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Ostrowski Type Inequalities via Some Exponentially s-Preinvex Functions on Time Scales with Applications

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Abstract: Integral inequalities concerned with convexity have many applications in several fields of mathematics in which symmetry plays an important role. In the theory of convexity, there exist strong connections between convexity and symmetry. If we are working on one of the concepts, then it can be applied to the other of them. In this paper, we establish some novel generalizations of Ostrowski type inequalities for exponentially s-preinvex and s-preinvex functions on time scale by using Hölder inequality and Montgomery Identity. We also obtain applications to some special means. These results are motivated by the symmetric results obtained in the recent article by Abbasi and Anwar in 2022 on Ostrowski type inequalities for exponentially s-convex functions and s-convex functions on time scale. Moreover, we discuss several special cases of the results obtained in this paper.

Keywords: Ostrowski type inequalities; s-preinvex functions; exponentially s-preinvex functions; time scales

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1. Introduction

The evolution of the theory of time scales was introduced by Hilger [1] in 1988, which initiated the survey of dynamic equations on time scales. This helps to demonstrate the results of differential inequality and again for difference inequality. Ahlbrandt et al. [2] derived a time scale T as a non-empty subset of the real numbers with the characteristic that every Cauchy sequence in T converges to a point of T, with the possible exception of Cauchy sequences converging to a finite infimum or finite supremum of T. Additionally, this concept has been studied by some authors, see, [3,4]. Time scale calculus has applications in several fields such as physics, biology, engineering, image processing, fluid dynamics, see [5–8]. If a function is defined on a time scale, we can consider the derivative and also the integral. For example: The time scale integral became an ordinary integral, Sum and Jackson integral when we consider time-scale as a set of real numbers, the set of all integers and the set of all integer powers of a fixed number, respectively. After that, many authors investigated the time scale versions of several aspects of the theory of dynamic inequalities that essentially depend on integral inequalities. Dinu [9] established the Hermite-Hadamard inequality for convex functions on time scales. Further, Lai et al. [10] obtained Hermite–Hadamard type inequality for the class of strongly convex function on time scales.

In 1938, Ostrowski gave a formula to evaluate the deviation of differentiable functions from its integral mean which is discussed in [11] named as the Ostrowski inequality, as follows:

Let $\psi : [m_1, m_2] \to \mathbb{R}$ be a differentiable mapping on (m_1, m_2) whose derivative $\psi' : (m_1, m_2) \to \mathbb{R}$ is bounded on (m_1, m_2), i.e., $\|\psi'\|_\infty = \sup_{h\in(m_1,m_2)} |\psi'(h)| < \infty$. Then, the following inequality holds:

$$\left|\psi(h) - \frac{1}{m_2 - m_1}\int_{m_1}^{m_2} \psi(g)dg\right|$$
$$\leq \sup_{m_1 \leq h \leq m_2} |\psi'(h)|(m_2 - m_1)\left[\frac{(h - \frac{(m_1+m_2)}{2})^2}{(m_2 - m_1)^2} + \frac{1}{4}\right], \forall\, h \in [m_1, m_2]. \tag{1}$$

This inequality is proved by using Montgomery identity as shown in [12]. Further, this identity on time scale was studied by M. Bohner and T. Matthews in [4]. The Ostrowski inequality has many applications in numerical analysis and in probability, many researchers have established generalizations, extensions and variants of inequality (1). We refer readers to [13–16]. This inequality is considered by many researchers as a function of bounded variation, Lipschitzian, monotonic, absolutely continuous and n-times differentiable mappings with error estimates with some special means together with some numerical quadrature rules. In 2019, Basci and Baleanu [17] gave new Ostrowski-type inequalities for both left and right sided fractional integrals of a function g with respect to another function ψ. Further Erden et al. [18] introduced some fractional Ostrowski-type inequalities for class of function L_P, L_∞, L_1 involving Riemann–Liouville fractional integrals for partially differentiable functions. Sarikaya and Filiz [19] introduced some Ostrowski-type integral inequalities for some differentiable mapping by using the Riemann–Liouville fractional integrals. In 2022, Hyder et al. [20] gave the Hermite–Hadamard inequality through generalized Riemann–Liouville fractional integral for a function with convex absolute values of derivative. In our paper, we establish Ostrowski type inequalities by using $\triangle$-integral for a differentiable function and its delta derivative is exponentially s-preinvex function and also bounded.

The concept of convexity has a great role in the field of integral inequality and mathematical analysis. Recently, several researchers have explored the close connection and interrelated work on convexity and symmetry. Hanson [21] established a new class of generalized convexity, which is known as invexity. In 1986, B. Isral and B. Mond [22] gave the concept of preinvex functions which is a special case of invexity. For more instances, see [23–25].

Recently, Abbasi and Anwar [26] investigated Ostrowski type inequalities for exponentially s-convex functions and s-convex functions on a time scale and also obtained several results which are essentially based on Ostrowski inequality.

The work is organized in the following way: In Section 2, we give some basic introduction into the time scales theory. In Section 3, we prove Hermite–Hadamard type inequality and Ostrowski type inequalities for exponentially s-preinvex functions and s-preinvex functions on time scales also we discuss some special cases when $T = \mathbb{R}$ then $\triangle$-integral became a classical integral. In Section 4, we obtain the applications to some special means. In Section 5, we present the conclusions of the present work.

2. Preliminaries

In this section, we give some definitions and results which is necessary for our main results.

Definition 1 ([26]). *A time scale (or measure chain) is a non-empty closed subset of the real numbers $\mathbb{R}$.*

The two most popular examples of time scale are $T = \mathbb{R}$ (set of real number) and $T = \mathbb{Z}$ (set of integers). Any (open or closed) interval I of $\mathbb{R}$, $I_T = I \cap T$ is called a time scale interval. Limit set $\{0\} \cup \{\frac{1}{n}\}$, $n = 1,2\ldots$, Cantor set, etc. are the examples of time scale. The forward and backward jumped operators σ, $\rho : T \to \mathbb{R}$ are defined by $\sigma(h) = inf\{\tau \in T : \tau > h\} \in T$, $\rho(h) = sup\{\tau \in T : \tau < h\} \in T$. Supplemented by $inf\emptyset = supT$ and $sup\emptyset = infT$, where $\emptyset$ denotes the empty set.

A point h is said to be right-scattered or left-scattered if $\sigma(h) > h$ and $\rho(h) < h$ respectively, h is said to be isolated if it is both right and left-scattered. If $\sigma(h) = h$ and $\rho(h) = h$, then the point h is called right dense and left dense, respectively, and it is said to be dense if left and right dense both.

Suppose $u_1 \in T$ is right-scattered minimum, then $T_k = T - \{u_1\}$, otherwise, $T_k = T$. Suppose $u_2 \in T$ is left-scattered maximum, then $T^k = T - \{u_2\}$, otherwise $T^k = T$. Moreover, $T_k^k = T_k \cap T^k$.

