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On Symmetric Additive Mappings and Their Applications

Shakir Ali [1,*], Turki Alsuraiheed [2], Vaishali Varshney [1] and Indah Emilia Wijayanti [3]

[1] Department of Mathematics, Faculty of Science, Aligarh Muslim University, Aligarh 202002, India; vaishalivarshney7625@gmail.com
[2] Department of Mathematics, College of Science, King Saud University, Riyadh 11451, Saudi Arabia
[3] Department of Mathematics, Universitas Gadjah Mada, Yogyakarta 55281, Indonesia
* Correspondence: shakir.ali.mm@amu.ac.in

Abstract: The key motive of this paper is to study symmetric additive mappings and discuss their applications. The study of these symmetric mappings makes it possible to characterize symmetric n-derivations and describe the structure of the quotient ring $\mathfrak{S}/\mathfrak{P}$, where $\mathfrak{S}$ is any ring and $\mathfrak{P}$ is a prime ideal of $\mathfrak{S}$. The symmetricity of additive mappings allows us to transfer ring theory results to functional analyses, particularly to C^*-algebras. Precisely, we describe the structures of C^*-algebras via symmetric additive mappings.

Keywords: additive symmetric mapping; biadditive mapping; derivation; symmetric n-derivation; symmetric generalized n-derivation; C^*-algebra; prime ideal

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1. Introduction

Symmetry, a timeless and universal concept in both mathematics and the natural sciences, serves as a foundational principle that reveals deep structures and relationships. Within the realm of mathematics, symmetry has emerged as a beautiful and powerful tool, leading to the development of various mathematical structures and theories. One exciting area of math is the study of something called "symmetric n -derivations". This concept is the extension of the notion of derivation. Over the years, researchers have extended the notion of derivations in various directions, such as generalized derivations, (α, β)-derivations, bi-derivations, higher derivations, symmetric n-derivations, etc., and have studied the structures of rings as well as additive mappings (viz., [1–6]). In this research article, we present a comprehensive investigation of symmetric generalized n-derivations, seeking to establish a theoretical connection between symmetric generalized n-derivations and other fundamental algebraic concepts. Throughout the discussion, we will consider $\mathfrak{S}$ to be an associative ring with $Z(\mathfrak{S})$ being its center, and $\mathcal{A}$ will represent C^*-algebra. However, $\mathcal{A}$ may not have unity with center $Z(\mathcal{A})$. A ring $\mathfrak{S}$ is said to be prime if $\varrho\mathfrak{S}\xi = \{0\}$ implies that either $\varrho = 0$ or $\xi = 0$, and semiprime if $\varrho\mathfrak{S}\varrho = \{0\}$ implies that $\varrho = 0$, where $\varrho, \xi \in \mathfrak{S}$. The symbols $[\varrho, \xi]$ and $\varrho \circ \xi$ denote the commutator, $\varrho\xi - \xi\varrho$, and the anti-commutator, $\varrho\xi + \xi\varrho$, respectively, for all $\varrho, \xi \in \mathfrak{S}$. A ring $\mathfrak{S}$ is said to be n-torsion-free if $n\varrho = 0$ implies that $\varrho = 0$ for all $\varrho \in \mathfrak{S}$. If $\mathfrak{S}$ is $n!$-torsion-free, then it is d-torsion-free for every divisor d of $n!$. Recall that an ideal $\mathfrak{P}$ of $\mathfrak{S}$ is said to be prime if $\mathfrak{P} \neq \mathfrak{S}$ and for $\varrho, \xi \in \mathfrak{S}$, $\varrho\mathfrak{S}\xi \subseteq \mathfrak{P}$ implies that $\varrho \in \mathfrak{P}$ or $\xi \in \mathfrak{P}$.

A map $\mathfrak{D} : \mathfrak{S} \times \mathfrak{S} \to \mathfrak{S}$ is said to be biadditive if it is additive in both arguments. A biadditive map $\mathfrak{D} : \mathfrak{S} \times \mathfrak{S} \to \mathfrak{S}$ is said to be symmetric if $\mathfrak{D}(\varrho, \xi) = \mathfrak{D}(\xi, \varrho)$ for all $\varrho, \xi \in \mathfrak{S}$. An additive mapping $d : \mathfrak{S} \to \mathfrak{S}$ is called a derivation if $d(\varrho\xi) = d(\varrho)\xi + \varrho d(\xi)$ holds for all $\varrho, \xi \in \mathfrak{S}$. Following [7], an additive mapping $g : \mathfrak{S} \longrightarrow \mathfrak{S}$ is said to be a generalized derivation on $\mathfrak{S}$ if there exists a derivation $d : \mathfrak{S} \longrightarrow \mathfrak{S}$, such that $g(\varrho\xi) = g(\varrho)\xi + \varrho d(\xi)$ holds for all $\varrho, \xi \in \mathfrak{S}$. A symmetric biadditive map is said to be a symmetric bi-derivation if $\mathfrak{D}(\varrho\xi, z) = \varrho\mathfrak{D}(\xi, z) + \mathfrak{D}(\varrho, z)\xi$ for all $\varrho, \xi, z \in \mathfrak{S}$. The concept of symmetric bi-derivation in rings was introduced by G. Maksa [8]. Suppose n is a fixed positive integer and $\mathfrak{S}^n = \mathfrak{S} \times \mathfrak{S} \times \cdots \times \mathfrak{S}$. A map $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ is said to be n-additive if it is additive in each argument. A map $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ is said to be symmetric if the relation $\mathfrak{D}(\varrho_1, \varrho_2, \ldots, \varrho_n) = \mathfrak{D}\left(\varrho_{\pi(1)}, \varrho_{\pi(2)}, \ldots, \varrho_{\pi(n)}\right)$ holds for all $\varrho_i \in \mathfrak{S}$ and every permutation $\{\pi(1), \pi(2), \ldots, \pi(n)\}$. If $\mathfrak{D}$ is both n-additive and symmetric, then it is called symmetric n-additive mapping. The concepts of derivation and symmetric bi-derivation were generalized by Park [9] as follows: a map $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ is said to be a symmetric n-derivation if $\mathfrak{D}$ is symmetric n-additive and $\mathfrak{D}(\varrho_1, \varrho_2, \cdots, \varrho_i\varrho_i', \cdots, \varrho_n) = \varrho_i\mathfrak{D}(\varrho_1, \varrho_2, \cdots, \varrho_i', \cdots, \varrho_n) + \mathfrak{D}(\varrho_1, \varrho_2, \cdots, \varrho_i, \cdots, \varrho_n)\varrho_i'$ holds for all $\varrho_i, \varrho_i' \in \mathfrak{S}$. A 1-derivation is a derivation and a 2-derivation is a symmetric bi-derivation while a 3-derivation is known as symmetric tri-derivation (viz., [3,6,10–14]). Let $n \geq 2$ be a fixed integer and a map $\mathscr{d} : \mathfrak{S} \to \mathfrak{S}$ defined by $\mathscr{d}(\varrho) = \mathfrak{D}(\varrho, \varrho, \ldots, \varrho)$ for all $\varrho \in \mathfrak{S}$, where $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ is a symmetric map, being the trace of $\mathfrak{D}$. If $\mathfrak{D}$ is symmetric and n-additive, then the trace $\mathscr{d}$ of $\mathfrak{D}$ satisfies the relation

$$\mathscr{d}(\varrho + \xi) = \mathscr{d}(\varrho) + \mathscr{d}(\xi) + \sum_{t=1}^{n-1} {}^nC_t\, \mathfrak{D}(\underbrace{\varrho, \ldots, \varrho}_{(n-t)-\text{times}}, \underbrace{\xi, \ldots, \xi}_{t-\text{times}})$$

for all $\varrho, \xi \in \mathfrak{S}$, where ${}^nC_t = \binom{n}{t}$.

Inspired by the idea of generalized derivation in the ring, Ashraf et al. [3] introduced the concept of symmetric generalized n-derivations in rings. Let $n \geq 1$ be a fixed positive integer. A symmetric n-additive map $\mathcal{G} : \mathfrak{S}^n \to \mathfrak{S}$ is known to be the symmetric generalized n-derivation if there exists a symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$, such that $\mathcal{G}(\varrho_1, \varrho_2, \ldots, \varrho_i\varrho_i', \ldots, \varrho_n) = \mathcal{G}(\varrho_1, \varrho_2, \ldots, \varrho_i, \ldots, \varrho_n)\varrho_i' + \varrho_i\mathfrak{D}(\varrho_1, \varrho_2, \ldots, \varrho_i', \ldots, \varrho_n)$ holds for all $\varrho_i, \varrho_i' \in \mathfrak{S}$. In fact, in [3], the authors proved that "for a fixed positive integer $n \geq 2$, let $\mathfrak{S}$ be an $n!$-torsion-free semiprime ring admitting a symmetric generalized n-derivation Ω with the associated n-derivation $\mathfrak{D}$, such that the trace ω of Ω is centralized on $\mathfrak{S}$. Then ω is commuting on $\mathfrak{S}$". Additionally, in [15], Ashraf et al. characterized the traces of symmetric generalized n-derivations. In fact, their results were motivated by the results from Hvala [16]. Basically, they proved that "for a fixed positive integer $n \geq 2$, let $\mathfrak{S}$ be an $n!$-torsion-free prime ring. Suppose that ω_1 and ω_2 are the traces of symmetric generalized n-derivations Ω_1, Ω_2, respectively and $\mathscr{d}_1 \neq 0$; $\mathscr{d}_2$ are the traces of associated derivations $\mathfrak{D}_1$ and $\mathfrak{D}_2$, respectively. If $\omega_1(\varrho)\omega_2(\xi) = \omega_2(\varrho)\omega_1(\xi)$ holds for all $\varrho, \xi \in \mathfrak{S}$, then there exists $\gamma \in C$, the extended centroid of $\mathfrak{S}$, such that $\mathscr{d}_2(\varrho) = \gamma\mathscr{d}_1(\varrho)$."

