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Quasi-Double Diagonally Dominant $\mathcal{H}$-Tensors and the Estimation Inequalities for the Spectral Radius of Nonnegative Tensors

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Abstract: In this paper, we study two classes of quasi-double diagonally dominant tensors and prove they are $\mathcal{H}$-tensors. Numerical examples show that two classes of $\mathcal{H}$-tensors are mutually exclusive. Thus, we extend the decision conditions of $\mathcal{H}$-tensors. Based on these two classes of tensors, two estimation inequalities for the upper and lower bounds for the spectral radius of nonnegative tensors are obtained.

Keywords: quasi-double diagonally dominant $\mathcal{H}$-tensor; $\mathcal{M}$-tensor; decision condition; nonnegative tensor; spectral radius; estimation inequality

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1. Introduction

Let $\mathbb{R}$ ($\mathbb{C}$) be the real (complex) field. Consider an m-th order n-dimensional tensor $\mathcal{A}$, which consists of n^m entries in $\mathbb{R}$:

$$\mathcal{A} = (a_{i_1 i_2 \cdots i_m}),\ a_{i_1 i_2 \cdots i_m} \in \mathbb{R},\ i_j = 1, 2, \cdots, n,\ j = 1, 2, \cdots, m.$$

Let $\mathbb{R}^n$ be the set of all n-dimensional real vectors, and let $\mathbb{R}^{[m,n]}$ ($\mathbb{C}^{[m,n]}$) be the set of all m-th order n-dimensional real (complex) tensors. A tensor $\mathcal{A}$ is called nonnegative if $a_{i_1 i_2 \cdots i_m} \geq 0$, and we denote this by $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$. $\mathbb{R}_+^n$ and $\mathbb{R}_{++}^n$ represent the sets of nonnegative and positive vectors in n-dimensional Euclidean space, respectively. We denote $\langle n \rangle = \{1, 2, \cdots, n\}$, $\mathrm{i} = \sqrt{-1}$.

In 2005, Lim [1] and Qi [2] defined the eigenvalues of a tensor, respectively.

Definition 1. *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$. If there are a complex number λ and a nonzero complex vector $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$, such that*

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

then λ is called an eigenvalue of $\mathcal{A}$, x is termed an eigenvector of $\mathcal{A}$ associated with λ, and $\mathcal{A}x^{m-1}$ and $x^{[m-1]}$ are vectors, whose i-th entries are

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2,\ldots,i_m=1}^{n} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}$$

and $(x^{[m-1]})_i = x_i^{m-1}$, respectively.

Specifically, (λ, x) is called an H-eigenpair if $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n$. The largest eigenvalue of tensor $\mathcal{A}$ is called the spectral radius, and we denote it by $\rho(\mathcal{A})$. We denote the set of eigenvalues of tensor $\mathcal{A}$ as $\sigma(\mathcal{A})$.

As a higher-dimensional generalization of matrices, tensors are used in many scientific fields, such as signal and image processing, continuum physics, data mining and processing, nonlinear optimization, elastic analysis in physics, and higher-order statistics [3–6]. The properties and criteria of $\mathcal{H}$-tensor ($\mathcal{M}$-tensor) were discussed in detail in [7–9], and the relevant results were given. There are many applications for the $\mathcal{H}$-tensor ($\mathcal{M}$-tensor); for example, the multilinear systems can be expressed as $\mathcal{A}x^{m-1} = b$, where $\mathcal{A}$ and $b \in \mathbb{R}^n$ are given, and x is to be solved. Examples of multilinear systems can be found in [10–13]. Consider the positive define of $g(x) = \sum_{i_1,i_2,\ldots,i_m=1}^{n} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}$; that is, when $\forall 0 \neq x \in \mathbb{R}^n$, $g(x) > 0$, the $\mathcal{M}$-tensor is also an important application [9]. The estimation of the upper and lower bounds for the spectral radius of a nonnegative tensor is an important element in the study of the spectral problem of nonnegative tensors [14,15], and the application of the relation between the $\mathcal{M}$-tensor and the nonnegative tensor gives an estimate of the upper and lower bounds for the spectral radius of the nonnegative tensor. By analyzing the tensor structure, two classes of quasi-double diagonally dominant tensors are given in this paper, and they are proved to be $\mathcal{H}$-tensors; at the same time, an inequality is given for the estimation of the upper and lower bounds for the spectral radius of the nonnegative tensor.

2. Preliminaries

In this section, we first recall some preliminary knowledge important to our work on nonnegative tensors.

Ref. [16] generalized the concept of irreducible matrices to irreducible tensors.

Definition 2 (**[16]**). *An m-th order n-dimensional tensor $\mathcal{A}$ is called reducible if there exists a nonempty proper index subset $J \subset \langle n \rangle$, such that*

$$a_{i_1 i_2 \cdots i_m} = 0, \quad \forall i_1 \in J, \quad \forall i_2, \ldots, i_m \notin J.$$

If $\mathcal{A}$ is not reducible, then $\mathcal{A}$ is irreducible.

Definition 3 (**[17]**). *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}_+^{[m,n]}$.*
(1) We call a nonnegative matrix $G(\mathcal{A})$ the representation associated with the nonnegative tensor $\mathcal{A}$, if the (i,j)-th element of $G(\mathcal{A})$ is defined to be the summation of $a_{i i_2 \cdots i_m}$ with indices $\{i_2 \cdots i_m\} \ni j$.
(2) We call $\mathcal{A}$ weakly reducible if its representation $G(\mathcal{A})$ is a reducible matrix, and we call it weakly primitive if $G(\mathcal{A})$ is a primitive matrix. If $\mathcal{A}$ is not weakly reducible, then it is called weakly irreducible.

Definition 4 (**[7,18]**). *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$, $D = diag(d_1, d_2, \cdots, d_n)$ be a positive diagonal matrix of order n; we define it as $(\mathcal{A}D^{m-1})_{i_1 i_2 \cdots i_m} = a_{i_1 i_2 \cdots i_m} d_{i_1} d_{i_2} \cdots d_{i_m}$.*

We use $\mathcal{I}$ to denote the m-th order n-dimensional unit tensor with entries

$$\mathcal{I}_{i_1 i_2 \cdots i_m} = \begin{cases} 1, & \text{if } i_1 = i_2 = \cdots = i_m, \\ 0, & \text{otherwise,} \end{cases}$$

and we define the following m-th order $\delta_{i_1 i_2 \cdots i_m}$ Kronecker delta

$$\delta_{i_1 i_2 \cdots i_m} = \begin{cases} 1, & \text{if } i_1 = i_2 = \cdots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$, and denote

$$\bar{r}_i(\mathcal{A}) = \sum_{i_2,\ldots,i_m=1}^{n} |a_{ii_2\cdots i_m}|,\ r_i(\mathcal{A}) = \bar{r}_i(\mathcal{A}) - |a_{i\cdots i}|, \quad i \in \langle n \rangle,$$

$$r_i^{[j]}(\mathcal{A}) = \sum_{\substack{i_2,\ldots,i_m=1 \\ j\in\{i_2,\ldots,i_m\}}}^{n} |a_{ii_2\cdots i_m}| - a_{j\cdots j},\ \bar{r}_i^{[j]}(\mathcal{A}) = r_i(\mathcal{A}) - r_i^{[j]}(\mathcal{A}), \quad i \neq j,\ i,j \in \langle n \rangle.$$

The study of the conditions for the determination of the $\mathcal{H}$-tensor is the basis for the application of the $\mathcal{H}$-tensor. The literature [7–9] provides some methods for the determination of the $\mathcal{H}$-tensor. In this paper, a different method is used to obtain a class of quasi-double diagonally dominant tensor by carefully analysing the structure of the tensor, and another class of quasi-double diagonally dominant tensor is discussed by analysing the digraph of the majorization matrix of the tensor.

In the following, we describe two classes of quasi-double diagonally dominant tensors, prove that they are nonsingular $\mathcal{H}$-tensors, and give several inequalities to estimate the spectral radius of nonnegative tensors based on the correspondence between the diagonal dominance of a tensor and the inclusion domain of its eigenvalues.

3. Two Classes of Quasi-Double Diagonally Dominant $\mathcal{H}$-Tensors

In this section, we describe two classes of quasi-double dominant $\mathcal{H}$-tensors and show that the two classes of tensors are not mutually inclusive.

Definition 5 ([8])**.** *Let $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$. If*

$$|a_{i\cdots i}| \geq r_i(\mathcal{A}),\ i \in \langle n \rangle, \tag{1}$$

then tensor $\mathcal{A}$ is called diagonally dominant. If (1) are all strictly inequalities, then tensor $\mathcal{A}$ is called strictly diagonally dominant. If tensor $\mathcal{A}$ is irreducible, and (1) holds at least one strict inequality, then tensor $\mathcal{A}$ is called irreducible diagonally dominant. If there is a positive diagonal matrix D, such that $\mathcal{A}D^{m-1}$ is strictly diagonally dominant, then tensor $\mathcal{A}$ is called generalized strictly diagonally dominant.

