{Q}}}}g_1^{(4)} + \tag{22}$$

$$\frac{9M_H^4\alpha}{16M_Z^2 mM_{Q\bar{Q}}\alpha_s}\frac{\left(\frac{1}{2}-2|e_q|\sin^2\theta_W\right)^2}{\sin^2 2\theta_W}g_1^{(5)}\Big]v_1^\sigma v_2^\lambda +$$

$$\Big[g_2^{(1)}+\frac{9}{16}r_1^2\frac{e_q^2\alpha}{\alpha_s}g_2^{(2)}+\sum_Q\frac{27\pi}{8}r_1^4\frac{e_q^2e_Q^2\alpha^2m_Q^2}{\alpha_s mM_{Q\bar{Q}}}g_{2,Q}^{(3)}+$$

$$\frac{9\pi e_q^2\alpha^2r_1^4M_ZM_W}{64\alpha_s mM_{Q\bar{Q}}}\cos\theta_W g_2^{(4)}+\frac{9M_H^4\alpha}{16M_Z^2mM_{Q\bar{Q}}\alpha_s}\frac{1}{\sin^2 2\theta_W}g_2^{(5)}\Big]g^{\lambda\sigma}.$$

We found it convenient to separate in square brackets in (22) the coefficients denoting the relative contribution of different decay mechanisms with respect to the quark–gluon mechanism.

The general expression for the decay rate (21) contains numerous parameters. There are some parameters, such as the masses of quarks and mesons, which were fixed and obtained in calculations within the framework of quark models. For this purpose, the requirement of the best correspondence between the performed calculations and the observed characteristics of mesons was used. Another part of the relativistic parameters that determine the magnitude of the relativistic corrections at the production of quark bound states can be obtained directly within the framework of the relativistic quark model. Such corrections are obtained by calculating the momentum integrals with the wave functions of the bound states of quarks.

The amplitudes of the pair production of vector charmonium and bottomonium in the Higgs boson decay are determined by a number of functions F_{VV}, which are presented in the form of an expansion in $|\mathbf{p}|/m$, $|\mathbf{q}|/m$ up to terms of the second order. After a series of algebraic transformations, it turns out to be convenient to express the relativistic corrections in terms of special relativistic parameters ω_n. In the case of S-states, ω_n are determined in terms of momentum integrals I_n, which can be calculated analytically as in [11,18] (see Equations (27) and (28) [11]).

There is another source of relativistic corrections, which is determined by the Hamiltonian of the interaction of heavy quarks forming a bound state. This leads to important changes in the wave functions of vector mesons. The exact form of the bound state wave functions $\Psi_0(\mathbf{q})$ is important to obtain more reliable predictions for the decay widths. In the nonrelativistic approximation, the Higgs boson decay width into a pair of quarkonia contains the fourth power of the nonrelativistic wave function at the origin for S-states. The value of the decay rate is very sensitive to small changes of $\tilde{R}(0)$. To take into account relativistic corrections in the vector meson wave functions, we describe the dynamics of heavy quarks by the QCD generalization of the standard Breit Hamiltonian in the center-of-mass reference frame as in our previous papers [9,11,19].

Using the effective Hamiltonian of quarkonia constructed on the basis of perturbative quantum chromodynamics with the allowance for nonperturbative effects, a model of heavy quarkonia for S-states is obtained. Its main elements were presented in our previous works [9,12,13,18]. It was used for the calculation of the relativistic parameters that enter in the decay width of the Higgs boson (21) [11].

The results of the numerical calculations of the decay widths are presented in Tables 1 and 2. Comparing the contributions of different meson pair production mechanisms, it is important to emphasize that their relative magnitude in the total result is determined by such important parameters as α and α_s and the particle mass ratio. Therefore, for example, in the case of the production of a pair of vector mesons, the decay amplitude has two contributions from the quark–gluon (Figure 1) and quark–photon (Figure 2) diagrams. On the one hand, the contribution of photon amplitudes is proportional to α, which leads to its decrease in comparison with the contribution from quark–gluon amplitudes. However, on the other hand, the quark–photon contribution contains an additional factor $e_Q^2r_1^2$, which leads to an increase in the decay widths.

Table 1. The Higgs boson decay widths in the nonrelativistic approximation and with the account for relativistic corrections.

Final State $(Q\bar{Q})(Q\bar{Q})$	**Nonrelativistic Decay Width** Γ_{nr} **in GeV**	**Relativistic Decay Width** Γ_{rel} **in GeV**
$J/\Psi + J/\Psi$	3.29×10^{-12}	0.69×10^{-12}
$\Upsilon + \Upsilon$	0.63×10^{-12}	0.74×10^{-12}

Table 2. The contributions of different mechanisms to the Higgs boson decay widths in GeV.

