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Multiple Soliton Solutions for Coupled Modified Korteweg–de Vries (mkdV) with a Time-Dependent Variable Coefficient

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Abstract: In this manuscript, we implement analytical multiple soliton wave and singular soliton wave solutions for coupled mKdV with a time-dependent variable coefficient. Based on the similarity transformation and Hirota bilinear technique, we construct both multiple wave kink and wave singular kink solutions for coupled mKdV with a time-dependent variable coefficient. We implement the Hirota bilinear technique to compute analytical solutions for the coupled mKdV system. Such calculations are made by using a software with symbolic computation software, for instance, Maple. Recently some researchers used Maple in order to show that the bilinear method of Hirota is a straightforward technique which can be used in the approach of differential, nonlinear models. We analyzed whether the experiments proved that the procedure is effective and can be successfully used for many other mathematical models used in physics and engineering. The results of this study display that the profiles of multiple-kink and singular-kink soliton types can be efficiently controlled by selecting the particular form of a similar time variable. The changes in the solitons based on the changes in the arbitrary function of time allows for more applications of them in applied sciences.

Keywords: nonlinear models; coupled mKdV; time-dependent variable coefficient; similarity transformation; Hirota bilinear technique

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1. Introduction

The most famous theoretical model of shallow water wave surfaces is the class of KdV equations. It is notable that these are considered in particular to be examples of solvable partial differential equations whose solutions can be precisely determined. These are integrable models that are resolved using the scattering wave transform technique. An impactful and active point of research is the mathematical theory based on the KdV model which was first discussed by Boussinesq (1877) and rediscovered by Korteweg and de Vries (1895). The first advertisement of solitary waves was made by Scott Russell, which contributed to Korteweg and de Vries using them in their KdV equation. The KdV equation contains many connections to a large number of natural phenomena examples, including quantum mechanics, plasma, and soliton theory.

In the theory dedicated to solitons, it is considered that a soliton, in other words, a solitary wave, is a certain classic nonlinear wave that is caused by the effects of dispersion and nonlinearity. With other formulation, in physics and mathematics, a soliton is a nonlinear, self-reinforcing, localized wave packet that is strongly stable, in that it preserves its shape while propagating freely, at constant velocity [1–4]. It keeps its amplitude, speed and form, even when it comes into contact with different solitons.

Zabusky and Kruskal [5] detected one of the strongest essential features of solitons, namely the nonlinear interaction in collisionless plasma for a diverse series of KdV type equations. Moreover, it was found that in the case of reciprocal collisions (except when the phases are shifted) solitons, so solitary waves, do not change.

The class of KdV equations have found utility in many branches of engineering and physics [6,7]. Among all their important characteristics, we should mention the particular aspects of the bending solutions as well as their interactions [5–7]. A bend is that solution for which the value at the limit at - infinity is 0, and the value at the limit at infinity is 2π. Solitons in an integrable system, modeled with KdV equations, have constant shapes and evolve with constant speed. However, due to some inhomogeneities, in concrete situations, we can deal with solitons that have a more complex evolution, in which they can change both form and velocity [6]. This can be useful when it is want a faster communication, or even a faster transport [7].

The solitary waves which have an infinite support appear as connections of the balance between the linear dispersion u_{xxx} and nonlinear convection $u\, u_x$ in the equations of KdV type $u_t + a\, u\, u_x + u_{xxx} = 0$.

In the last three decades, many nonlinear coupled differential models and systems have emerged; examples include mKdV, coupled KdV, and variant Boussinesq models, which appear in many different scientific applications. Nowadays, coupled evolution models are attracting significant research in the literature. The aims and objectives of these studies are divided in two directions: the first one is to determine the soliton and solitary wave solutions, and the other is to prove the integrability properties for these systems [1,2]. Many different techniques have been investigated and studied the evolution of nonlinear single and coupled models [8–10]. We will mention, for example, that some of these techniques that were used are the Backlund transformation approach, Hirota bilinear procuredre, Hietarinta method, Darboux transformation, symmetry method, Pfaffian technique, Painlevé analysis, inverse scattering method, and so on [11–16]. The bilinear Hirota procedure is significant and a rather heuristic technique [2–4,8,9]. This approach possesses a lot of powerful and huge features which make it a good fit for constructing multiple solitons and solitary waves for a large number of evolution nonlinear models.

The mKdV model represented the propagation of nonlinear waves in a polar symmetry model. The mKdV model is also discussed in electrodynamics, elastic media, and the propagation waves of quantized films. That is, it describes acoustic waves with non-harmonic lattices and Alfven waves in the non-collision of plasma. Furthermore, the mKdV model is different from the KdV model since it possesses cubic nonlinearity. The mKdV model is an integrable model since it is solved using inverse scattering and discusses the Painlevé property.

Interestingly, and also naturally, soliton waves have a permanent shape. Since a soliton is defined from a localized wave, a soliton's behavior either decomposes exponentially to zero, like solitons in the KdV model, or it approaches a constant at infinity, like solitons in the sine-Gordon model.

Optical soliton research is nicely flourishing since it can be applied to the new developments of optical communication models and data transmission [4,10,17–21], which include the dynamics of electrons in semiconductors, metal phase changes induced by light, and chemical reactions. An optical soliton is special form of ultrashort pulses, which enables us to keep its shape and velocity unchanged in long-distance transmissions.

With the help of symbolic arithmetic packages, we can compute many direct and indirect methods for establishing solutions to models of nonlinear integer and fractional differential equations [22–32]. The methods created by Malfliet and Wazwaz have been applied, developed, and extended to construct analytical solutions for a great number of nonlinear differential models in terms of the tan, tanh, cot, coth, sine, and cosine functions. Many other methods developed by mathematicians and physicists are used for analysis and to find other solutions for nonlinear differential models: for example, the elliptic

Jacobi procedure, the variation iteration technique, the spectral collocation method, the sub equation procedure, and the F-expansion method.

It is worth noting that for nonlinear independent differential models that contain constant coefficients, we obtain a classical soliton due to the resulting equilibrium between the nonlinear effects and the dispersion. However, nonlinear differential models containing time-dependent coefficients have attracted a lot of attention recently due to their very interesting features for potential future scientific applications.

In this manuscript, a coupled mKdV system with a time-dependent variable coefficient will be discussed for constructing multiple solitons and solitary waves.

This system appears in many concrete physical situations, such as the evolution of waves encountered in nonlinear optics or in fluids, water waves, or plasma. Also, the motion of acoustic ion waves, which occurs without collision, in the plasma: the reader is referred to [33–35] and the references therein. KdV-type equations are found in DNA soliton dynamics, in the dynamics of those fluxons in junctions with Josephson-type impurities. Also, in the propagation of spin waves that have a variable strength of interaction [36–40]. In addition, it appears in the analysis done to the oceans regarding the coast waves, in the analysis of bubbles and drops for different liquids, in atmospheric analyzes regarding different blockages, for example blocking the dipoles [33–38].

