

## Article

# Linearization of Second-Order Non-Linear Ordinary Differential Equations: A Geometric Approach

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**Abstract:** Using the coefficients of a system semilinear cubic in the first derivative second order differential equations one defines a connection in the space of the independent and dependent variables, which is specified modulo two free parameters. In this way, to any such equation one associates an affine space which is not necessarily Riemannian, that is, a metric is not required. If such a metric exists, then under the Cartan parametrization the geodesic equations of the metric coincide with the system of the considered semilinear equations. In the present work, we consider semilinear cubic in the first derivative second order differential equations whose Lie symmetry algebra is the  $sl(3, \mathbb{R})$ . The covariant condition for these equations is the vanishing of the curvature tensor. We demonstrate the method in the solution of the Painlevé-Ince equation and in a system of two equations. Because the approach is geometric, the number of equations in the system is not important besides the complication in the calculations. It is shown that it is possible to linearize an equation in this form using a different covariant condition, for example, assuming the space to be of constant non-vanishing curvature. Finally, it is shown that one computes the associated metric to a semilinear cubic in the first derivatives differential equation using the inverse transformation derived from the transformation of the connection.

**Keywords:** linearization; second order semilinear cubic in the first derivative equations; Lie classification;  $sl(3, \mathbb{R})$ ; non-linear equations; Painlevé-Ince equation; covariant condition



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## 1. Introduction

Generally speaking, the problem of linearization of a given system of non-linear equations is to find a change of variables which transforms the given system to a system of linear equations, whose solution is possible to be found. It appears that the first to study systematically the linearization of second-order ordinary differential equations was S. Lie [1], who showed that a second-order ordinary differential equation  $\ddot{x} = f(t, x, \dot{x})$  is linearizable to the equation  $\frac{d^2u}{dv^2} = 0$ , by a change of variables  $(x, t) \rightleftharpoons (u, v)$  if, and only if,  $f$  is a polynomial of third degree with respect to the first-order derivative  $\dot{x}$ , that is,

$$\ddot{x} + a\dot{x}^3 + b\dot{x}^2 + c\dot{x} + d = 0$$

provided the coefficients  $a(t, y)$ ,  $b(t, x)$ ,  $c(t, x)$ , and  $d(t, x)$  satisfy certain conditions (see e.g., [2,3]). He showed that it is linearizable by a change of the dependent variable  $x$  if, and only if,  $a = 0$ .

It is well known that a second-order ODE can have 0, 1, 2, 3, or 8 Lie point symmetries. It is also known that the ones which admit the 8 Lie point symmetries (generating the  $sl(3, \mathbb{R})$  algebra) satisfy the Lie linearization test (e.g., [4–7]). For the other type of ODEs, non-local transformations have been investigated (see e.g., [2,8–13]) and the linearization problem for these transformations is still under consideration.

In addition to the linearization of single non-linear equations, in the literature there have been considered systems of quadratic non-linear cubic equations of the form

$$\frac{d^2 x^\mu}{dt^2} + A_{\nu\rho}(x, t) \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \frac{dx^\rho}{dt} + B_{\nu\rho}^\mu(x, t) \frac{dx^\nu}{dt} \frac{dx^\rho}{dt} + C_\nu^\mu(x, t) \frac{dx^\nu}{dt} + D^\mu(x, t) = 0 \quad (1)$$

where the coefficients  $A_{\nu\rho}(x, t)$ ,  $B_{\nu\rho}^\mu(x, t)$ ,  $C_\nu^\mu(x, t)$ ,  $D^\mu(x, t)$  are functions of the variables  $x^\mu$ ,  $\mu, \nu, \rho = 1, 2, \dots, n-1$ , and  $t$ , and  $A_{\nu\rho}(x, t)$ ,  $B_{\nu\rho}^\mu(x, t)$  are symmetric in the lower indices. In a number of works [4,5], using a geometric approach, the authors defined a connection in the space of the variables  $\{x^i\} = \{x^\mu, t\}$ , where  $i = 1, 2, \dots, n$ , and computed the linearization conditions from the requirement that this connection is flat, i.e.,  $R_{jkl}^i = 0$ , where  $R_{jkl}^i$  is the curvature tensor of the connection. These conditions are given explicitly in [6]. As expected, for the case of one equation these conditions reduce to the Lie linearization conditions.

A different algebraic approach for the linearization of (1) when  $n = 3$  has been proposed in [14], where the inverse function theorem is used in order to eliminate one variable and essentially reduce the system to one equation where one may apply the original Lie conditions. This approach has been called the sequential linearizing problem and a theorem is given in order to be applicable.

A more general geometric approach, different from the previous, concerns the use of projective connections and the Thomas parameters. Details on this approach can be found in the extensive paper of Aminova [15] and the detailed references given therein. This approach is best for theoretical considerations rather than for general practical applications. Close to that work is the use of Lie symmetries of the ordinary differential equations [16,17]. It has been shown [18] that the Lie symmetry group of path (geodesic) equations in an affine space (not necessarily a Riemannian space) is generated by the special projective group of the space. For a space of constant curvature, this group has dimension  $n(n+2)$ ; for the Euclidean space  $E^2$ , it has dimension 8 (the  $SL(3, R)$ ); and for the Euclidean space  $E^3$ , it has dimension 15. A system of equations of the form (1) is linearizable to the form  $\ddot{y}^i = 0$  provided it is invariant under the generators of the projective group.

Finally, another approach [19,20] is to determine a metric whose geodesic equations reduce to the given semilinear cubic equation. This process has been called metrizable. This is possible due to the Cartan parametrization in which one eliminates the differentiation parameter and replaces it with one of the dependent variables. Not all semilinear cubic equations are metrizable. For dimension two, an algorithm has been developed which determines the conditions under which metrizable is possible and also provides the corresponding metric. This approach relies heavily on the use of algebraic computing and it appears that it becomes prohibitive for higher dimensions as well as for systems of semilinear cubic equations.

In the present paper, we re-examine the linearization problem of a system of equations of the form (1) using a firm geometric approach which generalizes the previous discussions in the following aspects:

- It assumes only a symmetric connection whose paths (not necessarily geodesics) are given by equations of the form (1), i.e., does not assume a metric and a Riemannian connection. However, if it is wished, such a metric can be computed a posteriori.
- It does not require that the final linearized equation is of the form  $\ddot{y}^i = 0$ , but any linear form that can be solved. Therefore, potentially, it can be applied to all types of equations irrespective of the dimension of the corresponding symmetry algebra.
- It is systematic, in the sense that it can be used in practice in a step wise manner in order to linearize a single equation or a given system of equations of the form (1).

Although the proposed linearization method does not require that the reduced system will be the simple system  $\ddot{y}^i = 0$  (in general, the method connects a non-linear second-order differential equation (SODE) with the autoparallels/paths of an affine space), in this first discussion we limit our considerations to systems of equations linearized to the form  $\ddot{y}^i = 0$ . By means of solved examples, we demonstrate the application of the method.

The organization of the paper is as follows.

