Article

On Height-Zero Characters in *p*-Constrained Groups

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Abstract: Consider G to be a finite group and p to be a prime divisor of the order $|G|$ in the group G. The main aim of this paper is to prove that the outcome in a recent paper of A. Laradji is true in the case of a p-constrained group. We observe that the generalization of the concept of Navarro's vertex for an irreducible character in a p-constrained group G is generally undefined. We illustrate this with a suitable example. Let $\phi \in Irr(G)$ have a positive height, and let there be an anchor group A_ϕ. We prove that if the normalizer $N_G(A_\phi)$ is p-constrained, then $O_{\acute{p}}(N_G(A_\phi)) \neq \{1_G\}$, where $O_{\acute{p}}(N_G(A_\phi))$ is the maximal normal $\acute{p}$ subgroup of $N_G(A_\phi)$. We use character theoretic methods. In particular, Clifford theory is the main tool used to accomplish the results.

Keywords: finite group; p-constrained groups; defect group; character; symmetric groups; automorphism of groups

MSC: 20C15; 20C20

1. Introduction

Fix a prime number p, and consider G to be a finite group. Let $Irr(G)$ be the set of all ordinary irreducible characters of G. Let B be a p block of G with defect group D. We write $Irr_0(B)$ to denote the set of all ordinary irreducible characters of G with a height of zero which belong to the p block B. Let S be a normal subgroup of G and $\theta \in Irr(S)$. We write $Irr(G|\theta) := \{\chi : \chi \in Irr(G) \text{ and } \langle Res_S^G(\chi), \theta \rangle \neq 0\}$ for the set of all ordinary irreducible characters of G which lie over θ and $Irr_0(B|\theta) := Irr(G|\theta) \cap Irr_0(B)$. For any term which is not defined here, the reader is referred to [1] and [2].

Throughout this paper, $(k, \mathcal{R}, F)$ is a p-modular system [3–5]. The system is composed of a complete discrete valuation ring $\mathcal{R}$ with a field of fractions k of a characteristic of zero. Let v_p be a valuation of a field k such that $v_p(p) = 1$. Then, we have $F = \mathcal{R}/J(\mathcal{R})$, which is the residue field of the characteristic p, where $J(\mathcal{R})$ refers to the Jacobson radical of the local ring $\mathcal{R}$.

In this introduction, we try to provide and give sufficient background for our subject. Consider G to be a finite group of the order $|G| = p^\alpha n$ such that $gcd(p, n) = 1$, $\alpha, n \in \mathbb{Z}^+$. Let $\phi \in Irr(G)$. If p^m is the greatest power of p which divides the positive integer $\frac{|G|}{\phi(1)}$, then $p^m = \frac{|G|_p}{\phi(1)_p}$, where r_p refers to the p part of an integer r. The nonnegative integer m is termed as the defect of the irreducible character ϕ, and we denote it by $def(\phi)$. If $def(\phi) = \alpha$, then we say ϕ is of a full defect. The greatest defect of irreducible character which belongs to the p block B is termed the defect number of B and is denoted by $def(B)$. The process of subtracting the defect number of ϕ from the defect number of B produces a height of ϕ. We write $h(\phi) = def(B) - def(\phi)$ to indicate the height of ϕ. Let $Cl(G)$ be the set of all conjugacy classes of G. Consider the center $Z(\mathcal{R}G)$. This is a commutative group

ring over $\mathcal{R}$ with the basis $\{\widehat{C} : C \in Cl(G)\}$ such that $\widehat{C} = \sum_{x \in C} x$ is the class sum of the conjugacy class C of G. For every $\phi \in Irr(G)$ and $g \in C_g$, we have

$$\omega_\phi(\widehat{C}_g) = \frac{|C_g|}{\phi(1)}\phi(g),$$

as an algebra homomorphism $\omega_\phi : Z(kG) \to k$, which is called a central character.

The most notable references that deal with block theory R are from Brauer [6,7], C. W. Curtis and I. Reiner, [8] and W. Feit [9]. The first description of a p block's defect group was provided by R. Brauer [6] and J. A. Green [10].

