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Some Properties of Certain Classes of Meromorphic Multivalent Functions Defined by Subordination

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Abstract: In this paper, we define two classes of meromorphic multivalent functions in the punctured disc $\mathbb{U}^* = \{w \in \mathbb{C} : 0 < |w| < 1\}$ by using the principle of subordination. We investigate a number of useful results including subordination results, some connections with a certain integral operator, sandwich properties, an inclusion relationship, and Fekete-Szegö inequalities for the functions belonging these classes. Our results are connected with those in several earlier works, which are related to this field of Geometric Function Theory (GFT) of Complex Analysis.

Keywords: analytic functions; meromorphic multivalent functions; subordination; superordination; Fekete–Szegö inequalities

1. Introduction

Let $\mathcal{A}[a, n]$ be the subclass of analytic functions $g(w)$ in $\mathbb{U} = \{w \in \mathbb{C} : |w| < 1\}$ of the following form:

$$g(w) = a + a_n w^n + a_{n+1} w^{n+1} + \ldots \ (a \in \mathbb{C}; w \in \mathbb{U}).$$

Furthermore, let $\mathcal{M}_p$ denote the class of all analytic functions $g(w)$ of the following form:

$$g(w) = w^{-p} + \sum_{k=1-p}^{\infty} a_k w^k \ \ (p \in \mathbb{N} = \{1, 2, 3, \ldots\}), \tag{1}$$

which are meromorphic p-valent in the punctured disc $\mathbb{U}^* = \mathbb{U}\backslash\{0\}$. If $g_1(w)$ and $g_2(w)$ are analytic in $\mathbb{U}$, we say that $g_1(w)$ is subordinate to $g_2(w)$ or $g_2(w)$ is superordinate to $g_1(w)$, written as, $g_1(w) \prec g_2(w)$, if there exists an analytic function $v(w)$ in $\mathbb{U}$ with $v(0) = 0$ and $|v(w)| < 1 (w \in \mathbb{U})$ such that

$$g_1(w) = g_2(v(w))(w \in \mathbb{U}).$$

In particular, if $g_2(w)$ is a univalent function in $\mathbb{U}$, we have the following equivalence (see [1–3]):

$$g_1(w) \prec g_2(w) \Leftrightarrow g_1(0) = g_2(0) \ \text{ and } \ g_1(\mathbb{U}) \subset g_2(\mathbb{U}).$$

Many subclasses of meromorphically multivalent functions have been introduced and investigated by several earlier authors (see, for example, [4–12]). Now, we introduce a certain class $\mathcal{M}_p^{\beta}(L, M)$ of meromorphic multivalent functions by using the principle of subordination.

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Citation: Seoudy, T.M.; Shammaky, A.E. Some Properties of Certain Classes of Meromorphic Multivalent Functions Defined by Subordination. *Symmetry* **2023**, *15*, 347. https://doi.org/10.3390/sym15020347

Academic Editor: Şahsene Altınkaya

Received: 22 December 2022
Revised: 16 January 2023
Accepted: 22 January 2023
Published: 27 January 2023

Definition 1. *For fixed parameters L and M with $-1 \leq M < L \leq 1$, we say that a function $g(w) \in \mathcal{M}_p$ is in $\mathcal{M}_p^{\beta}(L,M)$ if it satisfies the following condition:*

$$\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta} \prec \frac{1+Lw}{1+Mw}. \tag{2}$$

By using the above definition of subordination, (2) is equivalent to the following inequality:

$$\left|\frac{\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta}-1}{M\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta}-L}\right| < 1 \quad (w \in \mathbb{U}).$$

For convenience, we write $\mathcal{M}_p^{\beta}(1-2\sigma,-1) = \mathcal{M}_p^{\beta}(\sigma)(0 \leq \sigma < 1)$, where

$$\mathcal{M}_p^{\beta}(\sigma) = \left\{g \in \mathcal{M}_p : \Re\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta} > \sigma, 0 \leq \sigma < 1\right\}. \tag{3}$$

We note that

(i) Taking $\beta = 1$ in (2), the class $\mathcal{M}_p^{\beta}(L,M)$ reduces to $\mathcal{M}_p(L,M)$, where

$$\mathcal{M}_p(L,M) = \left\{g \in \mathcal{M}_p : -\frac{w^{p+1}g'(w)}{p} \prec \frac{1+Lw}{1+Mw}\right\};$$

(ii) Taking $\beta = 1$, $L = 1-2\sigma(0 \leq \sigma < 1)$ and $M = -1$ in the class $\mathcal{M}_p^{\beta}(L,M)$, we obtain

$$\mathcal{M}_p(\sigma) = \left\{g \in \mathcal{M}_p : \Re\left\{-\frac{w^{p+1}g'(w)}{p}\right\} > \sigma,\ 0 \leq \sigma < 1\right\}.$$

In order to establish our main results, we need the following definition and lemmas.

Definition 2. *[13] Denote by Π the set of all analytic functions g that are injective on $\bar{\mathbb{U}} \backslash \mathcal{E}(g)$, where*

$$\mathcal{E}(g) = \{\zeta \in \partial\mathbb{U} : \lim_{w \to \zeta} g(w) = \infty\},$$

and such that $g'(\zeta) \neq 0$ for $\zeta \in \bar{\mathbb{U}} \backslash \mathcal{E}(g)$.