Numerous authors have thoroughly investigated a wide range of identities involving traces of symmetric n-derivations, leading to the discovery of various interesting results (see, for example [3,10,15,17–19] and the associated references). Very recently, Ali et al. [10], explored some algebraic identities associated with the trace of symmetric n-derivations acting on prime ideal $\mathfrak{P}$ of $\mathfrak{S}$, but without imposing the assumption of prime on the ring under consideration. In fact, apart from proving some other interesting results, they extended the famous result [20] [Theorem 2] for the trace of symmetric n-derivations, which involves prime ideals. Precisely, they proved that for any fixed integer $n \geq 2$, let $\mathfrak{S}$ be a ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$, such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion-free. If there exists a non-zero symmetric n-derivation $\mathfrak{D}$ with trace $\mathscr{d}$ on $\mathfrak{S}$, such that $[\mathscr{d}(\varrho), \varrho] \in \mathfrak{P}$, for all $\varrho \in \mathfrak{S}$, then either $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain or $\mathscr{d}(\mathfrak{S}) \subseteq \mathfrak{P}$.

The main focus of our current research is to analyze the structure of the quotient ring $\mathfrak{S}/\mathfrak{P}$, where $\mathfrak{S}$ is any ring and $\mathfrak{P}$ is a prime ideal of $\mathfrak{S}$, which admits symmetric generalized n-derivations satisfying certain algebraic identities acting on prime ideals $\mathfrak{P}$. Moreover, in the last section, we discuss the applications of the theory of symmetric n-derivations. Finally, we establish corresponding results in the C^*-algebra settings.

2. The Results

The following auxiliary results are essential for proving the above-mentioned results:

Lemma 1 ([18] Lemma 2.3)**.** *For a fixed positive integer n, let $\mathcal{R}$ be a ring and $\mathcal{P}$ be a prime ideal of $\mathcal{R}$, such that $\mathfrak{S}/\mathcal{P}$ is $n!$-torsion free. Suppose that $\ell_1, \ell_2, \ldots, \ell_n \in \mathcal{R}$ satisfies $\alpha\ell_1 + \alpha^2\ell_2 + \cdots + \alpha^n\ell_n \in \mathcal{P}$ for $\alpha = 1, 2, \ldots, n$. Then $\ell_t \in \mathcal{P}$ for $t = 1, 2, \ldots, n$.*

Lemma 2 ([21])**.** *Let $\mathcal{R}$ be a ring and $\mathcal{P}$ be a prime ideal of $\mathcal{R}$. If one of the following conditions is satisfied, then $\mathcal{R}/\mathcal{P}$ is a commutative integral domain.*

(i) $[\varrho, \xi] \in \mathcal{P} \ \forall \ \varrho, \xi \in \mathcal{R}$

(ii) $\varrho \circ \xi \in \mathcal{P} \ \forall \ \varrho, \xi \in \mathcal{R}$.

Lemma 3 ([10] Theorem 1.4)**.** *For a fixed integer $n \geq 2$, let $\mathfrak{S}$ be a ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$, such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion-free and $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ be a nonzero symmetric n-derivation on $\mathfrak{S}$ with trace $\mathscr{d} : \mathfrak{S} \to \mathfrak{S}$. If $[\mathscr{d}(\varrho), \varrho] \in \mathfrak{P}$ for all $\varrho \in \mathfrak{S}$, then $\mathscr{d}(\mathfrak{S}) \subseteq \mathfrak{P}$ or $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain.*

Our first main result establishes a link between the derivation and symmetric generalized n-derivation. In simpler terms, we demonstrate the following result:

Theorem 1. *For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be a ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$. Let $\mathcal{G} : \mathfrak{S}^n \longrightarrow \mathfrak{S}$ be a nonzero symmetric generalized n-derivation with associated symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \longrightarrow \mathfrak{S}$ with traces $g : \mathfrak{S} \longrightarrow \mathfrak{S}$ of $\mathcal{G}$ and $\mathscr{d} : \mathfrak{S} \longrightarrow \mathfrak{S}$ of $\mathfrak{D}$. Next, let $\delta : \mathfrak{S} \longrightarrow \mathfrak{S}$ be a derivation on $\mathfrak{S}$. If $[\delta(\varrho), \varrho] - g(\varrho) \in \mathfrak{P} \ \forall \ \varrho \in \mathfrak{S}$, then we have one of the following assertions:*

(i) $\delta(\mathfrak{S}) \subseteq \mathfrak{P}$

(ii) *$\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain.*

Proof. By the assumption, we have

$$[\delta(\varrho), \varrho] - g(\varrho) \in \mathfrak{P} \ \forall \ \varrho \in \mathfrak{S}. \tag{1}$$

Replacing ϱ with $\varrho + m\xi$ for $1 \leq m \leq n-1$, $\xi \in \mathfrak{S}$, we obtain

$$[\delta(\varrho + m\xi), \varrho + m\xi] - g(\varrho + m\xi) \in \mathfrak{P} \ \forall \ \varrho, \xi \in \mathfrak{S}. \tag{2}$$

Continuing to solve, we obtain

$$\begin{aligned}[] [\delta(\varrho), \varrho] + [\delta(\varrho), m\xi] + [\delta(m\xi), \varrho] + [\delta(m\xi), m\xi] - g(\varrho) - \\ g(m\xi) - \sum_{t=1}^{n-1} {}^{n}C_t \mathcal{G}(\underbrace{\varrho, \ldots, \varrho}_{(n-t)-\text{times}}, \underbrace{m\xi, \ldots, m\xi}_{t-\text{times}}) \in \mathfrak{P} \ \forall \ \varrho, \xi \in \mathfrak{S}. \end{aligned} \tag{3}$$

Application of relation (1) yields that

$$m[\delta(\varrho), \xi] + m[\delta(\xi), \varrho] - \sum_{t=1}^{n-1} {}^{n}C_t \mathcal{G}(\underbrace{\varrho, \ldots, \varrho}_{(n-t)-\text{times}}, \underbrace{m\xi, \ldots, m\xi}_{t-\text{times}}) \in \mathfrak{P} \ \forall \ \varrho, \xi \in \mathfrak{S}.$$

This can be written as

$$mA_1(\varrho, \xi) + m^2A_2(\varrho, \xi) + \cdots + m^{n-1}A_{n-1}(\varrho, \xi) \in \mathfrak{P} \ \forall \ \varrho, \xi \in \mathfrak{S},$$

where $A_t(\varrho, \xi)$ represents the term in which ξ appears t-times.

On taking account of Lemma 1, we obtain

$$[\delta(\varrho), \xi] + [\delta(\xi), \varrho] - n\mathcal{G}(\varrho, \ldots, \varrho, \xi) \in \mathfrak{P} \ \forall \ \varrho, \xi \in \mathfrak{S}. \tag{4}$$

Substitute $\xi\varrho$ for ξ, we see that

$$[\delta(\varrho), \xi\varrho] + [\delta(\xi\varrho), \varrho] - n\mathcal{G}(\varrho, \ldots, \varrho, \xi\varrho) \in \mathfrak{P} \; \forall \; \varrho, \xi \in \mathfrak{S}$$

which on solving, we obtain

$$2\xi[\delta(\varrho), \varrho] - n\xi\mathscr{d}(\varrho) + [\xi, \varrho]\delta(\varrho) + \{[\delta(\varrho), \xi] + [\delta(\xi), \varrho] - n\mathcal{G}(\varrho, \ldots, \varrho, \xi)\}\varrho - n\xi\mathscr{d}(\varrho) \in \mathfrak{P} \; \forall \; \varrho, \xi \in \mathfrak{S}.$$

By (4), we have

$$2\xi[\delta(\varrho), \varrho] - n\xi\mathscr{d}(\varrho) + [\xi, \varrho]\delta(\varrho) - n\xi\mathscr{d}(\varrho) \in \mathfrak{P} \; \forall \; \varrho, \xi \in \mathfrak{S}. \tag{5}$$

We replace ξ with $r\xi$ in (5) and use (5) to obtain

$$[r, \varrho]\xi\delta(\varrho) \in \mathfrak{P} \; \forall \; \varrho, \xi, r \in \mathfrak{S}$$

or

$$[r, \varrho]\mathfrak{S}\delta(\varrho) \in \mathfrak{P} \; \forall \; \varrho, r \in \mathfrak{S}.$$

Considering the primeness of the ideal $\mathfrak{P}$, we obtain for all $\varrho \in \mathfrak{S}$

$$\text{either } [r, \varrho] \in \mathfrak{P} \text{ or } \delta(\varrho) \in \mathfrak{P}.$$

Consequently, $\mathfrak{S}$ is a union of two of its proper subgroups, H_1 and H_2, where

$$H_1 = \{\varrho \in \mathfrak{S} \mid [r, \varrho] \in \mathfrak{P}\} \text{ and } H_2 = \{\varrho \in \mathfrak{S} \mid \delta(\varrho) \in \mathfrak{P}\}.$$

Since a group cannot be a union of two of its proper subgroups, we are forced to conclude that either $\mathfrak{S} = H_1$ or $\mathfrak{S} = H_2$. Consider the first case, $\mathfrak{S} = H_1$, i.e., $[r, \varrho] \in \mathfrak{P}$. Using Lemma 2, we conclude that $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain. On the other hand, if $\mathfrak{S} = H_2$, then $\delta(\varrho) \in \mathfrak{P}$ for all $\varrho \in \mathfrak{S}$. Henceforward, we conclude that $\delta(\mathfrak{S}) \subseteq \mathfrak{P}$. This completes the proof of the theorem. □