Definition 6 ([9])**.** *For $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$, its comparison tensor, denoted by $\mathcal{M}_{\mathcal{A}} = (m_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$, is defined as*

$$m_{i_1i_2\cdots i_m} = \begin{cases} |a_{i\cdots i}|, & \text{if } i_1 = i_2 = \cdots = i_m, \\ -|a_{i_1i_2\cdots i_m}|, & \text{otherwise.} \end{cases}$$

Definition 7 ([7–9])**.** *Let $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$. Tensor $\mathcal{A}$ is said to be a $\mathcal{Z}$-tensor if it can be written as $\mathcal{A} = s\mathcal{I} - \mathcal{B}$, where $s > 0$, $\mathcal{B} \in \mathbb{R}_+^{[m,n]}$. Furthermore, if $s \geq \rho(\mathcal{B})$, then tensor $\mathcal{A}$ is said to be an $\mathcal{M}$-tensor, and if $s > \rho(\mathcal{B})$, then tensor $\mathcal{A}$ is said to be a nonsingular $\mathcal{M}$-tensor.*

Reference [6] also proved the following:

Theorem 1 ([9])**.** *If $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$ is a $\mathcal{Z}$-tensor, then tensor $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor if and only if $Re\lambda > 0$, $\forall \lambda \in \sigma(\mathcal{A})$.*

Definition 8 ([7,8])**.** *Let $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$. If the comparison tensor $\mathcal{M}_{\mathcal{A}}$ of tensor $\mathcal{A}$ is an $\mathcal{M}$-tensor, then tensor $\mathcal{A}$ is called an $\mathcal{H}$-tensor, and if comparison tensor $\mathcal{M}_{\mathcal{A}}$ is a nonsingular $\mathcal{M}$-tensor, then tensor $\mathcal{A}$ is called a nonsingular $\mathcal{H}$-tensor.*

Theorem 2 ([7,8])**.** *Let $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$. If tensor $\mathcal{A}$ is strictly diagonally dominant, irreducible diagonally dominant, or generalized strictly diagonally dominant, then tensor $\mathcal{A}$ is called a nonsingular $\mathcal{H}$-tensor.*

Theorem 3. *Let* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$. *If*
(i) $|a_{i\cdots i}| > r_i^{[i]}(\mathcal{A})$, $\forall i \in \langle n \rangle$,
(ii) $\left(|a_{i\cdots i}| - r_i^{[i]}(\mathcal{A})\right)\left(|a_{j\cdots j}| - \overline{r}_j^{[i]}(\mathcal{A})\right) > \overline{r}_i^{[i]}(\mathcal{A}) r_j^{[i]}(\mathcal{A}), \forall i, j \in \langle n \rangle,\ i \neq j$,
then $\mathcal{A}$ *is nonsingular; that is,* $0 \notin \sigma(\mathcal{A})$.

Proof. If $0 \in \sigma(\mathcal{A})$, then there exists $0 \neq x = (x_1, x_2, \cdots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$, such that

$$\mathcal{A}x^{m-1} = 0.$$

Assume $|x_{t_1}| \geq |x_{t_2}| \geq \cdots \geq |x_{t_{n-1}}| \geq |x_{t_n}| \geq 0$; therefore, $|x_{t_1}| \neq 0$, and we have

$$\sum_{i_2,\ldots,i_m=1}^{n} a_{t_1 i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} = 0. \tag{2}$$

Hence,

$$a_{t_1\cdots t_1} x_{t_1}^{m-1} = -\sum_{\substack{i_2,\ldots,i_m=1 \\ t_1 \in \{i_2,\ldots,i_m\} \\ \delta_{t_1,t_2,\ldots,t_m}=0}}^{n} a_{t_1 i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} - \sum_{\substack{i_2,\ldots,i_m=1 \\ t_1 \notin \{i_2,\ldots,i_m\}}}^{n} a_{t_1 i_2 \cdots i_m} x_{i_2} \cdots x_{i_m};$$

thus, we have

$$|a_{t_1\cdots t_1}||x_{t_1}|^{m-1} \leq r_{t_1}^{[t_1]}(\mathcal{A})|x_{t_1}|^{m-1} + \overline{r}_{t_1}^{[t_1]}(\mathcal{A})|x_{t_2}|^{m-1},$$

i.e.,

$$\left(|a_{t_1\cdots t_1}| - r_{t_1}^{[t_1]}(\mathcal{A})\right)|x_{t_1}|^{m-1} \leq \overline{r}_{t_1}^{[t_1]}(\mathcal{A})|x_{t_2}|^{m-1}. \tag{3}$$

Similarly, from (2), we have

$$|a_{t_2\cdots t_2}||x_{t_2}|^{m-1} \leq r_{t_2}^{[t_1]}(\mathcal{A})|x_{t_1}|^{m-1} + \overline{r}_{t_2}^{[t_1]}(\mathcal{A})|x_{t_2}|^{m-1},$$

i.e.,

$$\left(|a_{t_2\cdots t_2}| - \overline{r}_{t_2}^{[t_1]}(\mathcal{A})\right)|x_{t_2}|^{m-1} \leq r_{t_2}^{[t_1]}(\mathcal{A})|x_{t_1}|^{m-1}, \tag{4}$$

where $x_{t_2} \neq 0$; otherwise, from $x_{t_1} \neq 0$ and (3), we have $|a_{t_1\cdots t_1}| - r_{t_1}^{[t_1]}(\mathcal{A} \leq 0$, in contradiction with (i). In this way, from (i), (3), and (4), we have

$$\begin{aligned}&\left(|a_{t_1\cdots t_1}| - r_{t_1}^{[t_1]}(\mathcal{A})\right)\left(|a_{t_2\cdots t_2}| - \overline{r}_{t_2}^{[t_1]}(\mathcal{A})\right)|x_{t_1}|^{m-1}|x_{t_2}|^{m-1} \\ &\leq \overline{r}_{t_1}^{[t_1]}(\mathcal{A}) r_{t_2}^{[t_1]}(\mathcal{A})|x_{t_1}|^{m-1}|x_{t_2}|^{m-1},\end{aligned}$$

i.e.,

$$\left(|a_{t_1\cdots t_1}| - r_{t_1}^{[t_1]}(\mathcal{A})\right)\left(|a_{t_2\cdots t_2}| - \overline{r}_{t_2}^{[t_1]}(\mathcal{A})\right) \leq \overline{r}_{t_1}^{[t_1]}(\mathcal{A}) r_{t_2}^{[t_1]}(\mathcal{A}),$$

in contradiction with (ii). Therefore, $0 \notin \sigma(\mathcal{A})$. □

Theorem 4. *If* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$, *then* $\sigma(\mathcal{A}) \subseteq D(\mathcal{A}) \bigcup \tilde{D}(\mathcal{A})$, *where*

$$\begin{aligned}
D(\mathcal{A}) &= \bigcup_{i \in \langle n \rangle} D_i(\mathcal{A}),\ D_i(\mathcal{A}) = \left\{z \in \mathbb{C} \mid |z - a_{i\cdots i}| \leq r_i^{[i]}(\mathcal{A})\right\},\ i \in \langle n \rangle, \\
\tilde{D}(\mathcal{A}) &= \bigcup_{i \neq j} D_{ij}(\mathcal{A}), \\
D_{ij}(\mathcal{A}) &= \left\{z \in \mathbb{C} \mid \left(|z - a_{i\cdots i}| - r_i^{[i]}(\mathcal{A})\right)\left(|a_{j\cdots j}| - \overline{r}_j^{[i]}(\mathcal{A})\right) \leq \overline{r}_i^{[i]}(\mathcal{A}) r_j^{[i]}(\mathcal{A})\right\},\ i, j \in \langle n \rangle.
\end{aligned}$$

Proof. If λ is an eigenvalue of tensor $\mathcal{A}$, then $0 \in \sigma(\lambda\mathcal{I} - \mathcal{A})$. From Theorem 3, we know there is some $i_0 \in \langle n \rangle$, such that

$$|\lambda - a_{i_0 \cdots i_0}| \leq r_{i_0}^{[i_0]}(\mathcal{A}),$$

or there is some $i_0, j_0 \in \langle n \rangle$, such that

$$\left(|\lambda - a_{i_0 \cdots i_0}| - r_{i_0}^{[i_0]}(\mathcal{A})\right)\left(|\lambda - a_{j_0 \cdots j_0}| - \bar{r}_{j_0}^{[i_0]}(\mathcal{A})\right) \leq \bar{r}_{i_0}^{[i_0]}(\mathcal{A}) r_{j_0}^{[i_0]}(\mathcal{A}).$$