These studies have increasingly covered the time-dependent equations, considering that this type of equations is best to highlight the real features of a great variety of scientific approaches. For the coupled mKdV model with a time-dependent variable coefficient [40–43], the chosen model is:

$$\begin{aligned} &U_\tau + 6\varphi(\tau)UVU_x + \psi(\tau)U_{xxx} = 0, \\ &V_\tau + 6\varphi(\tau)VUV_x + \psi(\tau)V_{xxx} = 0, \end{aligned} \tag{1}$$

where the subscripts denote the partial differentiations with respect to the corresponding variable, and $\varphi(\tau)$, $\psi(\tau)$ are the arbitrary smooth functions (R-analytical functions) that symbolize the coefficients of the time variable τ, and are related by the condition $\varphi(\tau) = a\psi(\tau)$, with an arbitrary constant a. When $\psi(\tau) = 1$, system (1) can be simplified as the coupled mKdV system with constant coefficients [40].

Suppose that:

$$U(\tau,\ x) = u(t,\ x),\ V(\tau,\ x) = v(t,\ x), \tag{2}$$

where $t = t(\tau)$ is the real function of the one time variable τ if we choose $\psi(\tau) = \frac{\partial t}{\partial \tau}$ or $t = \int\limits_0^t \psi(\tau)d\tau$ where $\psi(\tau)$ is an arbitrary function of τ. Substituting Transformation (2) into System (1), we find that System (1) becomes:

$$\begin{aligned} &u_t + 6auvu_x + u_{xxx} = 0, \\ &v_t + 6avuv_x + v_{xxx} = 0. \end{aligned} \tag{3}$$

The main motivation of this study is to investigate multiple soliton waves and singular solutions for the desired coupled system in a simplified form (3).

Note that the similarity time variable has a general form $t = \int\limits_0^t \psi(\tau)d\tau$, which defines an infinite number of substitutions since $\psi(\tau)$ is an arbitrary function. The only necessary condition for the solitary wave to exist is that the time-dependent coefficient $\psi(\tau)$ must be Riemenn-integrable.

Remark 1. *The scaling of the time variable $t = t(\tau)$ has a general form but must be at least an absolutely continuous function. We will assume also that the time-dependent coefficients are at least Riemann-integrable functions, which is a necessary condition for the solitary wave to exist:*

This manuscript is arranged as follows: Section 2 explores multiple-soliton solutions for the coupled mKdV model. We derive multiple singular solutions for the coupled mKdV system in Section 3. In Section 4, we discuss the soliton interactions of the soliton solution. Section 5 is a discussion and summary.

2. Soliton Solution

Let us discuss the multiple-soliton solutions. Based on the Hirota bilinear procedure [1,2,8,9,40,42], we assume that:

$$U = e^{\theta_i},\ V = Ae^{\theta_i},\ \theta_i = k_i x - c_i t,\ t = \int_0^t \psi(\tau)d\tau, \tag{4}$$

where A is an arbitrary constant. Substituting in the highest linear term of Equation (3), we obtain the following dispersion relation:

$$c_i = k_i^3. \tag{5}$$

As a result, we can write:

$$\theta_i = k_i x - k_i^3 t. \tag{6}$$

So, the multiple-soliton solution of system (3) is:

$$u = r\frac{\partial}{\partial x}\left[\tan^{-1}\left(f\, g^{-1}\right)\right] = r\,\frac{f_x g - g_x f}{g^2 + f^2}, \tag{7}$$

$$v = r_1\,\frac{\partial}{\partial x}\left[\tan^{-1}\left(f\, g^{-1}\right)\right] = r_1\,\frac{f_x g - g_x f}{g^2 + f^2}, \tag{8}$$

where r and r_1 are constants can be computed, and the functions f and g for a single-soliton wave are defined by:

$$f = e^{\theta_1} = e^{k_1 x - k_1^3 t},\ g = 1, \tag{9}$$

Using the above result in (3) and solving for r and r_1, we have $r = c$ and $r_1 = \frac{4}{a\, c^2}$, where c is an arbitrary constant. So, the single-soliton solution takes the form:

$$U = \frac{c\, k_1 e^{k_1 x - k_1^3 t}}{1 + e^{2(k_1 x - k_1^3 t)}},\ V = \frac{4\, k_1 e^{k_1 x - k_1^3 t}}{ac\, \left(1 + e^{2(k_1 x - k_1^3 t)}\right)},\ t = \int_0^t \psi(\tau)d\tau. \tag{10}$$

The evolutional behavior of a soliton wave represented by u and v is given by Equation (10) with the selection $a = 1,\ c = 5,\ k_1 = 3$, so U and V can be written as:

$$U = \frac{15\, e^{3(x - 9\int_0^t \psi(\tau)d\tau)}}{1 + e^{6(x - 9\int_0^t \psi(\tau)d\tau)}},\ V = \frac{2.4\, e^{3(x - 9\int_0^t \psi(\tau)d\tau)}}{1 + e^{6(x - 9\int_0^t \psi(\tau)d\tau)}}. \tag{11}$$

Figure 1a represents the one-soliton solution of the function U when $\psi(t) = 1$, Figure 1b when $\psi(t) = \sin t$, Figure 1c when $\psi(t) = \sec^2 t$, and Figure 1d when $\psi(t) = \tan t \sec t$, which shows the diverse kinds of shapes of a one-soliton wave.