Theorem 2. *For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be a ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$, such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion-free. If $\mathfrak{S}$ admits a nonzero symmetric generalized n-derivation $\mathcal{G} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $g : \mathfrak{S} \to \mathfrak{S}$ associated with a symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \longrightarrow \mathfrak{S}$ with trace $\mathscr{d} : \mathfrak{S} \to \mathfrak{S}$ satisfying the condition $[g(\varrho), \varrho] \in \mathfrak{P} \; \forall \; \varrho \in \mathfrak{S}$, then one of the following holds:*

(i) $\mathscr{d}(\mathfrak{S}) \subseteq \mathfrak{P}$

(ii) $\mathfrak{S}/\mathfrak{P}$ *is a commutative integral domain.*

Proof. We have

$$[g(\varrho), \varrho] \in \mathfrak{P} \; \forall \; \varrho \in \mathfrak{S}. \tag{6}$$

We replace ϱ with $\varrho + m\xi$ for $\xi \in \mathfrak{S}$ and $1 \leq m \leq n-1$ leads to

$$[g(\varrho + m\xi), (\varrho + m\xi)] \in \mathfrak{P} \; \forall \; \varrho, \xi \in \mathfrak{S}.$$

As a consequence, we obtain

$$[g(\varrho),\varrho]+[g(\varrho),m\xi]+[g(m\xi),\varrho]+[g(m\xi),m\xi]+ \left[\sum_{t=1}^{n-1}{}^{n}C_{t}\mathcal{G}(\underbrace{\varrho,\ldots,\varrho}_{(n-t)-\text{times}},\underbrace{m\xi,\ldots,m\xi}_{t-\text{times}}),\varrho\right]+ \left[\sum_{t=1}^{n-1}{}^{n}C_{t}\mathcal{G}(\underbrace{\varrho,\ldots,\varrho}_{(n-t)-\text{times}},\underbrace{m\xi,\ldots,m\xi}_{t-\text{times}}),m\xi\right]\in\mathfrak{P}$$

$\forall\ \varrho,\xi\in\mathfrak{S}$. Using the relation (6), we obviously find that

$$m[g(\varrho),\xi]+m^{n}[g(\xi),\varrho]+\left[\sum_{t=1}^{n-1}{}^{n}C_{t}\mathcal{G}(\underbrace{\varrho,\ldots,\varrho}_{(n-t)-\text{times}},\underbrace{m\xi,\ldots,m\xi}_{t-\text{times}}),\varrho\right]+ \left[\sum_{t=1}^{n-1}{}^{n}C_{t}\mathcal{G}(\underbrace{\varrho,\ldots,\varrho}_{(n-t)-\text{times}},\underbrace{m\xi,\ldots,m\xi}_{t-\text{times}}),m\xi\right]\in\mathfrak{P}\ \forall\ \varrho,\xi\in\mathfrak{S}$$

and, thus,

$$mA_{1}(\varrho;\xi)+m^{2}A_{2}(\varrho;\xi)+\cdots+m^{n}A_{n}(\varrho;\xi)\in\mathfrak{P}\ \forall\ \varrho,\xi\in\mathfrak{S},$$

where $A_t(\varrho;\xi)$ represents the term in which ξ appears t-times.

The application of Lemma 1 yields

$$[g(\varrho),\xi]+n[\mathcal{G}(\varrho,\ldots,\varrho,\xi),\varrho]\in\mathfrak{P}\ \forall\ \varrho,\xi\in\mathfrak{S}. \tag{7}$$

Replacing ξ with $\xi\varrho$, we can see that

$$[g(\varrho),\xi\varrho]+n[\mathcal{G}(\varrho,\ldots,\varrho,\xi\varrho),\varrho]\in\mathfrak{P}\ \forall\ \varrho,\xi\in\mathfrak{S}.$$

After additional computation

$$\{[g(\varrho),\xi]+n[\mathcal{G}(\varrho,\ldots,\varrho,\xi),\varrho]\}\varrho+\xi[g(\varrho),\varrho]+n\xi[d(\varrho),\varrho]+n[\xi,\varrho]d(\varrho)\in\mathfrak{P}\ \forall\ \varrho,\xi\in\mathfrak{S}.$$

By using (7) and using the hypothesis, we obtain

$$n\xi[d(\varrho),\varrho]+n[\xi,\varrho]d(\varrho)\in\mathfrak{P}\ \forall\ \varrho,\xi\in\mathfrak{S}.$$

Since $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion-free, we obtain

$$\xi[d(\varrho),\varrho]+[\xi,\varrho]d(\varrho)\in\mathfrak{P}\ \forall\ \varrho,\xi\in\mathfrak{S}. \tag{8}$$

Again, replacing ξ with $z\xi$ and using (8), we obtain

$$[z,\varrho]\xi d(\varrho)\in\mathfrak{P}\ \forall\ \varrho,\xi,z\in\mathfrak{S}. \tag{9}$$

Next, we replace ϱ with $\varrho+mw_1$ for $1\leq m\leq n-1$, $w_1\in\mathfrak{S}$ to obtain

$$[z,\varrho+mw_{1}]\xi d(\varrho+mw_{1})\in\mathfrak{P}\ \forall\ \varrho,\xi,w_{1},z\in\mathfrak{S}.$$

After simplification, we find that

$$[z,\varrho]\xi d(\varrho)+[z,mw_1]\xi d(\varrho)+[z,\varrho]\xi d(mw_1)+[z,mw_1]\xi d(mw_1)+ [z,\varrho]\xi\sum_{t=1}^{n-1}{}^nC_t\mathfrak{D}(\underbrace{\varrho,\ldots,\varrho}_{(n-t)-\text{times}},\underbrace{mw_1,\ldots,mw_1}_{t-\text{times}})+ [z,mw_1]\xi\sum_{t=1}^{n-1}{}^nC_t\mathfrak{D}(\underbrace{\varrho,\ldots,\varrho}_{(n-t)-\text{times}},\underbrace{mw_1,\ldots,mw_1}_{t-\text{times}})\in\mathfrak{P}$$

$\forall\ \varrho,\xi,w_1,z\in\mathfrak{S}$. Application of (9) and Lemma 1 gives

$$[z,w_1]\xi d(\varrho)+n[z,\varrho]\xi\mathfrak{D}(\varrho,\ldots,\varrho,w_1)\in\mathfrak{P}\ \forall\ \varrho,\xi,z,w_1\in\mathfrak{S}. \tag{10}$$

Replacing ξ with $[z,\varrho]\xi$ and using (9) and torsion restriction, we obtain

$$[z,\varrho]^{2^1}\xi\mathfrak{D}(w_1,\varrho,\ldots,\varrho)\in\mathfrak{P}\ \forall\ \varrho,\xi,z,w_1\in\mathfrak{S}. \tag{11}$$

Now, replacing ϱ with $\varrho+mw_2$ in above for $1\le m\le n-1$, for all $w_2\in\mathfrak{S}$, we obtain

$$[z,\varrho+mw_2]^{2^1}\xi\mathfrak{D}(w_1,\varrho+mw_2,\ldots,\varrho+mw_2)\in\mathfrak{P}\ \forall\ \varrho,\xi,z,w_1,w_2\in\mathfrak{S}.$$

which, upon further solving, we obtain

$$mA_1(\varrho;\xi;w_1;w_2)+m^2A_2(\varrho;\xi;w_1;w_2)+\cdots+m^nA_n(\varrho;\xi;w_1;w_2)\in\mathfrak{P}\ \forall\ \varrho,\xi,w_1,w_2\in\mathfrak{S},$$

where $A_t(\varrho;\xi;w_1;w_2)$ denotes the sum of the terms in which w_2 appears t-times. Application of Lemma 1, we have

$$(n-1)[z,\varrho]^{2^1}\xi\mathfrak{D}(w_1,w_2,\varrho,\ldots,\varrho)+[z,w_2][z,\varrho]\mathfrak{D}(w_1,\varrho,\ldots,\varrho)+ [z,\varrho][z,w_2]\mathfrak{D}(w_1,\varrho,\ldots,\varrho)r[\xi,\varrho][\xi,w_2]\in\mathfrak{P}\ \forall\ \varrho,\xi,z,w_1,w_2\in\mathfrak{S}$$

For $\xi=[z,\varrho]^2\xi$ and using (11), we obtain

$$[z,\varrho]^{2^2}\xi\mathfrak{D}(w_1,w_2,\varrho,\ldots,\varrho)\in\mathfrak{P}\ \forall\ \varrho,\xi,z,w_1,w_2\in\mathfrak{S}$$

Proceeding in a similar manner, after some steps we arrive at

$$[z,\varrho]^{2^n}\xi\mathfrak{D}(w_1,w_2,\ldots,w_n)\in\mathfrak{P}\ \forall\ \varrho,\xi,z,w_i\in\mathfrak{S},\ 1\le i\le n.$$

Taking into account the prime of $\mathfrak{P}$, we either obtain $\mathfrak{D}(w_1,w_2,w_3,\ldots,w_n)\in\mathfrak{P}$ or $[z,\varrho]^{2^n}\in\mathfrak{P}$ for all $z,\varrho,w_i\in\mathfrak{S},\ 1\le i\le n$. Let us suppose $\mathfrak{D}(w_1,w_2,w_3,\ldots,w_n)\in\mathfrak{P}$. In particular, when $w_1=w_2=\cdots=w_n=w$, we obtain $d(w)\in\mathfrak{P}$ for all $w\in\mathfrak{S}$ and, hence, $d(\mathfrak{S})\subseteq\mathfrak{P}$. If for all $\varrho,z\in\mathfrak{S}$, $[z,\varrho]^{2^n}\in\mathfrak{P}$ implies $[z,\varrho]\in\mathfrak{P}$. Using Lemma 2, we conclude that $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain. □