Definition 2 ([26]). *Let $\psi : T \to \mathbb{R}$ be a function then $\psi^\sigma : T \to \mathbb{R}$ is defined by $\psi^\sigma(h) = \psi(\sigma(h))$ for $h \in T$, where $\sigma(h)$ is defined as above. We also say that $\psi : T \to \mathbb{R}$ is delta derivative function at $h \in T^k$ is defined to be the number $\psi^\Delta(h)$ (if it exists) satisfying the property that, for any $\epsilon > 0$ there is a neighborhood U of h such that $\Big|[\psi(\sigma(h)) - \psi(g)] - \psi^\Delta(h)[\sigma(h) - g]\Big| < \epsilon|\sigma(h) - g|$ for all $g \in U$.*
If $T = \mathbb{R}$, then the delta derivative $\psi^\Delta = \psi'$, where ψ' is the derivative from continuous calculus. If $T = \mathbb{Z}$, then the delta derivative $\psi^\Delta = \Delta\psi$, where $\Delta\psi$ is the forward difference operator from discrete calculus.

Definition 3 ([3]). *A function $\psi : T \to \mathbb{R}$ is continuous at right dense points of T and its left-sided limit exist at left dense points of T, then ψ is known to be rd-continuous. Denoted by $\psi \in C_{rd}$.*

Theorem 1. *Suppose $\psi : T \to \mathbb{R}$ to be an rd-continuous function. Then, ψ has an anti-derivative Ξ satisfying $\Xi^\Delta = \psi$.*

Proof. See Theorem 1.74 of [3]. □

Definition 4 ([3]). *If $\psi : T \to \mathbb{R}$ is an rd-continuous function and $m_1 \in T$, then we define the integral $\Xi(h) = \int_{m_1}^{h} \psi(\tau)\Delta\tau$ for $h \in T$.*
Therefore, for $\psi \in C_{rd}$, we have $\Xi(m_1 + \eta(m_2, m_1)) - \Xi(m_1) = \int_{m_1}^{m_1+\eta(m_2,m_1)} \psi(\tau)\Delta\tau$, where $\Xi^\Delta = \psi$.

Theorem 2. *If $m_1, m_2, m_3 \in T$, $\beta \in \mathbb{R}$ and $\psi_1, \psi_2 \in C_{rd}$, then*
(i) $\int_{m_1}^{m_2}(\psi_1(h) + \psi_2(h))\Delta h = \int_{m_1}^{m_2} \psi_1(h)\Delta h + \int_{m_1}^{m_2} \psi_2(h)\Delta h$.
(ii) $\int_{m_1}^{m_2} \beta\psi_1(h)\Delta h = \beta \int_{m_1}^{m_2} \psi_1(h)\Delta h$,
(iii) $\int_{m_1}^{m_2} \psi_1(h)\Delta h = -\int_{m_2}^{m_1} \psi_1(h)\Delta h$,
(iv) $\int_{m_1}^{m_2} \psi_1(h)\Delta h = \int_{m_1}^{m_3} \psi_1(h)\Delta h + \int_{m_3}^{m_2} \psi_1(h)\Delta h$,
(v) $\int_{m_1}^{m_1} \psi_1(h)\Delta h = 0$,
(vi) $\int_{m_1}^{m_2} \psi_1(h)\psi_2^\Delta(h)\Delta h = (\psi_1\psi_2)(m_2) - (\psi_1\psi_2)(l_1) - \int_{m_1}^{m_2} \psi_1^\Delta(h)\psi_2(\sigma(h))\Delta h$,

Proof. See Theorem 1.77 of [3]. □

Theorem 3. *(Hölder's Inequality) Let m_1,$m_2 \in T$ and ψ_1,$\psi_2 : T \to \mathbb{R}$ be rd-continuous. Then,*

$$\int_{m_1}^{m_2} |\psi_1(h)\psi_2(h)|\Delta h \leq \left(\int_{m_1}^{m_2} |\psi_1(h)|^p \Delta h\right)^{\frac{1}{p}} \left(\int_{m_1}^{m_2} |\psi_2(h)|^q \Delta h\right)^{\frac{1}{q}}, \tag{2}$$

where $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. See Theorem 6.13 of [3]. □

The Ostrowski inequality on time scale was discussed by M. Bohner and T. Matthews in [4], which is given as

Lemma 1. *Suppose m_1,m_2, g, $h \in T$, $m_1 < m_2$ and $\psi : [m_1, m_2] \to \mathbb{R}$ be differentiable. Then,*

$$\psi(h) = \frac{1}{m_2 - m_1} \int_{m_1}^{m_2} \psi^\sigma(g)\Delta g + \frac{1}{m_2 - m_1} \int_{m_1}^{m_2} \chi(h, g)\psi^\Delta(g)\Delta g, \tag{3}$$

where $\chi(h, g) = \begin{cases} g - m_1, & m_1 \leq g < h \\ g - m_2, & h \leq g \leq m_2 \end{cases}$

In 2013, Wang et al. [27] introduced a function known as s-preinvex function.

Definition 5. *Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}^n$. A function $\psi : S \to \mathbb{R}_0 = [0, \infty)$ is said to be s-preinvex with respect to η and $s \in (0, 1]$ if for every $m_1, m_2 \in S$ and $\tau \in [0, 1]$*

$$\psi(m_1 + \tau\eta(m_2, m_1)) \leq \tau^s \psi(m_2) + (1 - \tau)^s \psi(m_1). \tag{4}$$

Safdar and Attique [28] introduced the concept of exponentially s-preinvex function.

Definition 6. *Let $s \in (0, 1]$ and a real-valued mapping ψ on the invex set Ω is said to be exponentially s-preinvex with respect to $\eta(.,.)$, if the inequality*

$$\psi(m_1 + \tau\eta(m_2, m_1)) \leq (1 - \tau)^s \frac{\psi(m_1)}{e^{\alpha m_1}} + \tau^s \frac{\psi(m_2)}{e^{\alpha m_2}}, \tag{5}$$

holds for all $m_1, m_1 + \eta(m_2, m_1) \in \Omega$, $\tau \in [0, 1]$, and $\alpha \in \mathbb{R}$.

Condition C: Let $A \subseteq \mathbb{R}^n$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$ we say that the function η satisfies the condition C if for any $m_1, m_2 \in A$ and any $t \in [0, 1]$,

$$\begin{aligned} \eta(m_2, m_2 + t\eta(m_1, m_2)) &= -t\eta(m_1, m_2), \\ \eta(m_1, m_2 + t\eta(m_1, m_2)) &= (1 - t)\eta(m_1, m_2) \end{aligned} \tag{6}$$

and from condition C,

$$\eta(m_2 + t_2\eta(m_1, m_2), m_2 + t_1\eta(m_1, m_2)) = (t_2 - t_1)\eta(m_1, m_2) \tag{7}$$

3. Main Results

In this section, first, we prove the Hermite–Hadamard inequality for exponentially s-preinvex functions on time scale.