The Contribution Accounting for Relativistic Corrections		
Mechanism	$H \to J/\psi\, J/\psi$	$H \to \Upsilon\,\Upsilon$
Quark-gluon	0.36×10^{-15}	0.10×10^{-12}
Quark-photon	0.80×10^{-12}	0.16×10^{-12}
Quark loop	0.70×10^{-13}	0.37×10^{-12}
W-boson loop	0.74×10^{-13}	0.68×10^{-13}
ZZ	0.22×10^{-12}	1.45×10^{-12}
Total contribution	0.69×10^{-12}	0.74×10^{-12}

3. Numerical Results and Conclusions

The study of rare exclusive decay processes of the Higgs boson is an important task, which makes it possible to refine the values of the interaction parameters of particles in the Higgs sector. In this paper, we attempted to present a complete study of the decay processes $H \to J/\Psi J/\Psi$, $H \to \Upsilon\Upsilon$ in the Standard Model by considering the various decay mechanisms. To improve the calculation's accuracy, we took into account relativistic corrections, which were not previously considered in [7,10]. The numerical contribution of different decay mechanisms was analyzed, and it was shown that, in the case of the double production of charmoniums, the main decay mechanism is the quark–photon one in Figure 2, the ZZ-mechanism. However, it is also necessary to take into account other mechanisms to achieve high calculation accuracy (see also [17]). For bottomonium pair production all mechanisms give close contributions to the width, but the ZZ-mechanism is dominant. Their sum ultimately determines the full numerical result.

We distinguished two types of relativistic corrections. The corrections of the first type were determined by the relative momenta p and q in the production amplitude of two quarks and two antiquarks. The corrections of the second type were determined by the transformation law of the meson wave functions, which resulted in Expressions (3) and (4). The $\psi(\mathbf{p})$ wave functions entering into (3) and (4) in the rest frame of the bound state were found from the solution of the Schrödinger equation with a potential that includes relativistic corrections. Having an exact expression for the decay amplitude, we carried out a series of transformations with it, extracting second-order corrections in p and q. The model used is described in more detail in [9]. In our approach, all arising relativistic parameters ω_1, ω_2, $\tilde{R}(0)$ were determined on the basis of the relativistic quark model. Accounting for relativistic corrections in this work shows that such contributions lead to a significant change in the nonrelativistic results. The main parameter that greatly reduces the nonrelativistic results is $\tilde{R}(0)$, which enters in the decay width to the fourth power.

The decay branchings are equal: $Br(H \to J/\Psi J/\Psi) = 2.1 \times 10^{-10}$, $Br(H \to \Upsilon\Upsilon) = 2.3 \times 10^{-10}$. There are other amplitudes for the production of a pair of J/Ψ, Υ mesons, which were calculated, but not included in detail in the work. Therefore, for example, there is a pair production mechanism (HH mechanism), when the original Higgs boson turns into an HH pair, which then gives a pair of vector mesons in the final state. It is similar to the ZZ mechanism, but the amplitude structure is different, which results in a contribution to the decay width that is several orders of magnitude smaller than those given in Table 2. Despite the obvious difference in the amplitudes in Figures 1 and 2 in terms of the mass factor M^2/M_h^2, we included the amplitudes of Figure 1 in the consideration, in contrast to the work [10]. A distinctive feature of our calculations is that, when studying the contributions of the amplitudes with quark and boson loops, we took into account two structure functions A and B (11) in the loop tensor function [11].

In Table 2, we present separately the numerical values of the contributions from different decay mechanisms with an accuracy of two significant figures after the decimal point. First of all, the problem was to obtain the main contribution to the decay width. This was the contribution from the quark–photon mechanism in Figure 2 and the ZZ mechanism, which are one order of magnitude greater than the other contributions in the case of charmonium production. Its value is due to the structure of the decay amplitudes, in which small denominators $1/M^2$ appear from the photon propagators, in contrast to other amplitudes in which there is a factor $1/M_H^2$. In addition, the numerator in these amplitudes contains amplifying factors in powers of a large parameter r_1. In each considered decay mechanism, there are radiative corrections $O(\alpha_s)$ that were not taken into account. They represent a major source of theoretical uncertainty, which we estimated to be about 30% in α_s. Other available errors connected with the parameters of the Standard Model and higher-order relativistic corrections do not exceed 10%. On the whole, our complete numerical estimates of the decay widths agree in order of magnitude with the results [10].

The numerical values of the decay widths of Higgs boson are small in comparison with the total width of the Higgs boson $\Gamma = 3.2^{+2.8}_{-2.2} \times 10^{-3}$ GeV. Therefore, to observe rare decay processes with the pair production of charmonium (or bottomonium), it is necessary to increase the luminosity of the LHC. Rare exclusive decays of the Higgs boson could be investigated at other Higgs factories besides the LHC, which are still in the project stage. Various parameters of future colliders are currently being discussed. For example, at the pp-collider at the production of 4×10^{10} Higgs bosons per year, we can expect about 10 Higgs boson decay events into a pair of vector charmoniums.