Corollary 1. *For any fixed integer $n\ge 2$, let $\mathfrak{S}$ be a ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$, such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion-free. If $\mathfrak{S}$ admits a nonzero symmetric generalized n-derivation $\mathcal{G}:\mathfrak{S}^n\to\mathfrak{S}$ with trace $g:\mathfrak{S}\to\mathfrak{S}$ associated with a symmetric n-derivation $\mathfrak{D}:\mathfrak{S}^n\longrightarrow\mathfrak{S}$ with trace $d:\mathfrak{S}\to\mathfrak{S}$ satisfying the condition $g([\varrho,\xi])\pm[g(\varrho),\xi]\in\mathfrak{P}\ \forall\ \varrho,\xi\in\mathfrak{S}$, then one of the following holds:*

(i) $d(\mathfrak{S})\subseteq\mathfrak{P}$

(ii) $\mathfrak{S}/\mathfrak{P}$ *is a commutative integral domain.*

Theorem 3. *For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be a ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$, such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion-free. If $\mathfrak{S}$ admits a nonzero symmetric generalized n-derivation $\mathcal{G} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $g : \mathfrak{S} \to \mathfrak{S}$ associated with a symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \longrightarrow \mathfrak{S}$ with trace $\ell : \mathfrak{S} \to \mathfrak{S}$ satisfying any one of the following conditions:*

(i) $g(\varrho \circ \xi) \pm \ell(\varrho) \circ \xi \in \mathfrak{P}\ \forall\ \varrho, \xi \in \mathfrak{S}$
(ii) $g(\varrho \circ \xi) \pm [\ell(\varrho), \xi] \in \mathfrak{P}\ \forall\ \varrho, \xi \in \mathfrak{S}$
(iii) $g([\varrho, \xi]) \pm \ell(\varrho) \circ \xi \in \mathfrak{P}\ \forall\ \varrho, \xi \in \mathfrak{S}$

then we have one of the following assertions:

1. $\ell(\mathfrak{S}) \subseteq \mathfrak{P}$
2. *$\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain.*

Proof. (i) Assume that
$$g(\varrho \circ \xi) \pm \ell(\varrho) \circ \xi \in \mathfrak{P}\ \forall\ \varrho, \xi \in \mathfrak{S}.$$
On replacing ξ with $\xi + mz$, for $1 \leq m \leq n-1$, we obtain
$$g(\varrho \circ \xi + mz) \pm \ell(\varrho) \circ (\xi + mz) \in \mathfrak{P}\ \forall\ \varrho, \xi, z \in \mathfrak{S}.$$
By simplifying and applying the provided condition, we obtain the following:
$$\sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{\varrho \circ \xi, \ldots, \varrho \circ \xi}_{(n-t)-\text{times}}, \underbrace{\varrho \circ mz, \ldots, \varrho \circ mz}_{t-\text{times}}) \in \mathfrak{P}\ \forall\ \varrho, \xi, z \in \mathfrak{S}$$
which implies that
$$mA_1(\varrho, \xi, z) + m^2 A_2(\varrho, \xi, z) + \cdots + m^{n-1} A_{n-1}(\varrho, \xi, z) \in \mathfrak{P}$$
$\forall\ \varrho, \xi, z \in \mathfrak{S}$, where $A_t(\varrho, \xi, z)$ represents the term in which z appears t-times. Taking into account Lemma 1 and the torsion restriction, we find that
$$\mathcal{G}(\varrho \circ \xi, \ldots, \varrho \circ \xi, \varrho \circ z) \in \mathfrak{P}\ \forall\ \varrho, \xi, z \in \mathfrak{S}.$$
Replacing z with ξ, we obtain $g(\varrho \circ \xi) \in \mathfrak{P}$, then our hypothesis reduces to $\ell(\varrho) \circ \xi \in \mathfrak{P}$. Replace ξ with ξr to obtain $[\ell(\varrho), \xi]r + \xi(\ell(\varrho) \circ r) \in \mathfrak{P}$ and, hence, we obtain $[\ell(\varrho), \xi]r \in \mathfrak{P}$ for all $\varrho, \xi, r \in \mathfrak{S}$, i.e., $[\ell(\varrho), \xi]\mathfrak{S} \subseteq \mathfrak{P}$. Since $\mathfrak{P} \neq \mathfrak{S}$, then $[\ell(\varrho), \xi] \in \mathfrak{P}$. In particular, $[\ell(\varrho), \varrho] \in \mathfrak{P}$ for all $\varrho \in \mathfrak{S}$. Thus, by ref. [10] (Theorem 1.4), $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain or $\ell(\mathfrak{S}) \subseteq \mathfrak{P}$.

(ii) Proceeding in the same way as in (i), we conclude.

(iii) It is given that
$$g([\varrho, \xi]) \pm \ell(\varrho) \circ \xi \in \mathfrak{P}\ \forall\ \varrho, \xi \in \mathfrak{S}.$$
Replacing ξ with $\xi + mz$ for $1 \leq m \leq n-1$ and $z \in \mathfrak{S}$ in above, we obtain
$$g([\varrho, \xi + mz]) \pm \ell(\varrho) \circ (\xi + mz) \in \mathfrak{P}\ \forall\ \varrho, \xi, z \in \mathfrak{S}.$$
Using the given condition, we obtain
$$\sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{[\varrho, \xi], \ldots, [\varrho, \xi]}_{(n-t)-\text{times}}, \underbrace{[\varrho, mz], \ldots, [\varrho, mz]}_{t-\text{times}}) \in \mathfrak{P}\ \forall\ \varrho, \xi, z \in \mathfrak{S}$$
which implies that
$$mA_1(\varrho, \xi, z) + m^2 A_2(\varrho, \xi, z) + \cdots + m^{n-1} A_{n-1}(\varrho, \xi, z) \in \mathfrak{P}$$

$\forall\ \varrho, \xi, z \in \mathfrak{S}$, where $A_t(\varrho, \xi, z)$ represents the term in which z appears t-times.

In view of Lemma 1 and torsion restriction, we have

$$\mathcal{G}([\varrho, \xi], \ldots, [\varrho, \xi], [\varrho, z]) \in \mathfrak{P}\ \forall\ \varrho, \xi, z \in \mathfrak{S}.$$

Replacing z with ξ and using the given condition, we find that $\mathscr{d}(\varrho) \circ \xi \in \mathfrak{P}$ for all $\varrho, \xi \in \mathfrak{S}$. Replace ξ with ξr to obtain $[\mathscr{d}(\varrho), \xi]r + \xi(\mathscr{d}(\varrho) \circ r) \in \mathfrak{P}$ and, hence, we obtain $[\mathscr{d}(\varrho), \xi]r \in \mathfrak{P}$ for all $\varrho, \xi, r \in \mathfrak{S}$, i.e., $[\mathscr{d}(\varrho), \xi]\mathfrak{S} \subseteq \mathfrak{P}$. Since $\mathfrak{P} \neq \mathfrak{S}$, then $[\mathscr{d}(\varrho), \xi] \in \mathfrak{P}$. In particular, $[\mathscr{d}(\varrho), \varrho] \in \mathfrak{P}$ for all $\varrho \in \mathfrak{S}$. Thus, by ref. [10] (Theorem 1.4), $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain or $\mathscr{d}(\mathfrak{S}) \subseteq \mathfrak{P}$. □

Theorem 4. *For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be a ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$, such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion-free. If $\mathfrak{S}$ admits a nonzero symmetric generalized n-derivation $\mathcal{G} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $g : \mathfrak{S} \to \mathfrak{S}$ associated with a symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \longrightarrow \mathfrak{S}$ with trace $\mathscr{d} : \mathfrak{S} \to \mathfrak{S}$ satisfying the condition $g(\varrho) \circ g(\xi) \pm [g(\varrho), \xi] \pm [\varrho, g(\xi)] \in \mathfrak{P}$ for all $\varrho, \xi \in \mathfrak{S}$, then either $\mathscr{d}(\mathfrak{S}) \subseteq \mathfrak{P}$ or $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain.*

Proof. It is provided that

$$g(\varrho) \circ g(\xi) \pm [g(\varrho), \xi] \pm [\varrho, g(\xi)] \in \mathfrak{P}\ \forall\ \varrho, \xi \in \mathfrak{S}.$$

We substitute $\xi + mz$ in place of ξ for $1 \leq m \leq n-1$ to obtain

$$\begin{aligned} g(\varrho) \circ g(\xi) + g(\varrho) \circ g(mz) + g(\varrho) \circ \sum_{t=1}^{n-1} {}^nC_t\mathcal{G}(\underbrace{\xi, \ldots, \xi}_{(n-t)-\text{times}}, \underbrace{mz, \ldots, mz}_{t-\text{times}}) \\ \pm [g(\varrho), \xi] \pm [g(\varrho), mz] \pm [\varrho, g(\xi)] \pm [\varrho, g(mz)] \\ \pm [\varrho, \sum_{t=1}^{n-1} {}^nC_t\mathcal{G}(\underbrace{\xi, \ldots, \xi}_{(n-t)-\text{times}}, \underbrace{mz, \ldots, mz}_{t-\text{times}}) \in \mathfrak{P} \end{aligned}$$

$\forall\ \varrho, \xi, z \in \mathfrak{S}$. Through the utilization of the given condition, we obtain

$$g(\varrho) \circ \sum_{t=1}^{n-1} {}^nC_t\mathcal{G}(\underbrace{\xi, \ldots, \xi}_{(n-t)-\text{times}}, \underbrace{mz, \ldots, mz}_{t-\text{times}}) \pm [\varrho, \sum_{t=1}^{n-1} {}^nC_t\mathcal{G}(\underbrace{\xi, \ldots, \xi}_{(n-t)-\text{times}}, \underbrace{mz, \ldots, mz}_{t-\text{times}})] \in \mathfrak{P}$$

which implies that

$$mA_1(\varrho, \xi, z) + m^2A_2(\varrho, \xi, z) + \cdots + m^{n-1}A_{n-1}(\varrho, \xi, z) \in \mathfrak{P}$$

$\forall\ \varrho, \xi, z \in \mathfrak{S}$, where $A_t(\varrho, \xi, z)$ represents the term in which z appears t-times.