In Section 2, we show that the type of Equation (1) follows from the path equations of an affine space with a symmetric connection (not necessarily a Riemannian connection) if one uses one of the coordinates in order to eliminate the effect of parametrization. This parametrization is known as Cartan parametrization [15]. In Section 3, we discuss how the linearization is achieved by using the connection. In Section 4, we solve the well-known Painlevé-Ince equation. In Section 5, we consider a system of two equations. In Section 6, we show by a counter-example that the linearizing condition  $R_{jkl}^i = 0$  is not the only choice. In Section 7, we show how one obtains an a posteriori metric for a given equation. In Section 8, we draw our conclusions.

## 2. Cartan Parametrization

Consider an  $n$  dimensional affine space  $(M, \Gamma_{jk}^i)$ , where  $\Gamma_{jk}^i$  are the components of a symmetric affine connection. In that space, the equation of paths is

$$\frac{d^2 x^i}{dt^2} + \Gamma_{jk}^i \frac{dx^j}{dt} \frac{dx^k}{dt} = 0 \quad (2)$$

where  $t$  is a canonical parameter (not to be confused with the affine parameter) and Latin indices take the values  $1, 2, \dots, n$ . In order to eliminate the effect of the parameter, Cartan used one of the coordinates as a parameter and, subsequently, used the path equation of this coordinate to eliminate the parameter from the remaining path equations. This new parametrization is called the Cartan parametrization of paths [15].

We choose the new parameter to be the coordinate  $x^n$ . Then, the path equations for the remaining  $n - 1$  equations are

$$\frac{d^2 x^\alpha}{dt^2} + \Gamma_{\beta\gamma}^\alpha \frac{dx^\beta}{dt} \frac{dx^\gamma}{dt} + 2\Gamma_{\beta n}^\alpha \frac{dx^\beta}{dt} \frac{dx^n}{dt} + \Gamma_{nn}^\alpha \left( \frac{dx^n}{dt} \right)^2 = 0. \quad (3)$$

where the Greek indices take the values  $1, 2, \dots, n - 1$ .

To rewrite the path equations in terms of the parameter  $x^n$ , we use that  $\frac{dx^i}{dt} = \dot{x}^i \frac{dx^n}{dt}$  and

$$\frac{d^2 x^i}{dt^2} = \ddot{x}^i \left( \frac{dx^n}{dt} \right)^2 + \dot{x}^i \frac{d^2 x^n}{dt^2} \quad (4)$$

where a dot over a letter indicates derivation wrt  $x^n$ , e.g.,  $\dot{x}^i = \frac{dx^i}{dx^n}$ . Equation (2) for the coordinate  $x^n$  is

$$\frac{d^2 x^n}{dt^2} + \Gamma_{\beta\gamma}^n \frac{dx^\beta}{dt} \frac{dx^\gamma}{dt} + 2\Gamma_{\beta n}^n \frac{dx^\beta}{dt} \frac{dx^n}{dt} + \Gamma_{nn}^n \left( \frac{dx^n}{dt} \right)^2 = 0. \quad (5)$$

Replacing  $\frac{d^2 x^n}{dt^2}$  in (4) we find:

$$\frac{d^2 x^\alpha}{dt^2} = \ddot{x}^\alpha \left( \frac{dx^n}{dt} \right)^2 - \dot{x}^\alpha \left[ \Gamma_{\beta\gamma}^n \dot{x}^\beta \dot{x}^\gamma + 2\Gamma_{\beta n}^n \dot{x}^\beta + \Gamma_{nn}^n \right] \left( \frac{dx^n}{dt} \right)^2. \quad (6)$$

Substituting (6) in the path Equation (3), we obtain:

$$\ddot{x}^\alpha - \Gamma_{\beta\gamma}^\alpha \dot{x}^\beta \dot{x}^\gamma - \Gamma_{n\gamma}^\alpha \dot{x}^\gamma - \Gamma_{\beta n}^\alpha \dot{x}^\beta - \Gamma_{nn}^\alpha + \Gamma_{\beta\gamma}^\alpha \dot{x}^\beta \dot{x}^\gamma + \Gamma_{n\gamma}^\alpha \dot{x}^\gamma + \Gamma_{\beta n}^\alpha \dot{x}^\beta + \Gamma_{nn}^\alpha = 0. \quad (7)$$

Obviously, for the coordinate  $x^n$ , the path equation is  $\ddot{x}^n = 0$ .

We introduce the parameters:

$$a_{\beta\gamma}(x) = -\Gamma_{\beta\gamma}^n \quad (8)$$

$$b_{\beta\gamma}^\alpha(x) = \Gamma_{\beta\gamma}^\alpha - \Gamma_{n\beta}^n \delta_\gamma^\alpha - \Gamma_{n\gamma}^n \delta_\beta^\alpha \quad (9)$$

$$c_\beta^\alpha(x) = \Gamma_{n\beta}^\alpha + \Gamma_{\beta n}^\alpha - \Gamma_{nn}^n \delta_\beta^\alpha \quad (10)$$

$$d^\alpha(x) = \Gamma_{nn}^\alpha \quad (11)$$

and the path equation for the  $n - 1$  Equation (7) is written as follows:

$$\ddot{x}^\alpha + a_{\beta\gamma}(x) \dot{x}^\alpha \dot{x}^\beta \dot{x}^\gamma + b_{\beta\gamma}^\alpha(x) \dot{x}^\beta \dot{x}^\gamma + c_\beta^\alpha(x) \dot{x}^\beta + d^\alpha(x) = 0. \quad (12)$$

This is a system of ordinary differential equations of the form (1). From (8)–(11), it follows that the quantities  $a_{\beta\gamma}(x)$ ,  $b_{\beta\gamma}^\alpha(x)$ ,  $c_\beta^\alpha(x)$ , and  $d^\alpha(x)$  are not in general tensors.

We can unambiguously invert (8)–(11) and compute the quantities  $\Gamma_{jk}^i$  in terms of the coefficients  $a_{\beta\gamma}(x)$ ,  $b_{\beta\gamma}^\alpha(x)$ ,  $c_\beta^\alpha(x)$ , and  $d^\alpha(x)$  as follows:

$$\Gamma_{\beta\gamma}^n = -a_{\beta\gamma} \quad (13)$$

$$\Gamma_{nn}^\alpha = d^\alpha \quad (14)$$

$$\Gamma_{nn}^n = \frac{1}{n-1} (2\Gamma_{n\alpha}^\alpha - c_\beta^\beta) \quad (15)$$

$$\Gamma_{n\gamma}^n = \frac{1}{n} (\Gamma_{\beta\gamma}^\beta - b_{\alpha\gamma}^\alpha) \quad (16)$$

$$2\Gamma_{n\beta}^\alpha = c_\beta^\alpha + \frac{1}{n-1} (2\Gamma_{n\gamma}^\gamma - c_\gamma^\gamma) \delta_\beta^\alpha \quad (17)$$

$$\Gamma_{\beta\gamma}^\alpha = b_{\beta\gamma}^\alpha + \frac{1}{n} (\Gamma_{\sigma\beta}^\sigma \delta_\gamma^\alpha + \Gamma_{\sigma\gamma}^\sigma \delta_\beta^\alpha) - \frac{1}{n} (b_{\sigma\beta}^\sigma \delta_\gamma^\alpha + b_{\sigma\gamma}^\sigma \delta_\beta^\alpha). \quad (18)$$