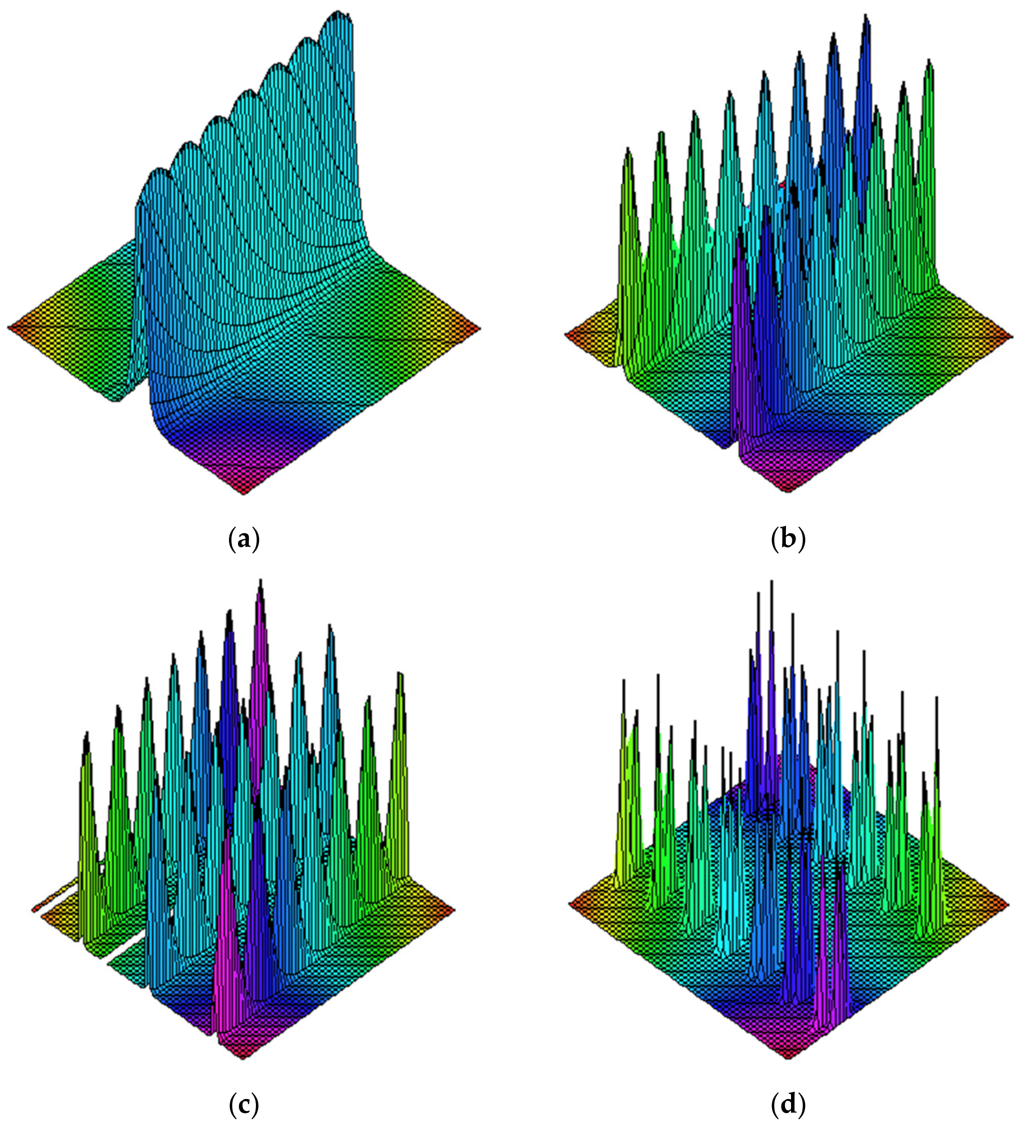


Figure 1. Evolution behavior of one-soliton solution (11) with $a = 1$, $c = 5$, $k_1 = 3$ (**a**) $\psi(t) = 1$ (**b**) $\psi(t) = \sin t$ (**c**) $\psi(t) = \sec^2 t$ (**d**) $\psi(t) = \tan t \sec t$.

By dividing U and V in Equation (11), we find the relationship between the two functions U and V as:

$$\frac{U}{V} = \frac{ac^2}{4}. \tag{12}$$

To construct the two-soliton wave, we assume that:

$$\begin{aligned} f &= e^{\theta_1} + e^{\theta_2}, \qquad g = 1 - a_{12}e^{\theta_1+\theta_2}, \\ \theta_1 &= k_1(x - k_1^2 t), \quad \theta_2 = k_2(x - k_2^2 t). \end{aligned} \tag{13}$$

Substituting (13) in (7) and (8), using the result in the coupled system (3), we have the phase shift:

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, \tag{14}$$

Hence, we can generalize this relation as:

$$a_{mn} = \frac{(k_m - k_n)^2}{(k_m + k_n)^2}, \qquad 1 \le m < n \le 3. \tag{15}$$

The two-soliton wave is constructed by substituting (12) and (11) into (7) and (8). The two functions U and V can take the forms:

$$\begin{aligned} &U = R\,\frac{\partial}{\partial x}\left[\tan^{-1}\left(\frac{f}{g}\right)\right] = c\frac{\partial}{\partial x}\left[\tan^{-1}\left(\frac{e^{\theta_1}+e^{\theta_2}}{1-a_{12}e^{\theta_1+\theta_2}}\right)\right],\\ &V = R_1\,\frac{\partial}{\partial x}\left[\tan^{-1}\left(\frac{f}{g}\right)\right] = \frac{4}{ac^2}\frac{\partial}{\partial x}\left[\tan^{-1}\left(\frac{e^{\theta_1}+e^{\theta_2}}{1-a_{12}e^{\theta_1+\theta_2}}\right)\right],\\ &a_{12} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2},\ t = \int_0^t \psi(\tau)d\tau,\ \theta_i\ = k_i(x-k_i^2t),\ i = 1,\ 2. \end{aligned} \tag{16}$$

It is desirable to point out that the coupled mKdV system (3) has not introduced any resonant phenomena since the term of the phase shift a_{12} in (14) does not have 0 or ∞ for $|k_1|\neq|k_2|$. The evolutional behavior of the two-soliton wave represented by U and V is given in Equation (16) with the selection $a = 1,\ c = 5,\ k_1 = 3,\ k_2 = 2$, so U and V take the forms:

$$\begin{aligned} &U = 5\frac{\partial}{\partial x}\left[\tan^{-1}\left(\frac{e^{3(x-9\int_0^t\psi(\tau)d\tau)}+e^{2(x-4\int_0^t\psi(\tau)d\tau)}}{1-\frac{1}{25}e^{3(x-9\int_0^t\psi(\tau)d\tau)+2(x-4\int_0^t\psi(\tau)d\tau)}}\right)\right],\\ &V = \frac{4}{25}\frac{\partial}{\partial x}\left[\tan^{-1}\left(\frac{e^{3(x-9\int_0^t\psi(\tau)d\tau)}+e^{2(x-4\int_0^t\psi(\tau)d\tau)}}{1-\frac{1}{25}e^{3(x-9\int_0^t\psi(\tau)d\tau)+2(x-4\int_0^t\psi(\tau)d\tau)}}\right)\right]. \end{aligned} \tag{17}$$

Figure 2a represents the two-soliton solution of the function U when $\psi(t) = 1$, Figure 2b when $\psi(t) = \sin t$, Figure 2c when $\psi(t) = \sec^2 t$, and Figure 2d when $\psi(t) = \tan t \sec t$, which shows different shapes of two-soliton solutions.

To construct a three-soliton wave, we suppose that:

$$\begin{aligned} &f = e^{\theta_3} + e^{\theta_2} + e^{\theta_1} - a_{123}e^{\theta_3+\theta_2+\theta_1}, \qquad g = 1 - a_{12}e^{\theta_2+\theta_1} - a_{13}e^{\theta_3+\theta_1} - a_{23}e^{\theta_3+\theta_2},\\ &\theta_i = k_i(x-k_i^2t), \quad i = 1,2,3, \quad a_{123} = a_{12}a_{13}a_{23}, \end{aligned} \tag{18}$$

with the phase shift terms a_{ij}, $1 \le i < j \le 3$, obtained above in (15). A three-soliton wave for the coupled mKdV model (3) is derived by substituting (18) into (7) and (8).

$$\begin{aligned} &U = c\frac{\partial}{\partial x}\left[\tan^{-1}\left(\frac{e^{\theta_3}+e^{\theta_2}+e^{\theta_1}-a_{123}e^{\theta_3+\theta_2+\theta_1}}{1-a_{12}e^{\theta_2+\theta_1}-a_{13}e^{\theta_3+\theta_1}-a_{23}e^{\theta_3+\theta_2}}\right)\right],\\ &V = \frac{4}{ac^2}\frac{\partial}{\partial x}\left[\tan^{-1}\left(\frac{e^{\theta_3}+e^{\theta_2}+e^{\theta_1}-a_{123}e^{\theta_3+\theta_2+\theta_1}}{1-a_{12}e^{\theta_2+\theta_1}-a_{13}e^{\theta_3+\theta_1}-a_{23}e^{\theta_3+\theta_2}}\right)\right],\\ &a_{ij} = \frac{(k_i-k_j)^2}{(k_i+k_j)^2},\ 1 \le i < j \le 3,\ \ \theta_i = k_i(x-k_i^2t),\ i = 1,2,3,\ t = \int_0^t \psi(\tau)d\tau. \end{aligned} \tag{19}$$