In the context of Lemma 1 and torsion restriction, we obtain

$$g(\varrho) \circ \mathcal{G}(\xi, \ldots, \xi, z) \pm [\varrho, \mathcal{G}(\xi, \ldots, \xi, z) \in \mathfrak{P}\ \forall\ \varrho, \xi, z \in \mathfrak{S}.$$

Replacing z with ξ and using the given condition, we find that $[g(\varrho), \xi] \in \mathfrak{P}$ for all $\varrho, \xi \in \mathfrak{S}$. In particular, $[g(\varrho), \varrho] \in \mathfrak{P}$ for all $\varrho \in \mathfrak{S}$. Thus, by Theorem 2, $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain or $\mathscr{d}(\mathfrak{S}) \subseteq \mathfrak{P}$. □

Corollary 2. *For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be a ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$, such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion-free. If $\mathfrak{S}$ admits a nonzero symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \longrightarrow \mathfrak{S}$ with trace $\mathscr{d} : \mathfrak{S} \to \mathfrak{S}$ satisfying the condition $\mathscr{d}(\varrho) \circ \mathscr{d}(\xi) \pm [\mathscr{d}(\varrho), \xi] \pm [\varrho, \mathscr{d}(\xi)] \in \mathfrak{P}\ \forall\ \varrho, \xi \in \mathfrak{S}$, then either $\mathscr{d}(\mathfrak{S}) \subseteq \mathfrak{P}$ or $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain.*

Theorem 5. *For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be a ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$, such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion-free. If $\mathfrak{S}$ admits a nonzero symmetric generalized n-derivation $\mathcal{G} : \mathfrak{S}^n \to \mathfrak{S}$*

with trace $g : \mathfrak{S} \to \mathfrak{S}$ associated with a symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \longrightarrow \mathfrak{S}$ with trace $d : \mathfrak{S} \to \mathfrak{S}$ satisfying the condition $g(\varrho) \circ g(\xi) \pm g(\varrho \circ \xi) \pm d(\varrho) \circ \xi \in \mathfrak{P}$ for all $\varrho, \xi \in \mathfrak{S}$, then one of the following assertions holds:

(i) $d(\mathfrak{S}) \subseteq \mathfrak{P}$

(ii) $\mathfrak{S}/\mathfrak{P}$ *is a commutative integral domain.*

Proof. Assume that

$$g(\varrho) \circ g(\xi) \pm g(\varrho \circ \xi) \pm d(\varrho) \circ \xi \in \mathfrak{P} \,\forall\, \varrho, \xi \in \mathfrak{S}.$$

We replace ξ with $\xi + mz$, for $1 \leq m \leq n-1$ to obtain

$$g(\varrho) \circ g(\xi + mz) \pm g(\varrho \circ (\xi + mz)) \pm d(\varrho) \circ (\xi + mz) \in \mathfrak{P} \,\forall\, \varrho, \xi, z \in \mathfrak{S}.$$

After simplification and employing the provided condition, we obtain

$$g(\varrho) \circ \sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{\xi, \ldots, \xi}_{(n-t)-\text{times}}, \underbrace{mz, \ldots, mz}_{t-\text{times}}) \pm \sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{\varrho \circ \xi, \ldots, \varrho \circ \xi}_{(n-t)-\text{times}}, \underbrace{\varrho \circ mz, \ldots, \varrho \circ mz}_{t-\text{times}}) \in \mathfrak{P} \,\forall\, \varrho, \xi, z \in \mathfrak{S}$$

which implies that

$$mA_1(\varrho, \xi, z) + m^2 A_2(\varrho, \xi, z) + \cdots + m^{n-1} A_{n-1}(\varrho, \xi, z) \in \mathfrak{P}$$

$\forall\, \varrho, \xi, z \in \mathfrak{S}$, where $A_t(\varrho, \xi, z)$ represents the term in which z appears t-times.

In light of Lemma 1 and torsion restriction, we obtain

$$g(\varrho) \circ \mathcal{G}(\xi, \ldots, \xi, z) \pm \mathcal{G}(\varrho \circ \xi, \ldots, \varrho \circ \xi, \varrho \circ z) \in \mathfrak{P} \,\forall\, \varrho, \xi, z \in \mathfrak{S}.$$

Replacing z with ξ and using the given condition, we find that $d(\varrho) \circ \xi \in \mathfrak{P}$ for all $\varrho, \xi \in \mathfrak{S}$. We replace ξ with ξr to obtain $[d(\varrho), \xi]r + \xi(d(\varrho) \circ r) \in \mathfrak{P}$ and, hence, we obtain $[d(\varrho), \xi]r \in \mathfrak{P}$ for all $\varrho, \xi, r \in \mathfrak{S}$, i.e., $[d(\varrho), \xi]\mathfrak{S} \subseteq \mathfrak{P}$. Since $\mathfrak{P} \neq \mathfrak{S}$, then $[d(\varrho), \xi] \in \mathfrak{P}$. In particular, $[d(\varrho), \varrho] \in \mathfrak{P}$ for all $\varrho \in \mathfrak{S}$. Thus, by ref. [10] (Theorem 1.4), $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain or $d(\mathfrak{S}) \subseteq \mathfrak{P}$. □

Theorem 6. *For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be a ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$, such that $\mathfrak{S}/\mathfrak{P}$ is $(n-1)!$-torsion-free. If $\mathfrak{S}$ admits a nonzero symmetric generalized n-derivation $\mathcal{G} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $g : \mathfrak{S} \to \mathfrak{S}$ associated with a symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \longrightarrow \mathfrak{S}$ with trace $d : \mathfrak{S} \to \mathfrak{S}$ satisfying $g(\varrho^2) \pm \varrho^2 \in \mathfrak{P} \,\forall\, \varrho \in \mathfrak{S}$, then $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain.*

Proof. Upon replacing ϱ with $\varrho + m\xi$, $\xi \in \mathfrak{S}$ for $1 \leq m \leq n-1$ in the given condition, we obtain

$$g(\varrho + m\xi)^2 \pm (\varrho + m\xi)^2 \in \mathfrak{P} \,\forall\, \varrho, \xi \in \mathfrak{S}.$$

Further solving, we have

$$g(\varrho^2) + g(m(\varrho\xi + \xi\varrho)) + \sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{\varrho^2, \ldots, \varrho^2}_{(n-t)-\text{times}}, \underbrace{m(\varrho\xi + \xi\varrho), \ldots, m(\varrho\xi + \xi\varrho)}_{t-\text{times}}) + g((m\xi)^2) + \sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{\varrho^2 + m(\varrho\xi + \xi\varrho), \ldots, \varrho^2 + m(\varrho\xi + \xi\varrho)}_{(n-t)-\text{times}}, \underbrace{(m\xi)^2, \ldots, (m\xi)^2}_{t-\text{times}}) \pm \varrho^2 \pm (m\xi)^2 \pm m(\varrho\xi + \xi\varrho) \in \mathfrak{P} \,\forall\, \varrho, \xi \in \mathfrak{S}.$$

In accordance with the given condition and Lemma 1, we obtain

$$n\mathcal{G}(\varrho^2,\ldots,\varrho^2,\varrho\xi+\xi\varrho)\pm(\varrho\xi+\xi\varrho)\in\mathfrak{P}\ \forall\ \varrho,\xi\in\mathfrak{S}.$$

Replacing ξ with ϱ, we find that

$$2ng(\varrho^2)\pm 2\varrho^2\in\mathfrak{P},$$

or

$$(2n-2)\varrho^2\in\mathfrak{P}.$$

The application of the torsion restriction gives that $\varrho^2\in\mathfrak{P}$. This implies that $\varrho\xi+\xi\varrho\in\mathfrak{P}$ for all $\varrho,\xi\in\mathfrak{S}$. Replacing ξ with ξz, we obtain $[\varrho,\xi]z\in\mathfrak{P}$. Replacing z with $r[\varrho,\xi]$, we obtain $[\varrho,\xi]r[\varrho,\xi]\in\mathfrak{P}$ for all $\varrho,\xi,r\in\mathfrak{S}$. Using the primeness of $\mathfrak{P}$, we obtain $[\varrho,\xi]\in\mathfrak{P}$ for all $\varrho,\xi\in\mathfrak{S}$. Hence, $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain by Lemma 2. □

Corollary 3. *For any fixed integer $n\geq 2$, let $\mathfrak{S}$ be any ring and P be a prime ideal of $\mathfrak{S}$, such that $\mathfrak{S}/\mathfrak{P}$ is $(n-1)!$-torsion-free. If $\mathfrak{S}$ admits a nonzero symmetric generalized n-derivation $\mathcal{G}:\mathfrak{S}^n\to\mathfrak{S}$ with trace $g:\mathfrak{S}\to\mathfrak{S}$ associated with a symmetric n-derivation $\mathfrak{D}:\mathfrak{S}^n\longrightarrow\mathfrak{S}$ with trace $\ell:\mathfrak{S}\to\mathfrak{S}$ satisfying $g(\varrho\circ\xi)\pm\varrho\circ\xi\in\mathfrak{P}\ \forall\ \varrho,\xi\in\mathfrak{S}$, then $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain.*