In order to find the degree that the coefficients  $a_{\beta\gamma}(x)$ ,  $b_{\beta\gamma}^\alpha(x)$ ,  $c_\beta^\alpha(x)$ , and  $d^\alpha(x)$  fix the quantities  $\Gamma_{jk}^i$  we count components. We have the following:

$$\text{For the } \Gamma_{jk}^i : \frac{n(n+1)}{2} \times n = \frac{n^2(n+1)}{2} = \frac{n^3+n^2}{2}$$

$$\text{For the coefficients } a_{\beta\gamma}(x), b_{\beta\gamma}^\alpha(x), c_\beta^\alpha(x), d^\alpha(x) : \frac{(n-1)n}{2} + (n-1) \frac{(n-1)n}{2} + (n-1)^2 + (n-1) = \frac{n^3+n^2-2n}{2}$$

The difference is:

$$\frac{n^3+n^2}{2} - \frac{n^3+n^2-2n}{2} = n.$$

We conclude that each set of coefficients  $a_{\beta\gamma}(x)$ ,  $b_{\beta\gamma}^\alpha(x)$ ,  $c_\beta^\alpha(x)$ ,  $d^\alpha(x)$  in an  $n$  dimensional manifold  $M$  defines an  $n$  parameter family of symmetric quantities  $\Gamma_{jk}^i$  (not necessarily connections!). It can be shown that this family has a geometric origin and, specifically, it is related to a projective structure in  $M$  [15]. However, this will not concern us here because our purpose is to linearize, i.e., to solve a given system of equations of the form (1) and not go into the geometric significance of each step. In the next section, we show how this is done.

### 3. Solving the System of Equation (12)

We assume that we are given a system of  $n - 1$  quadratic cubically semilinear equations of the form (12) and treat the variables  $x^\alpha, x^n$  as the coordinate functions in a manifold  $M$ . From the given system, we read the coefficients  $a_{\beta\gamma}(x)$ ,  $b_{\beta\gamma}^\alpha(x)$ ,  $c_\beta^\alpha(x)$ ,  $d^\alpha(x)$  and, using the relations, (13)–(18) we compute the  $n$  parameter family of quantities  $\Gamma_{jk}^i$ . As has been mentioned, the quantities  $\Gamma_{jk}^i$  so far do not have an explicit geometric meaning.

We recall that a geometric object is identified by its transformation law under coordinate transformations. Using this statement, from the  $n$  parameter family of the quantities

$\Gamma_{jk}^i$ , we select the members that are the components of the geometric object affine connection. This means that under a coordinate transformation  $x^i \rightleftharpoons q^i, i = 1, 2, \dots, n$ , with Jacobian  $J\left(\frac{\partial x^i}{\partial q^j}\right) \neq 0$  the selected quantities  $\Gamma_{jk}^i$  transform to  $\tilde{\Gamma}_{jk}^i$  according to the rule:

$$\Gamma_{jk}^i = \frac{\partial x^i}{\partial q^r} \frac{\partial q^l}{\partial x^j} \frac{\partial q^m}{\partial x^k} \tilde{\Gamma}_{lm}^r + \frac{\partial x^i}{\partial q^r} \frac{\partial^2 q^r}{\partial x^j \partial x^k}. \quad (19)$$

In order to secure that the system (19) has a solution, we have to check the integrability condition. It is well known [21] that this condition is the existence of the curvature tensor  $R_{jkl}^i$  of the connection  $\Gamma_{jk}^i$  (and  $\tilde{\Gamma}_{jk}^i$  because the two connections are related by a coordinate transformation and  $R_{jkl}^i$  is a tensor).

Up to this point, the required coordinate linearizing transformation cannot be determined. To do that, we have to determine a set of  $\Gamma_{jk}^i$  and  $\tilde{\Gamma}_{jk}^i$  and then apply the condition (19). For this purpose, we have to impose a second condition. This condition must satisfy the following requirements:

- It must be covariant, that is, expressed in terms of geometric objects; therefore, independent of the coordinate system.
- Must involve the curvature tensor in order for the system (19) to have a solution, that is, for the interpretation of the quantities  $\Gamma_{jk}^i$  to be the components of a connection. This explains why originally Lie and, subsequently, other authors (e.g., [2,4–6]) used the condition  $R_{jkl}^i = 0$  in order to realize the linearization process. It is seen that the use of a metric, hence of a Riemannian connection, is not necessary. However, as we shall show in Section 7, a metric is possible to be introduced and be computed.
- The geometry defined by the geometric object(s) of this condition must provide the coefficients  $\tilde{\Gamma}_{lm}^r$ .
- The transformed linear system must be solvable (which, in general, is the case).

The covariant condition leads to an overdetermined system of equations involving the  $n$  free parameters of the  $n$  parameter family of the quantities  $\Gamma_{jk}^i$  or, equivalently, the coefficients  $a_{\beta\gamma}(x)$ ,  $b_{\beta\gamma}^\alpha(x)$ ,  $c_\beta^\alpha(x)$ , and  $d^\alpha(x)$ . There are two possibilities:

- The overdetermined system has a solution, not necessarily unique. This solution fixes the quantities  $\Gamma_{jk}^i$ , while the quantities  $\tilde{\Gamma}_{lm}^r$  are determined independently by the considered covariant condition. Both these quantities are replaced in (19) in order to determine the coordinate transformation. Subsequently, one solves the linear system of second-order PDEs which contain the linearizing variables and uses the inverse coordinate transformation to find the solution of the given system (12).
- The overdetermined system of equations is not satisfied for any values of the free parameters of the  $n$  parameter family of the quantities  $\Gamma_{jk}^i$ . Then, one has to look for another covariant condition which will satisfy the conditions a.–d., above.

In this approach, the problem of linearizing the given system reduces to the problem of finding the appropriate covariant condition which will provide the necessary sets of  $\Gamma_{jk}^i$ ,  $\tilde{\Gamma}_{lm}^r$  within the family of connections defined by the given set of parameters  $a_{\beta\gamma}(x)$ ,  $b_{\beta\gamma}^\alpha(x)$ ,  $c_\beta^\alpha(x)$ ,  $d^\alpha(x)$ . We note that these sets do not exhaust all the members of the family.