Therefore, we have $\lambda \in D_{i_0}(\mathcal{A})$ or $\lambda \in D_{i_0 j_0}(\mathcal{A})$. □

Theorem 5. *Let* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$. *If*
(i) $|a_{i \cdots i}| > r_i^{[i]}(\mathcal{A}),\ \forall i \in \langle n \rangle$,
(ii) $\left(|a_{i \cdots i}| - r_i^{[i]}(\mathcal{A})\right)\left(|a_{j \cdots j}| - \bar{r}_j^{[i]}(\mathcal{A})\right) > \bar{r}_i^{[i]}(\mathcal{A}) r_j^{[i]}(\mathcal{A}), \forall i, j \in \langle n \rangle,\ i \neq j$,
then $\mathcal{M}_{\mathcal{A}}$ *is a nonsingular* $\mathcal{M}$*-tensor; that is,* $\mathcal{A}$ *is a nonsingular* $\mathcal{H}$*-tensor.*

Proof. Consider the comparison tensor $\mathcal{M}_{\mathcal{A}}$ of tensor $\mathcal{A}$. $\forall \lambda \in \sigma(\mathcal{M}_{\mathcal{A}}), Re\lambda > 0$. Otherwise, if there exists $\lambda_0 \in \sigma(\mathcal{M}_{\mathcal{A}}), Re\lambda_0 \neq 0$, then from (i), we have

$$|\lambda_0 - |a_{i \cdots i}|| = |(Im\lambda_0)\mathrm{i} + Re\lambda_0 - |a_{i \cdots i}|| \geq |Re\lambda_0 - |a_{i \cdots i}|| \geq |a_{i \cdots i}| > r_i^{[i]}(\mathcal{A}),\ \forall i \in \langle n \rangle.$$

From (ii), we have

$$|a_{j \cdots j}| - \bar{r}_j^{[i]}(\mathcal{A}) > 0,\ \forall j \in \langle n \rangle.$$

Hence, from (i) and (ii), we have

$$\begin{aligned}
&\left(|\lambda_0 - |a_{i \cdots i}| - r_i^{[i]}(\mathcal{A})\right)\left(|\lambda_0 - |a_{j \cdots j}|| - \bar{r}_j^{[i]}(\mathcal{A})\right) \\
&= \left(|(Im\lambda_0)\mathrm{i} + Re\lambda_0 - |a_{i \cdots i}|| - r_i^{[i]}(\mathcal{A})\right)\left(|(Im\lambda_0)\mathrm{i} + Re\lambda_0 - |a_{j \cdots j}|| - \bar{r}_j^{[i]}(\mathcal{A})\right) \\
&\geq \left(|Re\lambda_0 - |a_{i \cdots i}|| - r_i^{[i]}(\mathcal{A})\right)\left(Re\lambda_0 - |a_{j \cdots j}|| - \bar{r}_j^{[i]}(\mathcal{A})\right) \\
&\geq \left(|a_{i \cdots i}| - r_i^{[i]}(\mathcal{A})\right)\left(|a_{j \cdots j}| - \bar{r}_j^{[i]}(\mathcal{A})\right) \\
&> \bar{r}_i^{[i]}(\mathcal{A}) r_j^{[i]}(\mathcal{A}), \forall i, j \in \langle n \rangle,\ i \neq j.
\end{aligned}$$

Therefore, from Theorem 4, we know $\lambda_0 \notin \sigma(\mathcal{A})$, a contradiction with $\lambda_0 \in \sigma(\mathcal{A})$. Thus, there must be $Re\lambda_0 > 0$. Then, from Theorem 1, we know $\mathcal{M}_{\mathcal{A}}$ is a nonsingular $\mathcal{M}$-tensor; so, from Definition 8, we know tensor $\mathcal{A}$ is a nonsingular $\mathcal{H}$-tensor. □

Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$; its majorization matrix [19], we denote by $\hat{A} = (a_{ij}) \in \mathbb{C}^{n \times n}$, where $a_{ij} = a_{ij \cdots j},\ i, j \in \langle n \rangle,\ r_i(\hat{A}) = \sum\limits_{\substack{j=1 \\ j \neq i}} |a_{ij}|$. The digraph [20] of matrix $\hat{A}$ is denoted as $\Gamma(\hat{A})$, and the directed edge on $\Gamma(\hat{A})$ is denoted as e_{ij}, $\Gamma_i^+(\hat{A}) = \{j \in \langle n \rangle : a_{ij \cdots j} \neq 0\}$.

Theorem 6. *Let* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$. *If*
(i) $|a_{j \cdots j}|\left(|a_{i \cdots i}| - r_i(\mathcal{A}) + r_i(\hat{A})\right) > r_j(\mathcal{A}) r_i(\hat{A}),\ e_{ij} \in \Gamma(\hat{A})$,
(ii) $|a_{i \cdots i}| > r_i(\mathcal{A}),\ \Gamma_i^+(\hat{A}) = \varnothing$,
then $0 \notin \sigma(\mathcal{A})$.

Proof. If $0 \in \sigma(\mathcal{A})$, then there exists $0 \neq x = (x_1, x_2, \cdots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$, such that

$$\mathcal{A}x^{m-1} = 0. \tag{5}$$

Assume $|x_{t_1}| \geq |x_{t_2}| \geq \cdots \geq |x_{t_{n-1}}| \geq |x_{t_n}| \geq 0$, $a_{t_1 t_2 \cdots t_2} = \cdots = a_{t_1 t_{s-1} \cdots t_{s-1}} = 0$, $a_{t_1 t_s \cdots t_s} \neq 0$, $s \leq n$; therefore, $x_{t_1} \neq 0$, $e_{t_1 t_s} \in \Gamma(\hat{\mathcal{A}})$.

(1) If $\Gamma^+_{t_1}(\hat{\mathcal{A}}) = \emptyset$, then $r_{t_1}(\hat{\mathcal{A}}) = 0$. From (5), we have

$$\sum_{i_2,\ldots,i_m=1}^{n} a_{t_1 i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} = 0.$$

Hence, we have

$$|a_{t_1 \cdots t_1}||x_{t_1}|^{m-1} \leq \big(r_{t_1}(\mathcal{A}) - r_{t_1}(\hat{\mathcal{A}})\big)|x_{t_1}|^{m-1} + r_{t_1}(\hat{\mathcal{A}})|x_{t_s}|^{m-1} = r_{t_1}(\mathcal{A})|x_{t_1}|^{m-1},$$

i.e.,

$$|a_{t_1 \cdots t_1}| < r_{t_1}(\mathcal{A}).$$

This is in contradiction with (ii).

(2) If $\Gamma^+_{t_1}(\hat{\mathcal{A}}) \neq \emptyset$, we assume $a_{t_1 t_2 \cdots t_2} = \cdots = a_{t_1 t_{s-1} \cdots t_{s-1}} = 0$, $a_{t_1 t_s \cdots t_s} \neq 0$, $s \leq n$, then $e_{t_1 t_s} \in \Gamma(\hat{\mathcal{A}})$. We discuss this in two cases:

(2.1) Let $x_{t_s} \neq 0$; from (5), we have

$$\sum_{i_2,\ldots,i_m=1}^{n} a_{t_1 i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} = 0.$$

Hence, we have

$$|a_{t_1 \cdots t_1}||x_{t_1}|^{m-1} \leq \big(r_{t_1}(\mathcal{A}) - r_{t_1}(\hat{\mathcal{A}})\big)|x_{t_1}|^{m-1} + r_{t_1}(\hat{\mathcal{A}})|x_{t_s}|^{m-1},$$

i.e.,

$$\big(|a_{t_1 \cdots t_1}| - \big(r_{t_1}(\mathcal{A}) - r_{t_1}(\hat{\mathcal{A}})\big)\big)|x_{t_1}|^{m-1} \leq r_{t_1}(\hat{\mathcal{A}})|x_{t_s}|^{m-1}.$$

Similarly, from (5), we have

$$|a_{t_s \cdots t_s}||x_{t_s}|^{m-1} \leq r_{t_s}(\mathcal{A})|x_{t_1}|^{m-1}.$$

Thus,

$$\begin{aligned} &|a_{t_s \cdots t_s}|\big(|a_{t_1 \cdots t_1}| - \big(r_{t_1}(\mathcal{A}) - r_{t_1}(\hat{\mathcal{A}})\big)\big)|x_{t_s}|^{m-1}|x_{t_1}|^{m-1} \\ &\leq r_{t_1}(\hat{\mathcal{A}}) r_{t_s}(\mathcal{A})|x_{t_s}|^{m-1}|x_{t_1}|^{m-1}, \end{aligned}$$

i.e.,

$$|a_{t_s \cdots t_s}|\big(|a_{t_1 \cdots t_1}| - \big(r_{t_1}(\mathcal{A}) - r_{t_1}(\hat{\mathcal{A}})\big)\big) \leq r_{t_1}(\hat{\mathcal{A}}) r_{t_s}(\mathcal{A}),\ e_{t_1 t_s} \in \Gamma(\hat{\mathcal{A}}).$$