Let $\phi \in Irr(G)$. Then, ϕ can be extended to an algebra map $\phi : kG \to k$ in a unique way by the rule $\phi(\sum_{a \in G} \beta_a a) = \sum_{a \in G} \beta_a \phi(a)$. We consider the group algebra element

$$e_\phi = \frac{\phi(1)}{|G|} \sum_{a \in G} \phi(a^{-1})a,$$

where the unique primitive central idempotent in kG satisfies $\phi(e_\phi) \neq 0$. The algebra $\mathcal{R}Ge_\phi$ is a primitive G-interior $\mathcal{R}$ algebra [11] because the center $Z(kGe_\phi)$ contains the center $Z(\mathcal{R}Ge_\phi)$ as a subring.

In 1970, D. Wales [12] proved that if the normalizer of the nontrivial defect group D for a nonprincipal p block $N_G(D)$ is p-constrained, then $O_{\acute{p}}(N_G(D)) \neq \{1_G\}$, where $O_{\acute{p}}(N_G(D))$ is the largest normal $\acute{p}$ subgroup of $N_G(D)$. This result is essentially a restatement of [13] when the p-constrained is present.

The anchor group of an irreducible character ϕ of G was defined as the defect group of the primitive G-interior $\mathcal{R}$ algebra $\mathcal{R}Ge_\phi$ by R. Kessar, B. Külshammer, and M. Linckelmann in [14]. In this paper, we prove the relative version of the result for D. Wales [12], which states the following. Let $\phi \in Irr(G)$ with a positive height and an anchor group A_ϕ. If the normalizer $N_G(A_\phi)$ is p-constrained, then $O_{\acute{p}}(N_G(A_\phi)) \neq \{1_G\}$. A similar result holds for the normalizer $N_G(H)$ for $\{1_G\} \neq H \leq A_\phi$.

In 1987, G. Cliff, W. Plesken, and A. Weiss [15] proved that $2 \leq |Irr_0(B)|$ for any p block of G with a positive defect. In 1990, G. O. Michler [16] introduced another proof for this result using Brauer's main result [7] which states that $def(B) = 2\alpha - v_p(dim_F(B))$, where $dim_F(B)$ is the dimension of B as an F space.

Recall that the group G is said to be p-solvable if each of its composition factors is either a p group or a $\acute{p}$ group. We say $\phi \in Irr(G)$ is p-special if it satisfies the following: the degree of ϕ is a p number (a multiple of the prime number p), and if Q is a subnormal value of G and $\lambda \in Irr(Q)$ such that $\langle Res_Q^G(\phi), \lambda \rangle > 0$, then the determinant order of λ, $O(\lambda) = O(det(\lambda))$ is a p number. (Since $det(\lambda)$ is a linear character of G according to [17] (Exercise 2.3), then it is an element of the group of linear characters of G.) If we can factorize the irreducible character ϕ of G in a unique way such that $\phi = \phi_p \phi_{\acute{p}}$, where ϕ_p is p-special and $\phi_{\acute{p}}$ is $\acute{p}$-special, then ϕ is called p-factorable. For more information, see [18,19].

In 2002, G. Navarro [20] introduced the concept of a vertex for an ordinary irreducible character of a p-solvable group. In 2005, C.W. Eaton [21] extended the concept of Navarro vertices for ordinary irreducible characters of a p-solvable group to irreducible characters of any finite group belonging to the p blocks, with defect groups which are contained in a normal p-solvable subgroup.

In 2022, A. Laradji [22] proved that a relative version of the previous result for G. H. Cliff, W. Plesken, and A. Weiss in the case where G is a p-solvable group. A. Laradji in [22] proved the result in the case where G is a finite p-solvable group and S is a normal subgroup of G. Let B be a p block of G with a defect group D such that $|D \cap S| < |D|$. If $\theta \in Irr(S)$ with a height of zero, then the number of irreducible characters of a height of zero in $Irr(B)$ which lie over θ is greater than or equal to two.

A p-solvable group is characterized by the existence of normal subgroups. Thus, it is considered a suitable environment to use Clifford theory (see [23]). But we know that any

p-solvable element is a p-constrained group [24] (VI, 6.5) which again has many normal subgroups, and then we can use the tool of Clifford theory.