Lemma 1. *[14] Let $h(w)$ be an analytic and convex (univalent) function in $\mathbb{U}$ with $h(0) = 1$. Suppose also that $\varphi(w)$ given by*

$$\varphi(w) = 1 + c_1 w + c_2 w^2 + \ldots \tag{4}$$

in an analytic function in $\mathbb{U}$. If

$$\varphi(w) + \frac{w\varphi'(w)}{\delta} \prec h(w) \quad (\Re(\delta) \geq 0; \delta \neq 0), \tag{5}$$

then

$$\varphi(w) \prec \psi(w) = \delta w^{-\delta} \int_0^w t^{\delta-1} h(t)dt \prec h(w),$$

and ψ is the best dominant.

The Gaussian hypergeometric function ${}_2\mathcal{F}_1(\rho_1,\rho_2;\rho_3;w)$ is defined by

$${}_2\mathcal{F}_1(\rho_1,\rho_2;\rho_3;w) = 1 + \frac{\rho_1.\rho_2}{\rho_3}.\frac{w}{1!} + \frac{\rho_1(\rho_1+1).\rho_2(\rho_2+1)}{\rho_3(\rho_3+1)}.\frac{w^2}{2!} + \ldots, \tag{6}$$

$$\left(\rho_1, \rho_2, \rho_3 \in \mathbb{C};\ \rho_3 \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots\}\right).$$

Lemma 2. *[15] (Chapter 14): For $\rho_1, \rho_2, \rho_3 \in \mathbb{C}$ with $\rho_3 \notin \mathbb{Z}_0^-$,*

$$\int_0^1 t^{\rho_2 - 1} (1-t)^{\rho_3 - \rho_2 - 1} (1 - wt)^{-\rho_1}\, dt = \frac{\Gamma(\rho_2)\Gamma(\rho_3 - \rho_2)}{\Gamma(\rho_3)}\ {}_2\mathcal{F}_1(\rho_1, \rho_2; \rho_3; w) \tag{7}$$

$$(\Re(\rho_3) > \Re(\rho_2) > 0);$$

$${}_2\mathcal{F}_1(\rho_1, \rho_2; \rho_3; w) = (1-w)^{-\rho_1}\, {}_2\mathcal{F}_1\left(\rho_1, \rho_3 - \rho_2; \rho_3; \frac{w}{w-1}\right); \tag{8}$$

$${}_2\mathcal{F}_1(\rho_1, \rho_2; \rho_3; w) = {}_2\mathcal{F}_1(\rho_2, \rho_1; \rho_3; w). \tag{9}$$

Lemma 3. *[16] Let $q(w)$ be a convex univalent function in $\mathbb{U}$ such that*

$$\Re\left\{1 + \tfrac{wq''(w)}{q'(w)}\right\} > \max\left\{0, -\Re\left(\tfrac{1}{\varkappa}\right)\right\} \quad (\varkappa \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}).$$

If the function $\varphi(w)$ is analytic in $\mathbb{U}$ and

$$\varphi(w) + \varkappa w \varphi'(w) \prec q(w) + \varkappa w q'(w),$$

then $\varphi(w) \prec q(w)$ and $q(w)$ is the best dominant.

Lemma 4. *[13] Let $q(w)$ be convex univalent in $\mathbb{U}$ and $\kappa \in \mathbb{C}$. Further assume that $\Re(\kappa) > 0$. If*

$$\varphi(w) \in \mathcal{A}[q(0), 1] \cap \Pi,$$

and $\varphi(w) + \kappa w \varphi'(w)$ is univalent in $\mathbb{U}$, then

$$q(w) + \kappa w q'(w) \prec \varphi(w) + \kappa w \varphi'(w),$$

implies $q(w) \prec \varphi(w)$ and $q(w)$ is the best subordinant.

Lemma 5. *[17] Let $h(w) = 1 + c_1 w + c_2 w^2 + c_3 w^3 + \ldots \in \mathcal{P}$, i.e., let h be analytic fuction in $\mathbb{U}$ and satisfy $\Re\{h(w)\} > 0$ for w in $\mathbb{U}$, then*

$$\left|c_2 - v c_1^2\right| \leq 2 \max\{1, |2v - 1|\} \quad for\ all\ v \in \mathbb{C}. \tag{10}$$

The result is sharp for the functions given by $g(w) = \frac{1+w^2}{1-w^2}$ or $g(w) = \frac{1+w}{1-w}$.

In this paper, we study a number of useful properties including subordination results, sandwich properties, inclusion relationship and Fekete-Szegö inequalities for the function classes $\mathcal{M}_p^\beta(L, M)$ and $\mathcal{M}_p^\beta(\sigma)$, which are defined above. The results derived in the present paper will pave the way for the further study in the direction of the Geometric Function Theory (GFT). The recent developments in Geometric Function Theory (GFT) of Complex Analysis (especially in algebraic geometry, number theory, as well as in physics, hydrodynamics, hermodynamics, engineering, and quantum mechanics) play a crucial role in research in many disciplines, including in the concept of symmetry.