The evolutional behavior of the three-soliton wave represented by U and V is given in Equation (19) with the selection $a = 1,\ c = 5,\ k_1 = 3,\ k_2 = 2,\ k_3 = 1$, so U and V can be written as:

$$U = 5\frac{\partial}{\partial x}\left[\tan^{-1}\left(\frac{e^{3x-27\int_0^t\psi(\tau)d\tau}+e^{2x-8\int_0^t\psi(\tau)d\tau}+e^{x-\int_0^t\psi(\tau)d\tau}-\frac{1}{900}e^{6x-36\int_0^t\psi(\tau)d\tau}}{1-\frac{1}{25}e^{5x-35\int_0^t\psi(\tau)d\tau}-\frac{1}{4}e^{4x-28\int_0^t\psi(\tau)d\tau}-\frac{1}{9}e^{3x-9\int_0^t\psi(\tau)d\tau}}\right)\right], \tag{20}$$

$$V = \frac{4}{25}\frac{\partial}{\partial x}\left[\tan^{-1}\left(\frac{e^{3x-27\int_0^t\psi(\tau)d\tau}+e^{2x-8\int_0^t\psi(\tau)d\tau}+e^{x-\int_0^t\psi(\tau)d\tau}-\frac{1}{900}e^{6x-36\int_0^t\psi(\tau)d\tau}}{1-\frac{1}{25}e^{5x-35\int_0^t\psi(\tau)d\tau}-\frac{1}{4}e^{4x-28\int_0^t\psi(\tau)d\tau}-\frac{1}{9}e^{3x-9\int_0^t\psi(\tau)d\tau}}\right)\right]. \tag{21}$$

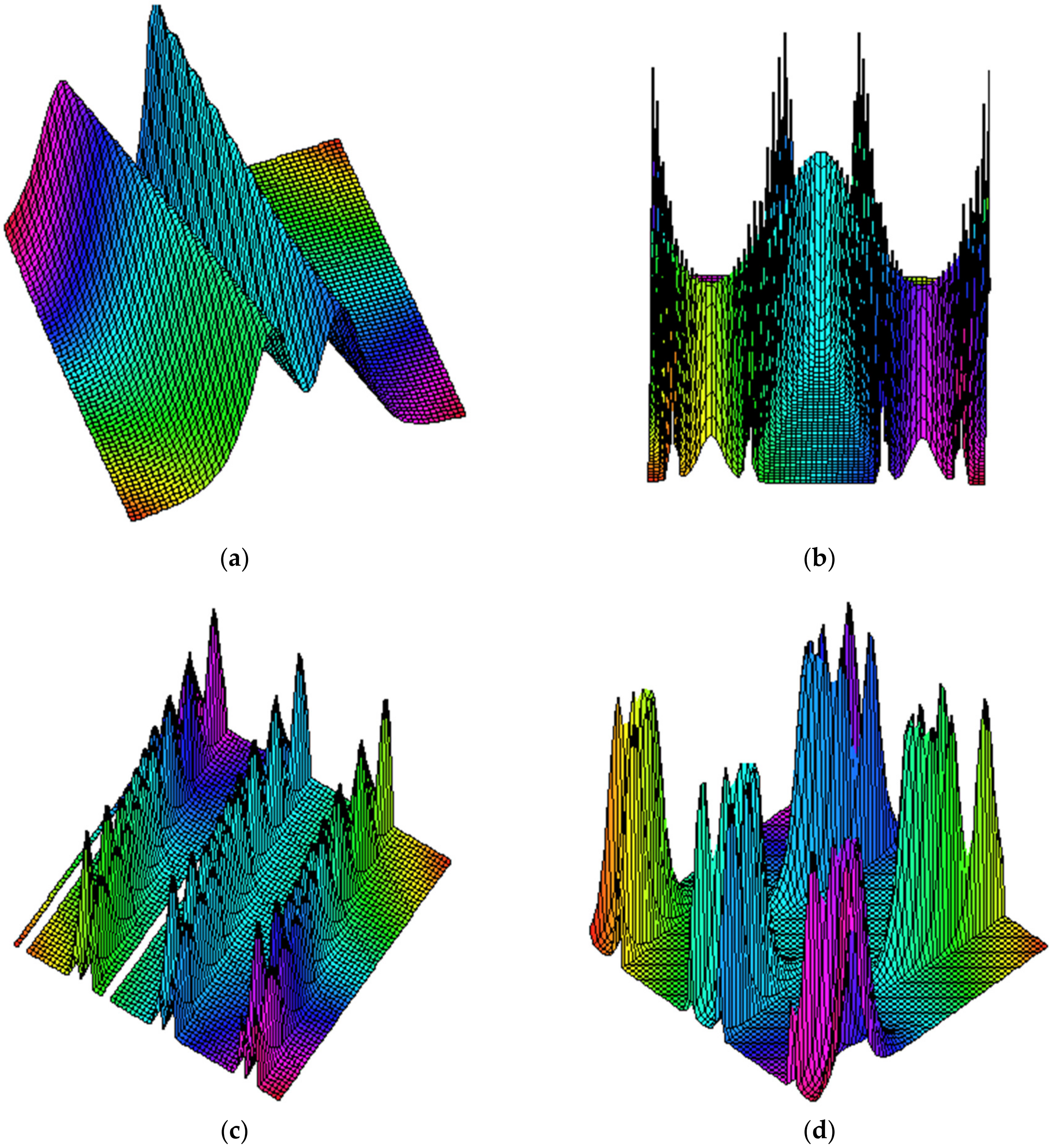


Figure 2. Evolution behavior of two-soliton solution (17) with $a = 1$, $c = 5$, $k_1 = 3$, $k_2 = 2$. (**a**) $\psi(t) = 1$ (**b**) $\psi(t) = \sin t$ (**c**) $\psi(t) = \sec^2 t$ (**d**) $\psi(t) = \tan t \sec t$.

Figure 3a represents the three-soliton solution of the function U when $\psi(t) = 1$, Figure 3b when $\psi(t) = \sin t$, Figure 3c when $\psi(t) = \sec^2 t$, and Figure 3d when $\psi(t) = \tan t \sec t$, which shows the different shapes of three-soliton solutions.