(2.2) If $a_{t_1 t_s \cdots t_s} \neq 0$, $t_1 \neq t_s$, $2 \leq s \leq n$, $|x_{t_s}| = 0$, then we have

$$\big(|a_{t_1 \cdots t_1}| - \big(r_{t_1}(\mathcal{A}) - r_{t_1}(\hat{\mathcal{A}})\big)\big)|x_{t_1}|^{m-1} \leq r_{t_1}(\hat{\mathcal{A}})|x_{t_s}|^{m-1} = 0;$$

thus,

$$\big(|a_{t_1 \cdots t_1}| - \big(r_{t_1}(\mathcal{A}) - r_{t_1}(\hat{\mathcal{A}})\big)\big) \leq 0.$$

Hence,

$$|a_{t_s \cdots t_s}|\big(|a_{t_1 \cdots t_1}| - \big(r_{t_1}(\mathcal{A}) - r_{t_1}(\hat{\mathcal{A}})\big)\big) \leq r_{t_1}(\hat{\mathcal{A}}) r_{t_s}(\mathcal{A}).$$

Combining (2.1) and (2.2), we know that the result contradicts with (ii). Recombining (1) and (2), we know $0 \notin \sigma(\mathcal{A})$. $\square$

Theorem 7. *If* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$, *then*

$$\sigma(\mathcal{A}) \subseteq \bigcup_{e_{ij}\in\Gamma(\hat{\mathcal{A}})} \left\{z \in \mathbb{C} : |z - a_{j\cdots j}|\big(|z - a_{i\cdots i}| - r_i(\mathcal{A}) + r_i(\hat{\mathcal{A}})\big) \leq r_j(\mathcal{A})r_i(\hat{\mathcal{A}})\right\}$$
$$\bigcup_{i\in\langle n\rangle} \bigcup_{\Gamma_i^+(\hat{\mathcal{A}})=\emptyset} \{z \in \mathbb{C} : |z - a_{i\cdots i}| \leq r_i(\mathcal{A})\}.$$

Proof. If λ is an eigenvalue of tensor $\mathcal{A}$, then $0 \in \sigma(\lambda\mathcal{I} - \mathcal{A})$. From Theorem 6, we know there is some $i_0, j_0 \in \langle n\rangle$, $e_{i_0j_0} \in \Gamma(\hat{\mathcal{A}})$, such that

$$|\lambda - a_{j_0\cdots j_0}|\big(|\lambda - a_{i_0\cdots i_0}| - r_{i_0}(\mathcal{A}) + r_{i_0}(\hat{\mathcal{A}})\big) \leq r_{j_0}(\mathcal{A})r_{i_0}(\hat{\mathcal{A}}),$$

or there exists $i_0 \in \langle n\rangle$, $\Gamma_i^+(\hat{\mathcal{A}}) = \emptyset$, such that

$$|\lambda - a_{i_0j_0\cdots j_0}| < r_{i_0}(\mathcal{A}).$$

□

Theorem 8. *Let* $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$. *If*
(i) $|a_{j\cdots j}|\big(|a_{i\cdots i}| - r_i(\mathcal{A}) + r_i(\hat{\mathcal{A}})\big) > r_j(\mathcal{A})r_i(\hat{\mathcal{A}})$, $e_{ij} \in \Gamma(\hat{\mathcal{A}})$,
(ii) $|a_{i\cdots i}| > r_i(\mathcal{A})$, $\Gamma_i^+(\hat{\mathcal{A}}) = \emptyset$,
then $\mathcal{M}_{\mathcal{A}}$ *is a nonsingular* $\mathcal{M}$*-tensor; that is,* $\mathcal{A}$ *is a nonsingular* $\mathcal{H}$*-tensor.*

Proof. Consider the comparison tensor $\mathcal{M}_{\mathcal{A}}$ of tensor $\mathcal{A}$. $\forall\lambda \in \sigma(\mathcal{M}_{\mathcal{A}})$. Similar to the proof of Theorem 5, we know $Re\lambda > 0$. Therefore, from Theorem 1, we know the comparison tensor $\mathcal{M}_{\mathcal{A}}$ of tensor $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor; so, $\mathcal{A}$ is a nonsingular $\mathcal{H}$-tensor. □

We give a simple example for Theorems 5 and 8, respectively.

Example 1. *Let* $\mathcal{A} \in \mathbb{R}_+^{[3,3]}$, *where*

$$\mathcal{A}(1,:,:) = \begin{pmatrix} a_{111} & a_{112} & a_{113} \\ a_{121} & a_{122} & a_{123} \\ a_{131} & a_{132} & a_{133} \end{pmatrix} = \begin{pmatrix} 6 & 0 & 1 \\ 0 & -1 & -1 \\ -1 & -1 & -1.5 \end{pmatrix},$$
$$\mathcal{A}(2,:,:) = \begin{pmatrix} a_{211} & a_{212} & a_{213} \\ a_{221} & a_{222} & a_{223} \\ a_{231} & a_{232} & a_{233} \end{pmatrix} = \begin{pmatrix} 0 & -0.5 & 0 \\ 0 & 5 & -1 \\ -1 & -1 & -1 \end{pmatrix},$$
$$\mathcal{A}(3,:,:) = \begin{pmatrix} a_{311} & a_{312} & a_{313} \\ a_{321} & a_{322} & a_{323} \\ a_{331} & a_{332} & a_{333} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & -1 & -1 \\ -1 & -1 & 7 \end{pmatrix}.$$

Clearly, tensor $\mathcal{A}$ is a $\mathcal{Z}$-tensor, due to $|a_{111}| = 6 \leq 6.5 = r_i(\mathcal{A})$; thus, $\mathcal{A}$ is not a strictly diagonally dominant tensor. By calculation, we have

$$|a_{111}| = 6 > 2 = r_1^{[1]}(\mathcal{A}),\ |a_{222}| = 5 > 2.5 = r_2^{[2]}(\mathcal{A}),\ |a_{333}| = 7 > 3r_3^{[3]}(\mathcal{A}),$$

$$\Big(|a_{111}| - r_1^{[1]}(\mathcal{A})\Big)\Big(|a_{222}| - \bar{r}_2^{[1]}(\mathcal{A})\Big) = (6-2)(5-3) > 4.5 \times 1.5 = \bar{r}_1^{[1]}(\mathcal{A})r_2^{[1]}(\mathcal{A}),$$
$$\Big(|a_{111}| - r_1^{[1]}(\mathcal{A})\Big)\Big(|a_{333}| - \bar{r}_3^{[1]}(\mathcal{A})\Big) = (6-2)(7-3) > 4.5 \times 3 = \bar{r}_1^{[1]}(\mathcal{A})r_3^{[1]}(\mathcal{A}),$$
$$\Big(|a_{222}| - r_2^{[2]}(\mathcal{A})\Big)\Big(|a_{111}| - \bar{r}_1^{[2]}(\mathcal{A})\Big) = (5-2.5)(6-3.5) > 2 \times 3 = \bar{r}_2^{[2]}(\mathcal{A})r_1^{[2]}(\mathcal{A}),$$
$$\Big(|a_{222}| - r_2^{[2]}(\mathcal{A})\Big)\Big(|a_{333}| - \bar{r}_3^{[2]}(\mathcal{A})\Big) = (5-2.5)(7-1) > 2 \times 5 = \bar{r}_2^{[2]}(\mathcal{A})r_3^{[2]}(\mathcal{A}),$$
$$\Big(|a_{333}| - r_3^{[3]}(\mathcal{A})\Big)\Big(|a_{111}| - \bar{r}_1^{[3]}(\mathcal{A})\Big) = (7-3)(6-1) > 3 \times 5.5 = \bar{r}_3^{[3]}(\mathcal{A})r_1^{[3]}(\mathcal{A}),$$

$$\left(|a_{333}|-r_3^{[3]}(\mathcal{A})\right)\left(|a_{222}|-\bar{r}_2^{[3]}(\mathcal{A})\right)=(7-3)(5-0.5)>3\times 4=\bar{r}_3^{[3]}(\mathcal{A})r_2^{[3]}(\mathcal{A}).$$

Conditions (i) and (ii) of Theorem 5 is satisfied; therefore, from Theorem 5, we know tensor $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor; so, $\mathcal{A}$ is a nonsingular $\mathcal{H}$-tensor.