2. Main Geometric Properties

Unless otherwise mentioned, we assume throughout this investigation that $-1 \leq M < L \leq 1$, $\beta, \gamma > 0$, $p \in \mathbb{N}$ and all powers are understood as principal values.

Theorem 1. *Let $g(w) \in \mathcal{M}_p$ defined by (1) satisfying the following subordination condition:*

$$(1+\gamma)\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta} + \frac{\gamma}{p}\left[\frac{wg''(w)}{g'(w)}+1\right]\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta} \prec \frac{1+Lw}{1+Mw}. \tag{11}$$

Then $g(w) \in \mathcal{M}_p^{\beta}(L,M)$ and

$$\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta} \prec \Omega(w) \prec \frac{1+Lw}{1+Mw}, \tag{12}$$

where the function $\Omega(w)$ given by

$$\Omega(w) = \begin{cases} \frac{L}{M} + \frac{(M-L)}{M(1+Mw)}\, {}_2\mathcal{F}_1\left(1,1;\frac{p\beta+\gamma}{\gamma};\frac{Mw}{1+Mw}\right) & (M \neq 0) \\ 1+\frac{p\beta L}{\gamma+p\beta}w & (M=0) \end{cases} \tag{13}$$

is the best dominant. Furthermore, $g(w) \in \mathcal{M}_p^{\beta}(\sigma)$, i.e,

$$\Re\left\{\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta}\right\} > \sigma \quad (w \in \mathbb{U}), \tag{14}$$

where

$$\sigma = \begin{cases} \frac{L}{M} + \frac{(M-L)}{M(1-M)}\, {}_2\mathcal{F}_1\left(1,1;\frac{p\beta}{\gamma}+1;\frac{M}{M-1}\right) & (M \neq 0) \\ 1-\frac{p\beta L}{\gamma+p\beta} & (M=0)\,. \end{cases} \tag{15}$$

The estimate in (14) is the best possible.

Proof. Let

$$\varphi(w) = \left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta} \quad (w \in \mathbb{U})\,. \tag{16}$$

Then, $\varphi(w)$ is analytic in $\mathbb{U}$ and is of the form (4). Differentiating (16) with respect to w, we obtain

$$(1+\gamma)\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta} + \frac{\gamma}{p}\left[\frac{wg''(w)}{g'(w)}+1\right]\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta} = \varphi(w) + \frac{\gamma}{p\beta}w\varphi'(w) \prec \frac{1+Lw}{1+Mw}\,. \tag{17}$$

Now, by using Lemma 1 for $\delta = \frac{p\beta}{\gamma}$, we obtain

$$\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta} \prec \Omega(w) = \frac{p\beta}{\gamma}w^{-\frac{p\beta}{\gamma}}\int_0^w t^{\frac{p\beta}{\gamma}-1}\left(\frac{1+Lt}{1+Mt}\right)dt = \frac{p\beta}{\gamma}\int_0^1 u^{\frac{p\beta}{\gamma}-1}\left(\frac{1+Lwu}{1+Mwu}\right)du. \tag{18}$$

By using Lemma 2 with $\rho_1 = 1, \rho_2 = \frac{p\beta}{\gamma}, \rho_3 = \frac{p\beta+\gamma}{\gamma}$ in (18), we obtain

$$\Omega(w) = \begin{cases} \frac{L}{M} + \frac{(M-L)}{M(1+Mw)}\, {}_2\mathcal{F}_1\left(1,1;\frac{p\beta+\gamma}{\gamma};\frac{Mw}{1+Mw}\right) & (M \neq 0) \\ 1+\frac{p\beta L}{\gamma+p\beta}w & (M=0). \end{cases}$$

This proves the assertion (12) of Theorem 1.

Next, in order to show the assertion (14) of Theorem 1, it suffices to prove that

$$\inf_{w\in\mathbb{U}}\{\Re(\Omega(w))\} = \Omega(-1)\,. \tag{19}$$

We here have

$$\Re\left(\tfrac{1+Lw}{1+Mw}\right) \geq \tfrac{1-Lr}{1-Mr} \quad (|w| \leq r < 1).$$

Putting

$$g(\zeta, w) = \tfrac{1+L\zeta w}{1+M\zeta w} \text{ and } d\nu(\zeta) = \tfrac{p\beta}{\gamma}\zeta^{\frac{p\beta}{\gamma}-1}\, d\zeta \quad (0 \leq \zeta \leq 1),$$

which is a positive measure on the interval $[0,1]$, we obtain

$$\Omega(w) = \int_0^1 g(\zeta, w) d\nu(\zeta),$$

so that

$$\Re\{\Omega(w)\} \geq \int_0^1 \left(\frac{1-L\zeta r}{1-M\zeta r}\right) d\nu(\zeta) = \Omega(-r) \quad (|w| \leq r < 1). \tag{20}$$

Letting $r \to 1^-$ in (20), we get the assertion (14) of Theorem 1. Finally, the estimate in (14) is the best possible as $\Omega(w)$ is the best dominant of (12). □