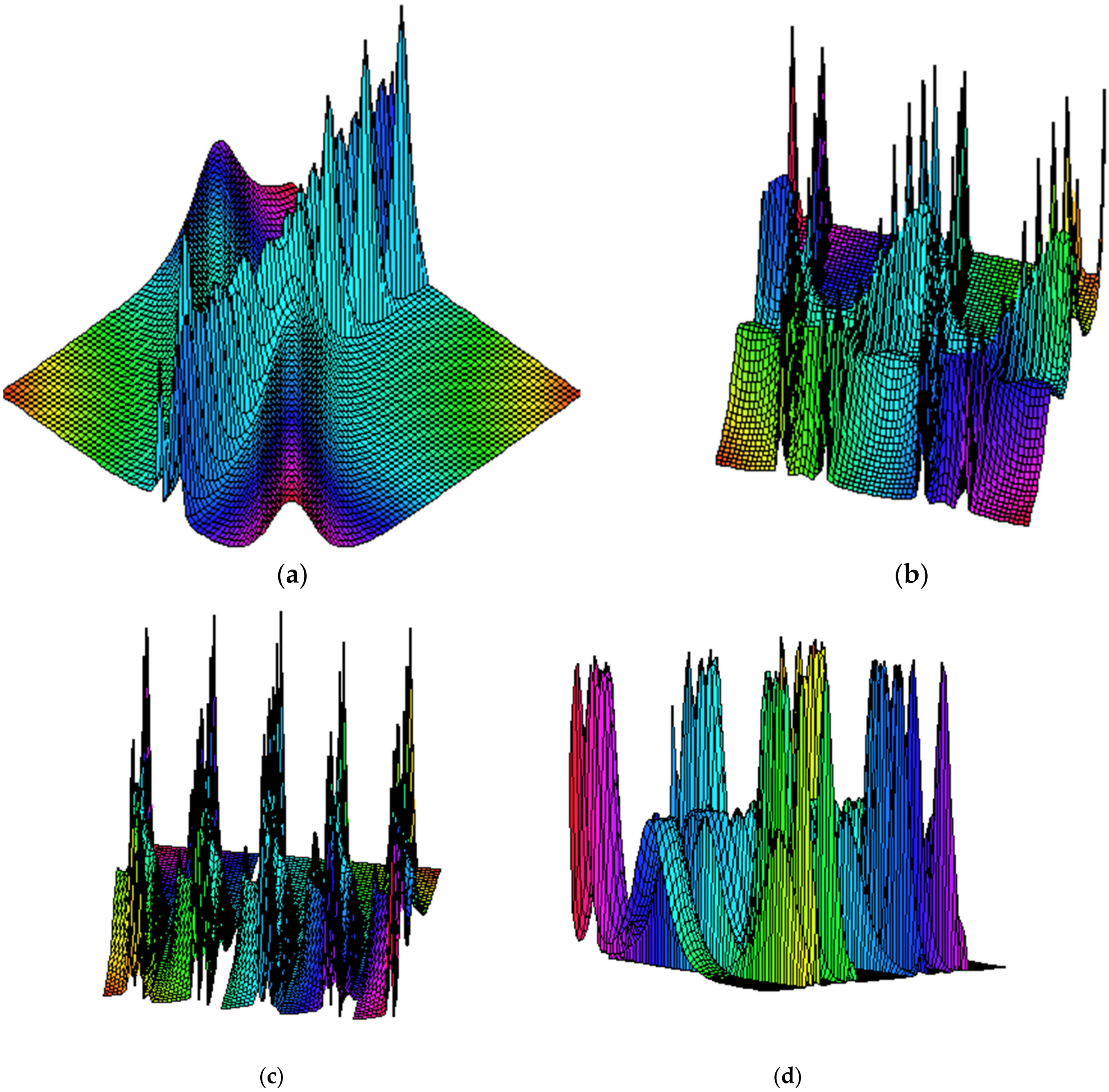


Figure 3. Evolution behavior of a three-soliton solution (20) with $a = 1,\ c = 5,\ k_1 = 3,\ k_2 = 2, k_3 = 1$. (**a**) $\psi(t) = 1$ (**b**) $\psi(t) = \sin t$ (**c**) $\psi(t) = \sec^2 t$ (**d**) $\psi(t) = \tan t \sec t$.

For constructing the four-soliton wave, we suppose that:

$$\begin{aligned}
&f = e^{\theta_4} + e^{\theta_3} + e^{\theta_2} + e^{\theta_1} - a_{1234}e^{\theta_4+\theta_3+\theta_2+\theta_1},\\
&g = 1 - a_{123}e^{\theta_1+\theta_2+\theta_3} - a_{134}e^{\theta_1+\theta_3+\theta_4} - a_{234}e^{\theta_2+\theta_3+\theta_4},\\
&\theta_1 = k_1x - k_1^3t,\ \theta_2 = k_2x - k_2^3t,\ \theta_3 = k_3x - k_3^3t,\\
&\theta_4 = k_4x - k_4^3t,\ a_{1234} = a_{12}a_{13}a_{14}a_{23}a_{24}a_{34},\ a_{123} = a_{12}a_{13}a_{23},\\
&a_{134} = a_{13}a_{14}a_{34},\ a_{234} = a_{23}a_{24}a_{34},\ \ a_{ij} = \left(\frac{k_i-k_j}{k_i+k_j}\right)^2,
\end{aligned} \tag{22}$$

with the phase shift terms a_{ij}, $1 \leq i < j \leq 4$, obtained above in (22). The four-soliton wave for the coupled mKdV model (3) is derived by substituting (22) into (7) and (8).

$$\begin{aligned}
U &= c\frac{\partial}{\partial x}\left[\tan^{-1}\left(\frac{e^{\theta_4}+e^{\theta_3}+e^{\theta_2}+e^{\theta_1}-a_{1234}e^{\theta_4+\theta_3+\theta_2+\theta_1}}{1-a_{123}e^{\theta_1+\theta_2+\theta_3}-a_{134}e^{\theta_1+\theta_3+\theta_4}-a_{234}e^{\theta_2+\theta_3+\theta_4}}\right)\right],\\
V &= \frac{4}{ac^2}\frac{\partial}{\partial x}\left[\tan^{-1}\left(\frac{e^{\theta_4}+e^{\theta_3}+e^{\theta_2}+e^{\theta_1}-a_{1234}e^{\theta_4+\theta_3+\theta_2+\theta_1}}{1-a_{123}e^{\theta_1+\theta_2+\theta_3}-a_{134}e^{\theta_1+\theta_3+\theta_4}-a_{234}e^{\theta_2+\theta_3+\theta_4}}\right)\right],\\
\theta_1 &= k_1x-k_1^3t,\ \theta_2 = k_2x-k_2^3t,\ \theta_3 = k_3x-k_3^3t,\ \theta_4 = k_4x-k_4^3t,\\
t &= \int_0^t \psi(\tau)d\tau,\ a_{1234} = a_{12}a_{13}a_{14}a_{23}a_{24}a_{34},\ a_{ijk} = a_{ij}a_{ik}a_{jk},\ 1 \leq i < j < k \leq 4.
\end{aligned} \tag{23}$$

Doing the same thing, an N-soliton wave can be constructed for a finite number N, where $N \geq 1$.