Example 2. *Let $\mathcal{A}\in\mathbb{R}_+^{[3,3]}$, where*

$$\mathcal{A}(1,:,:)=\begin{pmatrix}a_{111}&a_{112}&a_{113}\\a_{121}&a_{122}&a_{123}\\a_{131}&a_{132}&a_{133}\end{pmatrix}=\begin{pmatrix}5&-0.8&-0.5\\0&-2&-0.2\\-0.5&0&-2.2\end{pmatrix},$$
$$\mathcal{A}(2,:,:)=\begin{pmatrix}a_{211}&a_{212}&a_{213}\\a_{221}&a_{222}&a_{223}\\a_{231}&a_{232}&a_{233}\end{pmatrix}=\begin{pmatrix}-2&-0.4&-0.5\\-0.7&8.65&-0.6\\-0.5&-0.3&-1\end{pmatrix},$$
$$\mathcal{A}(3,:,:)=\begin{pmatrix}a_{311}&a_{312}&a_{313}\\a_{321}&a_{322}&a_{323}\\a_{331}&a_{332}&a_{333}\end{pmatrix}=\begin{pmatrix}-1.5&-0.5&-0.5\\-0.5&-1.5&-0.5\\-0.5&-0.5&8.45\end{pmatrix}.$$

Clearly, tensor $\mathcal{A}$ is a $\mathcal{Z}$-tensor, due to $|a_{111}|=5\leq 6.2=r_i(\mathcal{A})$; thus, tensor $\mathcal{A}$ is not strictly diagonally dominant. However, it is easy to verify that the condition of Theorem 8 is satisfied; therefore, $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor; that is, $\mathcal{A}$ is a nonsingular $\mathcal{H}$-tensor.

Remark 1. *The conditions of Theorems 3 and 8, which determine the $\mathcal{H}$-tensor, are not mutually inclusive. If Example 1 satisfies the conditions of Theorem 3, it is known to be an $\mathcal{H}$-tensor by applying Theorem 3; however,*

$$|a_{222}|\left(|a_{111}|-r_1(\mathcal{A})+r_1(\hat{\mathcal{A}})\right)=5\times(6-4)<4.5\times 2.5=r_2(\mathcal{A})r_1(\hat{\mathcal{A}}),\ e_{12}\in\Gamma(\hat{\mathcal{A}}).$$

Therefore, the conditions of Theorem 8 are not satisfied, and thus Theorem 8 can not determine it to be an $\mathcal{H}$-tensor.

Another example is Example 2, which satisfies the conditions of Theorem 8 and is known to be an $\mathcal{H}$-tensor by applying Theorem 8; however,

$$\left(|a_{222}|-r_2^{[2]}(\mathcal{A})\right)\left(|a_{111}|-\bar{r}_1^{[2]}(\mathcal{A})\right)=(8.65-2)(5-3.2)<4\times 3=\bar{r}_2^{[2]}(\mathcal{A})r_1^{[2]}(\mathcal{A}).$$

Therefore, the conditions of Theorem 3 are not satisfied, and thus Theorem 3 cannot be applied to determine that it is an $\mathcal{H}$-tensor.

4. Estimation Inequalities for the Spectral Radius of Nonnegative Tensors

Based on the two classes of $\mathcal{H}$-tensors given in Section 3, two estimation inequalities for the spectral radius of nonnegative tensors are given in this section. First, some basic results of the spectral radius are introduced.

Let $\mathcal{A}=(a_{i_1i_2\cdots i_m})$, $\mathcal{B}=(b_{i_1i_2\cdots i_m})\in\mathbb{R}_+^{[m,n]}$. If $a_{i_1i_2\cdots i_m}\leq b_{i_1i_2\cdots i_m}$, $i_1,i_2,\cdots,i_m\in\langle n\rangle$, then we denote $0\leq\mathcal{A}\leq\mathcal{B}$.

Theorem 9 (**[21]**)**.** *Let $\mathcal{A}=(a_{i_1i_2\cdots i_m})$, and $\mathcal{B}=(b_{i_1i_2\cdots i_m})\in\mathbb{R}_+^{[m,n]}$. If $0\leq\mathcal{A}\leq B$, then $\rho(\mathcal{A})\leq\rho(\mathcal{B})$. Specifically, $\rho(\mathcal{A})\geq a_{i\cdots i}, i\in\langle n\rangle$.*

For the spectral properties of general nonnegative tensors, Ref. [21] provided the following results.

Theorem 10. *If $\mathcal{A}=(a_{i_1i_2\cdots i_m})\in\mathbb{R}_+^{[m,n]}$, then $\rho(\mathcal{A})$ is the eigenvalue of $\mathcal{A}$, and there is a corresponding nonnegative eigenvector $x\in\mathbb{R}_+^n$.*

Theorem 11. *Let $\mathcal{A}$ be an m-th order n-dimensional nonnegative weakly irreducible tensor; then, there exists a unique positive eigenvector corresponding to the spectral radius up to a multiplicative constant.*

In [21], the upper and lower bounds for the spectral radius of a nonnegative tensor were given, which all depended only on the entries of $\mathcal{A}$.

Theorem 12. *If $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}_+^{[m,n]}$, then*

$$\min_{i \in \langle n \rangle} \bar{r}_i(\mathcal{A}) \leq \rho(\mathcal{A}) \leq \max_{i \in \langle n \rangle} \bar{r}_i(\mathcal{A}).$$

Based on Theorems 4 and 5 in Section 3, the following estimation inequalities for the upper and lower bounds for the spectral radius of nonnegative tensors are given.

Theorem 13. *Let $\mathcal{A} = \left(a_{i_1 i_2 \cdots i_m}\right) \in \mathbb{R}_+^{[m,n]}$; then,*

$$\min_{i \neq j, i,j \in \langle n \rangle} r_{ij}(\mathcal{A}) \leq \rho(\mathcal{A}) \leq \max\left\{ \max_{i \in \langle n \rangle} \{a_{i \cdots i} + r_i^{[i]}(\mathcal{A})\}, \max_{i \neq j, i,j \in \langle n \rangle} r_{ij}(\mathcal{A}) \right\},$$

where

$$\begin{aligned} r_{ij}(\mathcal{A}) =& \frac{1}{2}\left\{ a_{i \cdots i} + r_i^{[i]}(\mathcal{A}) + a_{j \cdots j} + \bar{r}_j^{[i]}(\mathcal{A}) \right. \\ & \left. + \left[\left(\left(a_{i \cdots i} + r_i^{[i]}(\mathcal{A}) \right) - \left(a_{j \cdots j} + \bar{r}_j^{[i]}(\mathcal{A}) \right) \right)^2 + 4\bar{r}_i^{[i]}(\mathcal{A}\mathcal{A}) r_j^{[i]}(\mathcal{A}) \right]^{\frac{1}{2}} \right\}. \end{aligned}$$

Proof. From Theorem 10, we have $\rho(\mathcal{A}) \in \sigma(\mathcal{A})$. From Theorem 4, we know there exists $i_0 \in \langle n \rangle$, satisfying

$$\rho(\mathcal{A}) \leq a_{i_0 \cdots i_0} + r_{i_0}^{[i_0]}(\mathcal{A}),$$

or there exists $i_0, j_0 \in \langle n \rangle, i_0 \neq j_0$, satisfying

$$\left(\rho(\mathcal{A}) - a_{i_0 \cdots i_0} - r_{i_0}^{[i_0]}(\mathcal{A}))(\rho(\mathcal{A}) - a_{j_0 \cdots j_0} - \bar{r}_{j_0}^{[i_0]}(\mathcal{A}) \right) \leq \bar{r}_{i_0}^{[i_0]}(\mathcal{A}) r_{j_0}^{[i_0]}(\mathcal{A}).$$

Therefore,

$$\rho(\mathcal{A}) \leq \max\left\{ \max_{i \in \langle n \rangle} \left\{ a_{i \cdots i} + r_i^{[i]}(\mathcal{A}) \right\}, \max_{i \neq j, i,j \in \langle n \rangle} r_{ij}(\mathcal{A}) \right\}.$$