Theorem 4. *Let T be a time scale and $H = [m_1, m_1 + \eta(m_2, m_1)]$. Let $\psi : H \to \mathbb{R}$ is exponentially s-preinvex function on H° and Δ-integrable as well. Then, for $m_1, m_1 + \eta(m_2, m_1) \in H$ with $m_1 < m_1 + \eta(m_2, m_1)$ and $\alpha \in \mathbb{R}$, we have*

$$\begin{aligned} 2^{s-1}\psi\Big(m_1 + \frac{\eta}{2}(m_2, m_1)\Big) &\leq \frac{1}{\eta(m_2, m_1)} \int_{m_1}^{m_1+\eta(m_2,m_1)} \frac{\psi(t)}{e^{\alpha t}} \Delta t \\ &\leq \frac{\psi(m_1)}{e^{\alpha m_1}} \int_0^1 \frac{(1-\tau)^s}{e^{\alpha(m_1+\tau\eta(m_2,m_1))}} \Delta\tau \\ &\quad + \frac{\psi(m_1 + \eta(m_2, m_1))}{e^{\alpha(m_1+\eta(m_2,m_1))}} \int_0^1 \frac{\tau^s}{e^{\alpha(m_1+\tau\eta(m_2,m_1))}} \Delta\tau. \end{aligned} \tag{8}$$

Proof. Since ψ is an exponential s-preinvex function, we have

$$2^s\psi\Big(c + \frac{\eta}{2}(d, c)\Big) \leq \frac{\psi(c)}{e^{\alpha c}} + \frac{\psi(d)}{e^{\alpha d}}.$$

Making use of change of variable $c = m_1 + \tau\eta(m_2, m_1)$ and $d = m_1 + (1 - \tau)\eta(m_2, m_1)$ with using condition C and taking $\Delta - integrable$ with respect to $\tau \in [0, 1]$, we obtain

$$2^s\psi\Big(m_1 + \frac{\eta}{2}(m_2, m_1)\Big) \leq \frac{2}{\eta(m_2, m_1)} \int_{m_1}^{m_1+\eta(m_2,m_1)} \frac{\psi(t)}{e^{\alpha t}} \Delta t$$

and

$$2^{s-1}\psi\Big(m_1 + \frac{\eta}{2}(m_2, m_1)\Big) \leq \frac{1}{\eta(m_2, m_1)} \int_{m_1}^{m_1+\eta(m_2,m_1)} \frac{\psi(t)}{e^{\alpha t}} \Delta t. \tag{9}$$

Now, we prove the second inequality

$$\frac{\psi(m_1+\tau\eta(m_2,m_1))}{e^{\alpha(m_1+\tau\eta(m_2,m_1))}} \leq \frac{(1-\tau)^s\frac{\psi(m_1)}{e^{\alpha m_1}}+\tau^s\frac{\psi(m_2)}{e^{\alpha m_2}}}{e^{\alpha(m_1+\tau\eta(m_2,m_1))}}$$

Taking $\Delta - integral$ with respect to $\tau \in [0,1]$, we obtain

$$\begin{aligned}\frac{1}{\eta(m_2,m_1)}\int_0^1 \frac{\psi(t)}{e^{\alpha t}}\Delta t \leq \frac{\psi(m_1)}{e^{\alpha m_1}}\int_0^1 \frac{(1-\tau)^s}{e^{\alpha(m_1+\tau\eta(m_2,m_1))}}\Delta\tau \\ + \frac{\psi(m_1+\eta(m_2,m_1))}{e^{\alpha(m_1+\eta(m_2,m_1))}}\int_0^1 \frac{\tau^s}{e^{\alpha(m_1+\tau\eta(m_2,m_1))}}\Delta\tau. \end{aligned} \tag{10}$$

Combining (9) and (10), we obtain inequality (8). □

Now, we will discuss Ostrowski inequality for exponentially s-preinvex function on time scale.

Theorem 5. *Let T be a time scale and $H \subseteq T$. Let $\psi : H \to \mathbb{R}$ be a differentiable function on H° such that $\psi^\Delta \in H$ for $m_1, m_1+\eta(m_2,m_1) \in K$ where $m_1 < m_1+\eta(m_2,m_1)$. If ψ^Δ is exponentially s-preinvex on $[m_1, m_1+\eta(m_2,m_1)]$ for $s \in (0,1]$ and $\sup_{m_1 \leq h \leq m_1+\eta(m_2,m_1)} |\psi^\Delta(h)| = M$, $h \in [m_1, m_1+\eta(m_2,m_1)]$. Then, the following inequality holds:*

$$\begin{aligned}&\left|\psi(h) - \frac{1}{\eta(m_2,m_1)}\int_{m_1}^{m_1+\eta(m_2,m_1)} \psi^\sigma(g)\Delta g\right| \\ &\leq \frac{M(h-m_1)^2}{\eta(m_2,m_1)}\int_0^1 \left(\frac{\tau(2-\tau)^s}{e^{\alpha h}} + \frac{\tau(\tau-1)^s}{e^{\alpha m_1}}\right)\Delta\tau \\ &+ \frac{M(h-m_2)^2}{\eta(m_2,m_1)}\int_0^1 \left(\frac{\tau(1-\tau)^s}{e^{\alpha h}} + \frac{\tau^{s+1}}{e^{\alpha(m_1+\eta(m_2,m_1))}}\right)\Delta\tau. \end{aligned} \tag{11}$$

Proof. Using Montgomery identity,

$$\begin{aligned}&\left|\psi(h) - \frac{1}{\eta(m_2,m_1)}\int_{m_1}^{m_1+\eta(m_2,m_1)} \psi^\sigma(g)\Delta g\right| \\ &= \left|\frac{1}{\eta(m_2,m_1)}\int_{m_1}^{m_1+\eta(m_2,m_1)} \chi(h,g)\psi^\Delta(g)\Delta g\right| \\ &\leq \frac{1}{\eta(m_2,m_1)}\left(\int_{m_1}^{h}(g-m_1)|\psi^\Delta(g)|\Delta g \right. \\ &\left. + \int_h^{m_1+\eta(m_2,m_1)} (g-(m_1+\eta(m_2,m_1)))|\psi^\Delta(g)|\Delta g\right). \end{aligned} \tag{12}$$

Making use of change of variables, we obtain

$$\begin{aligned}&\left|\psi(h) - \frac{1}{\eta(m_2,m_1)}\int_{m_1}^{m_1+\eta(m_2,m_1)} \psi^\sigma(g)\Delta g\right| \\ &\leq \frac{1}{\eta(m_2,m_1)}\int_0^1 (h-m_1)^2\tau\left|\psi^\Delta(h+(\tau-1)\eta(m_1,h)\right|\Delta\tau \\ &+ \frac{1}{\eta(m_2,m_1)}\int_0^1 \Big(h-(m_1+\eta(m_2,m_1))\Big)^2\tau\left|\psi^\Delta(h+\tau\eta(m_2,h))\right|\Delta\tau. \end{aligned} \tag{13}$$