*The Requirement  $\tilde{\Gamma}_{jk}^i = 0$*

The requirement  $\tilde{\Gamma}_{jk}^i = 0$  implies a. The condition  $R_{jkl}^i = 0$  and b. The transformed linear equation is  $\frac{d^2 q^a}{dt^2} = 0$  with solution  $q^a(t) = A_1^a t + A_2^a$  where  $A_1^a, A_2^a$  are arbitrary constants. Furthermore, the transformation rule (19) becomes

$$\Gamma_{jk}^i = \frac{\partial x^i}{\partial q^r} \frac{\partial^2 q^r}{\partial x^j \partial x^k} \implies \frac{\partial q^m}{\partial x^i} \Gamma_{jk}^i = \frac{\partial^2 q^m}{\partial x^j \partial x^k}. \quad (20)$$

In this relation, the unknown quantity is the  $\frac{\partial q^m}{\partial x^i}$ ; therefore, (in principle) it can be solved in order to determine the transformation  $q^i(x^a)$ . Then, using the solution  $q^a(t) = A_1^a t + A_2^a$  and taking the inverse transformation  $x^a(q^i)$ , one obtains the solution  $x^i(t)$  of the original system.

We demonstrate the above systematic method by solving a number of examples.

#### 4. The Generalized Painlevé-Ince Equation

We consider the second-order non-linear ODE

$$y'' + k_1 y y'^3 + k_2 y y' + k_3 y^3 = 0 \quad (21)$$

where  $y' = \frac{dy}{dx}$  and  $k_1, k_2, k_3$  are arbitrary constants (possibly zero). For the values  $k_1 = 0$ ,  $k_2 = 3$  and  $k_3 = 1$  Equation (21) is the well-known Painlevé-Ince equation. The modified form of this equation—resulting from (21) for  $k_1 = 0$ —has attracted a certain amount of attention in recent decades [22–25]. We shall require that (21) is linearizable to the reduced form  $\frac{d^2 q^i}{dt^2} = 0$ , that is, it admits the Lie symmetry algebra of  $sl(3, R)$  (however, see [26]), and we shall fix, accordingly, the values of the parameters  $k_1, k_2$ , and  $k_3$ . We demonstrate this using a systematic approach.

Step 1: Compute the quantities  $\Gamma_{jk}^i$ .

The ODE (21) is of the form (12) with  $n = 2$ ,  $x^1 = y$ ,  $x^2 = x$ ,  $a_{11} = k_1 y$ ,  $b_{11}^1 = 0$ ,  $c_1^1 = k_2 y$ , and  $d^1 = k_3 y^3$ . Replacing in the system of Equations (13)–(18), we find the non-vanishing connection coefficients:

$$\Gamma_{22}^1 = k_3 y^3, \quad \Gamma_{11}^2 = -k_1 y, \quad \Gamma_{11}^1 = 2\Gamma_{12}^2, \quad \Gamma_{22}^2 = 2\Gamma_{12}^1 - k_2 y \quad (22)$$

where the coefficients  $\Gamma_{12}^1$  and  $\Gamma_{12}^2$  are not specified (a two-parameter family as expected).

Step 2: Select the appropriate affine connection  $\Gamma_{jk}^i$ .

We require that there exist canonical coordinates  $q^i = (u, v)$  in which  $\tilde{\Gamma}_{jk}^i = 0$  (i.e., the connection is flat). The condition for this requirement is

$$R_{jkl}^i = 0 \implies \Gamma_{jl,k}^i - \Gamma_{jk,l}^i + \Gamma_{ck}^i \Gamma_{jl}^c - \Gamma_{cl}^i \Gamma_{jk}^c = 0$$

from which we obtain the following set of equations:

$$R_{112}^1 = 0 : \quad \Gamma_{12,y}^1 - 2\Gamma_{12,x}^2 + \Gamma_{12}^1 \Gamma_{12}^2 + k_1 k_3 y^4 = 0 \quad (23)$$

$$R_{212}^1 = 0 : \quad \Gamma_{12,x}^1 - \left(\Gamma_{12}^1\right)^2 + k_2 y \Gamma_{12}^1 - k_3 y^3 \Gamma_{12}^2 - 3k_3 y^2 = 0 \quad (24)$$

$$R_{112}^2 = 0 : \quad \Gamma_{12,y}^2 - \left(\Gamma_{12}^2\right)^2 + k_1 y \Gamma_{12}^1 - k_1 k_2 y^2 = 0 \quad (25)$$

$$R_{212}^2 = 0 : \quad \Gamma_{12,x}^2 - 2\Gamma_{12,y}^1 + \Gamma_{12}^1 \Gamma_{12}^2 + k_1 k_3 y^4 + k_2 = 0. \quad (26)$$

To simplify the notation, we set  $\Gamma_{12}^1 = A(x, y)$  and  $\Gamma_{12}^2 = B(x, y)$ . Then, Equations (23)–(26) become:

$$A_y - 2B_x + AB + k_1 k_3 y^4 = 0 \quad (27)$$

$$A_x - A^2 + k_2 y A - k_3 y^3 B - 3k_3 y^2 = 0 \quad (28)$$

$$B_y - B^2 + k_1 y A - k_1 k_2 y^2 = 0 \quad (29)$$

$$B_x - 2A_y + AB + k_1 k_3 y^4 + k_2 = 0. \quad (30)$$

Replacing  $A_y$  from (27) into (30) and  $B_x$  from (30) into (27), we obtain the following first order system of PDEs:

$$A_y - AB - k_1 k_3 y^4 - \frac{2k_2}{3} = 0 \quad (31)$$

$$A_x - A^2 + k_2 y A - k_3 y^3 B - 3k_3 y^2 = 0 \quad (32)$$

$$B_y - B^2 + k_1 y A - k_1 k_2 y^2 = 0 \quad (33)$$

$$B_x - AB - k_1 k_3 y^4 - \frac{k_2}{3} = 0. \quad (34)$$

This system is solved for two families of the parameters  $k_1, k_2$ , and  $k_3$  only when  $A = g(y)$  and  $B = f(y)$ . In this case, Equations (31)–(34) become:

$$\frac{dg}{dy} - fg - k_1 k_3 y^4 - \frac{2k_2}{3} = 0 \quad (35)$$

$$g^2 - k_2 y g + k_3 y^3 f + 3k_3 y^2 = 0 \quad (36)$$

$$\frac{df}{dy} - f^2 + k_1 y g - k_1 k_2 y^2 = 0 \quad (37)$$

$$fg + k_1 k_3 y^4 + \frac{k_2}{3} = 0. \quad (38)$$

Adding Equations (35) and (38), we find  $g = \frac{k_2}{3}y$ . Replacing this into the remaining Equations (36)–(38), we obtain:

$$k_3 y f + 3k_3 - \frac{2k_2^2}{9} = 0 \quad (39)$$

$$\frac{df}{dy} = f^2 + \frac{2k_1 k_2}{3} y^2 \quad (40)$$

$$k_2 y f + 3k_1 k_3 y^4 + k_2 = 0. \quad (41)$$

We consider two cases: (i)  $k_2 = 0$ , and (ii)  $k_2 \neq 0$ .

#### 4.1. Case $k_2 = 0$

In this case,  $g = 0 \implies A = 0 \implies \Gamma_{12}^1 = 0$ , and Equations (39)–(41) become

$$k_3(yf + 3) = 0 \quad (42)$$

$$\frac{df}{dy} = f^2 \quad (43)$$

$$k_1 k_3 = 0. \quad (44)$$

Solving Equation (43), we find that  $f = -\frac{1}{y} \implies \Gamma_{12}^2 = -\frac{1}{y}$ . Replacing this function into (42), we get  $k_3 = 0$  and Equation (44) is satisfied identically, leaving  $k_1$  free (in order to have non-trivial solutions, we take  $k_1 \neq 0$ ).