One of our main methods is character theory, which includes the restrictions, induction, orthogonality relations, and inner product of characters (see [1,17]). In addition, the most important theorem for studying character theory and the p blocks of finite groups is Clifford theory. Given a finite group G with its normal subgroup S and $\tau \in Irr(S)$, assume that $I_G(\tau) = \{a \in G | \tau^a = \tau\} < G$ is the inertia group of τ in G, Clifford theory could be employed for creating a bijection between any $\phi \in Irr(G|\tau)$ and $\eta \in Irr(I_G(\tau)|\tau)$ such that $\phi = Ind_{I_G(\tau)}^G(\eta)$ and $Res_{I_G(\tau)}^G(\eta) = e\tau$, where e is a nonnegative integer. The nonnegative integer e is said to be the ramification index of ϕ relative to S, which satisfies $Res_S^G(\phi) = e(\sum_{a \in [G/I_G(\tau)]} \tau^a)$. Here, τ^a is the conjugate character of τ such that $\tau^a(x) = \tau(axa^{-1})$ for all $x \in S$. (See [1,3,19] for more details about Clifford theory). The motivation of this paper is to prove that A. Laradji's result in [22] is true in the case of a p-constrained group. Also, we illustrate with an example that the generalization of the concept of Navarro's vertices in a p-constrained group is generally undefined.

The concept of a p-constrained group appeared in [25] (Section 8.1) and in [26] (VII, 13.3). In Section 2, we present the basic facts of a p-constrained group and the essential theories which were used in the proofs of the main results of this paper. In Section 3, we present the main results. In Section 4, titled Future Work, we propose several problems which extend many results from the class of p-solvable groups to the class of p-constrained groups. We end this paper with two sections: Section 5 contains the discussion about this topic, and Section 6 contains the conclusion that we came up with from our work.

2. Preliminaries

In this section, we provide some details about p-constrained groups. We offer the essential theories that we rely on to prove our main results.

Assume that G is a finite group and p is a prime divisor of $|G|$. Write $O_{\acute{p},p}(G)$ to mean the second term of the lower p series $\{1_G\} \leq O_{\acute{p}}(G) \leq O_{\acute{p},p}(G) \leq O_{\acute{p},p,\acute{p}}(G) \leq ...$ which are linked through the following relation:

$$O_{\acute{p},p}(G)/O_{\acute{p}}(G) = O_p(G/O_{\acute{p}}(G)),$$

where $O_p(G)$ is the maximal normal p subgroup of G and $O_{\acute{p}}(G)$ is the maximal normal $\acute{p}$ subgroup of G. Write $C_G(O_p(G))$ to mean the centralizer of $O_p(G)$ in G. We say that G is p-constrained if it satisfies $C_G(P \cap O_{\acute{p},p}(G)) \leq O_{\acute{p},p}(G)$, where P is any Sylow p subgroup of G. If $O_{\acute{p}}(G) = \{1_G\}$, then $O_{\acute{p},p}(G) = O_p(G)$. It follows that if $O_{\acute{p}}(G) = \{1_G\}$, then G is said to be p-constrained if it satisfies $C_G(O_p(G)) \leq O_p(G)$.

The following corollary is immediate from the definition of a p-constrained group:

Corollary 1. *If G is a finite group which possesses a normal p subgroup that contains its centralizer, then G is p-constrained and $O_{\acute{p}}(G) = \{1_G\}$.*

Modulo the maximal normal $\acute{p}$ subgroup of G, the quotient of a p-constrained group is p-constrained. The proof can be seen in [25] (Theorem 1.1 (ii), p. 269).

Theorem 1. *If G is a p-constrained group, then $G/O_{\acute{p}}(G)$ is p-constrained.*

Let M be a normal subgroup of G and $\overline{G} = G/M$. Assume that $\overline{\phi} \in Irr(\overline{G})$. We call the character ϕ the lift of $\overline{\phi}$ to G if it satisfies $\phi(a) = \overline{\phi}(aM)$ for $a \in G$. From Lemma 2.22 in [17], if $\overline{\phi} \in Irr(\overline{G})$, then $\phi \in Irr(G)$ if $M \leq ker\phi$. Consequently, we consider $Irr(\overline{G})$ to be a subset of $Irr(G)$. From [2] (p. 137), there exists a unique p block B of G that contains the p block of G/M, say $\overline{B}$; that is, $Irr(\overline{B}) \subseteq Irr(B)$. The following theorems appeared in [2] (Theorem 9.9 (C) and Theorem 9.1, respectively).