Using the definition of exponential s-preinvexity of ψ^Δ, we obtain

$$
\begin{aligned}
&\left|\psi(h)-\frac{1}{\eta(m_2,m_1)}\int_{m_1}^{m_1+\eta(m_2,m_1)}\psi^{\sigma}(g)\Delta g\right| \\
&\leq \frac{(h-m_1)^2}{\eta(m_2,m_1)}\int_0^1\left(\tau(2-\tau)^s\frac{|\psi^{\Delta}(h)|}{e^{\alpha h}}+\tau(\tau-1)^s\frac{|\psi^{\Delta}(m_1)|}{e^{\alpha m_1}}\right)\Delta\tau \\
&\quad+\frac{\big(h-(m_1+\eta(m_2,m_1))\big)^2}{\eta(m_2,m_1)}\int_0^1\left(\tau(1-\tau)^s\frac{|\psi^{\Delta}(h)|}{e^{\alpha h}}\right. \\
&\quad\left.+\tau(\tau)^s\frac{|\psi^{\Delta}(m_1+\eta(m_2,m_1))|}{e^{\alpha(m_1+\eta(m_2,m_1))}}\right)\Delta\tau \\
&\leq \frac{M(h-m_1)^2}{\eta(m_2,m_1)}\int_0^1\left(\frac{\tau(2-\tau)^s}{e^{\alpha h}}+\frac{\tau(\tau-1)^s}{e^{\alpha m_1}}\right)\Delta\tau \\
&\quad+\frac{M\big(h-(m_1+\eta(m_2,m_1))\big)^2}{\eta(m_2,m_1)}\int_0^1\left(\frac{\tau(1-\tau)^s}{e^{\alpha h}}+\frac{\tau^{s+1}}{e^{\alpha(m_1+\eta(m_2,m_1))}}\right)\Delta\tau.
\end{aligned}
$$

□

This completes the proof.

Remark 1. *If we consider α=0 in Theorem 5, we obtain the inequality (21).*

Theorem 6. *Suppose that $\psi : H \to \mathbb{R}$ is a differentiable mapping on H° such that $\psi^{\Delta} \in H$ for $m_1, m_1+\eta(m_2,m_1) \in H$ with $m_1 < m_1+\eta(m_2,m_1)$. If $|\psi^{\Delta}|^q$ is exponentially s-preinvex on $[m_1, m_1+\eta(m_2,m_1)]$ for some $s \in (0,1]$, $p,q > 1$ and $\frac{1}{p}+\frac{1}{q}=1$ $\sup_{m_1 \leq h \leq m_1+\eta(m_2,m_1)} |\psi^{\Delta}(h)| = M$, $h \in [m_1, m_1+\eta(m_2,m_1)]$, then the following inequality holds:*

$$
\begin{aligned}
&\left|\psi(h)-\frac{1}{\eta(m_2,m_1)}\int_{m_1}^{m_1+\eta(m_2,m_1)}\psi^{\sigma}(g)\Delta g\right| \\
&\leq \frac{M(h-m_1)^2}{\eta(m_2,m_1)}\Big(\int_0^1\tau^p\Delta\tau\Big)^{\frac{1}{p}}\left(\int_0^1\Big(\frac{(2-\tau)^s}{e^{\alpha h}}+\frac{(\tau-1)^s}{e^{\alpha m_1}}\Big)\Delta\tau\right)^{\frac{1}{q}} \\
&\quad+\frac{M(h-(m_1+\eta(m_2,m_1)))^2}{\eta(m_2,m_1)}\Big(\int_0^1\tau^p\Delta\tau\Big)^{\frac{1}{p}} \\
&\quad\times\left(\int_0^1\Big(\frac{\tau^s}{e^{\alpha h}}+\frac{(1-\tau)^s}{e^{\alpha(m_1+\eta(m_2,m_1))}}\Big)\Delta\tau\right)^{\frac{1}{q}}.
\end{aligned}
\tag{14}
$$

Proof. By the Montgomery identity, we have

$$
\begin{aligned}
&\left|\psi(h)-\frac{1}{\eta(m_2,m_1)}\int_{m_1}^{m_1+\eta(m_2,m_1)}\psi^{\sigma}(g)\Delta g\right| \\
&=\left|\frac{1}{\eta(m_2,m_1)}\int_{m_1}^{m_1+\eta(m_2,m_1)}\chi(h,g)\psi^{\Delta}(g)\Delta g\right| \\
&\leq \frac{1}{\eta(m_2,m_1)}\left(\int_{m_1}^{h}(g-m_1)|\psi^{\Delta}(g)|\Delta g\right. \\
&\quad\left.+\int_{h}^{m_1+\eta(m_2,m_1)}(g-(m_1+\eta(m_2,m_1)))|\psi^{\Delta}(g)|\Delta g\right).
\end{aligned}
$$

Making use of change of variables, we obtain

$$\begin{aligned}
&\left|\psi(h)-\frac{1}{\eta(m_2,m_1)}\int_{m_1}^{m_1+\eta(m_2,m_1)}\psi^{\sigma}(g)\Delta g\right| \\
&\le \frac{1}{\eta(m_2,m_1)}\int_0^1 (h-m_1)^2\tau\left|\psi^{\Delta}(h+(\tau-1)\eta(m_1,h)\right|\Delta\tau \\
&+\frac{1}{\eta(m_2,m_1)}\int_0^1 \Big(h-\big(m_1+\eta(m_2,m_1)\big)\Big)^2\tau\left|\psi^{\Delta}(h+\tau\eta(m_2,h))\right|\Delta\tau.
\end{aligned}$$

Using (2), we obtain

$$\begin{aligned}
&\left|\psi(h)-\frac{1}{\eta(m_2,m_1)}\int_{m_1}^{m_1+\eta(m_2,m_1)}\psi^{\sigma}(g)\Delta g\right| \\
&\le \frac{(h-m_1)^2}{\eta(m_2,m_1)}\left(\int_0^1\tau^p\Delta\tau\right)^{\frac{1}{p}}\left(\int_0^1\left|\psi^{\Delta}(h+(\tau-1)\eta(m_1,h))\right|^q\Delta\tau\right)^{\frac{1}{q}} \\
&+\frac{(h-(m_1+\eta(m_2,m_1)))^2}{\eta(m_2,m_1)}\left(\int_0^1\tau^p\Delta\tau\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1\left|\psi^{\Delta}(h+\tau\eta(m_2,h))\right|^q\Delta\tau\right)^{\frac{1}{q}}. \qquad (15)
\end{aligned}$$