On the other hand, if $\mathcal{A}$ is weakly irreducible, then it is known from Theorem 11 that there exists $x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}_{++}^n$, such that

$$\mathcal{A}x^{m-1} = \rho(\mathcal{A}) x^{[m-1]}. \tag{6}$$

Without loss of generality, suppose that $x_{t_1} \geq x_{t_2} \geq \cdots \geq x_{t_{n-1}} \geq x_{t_n} > 0$. From (6), we have

$$\begin{aligned} & (\rho(\mathcal{A}) - a_{t_n \cdots t_n}) x_{t_n}^{m-1} \\ =& \sum_{\substack{i_2, \cdots, i_m = 1 \\ \delta_{t_n i_2 \cdots i_m} = 0}}^{n} a_{t_n i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} \geq r_{t_n}^{[t_n]}(\mathcal{A}) x_{t_n}^{m-1} + \bar{r}_{t_n}^{[t_n]}(\mathcal{A}) x_{t_{n-1}}^{m-1}, \end{aligned}$$

and

$$\begin{aligned}
&(\rho(\mathcal{A}) - a_{t_{n-1}\cdots t_{n-1}})x_{t_{n-1}}^{m-1} \\
&= \sum_{\substack{i_2,\cdots,i_m=1 \\ \delta_{t_{n-1}i_2\cdots i_m}=0}}^{n} a_{t_{n-1}i_2\cdots i_m}x_{i_2}\cdots x_{i_m} \geq r_{t_{n-1}}^{[t_n]}(\mathcal{A})x_{t_n}^{m-1} + \bar{r}_{t_n}^{[t_n]}(\mathcal{A})x_{t_{n-1}}^{m-1}.
\end{aligned}$$

Thus, we have

$$\Big(\rho(\mathcal{A}) - a_{t_n\cdots t_n} - r_{t_n}^{[t_n]}(\mathcal{A})\Big)x_{t_n}^{m-1} \geq \bar{r}_{t_n}^{[t_n]}(\mathcal{A})x_{t_{n-1}}^{m-1}, \tag{7}$$

and

$$\Big(\rho(\mathcal{A}) - a_{t_{n-1}\cdots t_{n-1}} - \bar{r}_{t_n}^{[t_n]}(\mathcal{A})\Big)x_{t_{n-1}}^{m-1} \geq r_{t_{n-1}}^{[t_n]}(\mathcal{A})x_{t_n}^{m-1}. \tag{8}$$

So multiplying (7) with (8) gives

$$\begin{aligned}
&(\rho(\mathcal{A}) - a_{t_n\cdots t_n} - r_{t_n}^{[t_n]}(\mathcal{A}))(\rho(\mathcal{A}) - a_{t_{n-1}\cdots t_{n-1}} - \bar{r}_{t_n}^{[t_n]}(\mathcal{A}))x_{t_{n-1}}^{m-1}x_{t_n}^{m-1} \\
&\geq \bar{r}_{t_n}^{[t_n]}(\mathcal{A})r_{t_{n-1}}^{[t_n]}(\mathcal{A})x_{t_{n-1}}^{m-1}x_{t_n}^{m-1};
\end{aligned}$$

that is,

$$\begin{aligned}
&(\rho(\mathcal{A}) - a_{t_n\cdots t_n} - r_{t_n}^{[t_n]}(\mathcal{A}))(\rho(\mathcal{A}) - a_{t_{n-1}\cdots t_{n-1}} - \bar{r}_{t_n}^{[t_n]}(\mathcal{A})) \\
&\geq \bar{r}_{t_n}^{[t_n]}(\mathcal{A})r_{t_{n-1}}^{[t_n]}(\mathcal{A}).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\rho(\mathcal{A}) \geq \tfrac{1}{2}\{ & a_{t_n\cdots t_n} + r_{t_n}^{[t_n]}(\mathcal{A}) + a_{t_{n-1}\cdots t_{n-1}} + \bar{r}_{t_{n-1}}^{[t_n]}(\mathcal{A}) \\
& + [((a_{t_n\cdots t_n} + r_{t_n}^{[t_n]}(\mathcal{A})) - (a_{t_{n-1}\cdots t_{n-1}} + \bar{r}_{t_{n-1}}^{[t_n]}(\mathcal{A})))^2 \\
& + 4\bar{r}_{t_n}^{[t_n]}(\mathcal{A})r_{t_{n-1}}^{[t_n]}(\mathcal{A})]^{\frac{1}{2}}\} \\
& \geq \min_{i\neq j, i,j\in\langle n\rangle} r_{ij}(\mathcal{A}).
\end{aligned}$$

For general nonnegative tensors $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{R}_+^{[m,n]}$, we define

$$\mathcal{A}(\varepsilon) = (a_{i_1i_2\cdots i_m}(\varepsilon)) \in \mathbb{R}_+^{[m,n]}, \varepsilon > 0,$$

where $a_{i_1i_2\cdots i_m}(\varepsilon) = a_{i_1i_2\cdots i_m} + \varepsilon$; then, $\mathcal{A}(\varepsilon)$ is irreducible. Therefore, from the above proof, we have

$$\begin{aligned}
\rho(\mathcal{A}(\varepsilon)) \geq \tfrac{1}{2}\{ & a_{t_n\cdots t_n}(\varepsilon) + r_{t_n}^{[t_n]}(\mathcal{A}(\varepsilon)) + a_{t_{n-1}\cdots t_{n-1}}(\varepsilon) + \bar{r}_{t_{n-1}}^{[t_n]}(\mathcal{A}(\varepsilon)) \\
& + [((a_{t_n\cdots t_n}(\varepsilon) + r_{t_n}^{[t_n]}(\mathcal{A}(\varepsilon))) - (a_{t_{n-1}\cdots t_{n-1}}(\varepsilon) + \bar{r}_{t_{n-1}}^{[t_n]}(\mathcal{A}(\varepsilon))))^2 \\
& + 4\bar{r}_{t_n}^{[t_n]}(\mathcal{A}(\varepsilon))r_{t_{n-1}}^{[t_n]}(\mathcal{A}(\varepsilon))]^{\frac{1}{2}}\} \\
& \geq \min_{i\neq j} r_{ij}(\mathcal{A}(\varepsilon)).
\end{aligned}$$

Notice that $\mathcal{A}(\varepsilon), a_{i_1i_2\cdots i_n}(\varepsilon), r_{t_n}^{[t_n]}(A(\varepsilon)), \bar{r}_{t_{n-1}}^{[t_n]}(A(\varepsilon)), r_{t_n}^{[t_n]}(A(\varepsilon)), \bar{r}_{t_{n-1}}^{[t_n]}(A(\varepsilon)), r_{ij}(A(\varepsilon))$ are continuous functions of ε. Let $\varepsilon \to 0$; then,

$$\begin{aligned}
\rho(\mathcal{A}) \geq \tfrac{1}{2}\{ & a_{t_n\cdots t_n} + r_{t_n}^{[t_n]}(\mathcal{A}) + a_{t_{n-1}\cdots t_{n-1}} + \bar{r}_{t_{n-1}}^{[t_n]}(\mathcal{A}) \\
& + [((a_{t_n\cdots t_n} + r_{t_n}^{[t_n]}(\mathcal{A})) - (a_{t_{n-1}\cdots t_{n-1}} + \bar{r}_{t_{n-1}}^{[t_n]}(\mathcal{A})))^2 \\
& + 4\bar{r}_{t_n}^{[t_n]}(\mathcal{A})r_{t_{n-1}}^{[t_n]}(\mathcal{A})]^{\frac{1}{2}}\} \\
& \geq \min_{i\neq j, i,j\in\langle n\rangle} r_{ij}(\mathcal{A}).
\end{aligned}$$

□

Remark 2. *The inequality in the spectral radius of nonnegative tensors given by Theorem 13 is not a complete improvement of Theorem 12, and it can be combined with Theorem 12 to obtain further improved results.*

Theorem 14. *If $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}_{+}^{[m,n]}$, then*

$$\max\left\{\min_{i\in\langle n\rangle} \bar{r}_i(\mathcal{A}), \min_{i\neq j, i,j\in\langle n\rangle} r_{ij}(\mathcal{A})\right\} \leq \rho(\mathcal{A})$$
$$\leq \min\left\{\max_{i\in\langle n\rangle} \bar{r}_i(\mathcal{A}), \max_{i\in\langle n\rangle}\left\{a_{i\cdots i} + r_i^{[i]}(\mathcal{A})\right\}, \max_{i\neq j, i,j\in\langle n\rangle} r_{ij}(\mathcal{A})\right\},$$

where $r_{ij}(\mathcal{A})$, see Theorem 13.

Similarly, based on Theorems 7 and 8 in Section 3, we have the following estimation inequalities for the upper and lower bounds of the spectral radius of nonnegative tensors.