Theorem 2. *In accordance with the above notations, if M is a ṕ-normal subgroup of G and $Irr(\overline{B}) \subseteq Irr(B)$, then $Irr(\overline{B}) = Irr(B)$, and the set of all defect groups of $\overline{B}$ is of the form*

$$\{D_B M/M | D_B \text{ is any defect group of } B\}.$$

Theorem 3. *Let S be a normal subgroup of G. Let $\psi \in Irr(G)$ and $\theta \in Irr(S)$. Then, ψ lies over θ if and only if $\omega_\psi(\widehat{C}) = \omega_\theta(\widehat{C})$ for all conjugacy classes C of G which are contained in S.*

As is well known, a p block with a defect value of zero has only one irreducible ordinary character as well having only an irreducible Brauer character (see [1] (Chapter 3, Theorem 6.29) and [2] (Theorem 3.18)), while a p block with a positive defect appeared in [15] (Proposition 3.3).

Proposition 1. *Let B be a p block of G with a positive defect. Then, $|Irr_0(B)| \geq 2$.*

3. Main Results

In this section, we extend the result from [22] (Theorem 2.1) in the case of a p-constrained group. Then, we give some examples. Let $\phi \in Irr(G)$ with a positive height and an anchor group A_ϕ. We prove that if the normalizer $N_G(A_\phi)$ is p-constrained, then $O_{\acute{p}}(N_G(A_\phi)) \neq \{1_G\}$. A similar result holds for the normalizer $N_G(H)$, for $\{1_G\} \neq H \leq A_\phi$.

Theorem 4. *Suppose that G is a p-constrained group and S is a normal subgroup of G. Let B be a p block of G with a defect group D such that $|D \cap S| < |D|$. Let $\theta \in Irr(S)$ with a height of zero. Assume that $Irr_0(B|\theta) \neq \phi$. Then, $|Irr_0(B|\theta)| \geq 2$.*

Proof. Suppose that G is a p-constrained group. We have two cases for this group:

- **Case 1**: G is p-solvable. According to [24] (VI, 6.5), any p-solvable group is a p-constrained group. Then, the result from A. Laradji holds (Theorem 2.1 in [22]).
- **Case 2**: G is not p-solvable, and $O_{\acute{p}}(G) \neq \{1_G\}$. We use induction on the order of G. Since G is p-constrained, then under Theorem 1, $G/O_{\acute{p}}(G)$ is p-constrained. We write $M := O_{\acute{p}}(G)$ as the maximal normal $\acute{p}$ subgroup of G. Let $\overline{B}$ be a p block of $G/O_{\acute{p}}(G)$ and $\overline{B} \subseteq B$. According to Theorem 2, if $\overline{D}$ is a defect group of $\overline{B}$, then $\overline{D} \in \{D_B M/M | D_B \text{ is any defect group of } B\}$, and $Irr(B) = Irr(\overline{B})$. Without loss of generality, we may assume that $\overline{D} = DM/M$. We have $|D \cap S| < |D|$, which implies that $|DM/M \cap SM/M| < |DM/M|$. Since $D \cap M = \{1_G\}$, then under the second isomorphism theorem [27], $|D| = |D/D \cap M| = |DM/M|$ and $|DM/M \cap SM/M| = |(D \cap S)M/M| = |D \cap S||M|/|M| = |D \cap S|$ as $(D \cap S) \cap M = \{1_G\}$. Now, let $\theta \in Irr(S)$, and assume that $Irr_0(B|\theta) \neq \phi$. Since, under Theorem 2, $Irr(B) = Irr(\overline{B})$, then $Irr_0(\overline{B}|\theta) \neq \phi$. We have $G/O_{\acute{p}}(G)$ as a p-constrained group with an order less than that of G and $Irr_0(\overline{B}|\theta) \neq \phi$. Therefore, by induction, $|Irr_0(\overline{B}|\theta)| \geq 2$. Now, according to Theorem 2, $|Irr_0(B|\theta)| \geq 2$.
- **Case 3**: G is not p-solvable, and $O_{\acute{p}}(G) = \{1_G\}$. Then, from the definition of a p-constrained group, we have $C_G(O_p(G)) \leq O_p(G)$. Hence, G has a unique p block, namely the principal block B_0 from [9] (Chapter V, Corollary 3.11). thus, from Proposition 1, there are at least two irreducible characters χ, ψ with a height of zero which belong to B_0. Since we have $Irr_0(B_0|\theta) \neq \phi$, assume that $\chi \in Irr_0(B_0|\theta)$. It follows that $\langle Res_S^G(\chi), \theta \rangle \neq 0$.. Then, under Theorem 3, $\omega_\chi(\widehat{C}) = \omega_\theta(\widehat{C})$ for all conjugacy classes C of G which are contained in S. We know from [1] (Chapter 3, Theorem 6.24) that two irreducible characters χ, ψ belong to the same p block if and only if $\omega_\chi(\widehat{C}) \equiv \omega_\psi(\widehat{C}) \, mod(p)$ for all conjugacy classes C of G. Thus, $\omega_\psi(\widehat{C}) = \omega_\theta(\widehat{C})$. Then, from Theorem 3 again, $\langle Res_S^G(\psi), \theta \rangle \neq 0$. It follows that $|Irr_0(B_0|\theta)| \geq 2$.