Therefore, the original ODE (21) becomes

$$y'' + k_1 y y'^3 = 0 \quad (45)$$

and the associated connection coefficients (22) are

$$\Gamma_{22}^1 = \Gamma_{12}^1 = \Gamma_{22}^2 = 0, \quad \Gamma_{11}^1 = -\frac{2}{y}, \quad \Gamma_{11}^2 = -k_1 y, \quad \Gamma_{12}^2 = -\frac{1}{y}. \quad (46)$$

We note that all components of  $\Gamma_{jk}^i$  have been computed.



Step 3: Compute the linearizing coordinate transformation.

In the canonical coordinates  $q^i = (u, v)$  the  $\tilde{\Gamma}_{jk}^i = 0$ . Using the resulting transformation law (20) of the connection coefficients, we find the following system of equations:

$$u_{yy} = -\frac{2}{y}u_y - k_1yu_x, \quad u_{xx} = 0, \quad u_{xy} = -\frac{u_x}{y} \quad (47)$$

and

$$v_{yy} = -\frac{2}{y}v_y - k_1yv_x, \quad v_{xx} = 0, \quad v_{xy} = -\frac{v_x}{y}. \quad (48)$$

We note that the two sets of Equations (47) and (48) are similar. Therefore, it suffices to solve one of them. Solving the system of PDEs (47), we find the solutions

$$u = \frac{c_1}{y} + c_2 \left( \frac{k_1}{6}y^2 - \frac{x}{y} \right), \quad v = \frac{c_3}{y} + c_4 \left( \frac{k_1}{6}y^2 - \frac{x}{y} \right) \quad (49)$$

where  $c_1, c_2, c_3$ , and  $c_4$  are arbitrary constants.

The solutions (49) generate a well-defined transformation  $x^i = (y, x) \mapsto q^i = (u, v)$  iff

$$J = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial x} \end{vmatrix} = \frac{c_4c_1 - c_2c_3}{y^3} \neq 0.$$

For the choice  $c_1 = c_4 = 1$  and  $c_2 = c_3 = 0$ , we obtain the admissible transformation

$$u = \frac{1}{y}, \quad v = \frac{k_1}{6}y^2 - \frac{x}{y}. \quad (50)$$

In the canonical coordinates, the reduced form  $\frac{d^2q^i}{dt^2} = 0$  implies that

$$v = m_1u + m_2 \quad (51)$$

where  $m_1$  and  $m_2$  are arbitrary constants.

Replacing (51) into Equation (50), we find the cubic algebraic equation:

$$\frac{k_1}{6}y^3 - m_2y - x - m_1 = 0. \quad (52)$$

whose solution is

$$y(x) = k_1^{-1/3}h(x) - \frac{2m_3k_1^{-2/3}}{h(x)}, \quad y_{\pm}(x) = -\frac{1}{2} \left( k_1^{-1/3}h(x) - \frac{2m_3k_1^{-2/3}}{h(x)} \right) \pm \frac{\sqrt{3}}{2}i \left( k_1^{-1/3}h(x) + \frac{2m_3k_1^{-2/3}}{h(x)} \right) \quad (53)$$

where we rename the constant  $m_2 = -m_3$  and the function

$$h(x) = \left[ 3(x + m_1) + \sqrt{\frac{8m_3^3}{k_1} + 9(x + m_1)^2} \right]^{1/3}.$$

The functions (53) are the solutions of the ODE (45).

Note: The coordinate transformation given in [22] (Equation (22)) is not correct. There are also some minor misprints in the given solution (Equation (26)). Finally, the two complex solutions  $y_{\pm}(x)$  we found here are new.



#### 4.2. Case $k_2 \neq 0$

In this case, we have  $g = \frac{k_2}{3}y$  and Equation (41) gives  $f = -\frac{3k_1k_3}{k_2}y^3 - \frac{1}{y}$ . Replacing this function in the remaining Equations (39) and (40), we find that  $k_1 = 0$  and  $k_3 = \left(\frac{k_2}{3}\right)^2$ ; therefore,  $f = -\frac{1}{y}$ .

The original ODE (21) becomes the generalized Painlevé-Ince equation

$$y'' + k_2yy' + \left(\frac{k_2}{3}\right)^2y^3 = 0 \quad (54)$$

and the associated connection coefficients (22) are

$$\Gamma_{22}^1 = \left(\frac{k_2}{3}\right)^2y^3, \quad \Gamma_{12}^1 = -\Gamma_{22}^2 = \frac{k_2}{3}y, \quad \Gamma_{11}^1 = -\frac{2}{y}, \quad \Gamma_{11}^2 = 0, \quad \Gamma_{12}^2 = -\frac{1}{y}. \quad (55)$$

In the canonical coordinates  $q^i = (u, v)$ , the  $\tilde{\Gamma}_{jk}^i = 0$ . Using the resulting transformation law of the connection coefficients (20), we find the following system of similar equations:

$$u_{yy} = -\frac{2}{y}u_y, \quad u_{xx} = \left(\frac{k_2}{3}\right)^2y^3u_y - \frac{k_2}{3}yu_x, \quad u_{xy} = \frac{k_2}{3}yu_y - \frac{u_x}{y} \quad (56)$$

and

$$v_{yy} = -\frac{2}{y}v_y, \quad v_{xx} = \left(\frac{k_2}{3}\right)^2y^3v_y - \frac{k_2}{3}yv_x, \quad v_{xy} = \frac{k_2}{3}yv_y - \frac{v_x}{y}. \quad (57)$$

It is enough to solve the system of PDEs (56). The answer is

$$u = c_1\left(\frac{x^2}{2} - \frac{3x}{k_2y}\right) + c_2\left(x - \frac{3}{k_2y}\right), \quad v = c_3\left(\frac{x^2}{2} - \frac{3x}{k_2y}\right) + c_4\left(x - \frac{3}{k_2y}\right) \quad (58)$$

where  $c_1, c_2, c_3$ , and  $c_4$  are arbitrary constants.

The Jacobian of the transformation  $J = \frac{9(c_4c_1 - c_2c_3)}{k_2^2y^3}$  must be non-zero. Therefore, for the choice  $c_1 = c_4 = 1$  and  $c_2 = c_3 = 0$ , we obtain the admissible transformation

$$u = \frac{x^2}{2} - \frac{3x}{k_2y}, \quad v = x - \frac{3}{k_2y}. \quad (59)$$

In the canonical coordinates, the reduced form  $\frac{d^2q^i}{dt^2} = 0$  implies that

$$v = m_1u + m_2 \quad (60)$$

where  $m_1$  and  $m_2$  are arbitrary constants.