Corollary 4. *For any fixed integer $n\geq 2$, let $\mathfrak{S}$ be an $n!$-torsion-free semiprime ring. If $\mathfrak{S}$ admits a nonzero symmetric n-derivation $\mathfrak{D}:\mathfrak{S}^n\to\mathfrak{S}$ with trace $\ell:\mathfrak{S}\to\mathfrak{S}$ satisfying $\ell(\varrho\circ\xi)\pm\varrho\circ\xi=0,\ \forall\ \varrho,\xi\in\mathfrak{S}$, then $\mathfrak{S}$ is a commutative.*

Theorem 7. *For any fixed integer $n\geq 2$, let $\mathfrak{S}$ be a ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$, such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion-free. If $\mathfrak{S}$ admits a nonzero symmetric generalized n-derivation $\mathcal{G}:\mathfrak{S}^n\to\mathfrak{S}$ with trace $g:\mathfrak{S}\to\mathfrak{S}$, associated with a symmetric n-derivation $\mathfrak{D}:\mathfrak{S}^n\longrightarrow\mathfrak{S}$ with trace $\ell:\mathfrak{S}\to\mathfrak{S}$ satisfying one of the following conditions:*

(i) $[g(\varrho),g(\xi)]-[\varrho,\xi]\in\mathfrak{P}\ \forall\ \varrho,\xi\in\mathfrak{S}$
(ii) $[g(\varrho),g(\xi)]-[\xi,\varrho]\in\mathfrak{P}\ \forall\ \varrho,\xi\in\mathfrak{S}$.

then $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain.

Proof. (i) Given that

$$[g(\varrho),g(\xi)]-[\varrho,\xi]\in\mathfrak{P}\ \forall\ \varrho,\xi\in\mathfrak{S}. \tag{12}$$

Consider a positive integer m; $1\leq m\leq n-1$. Replacing ξ with $\xi+mz$, where $z\in\mathfrak{S}$ in (12), we obtain

$$[g(\varrho),g(\xi+mz)]-[\varrho,\xi+mz]\in\mathfrak{P}\ \forall\ \varrho,\xi,z\in\mathfrak{S}.$$

Upon further solving, we obtain

$$[g(\varrho),g(\xi)]+[g(\varrho),g(mz)]+[g(\varrho),\sum_{t=1}^{n-1}{}^{n}C_t\mathcal{G}(\underbrace{\xi,\ldots,\xi}_{(n-t)-\text{times}},\underbrace{mz,\ldots,mz}_{t-\text{times}})]-$$
$$[\varrho,\xi]-[\varrho,mz]\in\mathfrak{P}\ \forall\ \varrho,\xi,z\in\mathfrak{S}.$$

Taking into account the hypothesis, we see that

$$mA_1(\varrho,\xi,z)+m^2A_2(\varrho,\xi,z)+\cdots+m^{n-1}A_{n-1}(\varrho,\xi,z)\in\mathfrak{P}$$

where $A_t(\varrho,\xi,z)$ represents the term in which z appears t-times.
Using Lemma 1 and torsion restriction, we have

$$[g(\varrho),\mathcal{G}(\xi,\ldots,\xi,z)]\in\mathfrak{P}\ \forall\ \varrho,\xi,z\in\mathfrak{S}.$$

In particular, for $z = \xi$, we obtain

$$[g(\varrho), g(\xi)] \in \mathfrak{P} \,\forall\, \varrho, \xi \in \mathfrak{S}.$$

Now, using the given condition, we find that

$$[\varrho, \xi] \in \mathfrak{P} \,\forall\, \varrho, \xi \in \mathfrak{S}.$$

From Lemma 2, $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain.

(ii) It follows from the first implication with a slight modification.
□

The following are very interesting observations derived from Theorem 7.

Corollary 5. *For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be a ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$, such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion-free. If $\mathfrak{S}$ admits a nonzero symmetric generalized n-derivation $\mathcal{G} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $g : \mathfrak{S} \to \mathfrak{S}$, associated with a symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \longrightarrow \mathfrak{S}$ with trace $\mathscr{d} : \mathfrak{S} \to \mathfrak{S}$ satisfying one of the following conditions:*

(i) $g(\varrho)g(\xi) \pm \varrho\xi \in \mathfrak{P} \,\forall\, \varrho, \xi \in \mathfrak{S}$
(ii) $g(\varrho)g(\xi) \pm \xi\varrho \in \mathfrak{P} \,\forall\, \varrho, \xi \in \mathfrak{S}$.

then $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain.

Corollary 6. *For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be an $n!$-torsion-free semiprime ring and $\mathcal{G} : \mathfrak{S}^n \to \mathfrak{S}$ be a nonzero symmetric generalized n-derivation with trace $g : \mathfrak{S} \to \mathfrak{S}$ associated with a symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \longrightarrow \mathfrak{S}$ with trace $\mathscr{d} : \mathfrak{S} \to \mathfrak{S}$ satisfying one of the following conditions:*

(i) $[g(\varrho), g(\xi)] = [\varrho, \xi] \,\forall\, \varrho, \xi \in \mathfrak{S}$
(ii) $[g(\varrho), g(\xi)] = [\xi, \varrho] \,\forall\, \varrho, \xi \in \mathfrak{S}$.

then $\mathfrak{S}$ is commutative.

Proof. (i) Let us assume that

$$[g(\varrho), g(\xi)] + [\varrho, \xi] = 0 \,\forall\, \varrho, \xi \in \mathfrak{S}.$$

According to semi-primeness, there exists a family Γ of prime ideals $\mathfrak{P}$, such that $\bigcap_{\mathfrak{P} \in \Gamma} \mathfrak{P} = (0)$, thereby obtaining $[g(\varrho), g(\xi)] + [\varrho, \xi] \in \mathfrak{P}$ for all $\mathfrak{P} \in \Gamma$. Invoking the previous theorem, we conclude that $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain. Therefore, for all $\varrho, \xi \in \mathfrak{S}$, we have $[\varrho, \xi] \in \mathfrak{P}$ and since $\bigcap_{\mathfrak{P} \in \Gamma} \mathfrak{P} = (0)$. This implies that $[\varrho, \xi] = 0$. Hence, $\mathfrak{S}$ is commutative.

(ii) Similarly, if $[g(\varrho), g(\xi)] - [\xi, \varrho] = 0$ for all $\varrho, \xi \in \mathfrak{S}$, then the same reasoning proves the required result.
□

Corollary 7. *For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be an $n!$-torsion-free semiprime ring and $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ be a n-derivation of $\mathfrak{S}$ with trace $\mathscr{d} : \mathfrak{S} \to \mathfrak{S}$ satisfying any one of the following conditions:*

(i) $[\mathscr{d}(\varrho), \mathscr{d}(\xi)] = [\varrho, \xi] \,\forall\, \varrho, \xi \in \mathfrak{S}$
(ii) $[\mathscr{d}(\varrho), \mathscr{d}(\xi)] = [\xi, \varrho] \,\forall\, \varrho, \xi \in \mathfrak{S}$

then $\mathfrak{S}$ is commutative.

Theorem 8. *For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be a ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$, such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion-free. If $\mathfrak{S}$ admits a nonzero symmetric generalized n-derivation $\mathcal{G} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $g : \mathfrak{S} \to \mathfrak{S}$ associated with a symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \longrightarrow \mathfrak{S}$ with trace*

$d : \mathfrak{S} \to \mathfrak{S}$ satisfying the condition $g(\varrho \circ \xi) \pm [\varrho, \xi] \in \mathfrak{P}\ \forall\ \varrho, \xi \in \mathfrak{S}$, then $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain.

Proof. (i) Replacing ξ with $\xi + mz$ for $1 \leq m \leq n-1$, $z \in \mathfrak{S}$ in the given condition, we obtain

$$g(\varrho \circ (\xi + mz)) \pm [\varrho, \xi + mz] \in \mathfrak{P}\ \forall\ \varrho, \xi, z \in \mathfrak{S}.$$

Upon further solving and using the specified condition, we obtain

$$\sum_{t=1}^{n-1} {}^{n}C_t \mathcal{G}(\underbrace{\varrho \circ \xi, \ldots, \varrho \circ \xi}_{(n-t)-\text{times}}, \underbrace{\varrho \circ mz, \ldots, \varrho \circ mz}_{t-\text{times}}) \in \mathfrak{P}\ \forall\ \varrho, \xi, z \in \mathfrak{S}$$

which implies that

$$mA_1(\varrho, \xi, z) + m^2 A_2(\varrho, \xi, z) + \cdots + m^{n-1} A_{n-1}(\varrho, \xi, z) \in \mathfrak{P}$$

$\forall\ \varrho, \xi, z \in \mathfrak{S}$, where $A_t(\varrho, \xi, z)$ represents the term in which z appears t-times. Using Lemma 1 and using the fact that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion-free, we obtain

$$\mathcal{G}(\varrho \circ \xi, \ldots, \varrho \circ \xi, \varrho \circ z) \in \mathfrak{P}\ \forall\ \varrho, \xi, z \in \mathfrak{S}. \tag{13}$$

For $z = \xi$, we obtain $g(\varrho \circ \xi) \in \mathfrak{P}$ then our hypothesis reduces to $[\varrho, \xi] \in \mathfrak{P}$. Using Lemma 2, we obtain that $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain. □

Corollary 8. *For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be a ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$, such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion-free. If $\mathfrak{S}$ admits a nonzero symmetric n-derivation with trace $d : \mathfrak{S} \to \mathfrak{S}$. If d satisfies the condition $d(\varrho \circ \xi) \pm [\varrho, \xi] \in \mathfrak{P}\ \forall\ \varrho, \xi \in \mathfrak{S}$, then $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain.*

Corollary 9 ([22] Theorem 3.1)**.** *For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be an $n!$-torsion-free semiprime ring. If $\mathfrak{S}$ admits a nonzero symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $d : \mathfrak{S} \to \mathfrak{S}$ satisfying $d(\varrho \circ \xi) = [\varrho, \xi]$, $\forall\ \varrho, \xi \in \mathfrak{S}$, then $\mathfrak{S}$ is commutative.*

Theorem 9. *For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be a ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$, such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion-free. If $\mathfrak{S}$ admits a nonzero symmetric generalized n-derivation $\mathcal{G} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $g : \mathfrak{S} \to \mathfrak{S}$ associated with a symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $d : \mathfrak{S} \to \mathfrak{S}$ satisfying one of the following conditions:*

(i) $g([\varrho, \xi]) \pm g(\varrho) \pm [\varrho, \xi] \in \mathfrak{P}\ \forall\ \varrho, \xi \in \mathfrak{S}$
(ii) $g([\varrho, \xi]) \pm g(\xi) \pm [\varrho, \xi] \in \mathfrak{P}\ \forall\ \varrho, \xi \in \mathfrak{S}$

then $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain.