Taking $\beta = 1$ in Theorem 1, we obtain

Corollary 1. *Let $g(w) \in \mathcal{M}_p$ defined by (1), satisfying the following subordination condition:*

$$-\tfrac{w^{p+1}g'(w)}{p}\left\{1+\gamma+\tfrac{\gamma}{p}\left[\tfrac{wg''(w)}{g'(w)}+1\right]\right\} \prec \tfrac{1+Lw}{1+Mw}.$$

Then $g(w) \in \mathcal{M}_p(L, M)$ and

$$-\tfrac{w^{p+1}g'(w)}{p} \prec \Omega_1(w) \prec \tfrac{1+Lw}{1+Mw},$$

where $\Omega_1(w)$ given by

$$\Omega_1(w) = \begin{cases} \frac{L}{M} + \frac{(M-L)}{M(1+Mw)}\, {}_2\mathcal{F}_1\left(1, 1; \frac{p}{\gamma}+1; \frac{Mw}{1+Mw}\right) & (M \neq 0) \\ 1 + \frac{pL}{\gamma+p} w & (M = 0) \end{cases}$$

is the best dominant. Furthermore, $g(w) \in \mathcal{M}_p(\sigma_1)$, i.e,

$$\Re\left\{-\tfrac{w^{p+1}g'(w)}{p}\right\} > \sigma_1 \quad (w \in \mathbb{U}),$$

where

$$\sigma_1 = \begin{cases} \frac{L}{M} + \frac{(M-L)}{M(1-M)}\, {}_2\mathcal{F}_1\left(1, 1; \frac{p}{\gamma}+1; \frac{M}{M-1}\right) & (M \neq 0) \\ 1 - \frac{pL}{\gamma+p} & (M = 0)\,. \end{cases}$$

The above estimate is the best possible.

For the function $g(w) \in \mathcal{M}_p$, Kumar and Shukla [18] defined the integral operator $\mathcal{G}_{\mu,p}(g)(w) : \mathcal{M}_p \to \mathcal{M}_p$ as follows:

$$\mathcal{G}_{\mu,p}(g)(w) = \frac{\mu}{w^{\mu+p}} \int_0^w t^{\mu+p-1} g(t) dt \quad (\mu > 0; w \in \mathbb{U}). \tag{21}$$

From (21), we obtain

$$w\mathcal{G}''_{\mu,p}(g)(w) = \mu g'(w) - (\mu+p+1)\mathcal{G}'_{\mu,p}(g)(w). \tag{22}$$

Theorem 2. *If $g(w) \in \mathcal{M}_p$ satisfies the following subordination condition:*

$$(1-\gamma)\left[-\frac{w^{p+1}\mathcal{G}'_{\mu,p}(g)(w)}{p}\right]^{\beta} + \gamma\frac{g'(w)}{\mathcal{G}'_{\mu,p}(g)(w)}\left[-\frac{w^{p+1}\mathcal{G}'_{\mu,p}(g)(w)}{p}\right]^{\beta} \prec \frac{1+Lw}{1+Mw}, \quad (23)$$

where $\mathcal{G}_{\mu,p}(g)(w)$ is defined by (21), then $\mathcal{G}_{\mu,p}(g)(w) \in \mathcal{M}_p^{\beta}(L,M)$ and

$$\left[-\frac{w^{p+1}\mathcal{G}'_{\mu,p}(g)(w)}{p}\right]^{\beta} \prec \Phi(w) \prec \frac{1+Lw}{1+Mw},$$

where the function

$$\Phi(w) = \begin{cases} \frac{L}{M} + \frac{(M-L)}{M(1+Mw)}\,{}_2\mathcal{F}_1\left(1,1;\frac{\beta\mu+\gamma}{\gamma};\frac{Mw}{Mw+1}\right) & (M \neq 0) \\ 1 + \frac{\beta\mu}{\beta\mu+\gamma}Lw & (M = 0), \end{cases}$$

is the best dominant. Furthermore, $\mathcal{G}_{\mu,p}(g)(w) \in \mathcal{M}_p^{\beta}(\xi^*)$ and

$$\Re\left\{\left[-\frac{w^{p+1}\mathcal{G}'_{\mu,p}(g)(w)}{p}\right]^{\beta}\right\} > \xi^* \quad (w \in \mathbb{U}), \quad (24)$$

where

$$\xi^* = \begin{cases} \frac{L}{M} + \left(1-\frac{L}{M}\right)(1-M)^{-1}\,{}_2\mathcal{F}_1\left(1,1;\frac{\beta\mu+\gamma}{\gamma};\frac{M}{M-1}\right) & (M \neq 0) \\ 1 - \frac{\beta\mu}{\beta\mu+\gamma}L & (M = 0). \end{cases}$$

The above result is the best possible.