□

In the following examples, we verify our theorem above in the case of p-constrained groups. The wreath product (cyclic group (p), alternating group (5)) for $p = 2, 3, 5$ is a

typical example of a p-constrained group which is not p-solvable. The wreath product arises naturally from the symmetric group. We emphasize that there is a strong relationship between our work and the concept of symmetry. The information for these examples about the height of the irreducible character, the defect of the block, and the structure of the defect group are from the GAP [28].

Example 1. *Let $G :=$ wreath product (cyclic group (2), alternating group (5)) be a group of the order 1920. We set $p = 2$. We have $O_2(G) = \{1_G\}$ and $O_2(G) = C_2 \times C_2 \times C_2 \times C_2 \times C_2$ such that $C_G(O_2(G)) = O_2(G)$. Therefore, the group G is 2-constrained by the definition above. From [9] (Chapter V, Corollary 3.11.), there is only one 2-block B_0 of G. It contains 24 irreducible characters with $def(B_0) = 7$ and the defect group $D = C_2 \times (((C_2 \times C_2 \times C_2) : C_4) : C_2)$ of an order of 128. We have $S := (C_2 \times C_2 \times C_2 \times C_2) : A_5$ as the normal subgroup of G such that $D \cap S = ((C_2 \times C_2 \times C_2 \times C_2) : C_2) : C_2$ is a group of the order 64. Let $\theta \in Irr(S)$ have a height of zero. Then, there are exactly two irreducible characters of G of a height of zero over θ. Note that this group is 2-constrained, making it not a 2-solvable group because it has the alternating group A_5 as a non-Abelian composition factor of G for neither a 2-group nor a $\acute{2}$-group.*

Example 2. *Let $G :=$ wreath product (cyclic group (3), alternating group (5)) be a group of the order $14,580$. We set $p = 3$. The group G is 3-constrained. Because we see that $O_3(G) = \{1_G\}$, then under the definition of a p-constrained group, the unique largest normal 3 subgroup of G, $O_3(G) = C_3 \times C_3 \times C_3 \times C_3 \times C_3$, is self-centralizing; that is, $C_G(O_3(G)) = O_3(G)$. It follows that there is only one 3-block B_0 of G. It contains 72 irreducible characters with $def(B_0) = 6$, and the defect group $D = C_3 \times C_3 \times ((C_3 \times C_3 \times C_3) : C_3)$ is of the order 729. We have $S := (C_3 \times C_3 \times C_3 \times C_3) : A_5$ as the normal subgroup of G and $D \cap S = C_3 \times ((C_3 \times C_3 \times C_3) : C_3)$ as a group of the order 243. Let $\theta \in Irr(S)$ have a height of zero. Note that there are at least two irreducible characters of G of a height of zero over θ. Note that this group is 3-constrained, meaning it is not a 3-solvable group because it has the alternating group A_5 as a non-Abelian composition factor of G for neither a 3 group nor a $\acute{3}$ group.*