Using the definition of exponential s-preinvexity of $|\psi^{\Delta}|^q$, we have

$$\begin{aligned}
&\left|\psi(h)-\frac{1}{\eta(m_2,m_1)}\int_{m_1}^{m_1+\eta(m_2,m_1)}\psi^{\sigma}(g)\Delta g\right| \\
&\le \frac{(h-m_1)^2}{\eta(m_2,m_1)}\left(\int_0^1\tau^p\Delta\tau\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1\Big((2-\tau)^s\frac{|\psi^{\Delta}(h)|^q}{e^{\alpha h}}+(\tau-1)^s\frac{|\psi^{\Delta}(m_1)|^q}{e^{\alpha m_1}}\Big)\Delta\tau\right)^{\frac{1}{q}} \\
&+\frac{(h-(m_1+\eta(m_2,m_1)))^2}{\eta(m_2,m_1)}\left(\int_0^1\tau^p\Delta\tau\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1\Big(\tau^s\frac{|\psi^{\Delta}(h)|^q}{e^{\alpha h}}+(1-\tau)^s\frac{|\psi^{\Delta}(m_1+\eta(m_2,m_1))|^q}{e^{\alpha(m_1+\eta(m_2,m_1))}}\Big)\Delta\tau\right)^{\frac{1}{q}} \\
&\le \frac{M(h-m_1)^2}{\eta(m_2,m_1)}\left(\int_0^1\tau^p\Delta\tau\right)^{\frac{1}{p}}\left(\int_0^1\Big(\frac{(2-\tau)^s}{e^{\alpha h}}+\frac{(\tau-1)^s}{e^{\alpha m_1}}\Big)\Delta\tau\right)^{\frac{1}{q}} \\
&+\frac{M(h-(m_1+\eta(m_2,m_1)))^2}{\eta(m_2,m_1)}\left(\int_0^1\tau^p\Delta\tau\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1\Big(\frac{\tau^s}{e^{\alpha h}}+\frac{(1-\tau)^s}{e^{\alpha(m_1+\eta(m_2,m_1))}}\Big)\Delta\tau\right)^{\frac{1}{q}}. \qquad (16)
\end{aligned}$$

□

This completes the proof.

Remark 2. *If we take α=0 in Theorem 6, we obtain inequality (22).*

Theorem 7. *Suppose a differentiable mapping $\psi : H \to \mathbb{R}$ on H° such that $\psi^\Delta \in H$ for $m_1, m_1 + \eta(m_2, m_1) \in H$ with $m_1 < m_1 + \eta(m_2, m_1)$. If $|\psi^\Delta|^q$ is exponentially s-preinvex on $[m_1, m_1 + \eta(m_2, m_1)]$ for some $s \in (0, 1]$, $q > 1$ and $\sup_{m_1 \le h \le m_1 + \eta(m_2,m_1)} |\psi^\Delta(h)| = M$, $h \in [m_1, m_1 + \eta(m_2, m_1)]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\psi(h) - \frac{1}{\eta(m_2, m_1)} \int_{m_1}^{m_1+\eta(m_2,m_1)} \psi^\sigma(g)\Delta g\right| \\
&\le \frac{M(h - m_1)^2}{\eta(m_2, m_1)} \Big(\int_0^1 \tau\Delta\tau \Big)^{1-\frac{1}{q}} \left(\int_0^1 \Big(\frac{\tau(2-\tau)^s}{e^{\alpha h}} + \frac{\tau(\tau-1)^s}{e^{\alpha m_1}} \Big)\Delta\tau \right)^{\frac{1}{q}} \\
&\quad + \frac{M(h - (m_1 + \eta(m_2, m_1)))^2}{\eta(m_2, m_1)} \Big(\int_0^1 \tau\Delta\tau \Big)^{1-\frac{1}{q}} \\
&\quad \times \left(\int_0^1 \Big(\frac{\tau(1-\tau)^s}{e^{\alpha h}} + \frac{\tau^{s+1}}{e^{\alpha(m_1+\eta(m_2,m_1))}} \Big)\Delta\tau \right)^{\frac{1}{q}}.
\end{aligned} \tag{17}$$

Proof. By Montgomery identity, we have

$$\begin{aligned}
&\left|\psi(h) - \frac{1}{\eta(m_2, m_1)} \int_{m_1}^{m_1+\eta(m_2,m_1)} \psi^\sigma(g)\Delta g\right| \\
&= \left| \frac{1}{\eta(m_2, m_1)} \int_{m_1}^{m_1+\eta(m_2,m_1)} \chi(h, g)\psi^\Delta(g)\Delta g \right| \\
&\le \frac{1}{\eta(m_2, m_1)} \left(\int_{m_1}^{h} (g - m_1)|\psi^\Delta(g)|\Delta g \right. \\
&\quad \left. + \int_{h}^{m_1+\eta(m_2,m_1)} \big(g - (m_1 + \eta(m_2, m_1))\big)|\psi^\Delta(g)|\Delta g \right).
\end{aligned}$$

Making use of change of variables, we obtain

$$\begin{aligned}
&\left|\psi(h) - \frac{1}{\eta(m_2, m_1)} \int_{m_1}^{m_1+\eta(m_2,m_1)} \psi^\sigma(g)\Delta g\right| \\
&\le \frac{1}{\eta(m_2, m_1)} \int_0^1 (h - m_1)^2 \tau \big|\psi^\Delta(h + (\tau - 1)\eta(m_1, h)\big|\Delta\tau \\
&\quad + \frac{1}{\eta(m_2, m_1)} \int_0^1 \Big(h - \big(m_1 + \eta(m_2, m_1)\big)\Big)^2 \tau \big|\psi^\Delta(h + \tau\eta(m_2, h))\big|\Delta\tau.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left|\psi(h) - \frac{1}{\eta(m_2, m_1)} \int_{m_1}^{m_1+\eta(m_2,m_1)} \psi^\sigma(g)\Delta g\right| \\
&\le \frac{(h - m_1)^2}{\eta(m_2, m_1)} \Big(\int_0^1 \tau\Delta\tau \Big)^{1-\frac{1}{q}} \left(\int_0^1 \tau \Big|\psi^\Delta\big(h + (\tau - 1)\eta(m_1, h)\big)\Big|^q \Delta\tau \right)^{\frac{1}{q}} \\
&\quad + \frac{(h - (m_1 + \eta(m_2, m_1)))^2}{\eta(m_2, m_1)} \Big(\int_0^1 \tau\Delta\tau \Big)^{1-\frac{1}{q}} \left(\int_0^1 \tau \Big|\psi^\Delta\big(h + \tau\eta(m_2, h)\big)\Big|^q \Delta\tau \right)^{\frac{1}{q}}.
\end{aligned}$$