Proof. Defining $\varphi(w)$ by

$$\varphi(w) = \left[-\frac{w^{p+1}\mathcal{G}'_{\mu,p}(g)(w)}{p}\right]^{\beta} \quad (w \in \mathbb{U}), \quad (25)$$

we note that φ is analytic in $\mathbb{U}$ and is of the form (4). Differentiating (25) with respect to w and using the identity (22), we find that

$$\begin{aligned}(1-\gamma)&\left[-\frac{w^{p+1}\mathcal{G}'_{\mu,p}(g)(w)}{p}\right]^{\beta} + \gamma\frac{g'(w)}{\mathcal{G}'_{\mu,p}(g)(w)}\left[-\frac{w^{p+1}\mathcal{G}'_{\mu,p}(g)(w)}{p}\right]^{\beta} \\ &= \varphi(w) + \frac{\gamma}{\beta\mu}w\varphi'(w) \prec \frac{1+Lw}{1+Mw}.\end{aligned}$$

Employing the techniques that we used in proving Theorem 1 above, we can prove the remaining proof of Theorem 2. □

Setting $\beta = 1$ in Theorem 2, we obtain

Corollary 2. *If $g(w) \in \mathcal{M}_p$ satisfies the following subordination condition:*

$$-\left[(1-\gamma)\frac{w^{p+1}\mathcal{G}'_{\mu,p}(g)(w)}{p} + \gamma\frac{w^{p+1}g'(w)}{p}\right] \prec \frac{1+Lw}{1+Mw},$$

where $\mathcal{G}_{\mu,p}(g)(w)$ is given by (21), then

$$-\frac{w^{p+1}\mathcal{G}'_{\mu,p}(g)(w)}{p} \prec \Phi_1(w) \prec \frac{1+Lw}{1+Mw},$$

where $\Phi_1(w)$ given by

$$\Phi_1(w) = \begin{cases} \frac{L}{M} + \frac{(M-L)}{M(1+Mw)}\, {}_2\mathcal{F}_1\left(1,1;\frac{\mu+\gamma}{\gamma};\frac{Mw}{Mw+1}\right) & (M \neq 0) \\ 1 + \frac{\mu}{\mu+\gamma}Lw & (M = 0)\,, \end{cases}$$

is the best dominant. Furthermore,

$$\Re\left\{-\frac{w^{p+1}\mathcal{G}'_{\mu,p}(g)(w)}{p}\right\} > \xi^* \quad (w \in \mathbb{U}),$$

where

$$\xi^* = \begin{cases} \frac{L}{M} + \frac{(M-L)}{M(1-M)}\, {}_2\mathcal{F}_1\left(1,1;\frac{\mu+\gamma}{\gamma};\frac{M}{M-1}\right) & (M \neq 0) \\ 1 - \frac{\mu}{\mu+\gamma}L & (M = 0)\,. \end{cases}$$

The result is the best possible.

Theorem 3. *If $g(w) \in \mathcal{M}_p^{\beta}(\sigma)$ $(0 \leq \sigma < 1)$, then*

$$\Re\left\{(1+\gamma)\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta} + \frac{\gamma}{p}\left[\frac{wg''(w)}{g'(w)}+1\right]\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta}\right\} > \sigma \quad (|w| < R)\,, \tag{26}$$

where

$$R = \sqrt{1+\left(\frac{\gamma}{p\beta}\right)^2} - \frac{\gamma}{p\beta}. \tag{27}$$

Proof. Since $g(w) \in \mathcal{M}_p^{\beta}(\sigma)$, we write

$$\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta} = \sigma + (1-\sigma)u(w) \quad (w \in \mathbb{U})\,. \tag{28}$$

Then, $u(w)$ is analytic in $\mathbb{U}$, is of the form (4), and $\Re\{u(w)\} > 0$. Differentiating (28) with respect to w, we obtain

$$\frac{(1+\gamma)\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta} + \frac{\gamma}{p}\left(\frac{wg''(w)}{g'(w)}+1\right)\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta} - \sigma}{1-\sigma} = u(w) + \frac{\gamma}{p\beta}wu'(w)\,. \tag{29}$$

Now, by applying the following estimate (see [19,20])

$$\frac{|wu'(w)|}{\Re\{u(w)\}} \leq \frac{2r}{1-r^2} \qquad (|w| = r < 1)$$

in (29), we obtain

$$\Re\left\{\frac{(1+\gamma)\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta} + \frac{\gamma}{p}\left(\frac{wg''(w)}{g'(w)}+1\right)\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta} - \delta}{1-\delta}\right\} \geq \Re\{u(w)\}\,.\left(1 - \frac{2\gamma r}{p\beta(1-r^2)}\right). \tag{30}$$

Note that the right-hand side of (30) is positive provided that $r < R$, where R is given by (27). This shows the assertion (26) of Theorem 3.

In order to prove that the bound R is the best possible, we consider $g(w) \in \mathcal{M}_p$ defined by

$$\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta} - \sigma = (1-\sigma)\frac{1+w}{1-w} \quad (0 \leq \sigma < 1).$$

Noting that

$$\frac{(1+\gamma)\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta}+\frac{\gamma}{p}\left(\frac{wg''(w)}{g'(w)}+1\right)\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta}-\delta}{1-\delta}=\frac{p\beta(1-w^2)+2\gamma w}{p\beta(1-w)^2}=0$$

for $w = R\exp(i\pi)$, we complete the proof of Theorem 3. □

Theorem 4. *Let $q(w)$ be univalent function in $\mathbb{U}$ such that*

$$\Re\left(1+\frac{wq''(w)}{q'(w)}\right)>\max\left\{0,-\Re\left(\frac{p\beta}{\gamma}\right)\right\}. \tag{31}$$

If $g(w)\in\mathcal{M}_p$ satisfies the subordination condition

$$(1+\gamma)\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta}+\frac{\gamma}{p}\left[\frac{wg''(w)}{g'(w)}+1\right]\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta}\prec q(w)+\frac{\gamma}{p\beta}wq'(w), \tag{32}$$

then $\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta}\prec q(w)$ and $q(w)$ is the best dominant.