Example 3. *Let $G :=$ wreath product (cyclic group (5), alternating group (5)) be a group of the order $187,500$. We set $p = 5$. We see that the unique largest normal 5 subgroup of G is $O_5(G) = C_5 \times C_5 \times C_5 \times C_5 \times C_5$. This is a group of the order 3125 such that $C_G(O_5(G)) = O_5(G)$. Under Corollary 1, the group G is 5-constrained, and $O_5(G) = \{1_G\}$. We have only one 5-block B_0 of G which contains 337 irreducible characters with $def(B_0) = 6$, and the defect group $D = (((C_5 : C_5) : C_5) : C_5) : C_5$ is of the order $15,625$. We have $S := (C_5 \times C_5 \times C_5 \times C_5) : A_5$ as the normal subgroup of G and $D \cap S = (C_5 \times C_5 \times C_5 \times C_5) : C_5$ as a group of the order 3125. Let $\theta \in Irr(S)$ have a height of zero. Note that there are exactly five irreducible characters of G with a height of zero which lie over θ. Also, this group is 5-constrained, meaning it is not a 5-solvable group because it has the alternating group A_5, which is a non-Abelian composition factor of G for neither a 5 group nor a $\acute{5}$ group.*

Consider the p-solvable group G and $\psi \in Irr(G)$. A nucleus of ψ [23] is a unique (up to G conjugacy) canonical pair (H, ζ) such that $H \leq G$, $\zeta \in Irr(H)$ is p-factorable, and $Ind_H^G(\zeta) = \psi$. The nucleus is defined as a result of repeated construction standard inducing pairs. If ψ is p-factorable, then $(G, \psi) = (H, \zeta)$ is a nucleus of ψ. If it is not p-factorable, then let (S, η) be a pair under (G, ψ) such that S is a maximal normal subgroup of G and $\eta \in Irr(S)$ is p-factorable such that $\langle Res_S^G(\psi), \eta \rangle \neq 0$. Assume that $I_G(\eta) = \{a \in G | \eta^a = \eta\} < G$ is the inertia group of η in G. If $\phi \in Irr(I_G(\eta)|\eta)$, then under Clifford theory, $Ind_{I_G(\eta)}^G(\phi) \in Irr(G|\eta)$. Assume that ϕ is the Clifford correspondent of ψ. Hence, $Ind_{I_G(\eta)}^G(\phi) = \psi$. Then, by induction, there is a nucleus (H, ζ) for $(I_G(\eta), \phi)$. It follows that (H, ζ) is a nucleus for (G, ψ) since

$$Ind_H^G(\zeta) = Ind_{I_G(\eta)}^G(Ind_H^{I_G(\eta)}(\zeta)) = Ind_{I_G(\eta)}^G(\phi) = \psi.$$

We say that (V, α) is Navarro's vertex of ψ if there exists a nucleus (H, ζ) for ψ such that V is a Sylow p subgroup of H and $\alpha = Res_V^H(\zeta_p)$ (see [20]).

Note that in this case, G is a p-constrained finite group Navarro's vertex is generally undefined. In 2005, C. W. Eaton gave a generalization of the concept of Navarro's vertices to vertices of ordinary irreducible characters of any finite group, with the additional condition that the irreducible character belongs to a p block with a defect group contained in a normal p-solvable subgroup of G in [21] (Theorem 4.2).

Example 4. *Consider G to be a 2-constrained group in a case where $p = 2$.*

$$G := WreathProduct(CyclicGroup(2), \ AlternatingGroup(5)),$$

with an order of 1920. *In Example 1, G has only one 2-block, namely the principal 2-block B_0 with a defect group $D = C_2 \times (((C_2 \times C_2 \times C_2) : C_4) : C_2)$, which is a Sylow 2 subgroup of G of the order* 128. *Note that there is no a normal 2-solvable subgroup of G which contains a defect group of the 2-block B_0. Moreover, not all irreducible characters of G are 2-factorable. There are irreducible characters of a degree of* 10 *which are not 2-factorable, and they cannot be associated with a nucleus.*

Recall that the ordinary irreducible character χ with a height of zero, which belongs to a p block B, has an anchor equal to a defect group of the p block B from [14] (Theorem 1.3 (d)). As is well known, the principal p block contains the principal irreducible character (the trivial character), and it has a Sylow p subgroup of G as a defect group (see [1] p. 316).