Theorem 15. *If $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}_{+}^{[m,n]}$ is weakly irreducible, then*

$$\min\left\{\min_{e_{ij}\in\Gamma(\hat{A})} s_{ij}(\mathcal{A}), \min_{i\in\langle n\rangle, \Gamma_i^{+}(\hat{A})=\varnothing} \bar{r}_i(\mathcal{A})\right\} \leq \rho(\mathcal{A})$$
$$\leq \max\left\{\max_{e_{ij}\in\Gamma(\hat{A})} s_{ij}(\mathcal{A}), \max_{i\in\langle n\rangle, \Gamma_i^{+}(\hat{A})=\varnothing} \bar{r}_i(\mathcal{A})\right\},$$

where

$$s_{ij}(\mathcal{A}) = \frac{1}{2}\{a_{i\cdots i} + a_{j\cdots j} + r_i(A) - r_i(\hat{A})$$
$$+ \left[\left(a_{i\cdots i} - a_{j\cdots j} + r_i(A) - r_i(\hat{A})\right)^2 + 4\hat{r}_i(\mathcal{A}) r_j(\mathcal{A})\right]^{\frac{1}{2}}\}.$$

Proof. From Theorem 10, we have $\rho(\mathcal{A}) \in \sigma(\mathcal{A})$. From Theorem 7, we know there exists $i_0, j_0 \in \langle n\rangle, e_{i_0 j_0} \in \Gamma_i^{+}(\hat{A})$, satisfying

$$(\rho(\mathcal{A}) - a_{j_0\cdots j_0})(\rho(\mathcal{A}) - a_{i_0\cdots i_0} - r_{i_0}(\mathcal{A}) + r_{i_0}(\hat{A})) \leq r_{j_0}(\mathcal{A}) r_{i_0}(\hat{A}),$$

or there exists $i_0 \in \langle n\rangle, \Gamma_{i_0}^{+}(\hat{A}) = \varnothing$, satisfying

$$\rho(\mathcal{A}) - a_{i_0\cdots i_0} \leq r_{i_0}(\mathcal{A}).$$

Therefore, we have

$$\rho(\mathcal{A}) \leq \max\left\{\max_{e_{ij}\in\Gamma(\hat{A})} s_{ij}(\mathcal{A}), \max_{i\in\langle n\rangle, \Gamma_i^{+}(\hat{A})=\varnothing} \bar{r}_i(\mathcal{A})\right\}.$$

Next, we prove that the left-hand side of the inequality of the theorem holds.

Since $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}_{+}^{[m,n]}$ is weakly irreducible, and from Theorem 11, we have $\rho(\mathcal{A}) \in \sigma(\mathcal{A})$; therefore, there exists $x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}_{++}^{n}$, such that

$$\mathcal{A}x^{m-1} = \rho(\mathcal{A}) x^{[m-1]}. \tag{9}$$

Without loss of generality, suppose that $x_{t_1} \geq x_{t_2} \geq \cdots \geq x_{t_{n-1}} \geq x_{t_n} > 0$.

(1.1) If $\Gamma_{t_n}^{+}(\hat{A}) = \varnothing$, then $r_{t_n}(\hat{A}) = 0$. From (9), we have

$$\sum_{i_2,\cdots,i_m=1}^{n} a_{t_n i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} = \rho(\mathcal{A}) x_{t_n}^{m-1}. \tag{10}$$

Therefore,
$$\rho(\mathcal{A}) \geq \bar{r}_{t_n}(\mathcal{A}).$$

(1.2) If $\Gamma^+_{t_n}(\hat{\mathcal{A}}) \neq \varnothing$, assume $a_{t_n t_{n-1} \cdots t_{n-1}} = \cdots = a_{t_n t_{n-r-1} \cdots t_{n-r-1}} = 0, a_{t_n t_{n-r} \cdots t_{n-r}} \neq 0, r \leq n-1$; then, $e_{t_n t_{n-r}} \in \Gamma(\hat{\mathcal{A}})$. From (10), we have

$$\left(\rho(\mathcal{A}) - a_{t_1 \cdots t_1} - r_{t_n}(\mathcal{A}) + r_{t_n}(\hat{\mathcal{A}})\right) x_{t_n}^{m-1} \geq r_{t_n}(\hat{\mathcal{A}}) x_{t_{n-r}}^{m-1}.$$

Similarly, from
$$\sum_{i_2, \cdots, i_m = 1}^{n} a_{t_{n-r} i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} = \rho(\mathcal{A}) x_{t_{n-r}}^{m-1},$$
we obtain
$$(\rho(\mathcal{A}) - a_{t_{n-r} \cdots t_{n-r}}) x_{t_{n-r}}^{m-1} \geq r_{t_{n-r}}(\mathcal{A}) x_{t_n}^{m-1}.$$

Therefore, we have

$$\begin{aligned} &\left(\rho(\mathcal{A}) - a_{t_n \cdots t_n} - r_{t_n}(\mathcal{A}) + r_{t_n}(\hat{\mathcal{A}})\right)(\rho(\mathcal{A}) - a_{t_{n-r} \cdots t_{n-r}}) x_{t_n}^{m-1} x_{t_{n-r}}^{m-1} \\ &\geq r_{t_n}(\hat{\mathcal{A}}) r_{t_{n-r}}(\mathcal{A}) x_{t_n}^{m-1} x_{t_{n-r}}^{m-1}; \end{aligned}$$

that is,
$$\rho(\mathcal{A}) \geq s_{t_n t_{n-r}}(\mathcal{A}) \geq \min_{e_{ij} \in \Gamma(\hat{\mathcal{A}})} s_{ij}(\mathcal{A}).$$

□

The estimation of the spectral radius of a general nonnegative tensor has the following result.

Theorem 16. *If* $\mathcal{A} = \left(a_{i_1 i_2 \cdots i_m}\right) \in \mathbb{R}_+^{[m,n]}$*, then*

$$\min_{i \neq j} s_{ij}(\mathcal{A}) \leq \rho(\mathcal{A}) \leq \max\left\{ \max_{e_{ij} \in \Gamma(\hat{\mathcal{A}})} s_{ij}(\mathcal{A}), \max_{i \in \langle n \rangle, \Gamma_i^+(\hat{\mathcal{A}}) = \varnothing} \bar{r}_i(\mathcal{A}) \right\},$$

where $s_{ij}(\mathcal{A})$*, see Theorem 15.*

Proof. We only need to prove the inequality on the left. Let $\mathcal{A} = \left(a_{i_1 i_2 \cdots i_m}\right) \in \mathbb{R}_+^{[m,n]}$ be reducible but not weakly irreducible. We construct nonnegative tensors $\mathcal{A}(\varepsilon) = \left(a_{i_1 i_2 \cdots i_m}(\varepsilon)\right) \in \mathbb{R}_+^{[m,n]}, \varepsilon > 0$, where

$$a_{i_1 i_2 \cdots i_m}(\varepsilon) = \begin{cases} a_{i_1 i_2 \cdots i_m} + \varepsilon, & \text{if } \delta_{i_1 i_2 \cdots i_m} = 0, \\ a_{i_1 i_2 \cdots i_m}, & \text{otherwise}; \end{cases}$$

then, $\mathcal{A}(\varepsilon)$ is weakly irreducible. Similar to the proof of Theorem 15, and with $\rho(\mathcal{A}(\varepsilon))$ as a continuous function of ε, letting $\varepsilon \to 0$, we obtain

$$\rho(\mathcal{A}) \geq \min_{i \neq j} s_{ij}(\mathcal{A}).$$

□

The following results show that Theorem 15 is an improvement of Theorem 12.

Theorem 17. *If* $\mathcal{A} = \left(a_{i_1 i_2 \cdots i_m}\right) \in \mathbb{R}_+^{[m,n]}$*, then*

$$\min_{i\in\langle n\rangle}\bar{r}_i(\mathcal{A}) \leq \min\left\{\min_{e_{ij}\in\Gamma(\hat{\mathcal{A}})} s_{ij}(\mathcal{A}), \min_{i\in\langle n\rangle,\Gamma_i^+(\hat{\mathcal{A}})=\varnothing} \bar{r}_i(\mathcal{A})\right\} \leq \rho(\mathcal{A})$$
$$\leq \max\left\{\max_{e_{ij}\in\Gamma(\hat{\mathcal{A}})} s_{ij}(\mathcal{A}), \max_{i\in\langle n\rangle,\Gamma_i^+(\hat{\mathcal{A}})=\varnothing} \bar{r}_i(\mathcal{A})\right\} \leq \max_{i\in\langle n\rangle}\bar{r}_i(\mathcal{A}),$$

where $s_{ij}(\mathcal{A})$, see Theorem 15.