Author Contributions: Conceptualization, A.P.M. and F.A.M.; methodology, A.P.M. and F.A.M.; software, A.P.M. and F.A.M.; validation, A.P.M.; formal analysis, A.P.M. and F.A.M.; investigation, A.P.M. and F.A.M.; data curation, A.P.M.; writing—original draft preparation, A.P.M. and F.A.M.; writing—review and editing, A.P.M. and F.A.M.; visualization, F.A.M.; supervision, A.P.M.; project administration, A.P.M.; funding acquisition, A.P.M. and F.A.M. The authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: The work was supported by the Foundation for the Advancement of Theoretical Physics and Mathematics "BASIS" (Grant No. 22-1-1-23-1).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The work was supported by the Foundation for the Advancement of Theoretical Physics and Mathematics "BASIS" (Grant No. 22-1-1-23-1).

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Aad, G.; Abajyan, T.; Abbott, B.; Abdallah, J.; Khalek, S.A.; Abdelalim, A.A.; Aben, R.; Abi, B.; Abolins, M.; AbouZeid, O.S.; et al. Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC. *Phys. Lett. B* **2012**, *716*, 1–29.
2. Chatrchyan, S.; Khachatryan, V.; Sirunyan, A.M.; Tumasyan, A.; Adam, W.; Aguilo, E.; Bergauer, T.; Dragicevic, M.; Erö, J.; Fabjan, C.; et al. Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC. *Phys. Lett. B* **2012**, *716*, 30–61. [CrossRef]

3. Sirunyan, A.M.; Tumasyan, A.; Adam, W.; Ambrogi, F.; Bergauer, T.; Brandstetter, J.; Dragicevic, M.; Erö, J.; Escalante Del Valle, A.; Flechl, M.; et al. Search for Higgs and Z boson decays to J/Ψ or Y pairs in the four-muon final state in proton-proton collisions at $\sqrt{s} = 13$ TeV. *Phys. Lett. B* **2019**, *797*, 134811. [CrossRef]
4. CMS Collaboration. Search for Higgs boson decays into Z and J/Ψ and for Higgs and Z boson decays into J/Ψ or Y pairs in pp collisions at $\sqrt{s} = 13$ TeV. *arXiv* **2022**, arXiv:2206.03525.
5. Bander, M.; Soni, A. Decays of Higgs Scalars Into Vector Mesons and Photons. *Phys. Lett. B* **1979**, *82*, 411. [CrossRef]
6. Keung, W.Y. The decay of the Higgs boson into heavy quarkonium states. *Phys. Rev. D* **1983**, *27*, 2762. [CrossRef]
7. Kartvelishvili, V.; Luchinsky, A.V.; Novoselov, A.A. Double vector quarkonia production in exclusive Higgs boson decays. *Phys. Rev. D* **2009**, *79*, 114015. [CrossRef]
8. Belov, I.N.; Berezhnoy, A.V.; Dorokhov, A.E.; Likhoded, A.K.; Martynenko, A.P.; Martynenko, F.A. Higgs boson decay to paired Bc: Relativistic and one-loop corrections. *Nucl. Phys. A* **2021**, *1015*, 122285.
9. Faustov, R.N.; Martynenko, F.A.; Martynenko, A.P. Higgs boson decay to the pair of S- and P-wave B_c mesons. *Eur. Phys. J. A* **2022**, *58*, 4. [CrossRef]
10. Gao, D.N.; Gong, X. Higgs boson decays into a pair of heavy vector quarkonia. *Phys. Lett. B* **2022**, *832*, 137. [CrossRef]
11. Faustov, R.N.; Martynenko, F.A.; Martynenko, A.P. Relativistic corrections to paired production of charmonium and bottomonium in decays of the Higgs boson. *arXiv* **2022**, arXiv:2209.12321.
12. Dorokhov, A.E.; Faustov, R.N.; Martynenko, A.P.; Martynenko, F.A. Photonic production of a pair of B_c mesons. *Phys. Rev. D* **2020**, *102*, 016027. [CrossRef]
13. Elekina, E.N.; Martynenko, A.P. Relativistic effects in the double S- and P-wave charmonium production in e^+e^- annihilation. *Phys. Rev. D* **2010**, *81*, 054006. [CrossRef]
14. Bodwin, G.T.; Petrelli, A. Order-v^4 corrections to S-wave quarkonium decay. *Phys. Rev. D* **2002**, *66*, 094011. [CrossRef]
15. Brodsky, S.J.; Primack, J.R. The Electromagnetic Interactions of Composite Systems. *Ann. Phys.* **1969**, *52*, 315–365. [CrossRef]
16. Faustov, R.N. Relativistic wave function and form-factors of the bound system. *Ann. Phys.* **1973**, *78*, 176–189. [CrossRef]
17. Bodwin, G.T.; Chung, H.S.; Ee, J.H.; Lee, J.; Petriello, F. Relativistic corrections to Higgs boson decays to quarkonia. *Phys. Rev. D* **2014**, *90*, 113010. [CrossRef]
18. Karyasov, A.A.; Martynenko, A.P.; Martynenko, F.A. Relativistic corrections to the pair B_c-meson production in e^+e^- annihilation. *Nucl. Phys. B* **2016**, *911*, 36–51. [CrossRef]
19. Gupta, S.N.; Radford, S.F.; Repko, W.W. Quarkonium Spectra and Quantum Chromodynamics. *Phys. Rev. D* **1982**, *26*, 3305. [CrossRef]