Theorem 5. *Consider G to be a finite group. Let $\phi \in Irr(G)$ with a positive height and a nontrivial anchor group A_ϕ. Let Q be a subgroup of G which contains $C_G(A_\phi)$, with $O_p(Q) \neq \{1_G\}$. If Q is p-constrained, then $O_{\acute{p}}(Q) \neq \{1_G\}$.*

Proof. We have ϕ with a positive height. Hence, ϕ cannot be the principal irreducible character of G. Assume that $\phi \in Irr(G)$ belongs to a non-principal p block B of G. If A_ϕ is the defect group of the p block B, then the result holds from [12] (Theorem 1). Now, assume that the anchor of ϕ is a proper subgroup of the defect group D of the p block B. We have $C_G(A_\phi) \subseteq Q$, where $O_p(Q) \neq \{1_G\}$. From [14] (Theorem 1.2 (a)), since $A_\phi \leq D$, then $C_G(D) \subseteq C_G(A_\chi) \subseteq Q$. Again, from [12] (Theorem 1), if Q is p-constrained, then $O_{\acute{p}}(Q) \neq \{1_G\}$. $\square$

The following corollary is immediately from the above theorem:

Corollary 2. *Consider G to be a finite group. Let $\phi \in Irr(G)$ with positive height and a nontrivial anchor group A_ϕ. Suppose that $\{1_G\} \neq H \subseteq A_\phi$. Let Q be a subgroup of $N_G(H)$, contains $C_G(A_\phi)$, that is $C_G(A_\phi) \subseteq Q \subseteq N_G(H)$. If Q is a p-constrained, then $O_{\acute{p}}(Q) \neq \{1_G\}$.*

4. Future Work

The class of p-constrained groups is very large. It includes the class of p-solvable groups. So our project is to extend many and important results in the literature from the class of p-solvable groups to the class of p-constrained groups. In particular, according to recent work [29–31]. We raise the following questions:

Problem 1: Let G be p-constrained group and Q be a Sylow p-subgroup of G. Write $\acute{Q}$ to denote the commutator subgroup of Q. Then

$$|Irr(N_G(Q)/O_{\acute{p}}(N_G(Q))\acute{Q})| = |Irr_0(B_0(N_G(Q)))| = |Irr_0(B_0(G))|,$$

where $|Irr_0(B_0(G))|$ denotes to the number of irreducible ordinary characters of height zero in the principal p block of G, $|Irr_0(B_0(N_G(Q)))|$ denotes to the number of irreducible ordinary characters of height zero in the principal p block of $N_G(Q)$ and $|Irr(N_G(Q)/O_{\acute{p}}(N_G(Q))\acute{Q})|$ refers to the number of irreducible ordinary characters in the group $N_G(Q)/O_{\acute{p}}(N_G(Q))\acute{Q}$.

For more details on the commutator subgroup see [27,32,33].

Problem 2: If G is p-constrained group and its principal p block satisfies the Alperin-Mckay conjecture, then

$$2 \leq |Irr_0(B_0(G))|.$$

Recall that Alperin-Mckay conjecture states: Let B be a p block of G, with defect group D_B, and let b be the Brauer correspondence of B in $N_G(D_B)$. Then $|Irr_0(b)| = |Irr_0(B)|$. For more details on Brauer correspondence of blocks, see [4], Section 12.6.

Problem 3: Let G be p-constrained group and Q be its Sylow p-subgroup. For each linear character γ of Q, then

$$|Irr_{\acute{p}}(G \mid \gamma)| = |Irr_{\acute{p}}(N_G(Q) \mid \gamma)|.$$

Here $|Irr_{\acute{p}}(G \mid \gamma)| := |\{\psi \in Irr(G)|p \text{ does not divide } \psi(1) \text{ and } \langle Res_Q^G \psi, \gamma \rangle \neq 0\}|$.

Problem 4: Let G be p-constrained group and Q be its Sylow p-subgroup. If $\phi \in Irr(G)$, then ϕ_p is a sum of characters induces from p-subgroup P of G which contained in Q such that $\phi(1)_p = |Q : P|$.