Proof. (i) Given that

$$g([\varrho, \xi]) \pm g(\varrho) \pm [\varrho, \xi] \in \mathfrak{P}\ \forall\ \varrho, \xi \in \mathfrak{S}.$$

Replacing ϱ with $\varrho + mz$, where $z \in \mathfrak{S}$ and $1 \leq m \leq n-1$ in the given condition, we obtain

$$g([\varrho + mz, \xi]) \pm g(\varrho + mz) \pm [\varrho + mz, \xi] \in \mathfrak{P}\ \forall\ \varrho, \xi \in \mathfrak{S}$$

which upon solving and using the hypothesis, we obtain

$$\sum_{t=1}^{n-1} {}^nC_t\mathcal{G}(\underbrace{[\varrho,\xi],\ldots,[\varrho,\xi]}_{(n-t)-\text{times}},\underbrace{[mz,\xi],\ldots,[mz,\xi]}_{t-\text{times}})$$
$$\pm\sum_{t=1}^{n-1} {}^nC_t\mathcal{G}(\underbrace{\varrho,\ldots,\varrho}_{(n-t)-\text{times}},\underbrace{mz,\ldots,mz}_{t-\text{times}})\in\mathfrak{P}\ \forall\ \varrho,\xi,z\in\mathfrak{S}$$

which implies that

$$mA_1(\varrho,\xi,z)+m^2A_2(\varrho,\xi,z)+\cdots+m^{n-1}A_{n-1}(\varrho,\xi,z)\in\mathfrak{P}$$

$\forall\ \varrho,\xi,z\in\mathfrak{S}$, where $A_t(\varrho,\xi,z)$ represents the term in which z appears t-times. Making use of Lemma 1 and torsion restriction, we see that

$$\mathcal{G}([\varrho,\xi],\ldots,[\varrho,\xi],[z,\xi])\pm\mathcal{G}(\varrho,\ldots,\varrho,z)\in\mathfrak{P}\ \forall\ \varrho,\xi,z\in\mathfrak{S}.$$

Replace z with ϱ to obtain

$$g([\varrho,\xi])\pm g(\varrho)\in\mathfrak{P}\ \forall\ \varrho,\xi,z\in\mathfrak{S}.$$

Hence, by using the given condition, we find that $[\varrho,\xi]\in\mathfrak{P}$. Taking into account Lemma 2, we conclude that $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain.

(ii) It follows from the first implication with a slight modification.

□

Corollary 10. *For any fixed integer $n\geq 2$, let $\mathfrak{S}$ be a ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$, such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion-free. If $\mathfrak{S}$ admits a nonzero symmetric n-derivation $\mathfrak{D}:\mathfrak{S}^n\to\mathfrak{S}$ with trace $\mathscr{d}:\mathfrak{S}\to\mathfrak{S}$ satisfying one of the following conditions:*

(i) $\mathscr{d}([\varrho,\xi])\pm\mathscr{d}(\varrho)\pm[\varrho,\xi]\in\mathfrak{P}\ \forall\ \varrho,\xi\in\mathfrak{S}$

(ii) $\mathscr{d}([\varrho,\xi])\pm\mathscr{d}(\xi)\pm[\varrho,\xi]\in\mathfrak{P}\ \forall\ \varrho,\xi\in\mathfrak{S}$

then $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain.

Theorem 10. *For any fixed integer $n\geq 2$, let $\mathfrak{S}$ be a ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$, such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion-free. If $\mathfrak{S}$ admits a nonzero symmetric generalized n-derivation $\mathcal{G}:\mathfrak{S}^n\to\mathfrak{S}$ with trace $g:\mathfrak{S}\to\mathfrak{S}$ associated with a symmetric n-derivation $\mathfrak{D}:\mathfrak{S}^n\to\mathfrak{S}$ with trace $\mathscr{d}:\mathfrak{S}\to\mathfrak{S}$ satisfying one of the following conditions:*

(i) $g(\varrho)\circ g(\xi)\pm\varrho\circ\xi\in\mathfrak{P}\ \forall\ \varrho,\xi\in\mathfrak{S}$

(ii) $g(\varrho)\circ g(\xi)\pm[\varrho,\xi]\in\mathfrak{P}\ \forall\ \varrho,\xi\in\mathfrak{S}$

then $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain.

Proof. (i) We assume that

$$g(\varrho)\circ g(\xi)\pm\varrho\circ\xi\in\mathfrak{P}\ \forall\ \varrho,\xi\in\mathfrak{S}.$$

Replacing ξ with $\xi+mz$, where $z\in\mathfrak{S}$ and $1\leq m\leq n-1$ in the given condition, we obtain

$$g(\varrho)\circ g(\xi+mz)\pm\varrho\circ(\xi+mz)\in\mathfrak{P}\ \forall\ \varrho,\xi,z\in\mathfrak{S}$$

which upon solving and using the hypothesis, we have

$$g(\varrho)\circ\sum_{t=1}^{n-1} {}^nC_t\mathfrak{D}(\underbrace{\xi,\ldots,\xi}_{(n-t)-\text{times}},\underbrace{mz,\ldots,mz}_{t-\text{times}})\in\mathfrak{P}\ \forall\ \varrho,\xi,z\in\mathfrak{S}$$

which implies that

$$mA_1(\varrho,\xi,z)+m^2A_2(\varrho,\xi,z)+\cdots+m^{n-1}A_{n-1}(\varrho,\xi,z)\in\mathfrak{P}$$

$\forall\ \varrho,\xi,z\in\mathfrak{S}$, where $A_t(\varrho,\xi,z)$ represents the term in which z appears t-times. Making use of Lemma 1, we see that

$$n(g(\varrho)\circ\mathcal{G}(\xi,\ldots,\xi,z))\in\mathfrak{P}\ \forall\ \varrho,\xi,z\in\mathfrak{S}.$$

Since $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion-free, we obtain

$$g(\varrho)\circ\mathcal{G}(\xi,\ldots,\xi,z)\in\mathfrak{P}\ \forall\ \varrho,\xi,z\in\mathfrak{S}.$$

In particular, $z=\xi$, we obtain

$$g(\varrho)\circ g(\xi)\in\mathfrak{P}\ \forall\ \varrho,\xi\in\mathfrak{S}.$$

Hence, by using the given condition, we find that $\varrho\circ\xi\in\mathfrak{P}$. Taking into account Lemma 2, we conclude that $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain.

(ii) Proceeding in the same way as in (i), we conclude.

□

Corollary 11. *For any fixed integer $n\geq 2$, let $\mathfrak{S}$ be an $n!$-torsion-free semiprime ring and $\mathfrak{D}:\mathfrak{S}^n\to\mathfrak{S}$ be the n-derivation of $\mathfrak{S}$ with trace $d:\mathfrak{S}\to\mathfrak{S}$. If d satisfies any one of the following:*

(i) $d(\varrho)\circ d(\xi)\pm\varrho\circ\xi=0\ \forall\ \varrho,\xi\in\mathfrak{S}$

(ii) $d(\varrho)\circ d(\xi)\pm[\varrho,\xi]=0\ \forall\ \varrho,\xi\in\mathfrak{S}$

then $\mathfrak{S}$ is commutative.

Theorem 11. *For any fixed integer $n\geq 2$, let $\mathfrak{S}$ be a ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$, such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion-free. If $\mathfrak{S}$ admits a nonzero symmetric n-derivation $\mathfrak{D}:\mathfrak{S}^n\longrightarrow\mathfrak{S}$ with trace $d:\mathfrak{S}\longrightarrow\mathfrak{S}$ satisfying $d(\varrho)d(\xi)-\varrho\circ\xi\in\mathfrak{P}\ \forall\ \varrho,\xi\in\mathfrak{S}$ then either $d(\mathfrak{S})\subseteq\mathfrak{P}$ or $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain.*

Proof. Replacing ξ with $\xi+mz$ for $1\leq m\leq n-1$, $z\in\mathfrak{S}$ in the hypothesis, we obtain

$$d(\varrho)d(\xi+mz)-\varrho\circ(\xi+mz)\in\mathfrak{P}\ \forall\ \varrho,\xi,z\in\mathfrak{S}.$$

After simplification, it becomes

$$d(\varrho)\sum_{t=1}^{n-1}{}^nC_t\mathfrak{D}(\underbrace{\xi,\ldots,\xi}_{(n-t)-\text{times}},\underbrace{mz,\ldots,mz}_{t-\text{times}})\in\mathfrak{P}\ \forall\ \varrho,\xi,z\in\mathfrak{S}$$

which implies that

$$mA_1(\varrho,\xi,z)+m^2A_2(\varrho,\xi,z)+\cdots+m^{n-1}A_{n-1}(\varrho,\xi,z)\in\mathfrak{P}$$

$\forall\ \varrho,\xi,z\in\mathfrak{S}$ where $A_t(\varrho,\xi,z)$ represents the term in which z appears t-times.