Replacing the solution (60) into the relations (59), we find that the ODE (54) admits the solution

$$y(x) = \frac{3}{k_2} \frac{m_1x - 1}{\frac{m_1}{2}x^2 - x + m_2}. \quad (61)$$

Note: This result reduces to that of [22] (see Equations (22) and (23)) for  $k_2 = 3$ . We also note that Equations (22) and (23) of [22] refer to Equation (32) instead of Equation (19).

## 5. The Case of Systems of Equations

The geometric approach discussed in Section 3 is covariant and independent of the number of equations considered. In addition, it leads directly to the solution of the system, as we show in the following example.

### A System of Two Second-Order ODEs of the Form (12)

We solve the second-order cubically semi-linear system (see Example 4 in [5])

$$y'' + \frac{1}{x}y' + y'^2 + \left(\frac{x}{y} + \frac{x}{y^2}\right)y'^3 = 0 \quad (62)$$

$$z'' + \frac{1}{x}z' + z'^2 + 2y'z' + \left(\frac{x}{y} + \frac{x}{y^2}\right)y'^2z' = 0 \quad (63)$$

where  $f' \equiv \frac{df}{dx}$ .

Solution.

Step 1: Compute the quantities  $\Gamma_{jk}^i$ .

We have two equations of the form (12); therefore,  $n = 3$ . We set  $y = x^1$ ,  $z = x^2$ , and  $x = x^3$ . From the given system, we read the non-vanishing coefficients:

$$a_{11} = \frac{x}{y} + \frac{x}{y^2}, \quad b_{11}^1 = b_{22}^2 = b_{12}^2 = b_{21}^2 = 1, \quad c_1^1 = c_2^2 = \frac{1}{x}. \quad (64)$$

Replacing in the system of Equations (13)–(18), we find the non-vanishing connection coefficients:

$$\Gamma_{11}^1 = 2\Gamma_{12}^2 - 1, \quad \Gamma_{22}^2 = 2\Gamma_{12}^1 + 1, \quad \Gamma_{23}^2 = \Gamma_{13}^1, \quad \Gamma_{32}^3 = \Gamma_{12}^1, \quad \Gamma_{11}^3 = -\frac{x}{y} - \frac{x}{y^2}, \quad \Gamma_{13}^3 = \Gamma_{12}^2 - 1, \quad \Gamma_{33}^3 = 2\Gamma_{13}^1 - \frac{1}{x}$$

The coefficients  $\Gamma_{12}^1$ ,  $\Gamma_{13}^1$ , and  $\Gamma_{12}^2$  are the three free parameters ( $2 + 1$  as expected).

Step 2: Select the appropriate affine connection  $\Gamma_{jk}^i$ .

The condition

$$R_{jkl}^i = 0 \implies \Gamma_{jl,k}^i - \Gamma_{jk,l}^i + \Gamma_{ck}^i \Gamma_{jl}^c - \Gamma_{cl}^i \Gamma_{jk}^c = 0.$$

leads to an overdetermined system of first order PDEs. One solution of this system is (This solution is not unique. To find it we have assumed a certain functional form for the connection coefficients. The point is to find some solution in order to determine a linearization transformation. The general solution is not required)

$$\Gamma_{12}^1 = \Gamma_{13}^1 = 0, \quad \Gamma_{12}^2 = 1.$$

Using these values, we end up with the following non-vanishing connection coefficients:

$$\Gamma_{11}^1 = \Gamma_{22}^2 = \Gamma_{12}^2 = 1, \quad \Gamma_{11}^3 = -\frac{x}{y} - \frac{x}{y^2}, \quad \Gamma_{33}^3 = -\frac{1}{x}. \quad (65)$$

Step 3: Find the linearizing coordinate transformation.

Let  $q^i = (u, v, w)$  be the canonical coordinates in which  $\tilde{\Gamma}_{jk}^i = 0$ . Replacing  $\Gamma_{jk}^i$  from (65) into the transformation relation (20), we find the following system of equations:

$$q_{yy}^i = q_y^i - \frac{x}{y} \left(1 + \frac{1}{y}\right) q_x^i, \quad q_{yz}^i = q_z^i, \quad q_{zz}^i = q_z^i, \quad q_{xx}^i = -\frac{q_x^i}{x}, \quad q_{xz}^i = q_{xy}^i = 0.$$

The solution of this system of PDEs is

$$u = A_1 e^{y+z} + A_2 e^y + A_3 \ln(xy), \quad v = A_4 e^{y+z} + A_5 e^y + A_6 \ln(xy), \quad w = A_7 e^{y+z} + A_8 e^y + A_9 \ln(xy) \quad (66)$$

where  $A_I$  with  $I = 1, 2, \dots, 9$  are arbitrary constants. We note that the difference between the canonical variables is only in the constants!

The requirement of the non-vanishing of the Jacobian gives

$$J = \frac{e^{2y+z}}{x} [A_1(A_6A_8 - A_5A_9) + A_2(A_4A_9 - A_6A_7) + A_3(A_5A_7 - A_4A_8)] \neq 0.$$

One admissible choice is:  $A_3 = A_5 = A_7 = 1$  and  $A_1 = A_2 = A_4 = A_6 = A_8 = A_9 = 0$  because then  $J = \frac{e^{2y+z}}{x} \neq 0$ . For this choice, we find the coordinate transformation:

$$u = \ln(xy), \quad v = e^y, \quad w = e^{y+z} \quad (67)$$

which coincides with the transformation (5.6) of [5]. Inverting (67), we obtain

$$y = \ln v, \quad z = \ln \frac{w}{v}, \quad x = \frac{e^u}{\ln v}. \quad (68)$$

In the canonical coordinates, the reduced form  $\frac{d^2q^i}{dt^2} = 0$  implies that

$$u = c_1v + c_2, \quad w = c_3v + c_4$$

where  $c_1, c_2, c_3$ , and  $c_4$  are arbitrary constants. Replacing these results in (67), we find

$$c_1e^y + c_2 = \ln(xy), \quad z = \ln(c_3 + c_4e^{-y}).$$

If we solve the first relation we obtain the solution  $y = y(x)$ , which when replaced in the second relation gives the solution  $z = z(x)$ . The functions  $y(x), z(x)$  are the solutions of the system (62) and (63).

A special solution of this system is found for  $c_1 = 0$ . In this case, we have:

$$\ln(xy) = \text{const} \implies y(x) = \frac{c_5}{x}, \quad z(x) = \ln\left(c_3 + \frac{c_4}{e^{\frac{c_5}{x}}}\right) \quad (69)$$

where  $c_5$  is an arbitrary constant. It can be checked that the functions given in (69) solve the system (62) and (63).

## 6. The Uniqueness of the Covariant Condition

One might ask if the covariant condition which linearizes a quadratic semilinear system (12) is unique. The answer is 'no'. It is possible that different covariant conditions (within reason!) lead to the linearization of a given system of the form (12). Of course, in all cases, the final result, i.e., the solution of the system, is the same. We show this in the following example.