Proof. Let $\varphi(w)$ be given by (16). Combining (17) and (32), we obtain

$$\varphi(w)+\frac{\gamma}{p\beta}w\varphi'(w)\prec q(w)+\frac{\gamma}{p\beta}wq'(w). \tag{33}$$

Applying Lemma 3 on (33) with $\varkappa=\frac{\gamma}{p\beta}$, we easily obtain the assertion of Theorem 4. □

Putting $q(w)=\frac{1+Lw}{1+Mw}$ in Theorem 4, we obtain

Corollary 3. *Suppose that*

$$\Re\left(\frac{1-Mw}{1+Mw}\right)>\max\left\{0,-\Re\left(\frac{p\beta}{\gamma}\right)\right\}.$$

If $g(w)\in\mathcal{M}_p$ satisfies the following subordination condition:

$$(1+\gamma)\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta}+\frac{\gamma}{p}\left[\frac{wg''(w)}{g'(w)}+1\right]\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta}\prec\frac{1+Lw}{1+Mw}+\frac{\gamma}{p\beta}\frac{(L-M)w}{(1+Mw)^2},$$

then $\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta}\prec\frac{1+Lw}{1+Mw}$, and $\frac{1+Lw}{1+Mw}$ is the best dominant.

Theorem 5. *Let $\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta}\in\mathcal{A}[q(0),1]\cap\Pi$ such that*

$$(1+\gamma)\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta}+\frac{\gamma}{p}\left[\frac{wg''(w)}{g'(w)}+1\right]\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta}$$

be univalent function in $\mathbb{U}$. If $g(w)\in\mathcal{M}_p$ satisfies the superordination condition

$$q(w)+\frac{\gamma}{p\beta}wq'(w)\prec(1+\gamma)\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta}+\frac{\gamma}{p}\left[\frac{wg''(w)}{g'(w)}+1\right]\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta},$$

where $q(w)$ be convex univalent function in $\mathbb{U}$, then $q(w)\prec\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta}$ and $q(w)$ is the best subordinant.

Proof. Let $\varphi(w)$ be defined by (16). Then

$$\begin{aligned} q(w)+\frac{\gamma}{p\beta}wq'(w) &\prec (1+\gamma)\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta}+\frac{\gamma}{p}\left[\frac{wg''(w)}{g'(w)}+1\right]\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta} \\ &= \varphi(w)+\frac{\gamma}{p\beta}w\varphi'(w) \end{aligned}$$

An application of Lemma 4 yields the assertion of Theorem 5. □

Putting $q(w) = \frac{1+Lw}{1+Mw}$ in Theorem 5, we obtain

Corollary 4. *Let* $\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta} \in \mathcal{A}[1,1] \cap \Pi$ *such that*

$$(1+\gamma)\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta} + \frac{\gamma}{p}\left[\frac{wg''(w)}{g'(w)} + 1\right]\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta}$$

be univalent in $\mathbb{U}$. *If* $g(w) \in \mathcal{M}_p$ *satisfies the superordination condition*

$$\frac{1+Lw}{1+Mw} + \frac{\gamma}{p\beta}\frac{(L-M)w}{(1+Mw)^2} \prec (1+\gamma)\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta} + \frac{\gamma}{p}\left[\frac{wg''(w)}{g'(w)} + 1\right]\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta},$$

then $\frac{1+Lw}{1+Mw} \prec \left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta}$ *and* $\frac{1+Lw}{1+Mw}$ *is the best subordinant.*

By combining the above results of subordination and superordination, we easily obtain the following "Sandwich-type result".

Theorem 6. *Let* q_1 *be convex univalent in* $\mathbb{U}$ *and* q_2 *be univalent in* $\mathbb{U}$ *such that* q_2 *satisfies (31). If*

$$\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta} \in \mathcal{A}[q_1(0),1] \cap \Pi,$$

and

$$(1+\gamma)\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta} + \frac{\gamma}{p}\left[\frac{wg''(w)}{g'(w)} + 1\right]\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta}$$

be univalent in $\mathbb{U}$, *also*

$$q_1(w) + \frac{\gamma}{p\beta}wq_1'(w) \prec (1+\gamma)\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta} + \frac{\gamma}{p}\left[\frac{wg''(w)}{g'(w)} + 1\right]\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta} \prec q_2(w) + \frac{\gamma}{p\beta}wq_2'(w),$$

then

$$q_1(w) \prec \left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta} \prec q_2(w),$$

and $q_1(w)$ *and* $q_2(w)$ *are, respectively, the best subordinant and the best dominant.*