Applying the definition of exponential s-preinvexity of $|\psi^\Delta|^q$, we have

$$
\begin{aligned}
&\left|\psi(h)-\frac{1}{\eta(m_2,m_1)}\int_{m_1}^{m_1+\eta(m_2,m_1)}\psi^{\sigma}(g)\Delta g\right| \\
&\leq \frac{(h-m_1)^2}{\eta(m_2,m_1)}\Big(\int_0^1 \tau\Delta\tau\Big)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1\Big(\tau(2-\tau)^s\frac{|\psi^{\Delta}(h)|^q}{e^{\alpha h}}+\tau(\tau-1)^s\frac{|\psi^{\Delta}(m_1)|^q}{e^{\alpha m_1}}\Big)\Delta\tau\right)^{\frac{1}{q}} \\
&\quad+\frac{(h-(m_1+\eta(m_2,m_1)))^2}{\eta(m_2,m_1)}\Big(\int_0^1 \tau\Delta\tau\Big)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1\Big(\tau(1-\tau)^s\frac{|\psi^{\Delta}(h)|^q}{e^{\alpha h}}+\tau^{s+1}\frac{|\psi^{\Delta}(m_1+\eta(m_2,m_1))|^q}{e^{\alpha(m_1+\eta(m_2,m_1))}}\Big)\Delta\tau\right)^{\frac{1}{q}} \\
&\leq \frac{M(h-m_1)^2}{\eta(m_2,m_1)}\Big(\int_0^1 \tau\Delta\tau\Big)^{1-\frac{1}{q}}\left(\int_0^1\Big(\frac{\tau(2-\tau)^s}{e^{\alpha h}}+\frac{\tau(\tau-1)^s}{e^{\alpha m_1}}\Big)\Delta\tau\right)^{\frac{1}{q}} \\
&\quad+\frac{M(h-(m_1+\eta(m_2,m_1)))^2}{\eta(m_2,m_1)}\Big(\int_0^1 \tau\Delta\tau\Big)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1\Big(\frac{\tau(1-\tau)^s}{e^{\alpha h}}+\frac{\tau^{s+1}}{e^{\alpha(m_1+\eta(m_2,m_1))}}\Big)\Delta\tau\right)^{\frac{1}{q}}.
\end{aligned}
\tag{18}
$$

□

This completes the proof.

Corollary 1. *If we consider $T=\mathbb{R}$ in Theorem 7, we get the result for exponentially s-preinvex function.*

$$
\begin{aligned}
&\left|\psi(h)-\frac{1}{\eta(m_2,m_1)}\int_{m_1}^{m_1+\eta(m_2,m_1)}\psi(g)dg\right| \\
&\leq \frac{M(h-m_1)^2}{\eta(m_2,m_1)(2)^{1-\frac{1}{q}}}\left[\frac{2^{s+2}-(s+3)}{e^{\alpha h}(s+1)(s+2)}+\frac{1}{e^{\alpha m_1}(s+1)(s+2)}\right]^{\frac{1}{q}} \\
&\quad+\frac{M(h-(m_1+\eta(m_2,m_1)))}{\eta(m_2,m_1)(2)^{1-\frac{1}{q}}}\left[\frac{1}{e^{\alpha h}(s+1)(s+2)}+\frac{1}{e^{\alpha(m_1+\eta(m_2,m_1))}(s+2)}\right]^{\frac{1}{q}}.
\end{aligned}
\tag{19}
$$

Proof. By Montgomery identity, we have

$$
\begin{aligned}
&\left|\psi(h)-\frac{1}{\eta(m_2,m_1)}\int_{m_1}^{m_1+\eta(m_2,m_1)}\psi(g)dg\right| \\
&=\left|\frac{1}{\eta(m_2,m_1)}\int_{m_1}^{m_1+\eta(m_2,m_1)}\chi(h,g)\psi'(g)dg\right| \\
&\leq \frac{1}{\eta(m_2,m_1)}\left(\int_{m_1}^{h}(g-m_1)|\psi'(g)|dg\right. \\
&\quad\left.+\int_h^{m_1+\eta(m_2,m_1)}\Big(g-\big(m_1+\eta(m_2,m_1)\big)\Big)|\psi'(g)|dg\right).
\end{aligned}
$$

Using change of variable, we get

$$\begin{aligned}
&\left|\psi(h)-\frac{1}{\eta(m_2,m_1)}\int_{m_1}^{m_1+\eta(m_2,m_1)}\psi(g)dg\right| \\
&\leq \frac{1}{\eta(m_2,m_1)}\int_0^1 (h-m_1)^2\tau\left|\psi'(h+(\tau-1)\eta(m_1,h))\right|d\tau \\
&\quad+\frac{1}{\eta(m_2,m_1)}\int_0^1 \Big(h-(m_1+\eta(m_2,m_1))\Big)^2\tau\left|\psi'(h+\tau\eta(m_2,h))\right|d\tau.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left|\psi(h)-\frac{1}{\eta(m_2,m_1)}\int_{m_1}^{m_1+\eta(m_2,m_1)}\psi(g)dg\right| \\
&\leq \frac{(h-m_1)^2}{\eta(m_2,m_1)}\Big(\int_0^1 \tau d\tau\Big)^{1-\frac{1}{q}}\left(\int_0^1 \tau\left|\psi'(h+(\tau-1)\eta(m_1,h))\right|^q d\tau\right)^{\frac{1}{q}} \\
&\quad+\frac{(h-(m_1+\eta(m_2,m_1)))^2}{\eta(m_2,m_1)}\Big(\int_0^1 \tau d\tau\Big)^{1-\frac{1}{q}}\left(\int_0^1 \tau\left|\psi'(h+\tau\eta(m_2,h))\right|^q d\tau\right)^{\frac{1}{q}}.
\end{aligned}$$

Applying the definition of exponential s-preinvexity of $|\psi'|^q$, we have

$$\begin{aligned}
&\left|\psi(h)-\frac{1}{\eta(m_2,m_1)}\int_{m_1}^{m_1+\eta(m_2,m_1)}\psi(g)\Delta g\right| \\
&\leq \frac{(h-m_1)^2}{\eta(m_2,m_1)}\Big(\int_0^1 \tau d\tau\Big)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1 \Big(\tau(2-\tau)^s\frac{|\psi'(h)|^q}{e^{\alpha h}}+\tau(\tau-1)^s\frac{|\psi'(m_1)|^q}{e^{\alpha m_1}}\Big)d\tau\right)^{\frac{1}{q}} \\
&\quad+\frac{(h-(m_1+\eta(m_2,m_1)))^2}{\eta(m_2,m_1)}\Big(\int_0^1 \tau d\tau\Big)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1 \Big(\tau(1-\tau)^s\frac{|\psi'(h)|^q}{e^{\alpha h}}+\tau^{s+1}\frac{|\psi'(m_1+\eta(m_2,m_1))|^q}{e^{\alpha(m_1+\eta(m_2,m_1))}}\Big)d\tau\right)^{\frac{1}{q}} \\
&\leq \frac{M(h-m_1)^2}{\eta(m_2,m_1)}\Big(\int_0^1 \tau d\tau\Big)^{1-\frac{1}{q}}\left(\int_0^1 \Big(\frac{\tau(2-\tau)^s}{e^{\alpha h}}+\frac{\tau(\tau-1)^s}{e^{\alpha m_1}}\Big)d\tau\right)^{\frac{1}{q}} \\
&\quad+\frac{M(h-(m_1+\eta(m_2,m_1)))^2}{\eta(m_2,m_1)}\Big(\int_0^1 \tau d\tau\Big)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1 \Big(\frac{\tau(1-\tau)^s}{e^{\alpha h}}+\frac{\tau^{s+1}}{e^{\alpha(m_1+\eta(m_2,m_1))}}\Big)d\tau\right)^{\frac{1}{q}} \\
&=\frac{M(h-m_1)^2}{\eta(m_2,m_1)(2)^{1-\frac{1}{q}}}\left[\frac{2^{s+2}-(s+3)}{e^{\alpha h}(s+1)(s+2)}+\frac{1}{e^{\alpha m_1}(s+1)(s+2)}\right]^{\frac{1}{q}} \\
&\quad+\frac{M\big(h-(m_1+\eta(m_2,m_1))\big)^2}{\eta(m_2,m_1)(2)^{1-\frac{1}{q}}}\left[\frac{1}{e^{\alpha h}(s+1)(s+2)}+\frac{1}{e^{\alpha(m_1+\eta(m_2,m_1))}(s+2)}\right]^{\frac{1}{q}}. \qquad (20)
\end{aligned}$$