*Example: The 2-Dimensional Sphere*

We solve the second order semilinear equation

$$y'' - \frac{x}{2\kappa^2 + \frac{r^2}{2}}y'^3 + \frac{y}{2\kappa^2 + \frac{r^2}{2}}y'^2 - \frac{x}{2\kappa^2 + \frac{r^2}{2}}y' + \frac{y}{2\kappa^2 + \frac{r^2}{2}} = 0 \quad (70)$$

where  $f' \equiv \frac{df}{dx}$ ,  $r^2 = x^2 + y^2$  and  $\kappa \neq 0$  is a constant.

*Solution.*

This ODE is of the form (12) with  $n = 2$ ,  $x^1 = y$ ,  $x^2 = x$ ,  $a_{11} = c_1^1 = -\frac{x}{2\kappa^2 + \frac{r^2}{2}}$ , and  $b_{11}^1 = d^1 = \frac{y}{2\kappa^2 + \frac{r^2}{2}}$ . Replacing these values in the system of Equations (13)–(18), we find the following non-vanishing connection coefficients:

$$\Gamma_{22}^1 = \frac{y}{2\kappa^2 + \frac{r^2}{2}}, \quad \Gamma_{11}^2 = \frac{x}{2\kappa^2 + \frac{r^2}{2}}, \quad \Gamma_{11}^1 = 2\Gamma_{12}^2 + \frac{y}{2\kappa^2 + \frac{r^2}{2}}, \quad \Gamma_{22}^2 = 2\Gamma_{12}^1 + \frac{x}{2\kappa^2 + \frac{r^2}{2}} \quad (71)$$

where the coefficients  $\Gamma_{12}^1$  and  $\Gamma_{12}^2$  are the free parameters. In order to determine the free parameters, we use two different covariant conditions.

**First covariant condition.**

We require  $R^i_{jkl} = 0$ . This implies the following set of equations

$$R^1_{112} = 0 : \Gamma_{12,y}^1 - 2\Gamma_{12,x}^2 + \Gamma_{12}^1\Gamma_{12}^2 = 0 \quad (72)$$

$$R^1_{212} = 0 : \Gamma_{12,x}^1 - \left(\Gamma_{12}^1\right)^2 - \frac{x}{2\kappa^2 + \frac{r^2}{2}}\Gamma_{12}^1 - \frac{y}{2\kappa^2 + \frac{r^2}{2}}\Gamma_{12}^2 - \frac{1}{2\kappa^2 + \frac{r^2}{2}} = 0 \quad (73)$$

$$R^2_{112} = 0 : \Gamma_{12,y}^2 - \left(\Gamma_{12}^2\right)^2 - \frac{x}{2\kappa^2 + \frac{r^2}{2}}\Gamma_{12}^1 - \frac{y}{2\kappa^2 + \frac{r^2}{2}}\Gamma_{12}^2 - \frac{1}{2\kappa^2 + \frac{r^2}{2}} = 0 \quad (74)$$

$$R^2_{212} = 0 : \Gamma_{12,x}^2 - 2\Gamma_{12,y}^1 + \Gamma_{12}^1\Gamma_{12}^2 = 0. \quad (75)$$

From Equations (72) and (75), we find that  $\Gamma_{12}^1 = 0$  and  $\Gamma_{12}^2 = f(y)$ . Replacing these results in the remaining Equations (73) and (74), we obtain  $f(y) = -\frac{1}{y}$ . Therefore, the connection coefficients (71) become

$$\Gamma_{22}^1 = \frac{y}{2\kappa^2 + \frac{r^2}{2}}, \Gamma_{12}^1 = 0, \Gamma_{11}^1 = \frac{y}{2\kappa^2 + \frac{r^2}{2}} - \frac{2}{y}, \Gamma_{11}^2 = \Gamma_{22}^2 = \frac{x}{2\kappa^2 + \frac{r^2}{2}}, \Gamma_{12}^2 = -\frac{1}{y}. \quad (76)$$

In the canonical coordinates  $q^i = (u, v)$ , the connection coefficients  $\tilde{\Gamma}_{jk}^i = 0$ . Using the transformation law of the connection coefficients, we find the following system of similar equations:

$$u_{yy} = \left(\frac{y}{2\kappa^2 + \frac{r^2}{2}} - \frac{2}{y}\right)u_y + \frac{x}{2\kappa^2 + \frac{r^2}{2}}u_x, \quad u_{xx} = \frac{y}{2\kappa^2 + \frac{r^2}{2}}u_y + \frac{x}{2\kappa^2 + \frac{r^2}{2}}u_x, \quad u_{xy} = -\frac{u_x}{y} \quad (77)$$

and

$$v_{yy} = \left(\frac{y}{2\kappa^2 + \frac{r^2}{2}} - \frac{2}{y}\right)v_y + \frac{x}{2\kappa^2 + \frac{r^2}{2}}v_x, \quad v_{xx} = \frac{y}{2\kappa^2 + \frac{r^2}{2}}v_y + \frac{x}{2\kappa^2 + \frac{r^2}{2}}v_x, \quad v_{xy} = -\frac{v_x}{y}. \quad (78)$$

It is enough to solve the system of PDEs (77). The answer is

$$u = A_1\frac{x}{y} + A_2\frac{r^2 - 4\kappa^2}{y}, \quad v = A_3\frac{x}{y} + A_4\frac{r^2 - 4\kappa^2}{y} \quad (79)$$

where  $A_1, A_2, A_3$ , and  $A_4$  are arbitrary constants.

The Jacobian of the coordinate transformation  $(y, x) \rightleftharpoons (u, v)$  is  $J = (A_2A_3 - A_1A_4)\frac{r^2 + 4\kappa^2}{y^3}$ . The requirement  $J \neq 0$  is satisfied for the choice  $A_1 = A_4 = 1$  and  $A_2 = A_3 = 0$ . For this choice we have the admissible coordinate transformation

$$u = \frac{x}{y}, \quad v = \frac{r^2 - 4\kappa^2}{y}. \quad (80)$$

In the canonical coordinates, the reduced form of the equation is  $\frac{d^2v^2}{du^2} = 0$  from which follows

$$v = m_1u + 2m_2 \quad (81)$$

where  $m_1$  and  $m_2$  are arbitrary constants.

Replacing (81) into (80), we obtain the quadratic algebraic equation

$$y^2 - 2m_2y + x^2 - m_1x - 4\kappa^2 = 0$$

whose solution is

$$y(x) = m_2 \pm \sqrt{m_2^2 + 4\kappa^2 + m_1x - x^2}. \quad (82)$$

This is the solution of the ODE (70).

### Second covariant condition.

Instead of the condition  $R^i_{jkl} = 0$ , we assume that the space with coordinates  $x^i = (y, x)$  is a 2-dimensional (2d) space of positive constant curvature, i.e., the 2d sphere. The covariant condition is

$$R_{ijkl} = Kg_{ijkl}$$

where  $g_{ijkl} = g_{ik}g_{jl} - g_{il}g_{jk}$ ,  $g_{ij} = \frac{1}{(1+\frac{K}{4}r^2)^2}\delta_{ij}$  and  $K > 0$  is the value of the constant sectional curvature. The connection coefficients (Christoffel symbols) are computed from the metric  $g_{ij}$  as follows

$$\Gamma_{22}^1 = -\Gamma_{11}^1 = -\Gamma_{12}^2 = \frac{Ky}{2 + \frac{K}{2}r^2}, \quad \Gamma_{11}^2 = -\Gamma_{22}^2 = -\Gamma_{12}^1 = \frac{Kx}{2 + \frac{K}{2}r^2}. \quad (83)$$

Comparing these quantities with the quantities (71), we find  $K = \frac{1}{\kappa^2} > 0$ .