Proof. Without loss of generality, suppose that for any $i,j\in\langle n\rangle, i\neq j, e_{ij}\in\Gamma_i^+(\hat{\mathcal{A}}), \bar{r}_i(\mathcal{A})\geq \bar{r}_j(A)$, we have

$$\begin{aligned}
&\left(a_{i\cdots i}-a_{j\cdots j}+r_i(\mathcal{A})-r_i(\hat{\mathcal{A}})\right)^2+4r_i(\hat{\mathcal{A}})r_j(\mathcal{A})\\
\leq&\left(a_{i\cdots i}-a_{j\cdots j}+r_i(\mathcal{A})-r_i(\hat{\mathcal{A}})\right)^2+4r_i(\hat{\mathcal{A}})\left(\bar{r}_i(\mathcal{A})-a_{j\cdots j}\right)\\
=&\left(a_{i\cdots i}-a_{j\cdots j}+r_i(\mathcal{A})-r_i(\hat{\mathcal{A}})\right)^2+4r_i(\hat{\mathcal{A}})\left(a_{i\cdots i}-a_{j\cdots j}+r_i(\mathcal{A})\right)\\
=&\left(a_{i\cdots i}-a_{j\cdots j}+r_i(\mathcal{A})+r_i(\hat{\mathcal{A}})\right)^2.
\end{aligned}$$

When $a_{i\cdots i}-a_{j\cdots j}+r_i(A)+r_i(\hat{A})\geq 0$, we have

$$\begin{aligned}
s_{ij}(\mathcal{A})=&\frac{1}{2}\left\{a_{i\cdots i}+a_{j\cdots j}+r_i(\mathcal{A})-r_i(\hat{\mathcal{A}})+\sqrt{\left(a_{i\cdots i}-a_{j\cdots j}+r_i(\mathcal{A})\right)^2+4r_i(\hat{\mathcal{A}})r_j(\mathcal{A})}\right\}\\
\leq&\frac{1}{2}\left\{a_{i\cdots i}+a_{j\cdots j}+r_i(\mathcal{A})-r_i(\hat{\mathcal{A}})+a_{i\cdots i}-a_{j\cdots j}+r_i(\mathcal{A})+r_i(\hat{\mathcal{A}})\right\}\\
=&\bar{r}_i(\mathcal{A})\leq\max_{i\in\langle n\rangle}\bar{r}_i(\mathcal{A}).
\end{aligned}$$

When $a_{i\cdots i}-a_{j\cdots j}+r_i(\mathcal{A})+r_i(\hat{\mathcal{A}})<0$, we have

$$\begin{aligned}
s_{ij}(\mathcal{A})=&\frac{1}{2}\left\{a_{i\cdots i}+a_{j\cdots j}+r_i(\mathcal{A})-r_i(\hat{\mathcal{A}})+\sqrt{\left(a_{i\cdots i}-a_{j\cdots j}+r_i(\mathcal{A})\right)^2+4r_i(\hat{\mathcal{A}})r_j(\mathcal{A})}\right\}\\
\leq&\frac{1}{2}\left\{a_{i\cdots i}+a_{j\cdots j}+r_i(\mathcal{A})-r_i(\hat{\mathcal{A}})-\left(a_{i\cdots i}-a_{j\cdots j}+r_i(\mathcal{A})+r_i(\hat{\mathcal{A}})\right)\right\}\\
=&a_{j\cdots j}-r_i(\hat{\mathcal{A}})\leq\max_{i\in\langle n\rangle}\bar{r}_i(\mathcal{A}).
\end{aligned}$$

From Theorem 12, we have

$$\rho(\mathcal{A})\leq\max\left\{\max_{e_{ij}\in\Gamma(\hat{\mathcal{A}})} s_{ij}(\mathcal{A}), \max_{i\in\langle n\rangle,\Gamma_i^+(\hat{\mathcal{A}})=\varnothing} \bar{r}_i(\mathcal{A})\right\}\leq\max_{i\in\langle n\rangle}\bar{r}_i(\mathcal{A}).$$

Similar to the above proof of the theorem, we have

$$\min_{i\in\langle n\rangle}\bar{r}_i(\mathcal{A})\leq\min\left\{\min_{e_{ij}\in\Gamma(\hat{\mathcal{A}})} s_{ij}(\mathcal{A}), \min_{i\in\langle n\rangle,\Gamma_i^+(\hat{\mathcal{A}})=\varnothing} \bar{r}_i(\mathcal{A})\right\}\leq\rho(\mathcal{A}).$$

□

Example 3. *Let*

$$A(1,:,:)=\begin{pmatrix}a_{111}&a_{112}&a_{113}\\a_{121}&a_{122}&a_{123}\\a_{131}&a_{132}&a_{133}\end{pmatrix}=\begin{pmatrix}5&1&3\\2&2&4\\3&6&2\end{pmatrix},$$

$$A(2,:,:) = \begin{pmatrix} a_{211} & a_{212} & a_{213} \\ a_{221} & a_{222} & a_{223} \\ a_{231} & a_{232} & a_{233} \end{pmatrix} = \begin{pmatrix} 3 & 4 & 5 \\ 3 & 8 & 2 \\ 3 & 4 & 1 \end{pmatrix},$$
$$A(3,:,:) = \begin{pmatrix} a_{311} & a_{312} & a_{313} \\ a_{321} & a_{322} & a_{323} \\ a_{331} & a_{332} & a_{333} \end{pmatrix} = \begin{pmatrix} 2 & 5 & 6 \\ 4 & 2 & 6 \\ 0 & 3 & 8 \end{pmatrix}.$$

Thus, $\rho(\mathcal{A}) = 32.1135$. From Theorem 17, we obtain

$$\begin{aligned} &\bar{r}_1(\mathcal{A}) = 28, \bar{r}_2(\mathcal{A}) = 32, \bar{r}_3(\mathcal{A}) = 26, \\ &a_{111} + \bar{r}_1^{[1]}(\mathcal{A}) = 14, a_{222} + \bar{r}_2^{[2]}(\mathcal{A}) = 21, a_{333} + \bar{r}_3^{[3]}(\mathcal{A}) = 23, \\ &r_{12}(\mathcal{A}) \approx 29.9353, r_{13}(\mathcal{A}) \approx 30.1285, r_{21}(\mathcal{A}) \approx 30.4536, \\ &r_{23}(\mathcal{A}) \approx 28.2082, r_{31}(\mathcal{A}) \approx 33.1208, r_{32}(\mathcal{A}) \approx 34.2829. \end{aligned}$$

Therefore,

$$28.2082 \le \rho(\mathcal{A}) \le 34.2829.$$

From Theorem 12,

$$28 \le \rho(A) \le 36.$$

Example 4. *Let*

$$A(1,:,:) = \begin{pmatrix} a_{111} & a_{112} & a_{113} \\ a_{121} & a_{122} & a_{123} \\ a_{131} & a_{132} & a_{133} \end{pmatrix} = \begin{pmatrix} 3 & 1 & 3 \\ 2 & 2 & 5 \\ 3 & 6 & 1 \end{pmatrix},$$
$$A(2,:,:) = \begin{pmatrix} a_{211} & a_{212} & a_{213} \\ a_{221} & a_{222} & a_{223} \\ a_{231} & a_{232} & a_{233} \end{pmatrix} = \begin{pmatrix} 0 & 2 & 5 \\ 2 & 5 & 4 \\ 6 & 5 & 0 \end{pmatrix},$$
$$A(3,:,:) = \begin{pmatrix} a_{311} & a_{312} & a_{313} \\ a_{321} & a_{322} & a_{323} \\ a_{331} & a_{332} & a_{333} \end{pmatrix} = \begin{pmatrix} 3 & 4 & 6 \\ 1 & 5 & 2 \\ 2 & 1 & 7 \end{pmatrix}.$$

We know that $\mathcal{A}$ is weakly irreducible, and

$$\Gamma(\hat{\mathcal{A}}) = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 5 & 0 \\ 3 & 5 & 7 \end{pmatrix}.$$

Thus, $\rho(A) = 28.8482$. From Theorem 15, we obtain

$$\begin{aligned} &\bar{r}_1(\mathcal{A}) = 26, \bar{r}_2(\mathcal{A}) = 29, \bar{r}_3(\mathcal{A}) = 31, \\ &s_{13}(\mathcal{A}) \approx 26.9146, s_{31}(\mathcal{A}) \approx 29.8523, s_{32}(\mathcal{A}) \approx 30.5227. \end{aligned}$$

Therefore,

$$26.3693 \le \rho(\mathcal{A}) \le 30.5227.$$

From Theorem 12,

$$26 \le \rho(\mathcal{A}) \le 31.$$

5. Conclusions

In this paper, by systematically analyzing the structure of tensors, a new classification method was used to define a class of quasi-double diagonally dominant tensors, and another class of quasi-double diagonally dominant tensors was defined by applying the digraph of the majorization matrix of a tensor, proving that they were $\mathcal{H}$-tensors and further extending the determination conditions of $\mathcal{H}$-tensors. Moreover, inequalities for estimating the upper and lower bounds for the spectral radius (the largest H-eigenvalue) of nonnegative tensors were given based on the relationship between the diagonal dominance of the tensor ($\mathcal{H}$-tensor) and the inclusion domain of the eigenvalues of the tensor, and these inequalities improved the Perron–Frobenius inequality for estimating the upper and lower bounds for the spectral radius of nonnegative tensors. This paper provides new ways of thinking to provide more refined determination conditions for the $\mathcal{H}$-tensor and to improve the inequalities for estimating the upper and lower bounds of the spectral radius of the nonnegative tensor.

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