□

This completes the proof.

Theorem 8. *Suppose T be a time scale and $H = [m_1, m_1 + \eta(m_2, m_1)] \subseteq T$ such that $m_1 < m_1 + \eta(m_2, m_1) \in T$. Consider $\psi : H \to \mathbb{R}$ be a delta differentiable on H° such that $\psi^\Delta \in H$, for $m_1, m_1 + \eta(m_2, m_1) \in H$ with $m_1 < m_1 + \eta(m_2, m_1)$. If $|\psi^\Delta|$ is s-preinvex on H for some fixed $s \in (0, 1]$ and $\sup_{m_1 \le h \le m_1 + \eta(m_2, m_1)} |\psi^\Delta(h)| = M$ for $h \in H$, then following inequality holds:*

$$\begin{aligned}
&\left|\psi(h) - \frac{1}{\eta(m_2, m_1)} \int_{m_1}^{m_1+\eta(m_2,m_1)} \psi^\sigma(g)\Delta g\right| \\
&\le \frac{M(h - m_1)^2}{\eta(m_2, m_1)} \int_0^1 \Big(\tau(\tau - 1)^s + \tau(2 - \tau)^s\Big)\Delta\tau \\
&\quad + \frac{M\big(h - (m_1 + \eta(m_2, m_1))\big)^2}{\eta(m_2, m_1)} \int_0^1 \Big(\tau^{s+1} + \tau(\tau - 1)^s\Big)\Delta\tau.
\end{aligned} \tag{21}$$

Proof. The proof is related to Theorem 5 only difference is to use definition of s-preinvex function $|\psi^\Delta|$ instead of exponential s-preinvexity. □

Theorem 9. *Suppose T be a time scale and $H = [m_1, m_1 + \eta(m_1, m_1 + \eta(m_2, m_1))] \subseteq T$ such that $m_1 < m_1 + \eta(m_2, m_1) \in T$. Let $\psi : H \to \mathbb{R}$ be a delta differentiable on H° such that $\psi^\Delta \in H$, for $m_1, m_1 + \eta(m_2, m_1) \in H$ with $m_1 < m_1 + \eta(m_2, m_1)$. If $|\psi^\Delta|^q$ is s-preinvex on H for some fixed $s \in (0, 1]$, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\sup_{m_1 \le h \le m_1 + \eta(m_2, m_1)} |\psi^\Delta(h)| = M$ for $h \in H$, then following inequality holds:*

$$\begin{aligned}
&\left|\psi(h) - \frac{1}{\eta(m_2, m_1)} \int_{m_1}^{m_1+\eta(m_2,m_1)} \psi^\sigma(g)\Delta g\right| \\
&\le \frac{M(h - m_1)^2}{\eta(m_2, m_1)} \Big(\int_0^1 \tau^p \Delta\tau\Big)^{\frac{1}{p}} \left(\int_0^1 \Big((\tau - 1)^s + (2 - \tau)^s\Big)\Delta\tau\right)^{\frac{1}{q}} \\
&\quad + \frac{M(h - (m_1 + \eta(m_2, m_1)))^2}{\eta(m_2, m_1)} \Big(\int_0^1 \tau^p \Delta\tau\Big)^{\frac{1}{p}} \left(\int_0^1 \Big(\tau^s + (1 - \tau)^s\Big)\Delta\tau\right)^{\frac{1}{q}}.
\end{aligned} \tag{22}$$

Proof. The proof is related to Theorem 6 but in place of definition of exponential s-preinvexity, we use s-preinvexity of $|\psi^\Delta|^q$. □

Theorem 10. *Suppose T is a time scale and $H = [m_1, m_1 + \eta(m_2, m_1)] \subseteq T$ such that $m_1 < m_1 + \eta(m_2, m_1) \in T$. Let $\psi : H \to \mathbb{R}$ be a delta differentiable on H° such that $\psi^\Delta \in H$, for $m_1, m_1 + \eta(m_2, m_1) \in H$ with $m_1 < m_1 + \eta(m_2, m_1)$. If $|\psi^\Delta|^q$ is s-preinvex on H for some fixed $s \in (0, 1]$, $q > 1$ and $\sup_{m_1 \le h \le m_1 + \eta(m_2, m_1)} |\psi^\Delta(h)| = M$ for $h \in H$, then the following inequality holds:*

$$\begin{aligned}
&\left|\psi(h) - \frac{1}{\eta(m_2, m_1)} \int_{m_1}^{m_1+\eta(m_2,m_1)} \psi^\sigma(g)\Delta g\right| \\
&\le \frac{M(h - m_1)^2}{\eta(m_2, m_1)} \Big(\int_0^1 \tau\Delta\tau\Big)^{1-\frac{1}{q}} \left(\int_0^1 \Big(\tau(\tau - 1)^s + \tau(2 - \tau)^s\Big)\Delta\tau\right)^{\frac{1}{q}} \\
&\quad + \frac{M(h - (m_1 + \eta(m_2, m_1)))^2}{\eta(m_2, m_1)} \Big(\int_0^1 \tau\Delta\tau\Big)^{1-\frac{1}{q}} \left(\int_0^1 \Big(\tau^{s+1} + \tau(1 - \tau)^s\Big)\Delta\tau\right)^{\frac{1}{q}}.
\end{aligned} \tag{23}$$

Proof. The proof is related to Theorem 7, but we use definition of s-preinvexity of $|\psi^\Delta|^q$ instead of exponential s-preinvexity. □

4. Applications

Suppose there are some special means of two positive numbers m_1, $m_1+\eta(m_2,m_1)$ with $m_1 < m_1+\eta(m_2,m_1)$.