Let $IBr(G)$ be the set of all Brauer irreducible characters of G. We write ψ^0 refers to the restriction of ψ to the set of all p-regular elements of G, that is (p does not divide the order of the elements). One of the most prominent theories that distinguish p-solvable group is the Fong-Swan theorem in [1], Theorem 7.5 and Wolf theorem [2], Theorem 10.3. We raise the following questions.

Problem 5: Consider G to be p-constrained group. If $\phi \in IBr(G)$, then there is $\psi \in Irr(G)$ that satisfies $\psi^0 = \phi$.

Problem 6: Let Q be normal subgroup of G. Assume that G/Q is p-constrained group and $\phi \in Irr(Q)$ with $\phi^0 \in IBr(Q)$ and $I_G(\phi) = I_G(\phi^0)$. Suppose that $O(\phi)\phi(1)$ does not divisible by p. If $\gamma \in IBr(G|\phi^0)$, then there is $\psi \in Irr(G|\phi)$ that satisfies $\psi^0 = \gamma$.

Remark 1. *We can think of the above problems in different ways. The first one is to get a theorem that extends the result from p-solvable groups to p-constrained groups. The second one is to look at an example that distinguishes between two classes of groups.*

5. Discussion

In this work, let $\phi \in Irr(G)$ with positive height and an anchor group A_ϕ. We prove that if the normalizer $N_G(A_\phi)$ is p-constrained, then $O_{\acute{p}}(N_G(A_\phi)) \neq \{1_G\}$. The same discussion holds for the normalizer $N_G(H)$, for $\{1_G\} \neq H \leq A_\phi$. This result is the relative version of the result for D. Wales in [12] which states: if the normalizer of the nontrivial defect group D for a nonprincipal p block, $N_G(D)$ is a p-constrained then $O_{\acute{p}}(N_G(D)) \neq \{1_G\}$. Since the anchor of irreducible character is the defect group of the primitive G-interior $\mathcal{R}$-algebra $\mathcal{R}Ge_\phi$, the previous conclusion is logical. The main result of this paper is the generalization of A. Laradji's result in [22] to the case that G is a finite p-constrained group and S is a normal subgroup of G: let B be a p block of G with defect group D such that $|D \cap S| < |D|$. If $\theta \in Irr(S)$ with height zero then the number of irreducible characters of height zero in $Irr(B)$ which lie over θ is greater than or equal to 2. We have applied this result on some examples for p-constrained group which is not p-solvable group. A p-solvable group is characterized by the existence of normal subgroups. Thus, it is considered a suitable environment to use Clifford theory. But we know that any p-solvable is p-constrained group which again has many normal subgroups and then we can use the tool of Clifford theory. If G is a finite p-constrained which is not p-solvable group, then we have either $O_{\acute{p}}(G) \neq \{1_G\}$ or $O_{\acute{p}}(G) = \{1_G\}$. In the first case to prove this result we used the method "induction on the order of G". In the second case we used the group properties also the block theory. While the generalization the concept of Navarro's vertex for an irreducible character of a p-solvable group to a p-constrained group is fails in

general. The outcomes of work are important because contains generalize the related theory of p-solvable group to a p-constrained group. We plan to study the necessary conditions to generalize the concept of Navarro's vertex in a p-constrained group. We plan to generalize properties of an irreducible character of a p-solvable group to a p-constrained group.

6. Conclusions

Consider G to be a p-constrained group and S to be a normal subgroup of G and $\theta \in Irr(S)$ with height zero. Let B be a p block of G with defect group D such that $|D \cap S| < |D|$. Our research shows that the number of $Irr_0(B|\theta)$ is greater than or equal to 2. We have applied this result on some examples for p-constrained group. We have introduced the example to show that Navarro's vertex for an irreducible character in a p-constrained group G is generally undefined. For ϕ is an irreducible character of a finite group G with positive height and a nontrivial anchor group A_ϕ. We have proved that if A_ϕ has a p-constrained normalizer group then $O_{\not p}(N_G(A_\phi))$ is nontrivial. A similar result holds for the normalizer $N_G(H)$, for $\{1_G\} \neq H \leq A_\phi$. The theories that we extended above which have been developed before by other influential scientists, see [12,22]. In the future we will try to establish the necessary conditions for the generalization of the concept of Navarro's vertex for an irreducible character of a p-solvable group and some of its characteristics to a p-constrained group.

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