Using Lemma 1 and the fact that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion-free, we have

$$d(\varrho)\mathfrak{D}(\xi,\ldots,\xi,z)\in\mathfrak{P}\ \forall\ \varrho,\xi,z\in\mathfrak{S}. \tag{14}$$

Replace z with zr in the above relation and using the above relation, we have

$$d(\varrho)z\mathfrak{D}(\xi,\ldots,\xi,r)\in\mathfrak{P}\ \forall\ \varrho,\xi,z,r\in\mathfrak{S}. \tag{15}$$

Also,
$$z\mathscr{d}(\varrho)\mathfrak{D}(\xi,\ldots,\xi,r) \in \mathfrak{P}\ \forall\ \varrho,\xi,z,r \in \mathfrak{S}. \tag{16}$$
Using (15) and (16), we obtain
$$[\mathscr{d}(\varrho),z]\mathfrak{D}(\xi,\ldots,\xi,r) \in \mathfrak{P}\ \forall\ \varrho,\xi,r \in \mathfrak{S}.$$
By writing $r\xi$ instead of r, we obtain
$$[\mathscr{d}(\varrho),z]r\mathscr{d}(\xi) \in \mathfrak{P}.$$
In particular for $z=\varrho$,
$$[\mathscr{d}(\varrho),\varrho]r\mathscr{d}(\xi) \in \mathfrak{P}.$$
Since $\mathfrak{P}$ is prime, it follows that either $[\mathscr{d}(\varrho),\varrho] \in \mathfrak{P}$ or $\mathscr{d}(\xi) \in \mathfrak{P}$ for all $\varrho,\xi \in \mathfrak{S}$. Using Lemma 3, we conclude that $\mathscr{d}(\mathfrak{S}) \subseteq \mathfrak{P}$ or $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain. □

Corollary 12 ([22] Theorem 2.11)**.** *For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be an $n!$-torsion-free semiprime ring. If $\mathfrak{S}$ admits a nonzero symmetric n-derivation $\mathfrak{D}:\mathfrak{S}^n \to \mathfrak{S}$ with trace $\mathscr{d}:\mathfrak{S}\to\mathfrak{S}$ satisfying $\mathscr{d}(\varrho)\mathscr{d}(\xi) = \varrho\circ\xi$, $\forall\ \varrho,\xi \in \mathfrak{S}$, then $\mathscr{d}$ commutes on $\mathfrak{S}$.*

The following example shows that the "primeness of $\mathfrak{P}$" condition in Theorems 2–5 cannot be omitted.

Example 1. *Consider the ring $\mathfrak{S} = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \mid a,b,c \in \mathbb{Z} \right\}$. Let $\mathfrak{P} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$ be an ideal of $\mathfrak{S}$. Let us define $\mathcal{G} = \mathfrak{D}:\mathfrak{S}^n \to \mathfrak{S}$ with $\mathcal{G}:(A_1,A_2,\ldots,A_n) = \left\{ \begin{bmatrix} 0 & 0 & a_1a_2\cdots a_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid a,b,c \in \mathbb{Z} \right\}$ with trace $g = \mathscr{d}:\mathfrak{S}\to\mathfrak{S}$ define by $g\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & a^n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. One can easily see that $\mathcal{G}$ is a symmetric generalized n-derivation and $[g(\varrho),\varrho] \in \mathfrak{P}$, $g(\varrho\circ\xi) \pm \mathscr{d}(\varrho)\circ\xi \in \mathfrak{P}$, $g(\varrho\circ\xi) \pm [\mathscr{d}(\varrho),\xi] \in \mathfrak{P}$, $g([\varrho,\xi]) \pm \mathscr{d}(\varrho)\circ\xi \in \mathfrak{P}$, $g(\varrho)\circ g(\xi) \pm [g(\varrho),\xi] \pm [\varrho,g(\xi)] \in \mathfrak{P}$ and $g(\varrho)\circ g(\xi) \pm g(\varrho\circ\xi) \pm \mathscr{d}(\varrho)\circ\xi \in \mathfrak{P}$. However, $g(\mathfrak{S}) \nsubseteq \mathfrak{P}$ and $\mathfrak{S}/\mathfrak{P}$ is noncommutative. Also, we observe that $\mathfrak{P}$ is not a prime ideal of $\mathfrak{S}$ as $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathfrak{S} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \subseteq \mathfrak{P}$, but $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \notin \mathfrak{P}$ and $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \notin \mathfrak{P}$.*

3. Applications

A Banach algebra is a linear associate algebra which, as a vector space, is a Banach space with norm $||.||$ satisfying the multiplicative inequality; $||xy|| \leq ||x||||y||$ for all x and y in $\mathcal{A}$. An involution on an algebra $\mathcal{A}$ is a linear map $x \mapsto x^*$ of $\mathcal{A}$ into itself, such that the following conditions hold: (i) $(xy)^* = y^*x^*$, (ii) $(x^*)^* = x$, and (iii) $(x+\lambda y)^* = x^* + \bar{\lambda}y^*$ for all $x,y \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, the field of complex numbers, where $\bar{\lambda}$ is the conjugate of λ. An algebra equipped with an involution is called an $*$-algebra or algebra with involution. A Banach $*$-algebra is a Banach algebra $\mathcal{A}$ together with an isometric involution $||x^*|| = ||x||$ for all $x \in \mathcal{A}$. A C^*-algebra $\mathcal{A}$ is a Banach $*$-algebra with the additional norm condition $||x^*x|| = ||x||^2$ for all $x \in \mathcal{A}$. A subset $\mathcal{I}$ of $\mathcal{A}$ is a closed ideal if it is an ideal and it is closed in the topology induced by the norm on $\mathcal{A}$ [23]. If $\mathcal{I}$ is a closed ideal of the C^*-algebra $\mathcal{A}$, then the quotient $\mathcal{A}/\mathcal{I}$ is a C^*-algebra under its usual operations and the quotient norm [24].

In this section, we explore the applications of our results proven in Section 2, in the context of functional analysis, particularly in C^*-algebras. In fact, we elucidate the structures of C^*-algebras through the utilization of symmetric n-derivations and generalized symmetric n-derivations. We begin with the following result:

Theorem 12. *For any fixed integer $n \geq 2$, let $\mathcal{A}$ be a C^*-algebra and $\mathcal{P}$ be a closed prime ideal of $\mathcal{A}$. If $\mathcal{A}$ admits a nonzero symmetric generalized n-derivation $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ with trace $g : \mathcal{A} \to \mathcal{A}$ associated with a symmetric n-derivation $\mathfrak{D} : \mathcal{A}^n \longrightarrow \mathcal{A}$ with trace $\mathscr{d} : \mathcal{A} \to \mathcal{A}$ satisfying the condition $[g(\varrho), \varrho] \in \mathcal{P}\ \forall\ \varrho \in \mathcal{A}$, then one of the following holds:*

1. $\mathscr{d}(\mathcal{A}) \subseteq \mathcal{P}$
2. $\mathcal{A}/\mathcal{P}$ *is commutative.*

Proof. Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{P}$ be a closed prime ideal of $\mathcal{A}$. We are given that $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ is a symmetric generalized n-derivation with trace $g : \mathcal{A} \to \mathcal{A}$ associated with a symmetric n-derivation $\mathfrak{D} : \mathcal{A}^n \longrightarrow \mathcal{A}$ with trace $\mathscr{d} : \mathcal{A} \to \mathcal{A}$, such that $[g(\varrho), \varrho] \in \mathcal{P}$ $\forall\ \varrho \in \mathcal{A}$. It is well-known that every closed prime ideal is indeed a prime ideal and, hence, the application of Theorem 2 yields the required result. □

Corollary 13. *For any fixed integer $n \geq 2$, let $\mathcal{A}$ be a C^*-algebra. If $\mathcal{A}$ admits a nonzero symmetric n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ with trace $\mathscr{d} : \mathcal{A} \to \mathcal{A}$ satisfying $\mathscr{d}(\varrho \circ \xi) \pm \varrho \circ \xi = 0$, $\forall\ \varrho, \xi \in \mathcal{A}$, then $\mathcal{A}$ is commutative.*

Proof. As a consequence of Corollary 4, and using the fact that every C^*-algebra is a semiprime ring, we conclude. □

Corollary 14. *For any fixed integer $n \geq 2$, let $\mathcal{A}$ be a C^*-algebra. If $\mathcal{A}$ admits a nonzero symmetric n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ with trace $\mathscr{d} : \mathcal{A} \to \mathcal{A}$ satisfying $\mathscr{d}(\varrho)\mathscr{d}(\xi) = \varrho \circ \xi$, $\forall\ \varrho, \xi \in \mathcal{A}$, then $\mathcal{A}$ is commutative.*

Proof. Direct application of Corollary 12, yields the required result. □

4. Conclusions

In conclusion, this paper has successfully achieved its primary objective, which was to investigate symmetric additive mappings and explore their applications in various mathematical contexts. This research has effectively established a significant connection between the structural properties of quotient rings and the behavior of traces of symmetric generalized n-derivations fulfilling certain algebraic identities involving prime ideals of an arbitrary ring $\mathfrak{S}$. The bridging of concepts from rings to functional analysis, particularly to C^*-algebras, has allowed us to provide a detailed description of the structural aspects of C^*-algebras through the lens of symmetric n-derivations. Through a comprehensive analysis and rigorous investigation, this study provided persuasive and substantial evidence of the interplay between these fundamental algebraic concepts.

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