It is well known that the geodesics on a 2d sphere are great circles of the general form

$$\cot \theta = \lambda_1 \cos \phi + \lambda_2 \sin \phi \quad (84)$$

where  $\lambda_1, \lambda_2$  are arbitrary constants and we have applied the transformation  $(y, x) \rightleftharpoons (\theta, \phi)$  defined by the coordinate transformation

$$y = 2\kappa \cot \frac{\theta}{2} \cos \phi, \quad x = 2\kappa \cot \frac{\theta}{2} \sin \phi.$$

From the transformation equations, we find that

$$\cot \frac{\theta}{2} = \pm \frac{\sqrt{y^2 + x^2}}{2\kappa}, \quad \cos \phi = \pm \frac{y}{\sqrt{y^2 + x^2}} \quad \text{and} \quad \sin \phi = \pm \frac{x}{\sqrt{y^2 + x^2}}.$$

Replacing in (84), we get the quadratic algebraic equation

$$\frac{y^2}{4\kappa} - \lambda_1 y + \frac{x^2}{4\kappa} - \lambda_2 x - \kappa = 0 \quad (85)$$

whose solution is

$$y(x) = 2\kappa\lambda_1 \pm \sqrt{4\kappa^2(1 + \lambda_1^2) + 4\kappa\lambda_2 x - x^2}. \quad (86)$$

Renaming the constants as  $m_2 = 2\kappa\lambda_1$  and  $m_1 = 4\kappa\lambda_2$ , the solution  $y(x)$  is written as

$$y(x) = m_2 \pm \sqrt{m_2^2 + 4\kappa^2 + m_1 x - x^2} \quad (87)$$

which coincides with the solution (82) found from the flat space condition  $R^i_{jkl} = 0$ .

The difference between the two covariant conditions is that in the latter, one does not have to solve a system of PDEs but simply has to fix the constant curvature in terms of the constants of the given equation. Therefore, the solution of the equation is significantly simplified.

## 7. The Associated Riemannian Structure

We derived the coordinate transformation, which brings the second-order semilinear system (12) into the ‘canonical form’  $\frac{d^2 q^i}{dt^2} = 0$ , assuming that the quantities  $\Gamma^i_{jk}$ , which are determined by the coefficients of the system, are the components of a symmetric connection. Nowhere have we used a metric. However, it is possible that we associate a metric with the system, in which case the connection coefficients become the components of the corresponding Riemannian connection. This is done as follows.

We consider the canonical equations as the geodesic equations of the Euclidian metric  $\delta_{ij}$ . The metric, as a geometric object, is identified by the transformation equations

$$g_{ij} = \frac{\partial q^k}{\partial x^i} \frac{\partial q^l}{\partial x^j} \tilde{g}_{kl}.$$

Therefore, if we apply the inverse transformation from the canonical coordinates to the original coordinates, we determine the components of the metric in the original coordinate system where the system of equations is given. The connection coefficients computed for the transformed metric are the same with the quantities  $\Gamma_{jk}^i$  determined from the covariant condition.

We show how this works in the case of the system of equations considered in Section 5. Taking  $q^i = (u, v, w)$ ,  $x^i = (y, z, x)$  and  $\tilde{g}_{kl} = \delta_{kl}$ , we find

$$g_{ij} = \frac{\partial q^k}{\partial x^i} \frac{\partial q^l}{\partial x^j} \delta_{kl} = \sum_{k=1}^3 \frac{\partial q^k}{\partial x^i} \frac{\partial q^k}{\partial x^j} = \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} + \frac{\partial v}{\partial x^i} \frac{\partial v}{\partial x^j} + \frac{\partial w}{\partial x^i} \frac{\partial w}{\partial x^j}.$$

Using the coordinate transformation (67), we compute

$$g_{ij} = \begin{pmatrix} \frac{1}{y^2} + e^{2y}(1 + e^{2z}) & e^{2(y+z)} & \frac{1}{xy} \\ e^{2(y+z)} & e^{2(y+z)} & 0 \\ \frac{1}{xy} & 0 & \frac{1}{x^2} \end{pmatrix}.$$

It is an easy exercise to show that the connection coefficients of this metric are precisely the connection coefficients (65).

## 8. Conclusions

We developed a geometric method which potentially could be used in order to linearize a second-order non-linear equation or a system of such equations. This method is covariant in the sense that uses geometric object(s) in order to define the required linearization condition. For equations of the type (1), the natural geometric object to consider is the connection, and, consequently, the linearizing condition must involve the curvature tensor. We have shown that every equation of this type defines an  $(N + 1)$ -parameter family of quantities  $\Gamma_{jk}^i$ , where  $N$  is the number of equations. Because the linearization condition involves only geometric objects and is not affected by the coordinate transformations, it therefore has no effect on the solution of the system of equations. Its role is:

- To provide an overdetermined system involving the quantities  $\Gamma_{jk}^i$  whose ‘solution’ fixes all values of  $\Gamma_{jk}^i$  for the given system;
- To produce the transformed quantities  $\tilde{\Gamma}_{jk}^i$  in the canonical coordinates.

When these are done, then, from the transformation equation of the connection, one is possible to determine the coordinate transformation which linearizes the given system of equations.

We have considered a number of rather simple and well-known examples that use Lie symmetry algebra with  $sl(3, R)$  in order to show how this approach is working in practice. For these equations, one covariant condition is  $R_{bcd}^a = 0$ . We have shown that this condition is not unique. Indeed, in a Riemannian space of constant curvature the covariant condition  $R_{abcd} = Kg_{abcd}$ , where  $K \neq 0$  is the constant sectional curvature of the space and  $g_{ab}$  is the metric, also leads to the linearization of the equation. The reason for this is due to the Beltrami theorem [21] (for Riemannian spaces), which says that the geodesics of two spaces of constant curvature, of not necessarily the same  $K$ , are in 1–1 correspondence. Because the linearization is taking place in an affine space with a symmetric affine connection, the metric is not required. However, if one wishes, it is

possible to use the inverse coordinate transformation and the proper tensor transformation to compute the metric.

The open problem is to find covariant conditions for the linearization of equations whose Lie symmetry algebra is not the  $sl(3, R)$ . As it has been shown [27,28], non-point transformations achieve the linearization of certain equations of this type. Therefore, it is reasonable to assume that a proper covariant condition could be in the jet bundle of the affine space spanned by the variables of the equation. This is not an easy exercise, but if it is achieved, then it will be very useful for the solution of many interesting and important non-linear equations.

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