(i) The arithmetic mean:

$$A(m_1, m_1+\eta(m_2,m_1)) = \frac{m_1+m_1+\eta(m_2,m_1)}{2} = \frac{2m_1+\eta(m_2,m_1)}{2}. \tag{24}$$

(ii) The Harmonic mean:

$$H(m_1, m_1+\eta(m_2,m_1)) = \frac{2m_1(m_1+\eta(m_2,m_1))}{m_1+m_1+\eta(m_2,m_1)} = \frac{2m_1(m_1+\eta(m_2,m_1))}{2m_1+\eta(m_2,m_1)}, \\ m_1>0, m_1+\eta(m_2,m_1)>0. \tag{25}$$

(iii) The identric mean:

$$I(m_1, m_1+\eta(m_2,m_1)) = \frac{1}{e}\left(\frac{(m_1+\eta(m_2,m_1))^{m_1+\eta(m_2,m_1)}}{(m_1)^{m_1}}\right)^{\frac{1}{\eta(m_2,m_1)}}, \tag{26}$$

where m_1, $m_1+\eta(m_2,m_1)>0$.

(iv) The log-mean:

$$\begin{aligned} L = L(m_1, m_1+\eta(m_2,m_1)) &= \frac{m_1+\eta(m_2,m_1)-m_1}{ln(m_1+\eta(m_2,m_1))-ln(m_1)} \\ &= \frac{\eta(m_2,m_1)}{ln(m_1+\eta(m_2,m_1))-ln(m_1)}. \end{aligned} \tag{27}$$

(v) The p-logarithmic mean:

$$L_p(m_1, m_1+\eta(m_2,m_1)) = \left(\frac{(m_1+\eta(m_2,m_1))^{p+1}-m_1^{p+1}}{(p+1)(\eta(m_2,m_1))}\right)^{\frac{1}{p}}, \tag{28}$$
$$p \in \mathbb{R}|\{-1,0\}$$

Proposition 1. *Let* $0<m_1<m_1+\eta(m_2,m_1)$*,* $q\geq 1$*,* $0<s<1$*, then we have*

$$\begin{aligned} &\left|A^s(m_1, m_1+\eta(m_2,m_1)) - L_s^s(m_1, m_1+\eta(m_2,m_1))\right| \\ &\leq \frac{M}{\eta(m_2,m_1)(2)^{1-\frac{1}{q}}}\left[(h-m_1)^2\left(\frac{2^{s+2}-(s+3)}{e^{\alpha h}(s+1)(s+2)}+\frac{1}{e^{\alpha m_1}(s+1)(s+2)}\right)^{\frac{1}{q}}\right. \\ &\left.+(h-(m_1+\eta(m_2,m_1)))^2\left(\frac{1}{e^{\alpha h}(s+1)(s+2)}+\frac{1}{e^{\alpha(m_1+\eta(m_2,m_1))}(s+2)}\right)^{\frac{1}{q}}\right]. \end{aligned}$$

Proof. The result is satisfied if we consider $h=\frac{(m_1+m_1+\eta(m_2,m_1))}{2}$ that is $h=\frac{2m_1+\eta(m_2,m_1)}{2}$ in (19) with exponentially s-preinvex function $\psi:(0,\infty)\to\mathbb{R}$, $\psi(h)=h^s$ for all $\alpha\leq -1$. □

Proposition 2. *Let* $0<m_1<m_1+\eta(m_2,m_1)$*,* $q\geq 1$ *and* $0<s<1$*. Then, we have*

$$\Big|lnA(m_1, m_1+\eta(m_2,m_1)) - lnI(m_1, m_1+\eta(m_2,m_1))\Big|$$
$$\leq \frac{M}{\eta(m_2,m_1)(2)^{1-\frac{1}{q}}}\Bigg[(h-m_1)^2\left(\frac{2^{s+2}-(s+3)}{e^{\alpha h}(s+1)(s+2)}+\frac{1}{e^{\alpha m_1}(s+1)(s+2)}\right)^{\frac{1}{q}}$$
$$+(h-(m_1+\eta(m_2,m_1)))^2\left(\frac{1}{e^{\alpha h}(s+1)(s+2)}+\frac{1}{e^{\alpha(m_1+\eta(m_2,m_1))}(s+2)}\right)^{\frac{1}{q}}\Bigg].$$

Proof. The result is satisfied if we consider $h = \frac{(m_1+m_1+\eta(m_2,m_1))}{2}$ that is $h = \frac{2m_1+\eta(m_2,m_1)}{2}$ in (19) with exponentially s-preinvex function $\psi : (0,\infty) \to \mathbb{R}$, $\psi(h) = ln(h)$ for all $\alpha \leq -1$. □

Proposition 3. *Let $0 < m_1 < m_1+\eta(m_2,m_1)$, $q \geq 1$ and $0 < s < 1$. Then, we have*

$$\Big|H(m_1, m_1+\eta(m_2,m_1)) - L^{-1}(m_1, m_1+\eta(m_2,m_1))\Big|$$
$$\leq \frac{M}{\eta(m_2,m_1)(2)^{1-\frac{1}{q}}}\Bigg[(h-m_1)^2\left(\frac{2^{s+2}-(s+3)}{e^{\alpha h}(s+1)(s+2)}+\frac{1}{e^{\alpha m_1}(s+1)(s+2)}\right)^{\frac{1}{q}}$$
$$+(h-(m_1+\eta(m_2,m_1)))^2\left(\frac{1}{e^{\alpha h}(s+1)(s+2)}+\frac{1}{e^{\alpha(m_1+\eta(m_2,m_1))}(s+2)}\right)^{\frac{1}{q}}\Bigg].$$

Proof. The result is satisfied if we consider $h = \frac{(m_1+m_1+\eta(m_2,m_1))}{2m_1(m_1+\eta(m_2,m_1))}$ that is $h = \frac{2m_1+\eta(m_2,m_1)}{2m_1(m_1+\eta(m_2,m_1))}$ in (19) with exponentially s-preinvex function $\psi : (0,\infty) \to \mathbb{R}$, $\psi(h) = \frac{1}{h}$ for all $\alpha \leq -1$. □

5. Conclusions

Ostrowski inequalities are of great importance while studying the error bounds of different numerical quadrature rules, for example, the midpoint rule, Simpson's rule, the Trapezoidal rule and other generalized Riemann types. In this article, by generalizing the inequalities [26], we consider the new integral inequality of Hermite–Hadamard for exponentially s-preinvex functions on time scale and some novel refinements of Ostrowski type inequalities for exponentially s-preinvex functions and s-preinvex functions on time scales and some of our results unify continuous and discrete analysis in the literature and we discuss some special cases when T= $\mathbb{R}$ then Δ-integral became a classical integral. In our results, if we take $s = 1$, then our results reduce to the results for preinvex function. We have also obtained applications to some special means. In the future research, the interested reader can search various new interesting inequalities from our results. Moreover, they can investigate (using our technique) applications to special means for various s-preinvex functions.

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