3. A Singular Soliton Wave

In this section, we will derive multiple singular solutions for a coupled mKdV system (3) using the Hirota bilinear procedure [1,2,8,9,40,42]. A singular solution for a coupled mKdV system (3) can be written with this formula:

$$\begin{aligned}
u &= R\tfrac{\partial}{\partial x}\left[\ln\left(\tfrac{f}{g}\right)\right] = R\,\tfrac{gf_x-fg_x}{f\,g},\\
v &= R_1\,\tfrac{\partial}{\partial x}\left[\ln\left(\tfrac{f}{g}\right)\right] = R_1\,\tfrac{gf_x-fg_x}{f\,g},
\end{aligned} \tag{24}$$

where the r and r_1 constants can be calculated, and the functions f and g for a singular soliton wave can be written as:

$$f = 1+\sum_{n=1}^{N} f_n,\quad g = 1-\sum_{n=1}^{N} g_n. \tag{25}$$

As introduced in the above, the relative of dispersion is calculated using:

$$c_i = k_i^3, \tag{26}$$

As a result, we get:

$$\theta_i = k_ix - k_i^3t. \tag{27}$$

The results can have a new definition as follows:

$$f = 1+e^{\theta_1},\quad g = 1-e^{\theta_1}.\quad \theta_1 = k_1x-k_1^3t. \tag{28}$$

Substituting (28) into (3) and resolving the results for R and R_1, we have:

$$R = c,\quad R_1 = -\frac{1}{a\,c^2}, \tag{29}$$

with c being an arbitrary constant. So, the singular soliton solution is:

$$u = \frac{2c\,k_1e^{k_1x-k_1^3t}}{1-e^{k_1x-k_1^3t}},\qquad v = -\frac{2\,k_1e^{k_1x-k_1^3t}}{a\,c\,(1-e^{k_1x-k_1^3t})}. \tag{30}$$

From the last equation, we obtain:

$$\frac{u}{v} = -a\,c^2. \tag{31}$$

To constructing a singular two-soliton wave, we have:

$$\begin{array}{ll} f = 1 + e^{\theta_2} + e^{\theta_1} + a_{12}e^{\theta_1+\theta_2}, & g = 1 - e^{\theta_2} - e^{\theta_1} + b_{12}e^{\theta_1+\theta_2}, \\ \theta_n = k_n x - k_n^3 t, \quad n = 1,\ 2. & \end{array} \tag{32}$$

Using (32) in (24) and resolving the outcome into (3), we have the phase shift:

$$a_{12} = b_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, \tag{33}$$

and then we obtain:

$$a_{mn} = b_{mn} = \frac{(k_m - k_n)^2}{(k_m + k_n)^2}, \qquad 1 \le m < n \le 2. \tag{34}$$

On behalf of the singular two-soliton wave, the two functions f and g take the form:

$$\begin{aligned} f &= 1 + e^{k_1(x-k_1^2 t)} + e^{k_2(x-k_2^2 t)} + \tfrac{(k_1-k_2)^2}{(k_1+k_2)^2} \quad e^{[(k_1+k_2)x-(k_1^3+k_2^3)t]}, \\ g &= 1 - e^{k_1(x-k_1^2 t)} - e^{k_2(x-k_2^2 t)} + \tfrac{(k_1-k_2)^2}{(k_1+k_2)^2} \quad e^{[(k_1+k_2)x-(k_1^3+k_2^3)t]}. \end{aligned} \tag{35}$$

A singular two-soliton wave is determined by substituting (35) into (24).

$$\begin{aligned} u &= c\tfrac{\partial}{\partial x}\left[\ln\left(\tfrac{1+a_{12}e^{\theta_1+\theta_2}+e^{\theta_2}+e^{\theta_1}}{1+b_{12}e^{\theta_1+\theta_2}-e^{\theta_2}-e^{\theta_1}}\right)\right], \\ v &= -\tfrac{1}{a\ c^2}\tfrac{\partial}{\partial x}\left[\ln\left(\tfrac{1+a_{12}e^{\theta_1+\theta_2}+e^{\theta_1}+e^{\theta_2}}{1+b_{12}e^{\theta_1+\theta_2}-e^{\theta_1}-e^{\theta_2}}\right)\right], \\ a_{12} &= b_{12} = \tfrac{(k_1-k_2)^2}{(k_1+k_2)^2}, \quad \theta_n = k_n(x - k_n^2 t), \quad n = 1, 2. \end{aligned} \tag{36}$$

To obtain a singular three-soliton wave, we can proceed similarly and set:

$$\begin{aligned} f &= 1 + a_{12}e^{\theta_1+\theta_2} + a_{13}e^{\theta_1+\theta_3} + a_{23}e^{\theta_2+\theta_3} + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + f_3, \\ g &= 1 + a_{12}e^{\theta_1+\theta_2} + a_{13}e^{\theta_1+\theta_3} + a_{23}e^{\theta_2+\theta_3} - e^{\theta_1} - e^{\theta_2} - e^{\theta_3} + g_3, \end{aligned} \tag{37}$$

Substituting (37) into (24) and using the result in (3), we obtain:

$$f_3 = b_{123}e^{\theta_1+\theta_2+\theta_3}, \quad g_3 = -b_{123}e^{\theta_1+\theta_2+\theta_3}, \quad b_{123} = a_{12}a_{13}a_{23}. \tag{38}$$

In the three-soliton singular solution, the functions f and g take the form:

$$\begin{aligned} f &= 1 + e^{k_1(x-k_1^2 t)} + e^{k_2(x-k_2^2 t)} + e^{k_6(x-k_2^2 t)} + \frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{[(k_1+k_2)x-(k_1^3+k_2^3)t]} \\ &\quad + \frac{(k_1-k_3)^2}{(k_1+k_3)^2}e^{[(k_1+k_3)x-(k_1^3+k_3^3)t]} + \frac{(k_2-k_3)^2}{(k_2+k_3)^2}e^{[(k_2+k_3)x-(k_2^3+k_3^3)t]} \\ &\quad + \frac{(k_1-k_2)^2}{(k_1+k_2)^2}\frac{(k_1-k_3)^2}{(k_1+k_3)^2}\frac{(k_2-k_3)^2}{(k_2+k_3)^2}e^{[(k_1+k_2+k_3)x-(k_1^3+k_2^3+k_3^3)t]}, \\ g &= 1 - e^{k_1(x-k_1^2 t)} - e^{k_2(x-k_2^2 t)} - e^{k_3(x-k_2^2 t)} + \frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{[(k_1+k_2)x-(k_1^3+k_2^3)t]} \\ &\quad + \frac{(k_1-k_3)^2}{(k_1+k_3)^2}e^{[(k_1+k_3)x-(k_1^3+k_3^3)t]} + \frac{(k_2-k_3)^2}{(k_2+k_3)^2}e^{[(k_2+k_3)x-(k_2^3+k_3^3)t]} \\ &\quad - \frac{(k_1-k_2)^2}{(k_1+k_2)^2}\frac{(k_1-k_3)^2}{(k_1+k_3)^2}\frac{(k_2-k_3)^2}{(k_2+k_3)^2}e^{[(k_1+k_2+k_3)x-(k_1^3+k_2^3+k_3^3)t]}. \end{aligned} \tag{39}$$

The three singular solution can be calculated by substituting the two functions f and g into (24).

4. Soliton Interactions

Many applications of soliton interactions can be obtained from fiber optics. As we all know, two solitons interacting is important. By studying the soliton interactions, we can effectively enhance the communication capacity and improve the stability of a system [43]. It is therefore essential to distinguish how to prevent two nearby solitons from moving each other. The nonlinear collaboration of two solitons has been deliberated widely in recent years. We know that based on their physical states, two solitons move each other individually when they are sufficiently close that their tails intersect. So, the two solitons may attract (come close to) or repel (move away from) each other depending on the parameters (wave numbers, phases, and velocity) or the initial conditions.