Theorem 7. *Suppose that* $g(w), h(w) \in \mathcal{M}_p$ *satisfy the following inequalities:*

$$\Re\left\{\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta}\right\} > 0 \quad (w \in \mathbb{U}).$$

If

$$\left|\frac{g'(w)}{h'(w)} - 1\right| < 1 \quad (w \in \mathbb{U}),$$

then

$$-\Re\left\{1 + \frac{wg''(w)}{g'(w)}\right\} > 0 \quad (|w| < R_0),$$

where

$$R_0 = \frac{\sqrt{(\beta+2)^2 + 4p\beta^2(p+1)} - (\beta+2)}{2\beta(p+1)}.$$

Proof. Letting

$$\phi(w) = \frac{g'(w)}{h'(w)} - 1 = t_1 w + t_2 w^2 + \ldots.., \tag{34}$$

since $\phi(w)$ is analytic function in $\mathbb{U}$ with $\phi(0) = 0$ and $|\phi(w)| \leq |w|$ $(w \in \mathbb{U})$. Then, by using the Schwarz's lemma (see [21]), we obtain

$$\phi(w) = \frac{g'(w)}{h'(w)} - 1 = w\Psi(w),$$

where $\Psi(w)$ is analytic function in $\mathbb{U}$ and $|\Psi(w)| \leq 1$ $(w \in \mathbb{U})$. Hence, (34) leads us to

$$wg'(w) = wh'(w)(1 + w\,\Psi(w)) \quad (w \in \mathbb{U})\,. \tag{35}$$

Differentiating (35) logarithmically with respect to w, we obtain

$$1 + \frac{wg''(w)}{g'(w)} = 1 + \frac{wh''(w)}{h'(w)} + \frac{w\{\Psi(w) + w\Psi'(w)\}}{1 + w\Psi(w)}\,. \tag{36}$$

With

$$\varphi(w) = \left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta},$$

we see that $\varphi(w)$ is analytic function in $\mathbb{U}$, is of the form (4), $\Re\{\varphi(w)\} > 0$ $(w \in \mathbb{U})$ and

$$-\left\{1 + \frac{wg''(w)}{g'(w)}\right\} = p - \frac{1}{\beta}\frac{w\varphi'(w)}{\varphi(w)}\,,$$

so that we find from (36) that

$$-\Re\left\{1 + \frac{wg''(w)}{g'(w)}\right\} \geq p - \frac{1}{\beta}\left|\frac{w\varphi'(w)}{\varphi(w)}\right| - \left|\frac{w\{\Psi(w) + w\Psi'(w)\}}{1 + w\Psi(w)}\right|. \tag{37}$$

Now, using the following known estimates (see [22]):

$$\left|\frac{w\varphi'(w)}{\varphi(w)}\right| \leq \frac{2r}{1-r^2} \quad (|w| = r < 1)$$

and

$$\left|\frac{\Psi(w)+w\Psi'(w)}{1+w\Psi(w)}\right| \leq \frac{1}{1-r} \quad (|w| = r < 1)$$

in (37), we obtain

$$-\Re\left\{1 + \frac{wg''(w)}{g'(w)}\right\} \geq \frac{p\beta-(\beta+2)r-\beta(p+1)r^2}{\beta[1-r^2]}(|w| = r < 1)\,,$$

which is certainly positive, provided that $r < R_0$, R_0 being defined as in Theorem 7. □

Now, employing the same techniques used in [23,24], we study the Fekete–Szegö problems for the classes $\mathcal{M}_p^{\beta}(L, M)$ and $\mathcal{M}_p^{\beta}(\sigma)$.

Theorem 8. *If $g(w) \in \mathcal{M}_p^{\beta}(L, M)$ given by (1), then*

$$\left|a_{2-p} - \mu a_{1-p}^2\right| \leq \frac{p(L-M)}{(p-2)\beta}\max\left\{1; \left|M + \left(\frac{\beta-1}{2} + \frac{p(p-2)}{(p-1)^2}\right)\frac{(L-M)\mu}{\beta}\right|\right\}\ (p \neq 1, 2). \tag{38}$$

Proof. If $g(w) \in \mathcal{M}_p^{\beta}(L, M)$, then there is an analytic in $\mathbb{U}$ with $\nu(0) = 0$ and $|\nu(w)| < 1$ in $\mathbb{U}$ such that

$$\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta} = \frac{1 + L\nu(w)}{1 + M\nu(w)}. \tag{39}$$

If we define the function $h(w)$ by

$$h(w) = \frac{1 + \nu(w)}{1 - \nu(w)} = 1 + c_1 w + c_2 w^2 + ..., \tag{40}$$

we see that $\Re\{h(w)\} > 0$ and $h(0) = 1$. Therefore,

$$\frac{1 + L\nu(w)}{1 + M\nu(w)} = 1 + \frac{(L-M)}{2}c_1 w + \frac{(L-M)}{2}\left[c_2 - \frac{(1+M)}{2}c_1^2\right]w^2 + ... \,. \tag{41}$$