To discuss the interaction of two solitons, we consider the two solitons' peaks to be disconnected in time. Generally speaking, as much as possible, we assume that their wave numbers, phases, and velocity change such that:

$$u_i = \frac{c\, k_{1i} e^{k_{1i}x - k_{1i}^3 t + \varphi_i}}{1 + e^{2(k_{1i}x - k_{1i}^3 t) + \varphi_i}}, \qquad v_i = \frac{4\, k_{1i} e^{k_{1i}x - k_{1i}^3 t + \varphi_i}}{a\, c\, (1 + e^{2(k_{1i}x - k_{1i}^3 t) + \varphi_i})}, \qquad t = \int_0^t \psi(\tau) d\tau, \tag{40}$$

where φ_i represents the phase difference with $i = 1$ or 2. We will study the interaction between two solitons by using the simple linear sum of two wave solitons u_1, u_2 and v_1, v_2 that satisfy the coupled mKdV system individually. It is clear that the total field $u = u_1 + u_2$ and $v = v_1 + v_2$ satisfies the perturbed coupled mKdV system, which is satisfied by each soliton. When $i = 1$, we have:

$$\begin{aligned} u_{1t} + 6au_1v_1u_{1x} + u_{1xxx} &= -6a(u_1 + u_2)(v_1 + v_2)(u_{1x} + u_{2x}) + 6a(u_1v_1u_{1x} + u_2v_2u_{2x}), \\ v_{1t} + 6av_1u_1v_{1x} + v_{1xxx} &= -6a(v_1 + v_2)(u_1 + u_2)(v_{1x} + v_{2x}) + 6a(u_1v_1u_{1x} + u_2v_2u_{2x}), \end{aligned} \tag{41}$$

when $i = 2$, we have:

$$\begin{aligned} u_{2t} + 6au_2v_2u_{2x} + u_{2xxx} &= -6a(u_1 + u_2)(v_1 + v_2)(u_{1x} + u_{2x}) + 6a(u_1v_1u_{1x} + u_2v_2u_{2x}), \\ v_{2t} + 6av_2u_2v_{2x} + v_{2xxx} &= -6a(v_1 + v_2)(u_1 + u_2)(v_{1x} + v_{2x}) + 6a(u_1v_1u_{1x} + u_2v_2u_{2x}). \end{aligned} \tag{42}$$

So, the nonlinear interaction between two neighboring solitons occurs based on the two terms on the right side, which act as perturbation terms.

In the same manner, in discussing the interaction between three solitons, we assume that their wave numbers, phases, and velocity differ such that:

$$u_i = \frac{c\, k_{1i} e^{k_{1i}x - k_{1i}^3 t + \varphi_i}}{1 + e^{2(k_{1i}x - k_{1i}^3 t) + \varphi_i}}, \qquad v_i = \frac{4\, k_{1i} e^{k_{1i}x - k_{1i}^3 t + \varphi_i}}{a\, c\, (1 + e^{2(k_{1i}x - k_{1i}^3 t) + \varphi_i})}, \qquad t = \int_0^t \psi(\tau) d\tau, \tag{43}$$

where φ_i represents the phase difference with $i = 1, 2,$ or 3. Since u_1, u_2, u_3 and v_1, v_2, v_3 satisfy the coupled mKdV system individually, it is clear that the total field $u = u_1 + u_2 + u_3$ and $v = v_1 + v_2 + v_3$ satisfies the perturbed coupled mKdV system. When $i = 1$, we have:

$$\begin{aligned} u_{1t} + 6au_1v_1u_{1x} + u_{1xxx} = \quad & -6a(u_1 + u_2 + u_3)(v_1 + v_2 + v_3)(u_{1x} + u_{2x} + u_{3x}) \\ & +6a(u_1v_1u_{1x} + u_2v_2u_{2x} + u_3v_3u_{3x}), \\ v_{1t} + 6av_1u_1v_{1x} + v_{1xxx} = \quad & -6a(v_1 + v_2 + v_3)(u_1 + u_2 + u_3)(v_{1x} + v_{2x} + v_{3x}) \\ & +6a(u_1v_1u_{1x} + u_2v_2u_{2x} + u_3v_3u_{3x}), \end{aligned} \tag{44}$$

when $i = 2$, we have:

$$\begin{aligned} u_{2t} + 6au_2v_2u_{2x} + u_{2xxx} = & \; -6a(u_1+u_2+u_3)(v_1+v_2+v_3)(u_{1x}+u_{2x}+u_{3x}) \\ & +6a(u_1v_1u_{1x}+u_2v_2u_{2x}+u_3v_3u_{3x}), \\ v_{2t} + 6av_2u_2v_{2x} + v_{2xxx} = & \; -6a(v_1+v_2+v_3)(u_1+u_2+u_3)(v_{1x}+v_{2x}+v_{3x}) \\ & +6a(u_1v_1u_{1x}+u_2v_2u_{2x}+u_3v_3u_{3x}). \end{aligned} \tag{45}$$

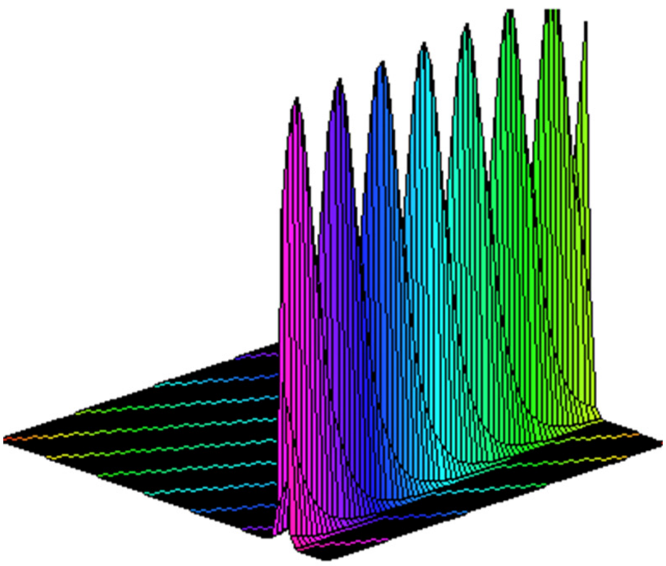
Also, when $i = 3$, we have:

$$\begin{aligned} u_{3t} + 6au_3v_3u_{3x} + u_{3xxx} = & \; -6a(u_1+u_2+u_3)(v_1+v_2+v_3)(u_{1x}+u_{2x}+u_{3x}) \\ & +6a(u_1v_1u_{1x}+u_2v_2u_{2x}+u_3v_3u_{3x}), \\ v_{3t} + 6av_3u_3v_{3x} + v_{3xxx} = & \; -6a(v_1+v_2+v_3)(u_1+u_2+u_3)(v_{1x}+v_{2x}+v_{3x}) \\ & +6a(u_1v_1u_{1x}+u_2v_2u_{2x}+u_3v_3u_{3x}). \end{aligned} \tag{46}$$

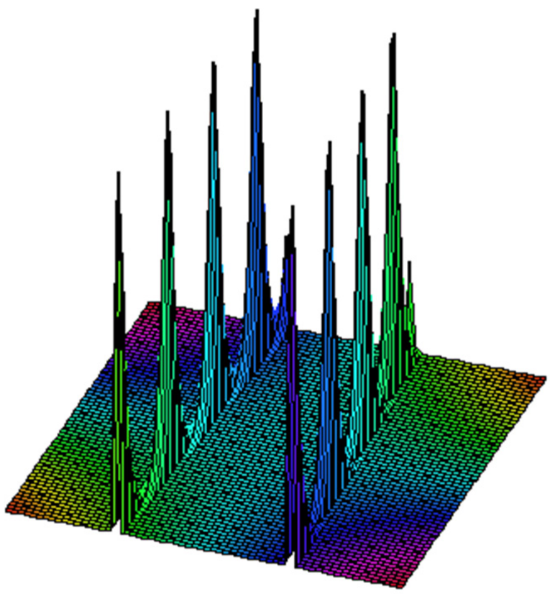
The evolutional behavior of the soliton interaction wave is represented by $u = u_1 + u_2$ given above with the selection $c = 2,\ a = 1,\ k_{11} = 4, k_{12} = 3,\ \varphi_1 = -100$, and $\varphi_2 = 100$, so u takes the form:

$$u = \frac{8e^{4x-64\int_0^t \psi(\tau)d\tau-100}}{1+e^{4x-64\int_0^t \psi(\tau)d\tau-100}} + \frac{6e^{3x-27\int_0^t \psi(\tau)d\tau+100}}{1+e^{3x-27\int_0^t \psi(\tau)d\tau+100}}, \tag{47}$$

Figure 4a represents when $\psi(t) = 1$, Figure 4b represents when $\psi(t) = t$, Figure 4c represents when $\psi(t) = e^t$, Figure 4d represents when $\psi(t) = \sin t$, Figure 4e represents when $\psi(t) = \tan t$, Figure 4f represents when $\psi(t) = \tanh t$, and Figure 1 shows the different types of interactions between two solitons.



(a)



(b)

Figure 4. *Cont.*

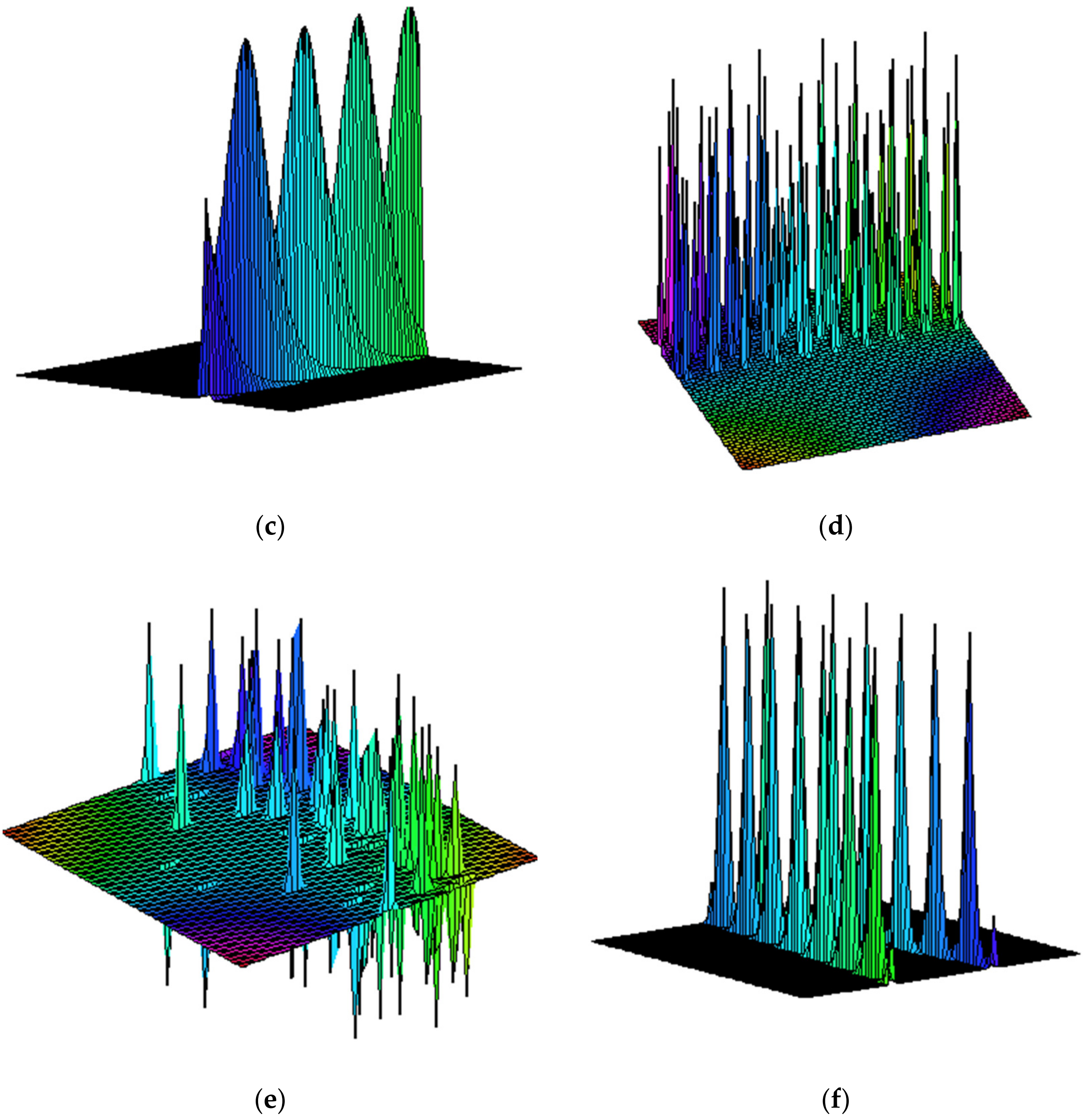


Figure 4. Evolution behavior of the interaction of a two-soliton solution with $c = 2$, $a = 1$, $k_{11} = 4$, $k_{12} = 3$, $\varphi_1 = -100$, $\varphi_2 = 100$, (**a**) $\psi(t) = 1$ (**b**) $\psi(t) = t$ (**c**) $\psi(t) = e^t$ (**d**) $\psi(t) = \sin t$ (**e**) $\psi(t) = \tan t$ (**f**) $\psi(t) = \tanh t$.

5. Discussion and Summery

In this paper, we study a coupled mKdV equation with a time-dependent coefficient which passes multiple soliton waves. We implement the bilinear Hirota technique to compute analytical solutions to the desired coupled mKdV system. We construct multiple kink soliton and singular kink soliton waves for the system under study. Such calculations are made by using a software with symbolic computation software, for instance, Maple. Recently some researchers used Maple in order to show that the bilinear method of Hirota is a straightforward technique which can be used in the approach of differential, nonlinear models. All our results have fruitful features because they can be extended to different physical structures, given that different functions have been chosen arbitrarily. The proper choice of constants and functions allows us to draw significant physical profiles. The respective solutions refer to profiles in a vast area, considering them as singular-sink solutions or sink soliton solutions. N-soliton solutions are obtained and the interactions between solitons are analyzed mathematically and graphically. It was observed that solitary waves existed as long as the time-dependent coefficient was Riemann-integrable. We analyzed whether the experiments proved that the procedure is effective and can be successfully used for many other mathematical models used in physics and engineering. The determined solutions can be of powerful significance for the clarification of certain practical physical problems. These results in the present study could be used to understand related physical phenomena in nonlinear optics and relevant fields. At any rate, we are encouraged by the results of this paper to further study the problem of solitons for models that depend on time, such as (1).

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