Now by substituting (41) in (39), we have

$$\left[-\frac{w^{p+1}g'(w)}{p}\right]^{\beta}=1+\frac{(L-M)}{2}c_1w+\frac{(L-M)}{2}\left[c_2-\frac{(1+M)}{2}c_1^2\right]w^2+\dots.$$

From the above equation, we obtain

$$\frac{(p-1)\beta}{p}a_{1-p}=\frac{(L-M)}{2}c_1 \tag{42}$$

and

$$\frac{(p-2)\beta}{p}a_{2-p}+\frac{(p-1)^2\beta(\beta-1)}{2p^2}a_{1-p}^2=\frac{(L-M)}{2}\left[c_2-\frac{(1+M)}{2}c_1^2\right]. \tag{43}$$

Thus,

$$a_{1-p}=\tfrac{p(L-M)}{2(p-1)\beta}c_1$$

and

$$a_{2-p}=\tfrac{p(L-M)}{2(p-2)\beta}\left[c_2-\tfrac{1}{2}\left(1+M+\tfrac{(\beta-1)(L-M)}{2\beta}\right)c_1^2\right],$$

Therefore, we have

$$a_{2-p}-\mu a_{1-p}^2=\frac{p(L-M)}{2(p-2)\beta}\left\{c_2-vc_1^2\right\}, \tag{44}$$

where

$$v=\frac{1}{2}\left[1+M+\left(\frac{\beta-1}{2}+\frac{p(p-2)}{(p-1)^2}\right)\frac{(L-M)\mu}{\beta}\right]. \tag{45}$$

Our result now follows from Lemma 5. This completes the proof of Theorem 7. □

Remark 1. *(i) Taking $p=1$ in (42) and (43), we have $c_1=0$ and $a_1=-\frac{L-M}{2\beta}c_2$. Thus*

$$|a_1|\le\tfrac{L-M}{\beta}.$$

(ii) Taking $p=2$ in (42) and (43), we have

$$|a_{-1}|\le\tfrac{2\sqrt{2}(L-M)}{\sqrt{|\beta[(2+L+M)\beta-L+M]|}}.$$

Putting $\beta=1$ in Theorem 8 and Remark 1, we obtain

Corollary 5. *If $g(w)\in\mathcal{M}_p(L,M)$ given by (1), then*

$$\begin{aligned}
|a_1| &\le L-M \quad (p=1);\\
|a_{-1}| &\le \frac{2(L-M)}{\sqrt{1+M}} \quad (p=2;M\neq-1);\\
\left|a_{2-p}-\mu a_{1-p}^2\right| &\le \frac{p(L-M)}{(p-2)}\max\left\{1;\left|M+\frac{p(p-2)(L-M)}{(p-1)^2}\mu\right|\right\}\ (p\neq1,2).
\end{aligned}$$

Putting $L=1-2\sigma(0\le\sigma<1)$ and $M=-1$ in Theorem 8 and Remark 1, we obtain

Corollary 6. *If $g(w) \in \mathcal{M}_p^{\beta}(\sigma)$ given by (1), then*

$$\begin{aligned} |a_1| &\leq \frac{2(1-\sigma)}{\beta} \quad (p=1); \\ |a_{-1}| &\leq \frac{4(1-\sigma)}{\sqrt{|\beta(\beta-1)(1-\sigma)|}} \quad (p=2; \beta \neq 1); \\ \left|a_{2-p} - \mu a_{1-p}^2\right| &\leq \frac{2p(1-\sigma)}{(p-2)\beta} \max\left\{1; \left|1 - \left(\beta - 1 + \frac{2p(p-2)}{(p-1)^2}\right)\frac{(1-\sigma)\mu}{\beta}\right|\right\} \ (p \neq 1,2). \end{aligned}$$

3. Conclusions

In our present investigation, we have defined some classes $\mathcal{M}_p^{\beta}(L, M)$ and $\mathcal{M}_p^{\beta}(\sigma)$ of meromorphic multivalent functions by using the principle of subordination. Furthermore, we have derived the subordination results, sandwich properties, inclusion relationship, and Fekete–Szegö inequalities for the functions belonging to these classes.

Author Contributions: Conceptualization, T.M.S. and A.E.S.; methodology, T.M.S. and A.E.S.; software, T.M.S. and A.E.S.; validation, T.M.S. and A.E.S.; formal analysis, T.M.S. and A.E.S.; investigation, T.M.S. and A.E.S.; resources, T.M.S. and A.E.S.; data curation, T.M.S. and A.E.S.; writing—original draft preparation, T.M.S. and A.E.S.; writing—review and editing, T.M.S. and A.E.S.; visualization, T.M.S. and A.E.S.; supervision, T.M.S. and A.E.S.; project administration, T.M.S. and A.E.S.; funding acquisition, T.M.S. and A.E.S. All authors have read and agreed to the published version of the manuscript.

Funding: The authors would like to thank the Deanship of Scientific Research at Umm Al-Qura University for supporting this work by Grant Code: (23UQU4350561DSR01).

Data Availability Statement: Not applicable.

Acknowledgments: The authors are thankful to the referees for their valuable comments which helped in improving the paper.

Conflicts of Interest: The authors declare no conflict of interest.

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