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Log-Extended Exponential–Geometric Distribution: Moments and Inference Based on Generalized Order Statistics

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Abstract: In this paper, we provide explicit expressions as well as recurrence relations for the single and product moments of generalized order statistics from the log-extended exponential–geometric distribution. These relations are utilized to discuss the special cases of generalized order statistics, and some numerical computations are carried out. Further, we use these results to obtain the best linear unbiased estimators for the location and scale parameters based on progressively Type-II right censored samples. Finally, a real data application is presented.

Keywords: best linear unbiased estimators; generalized order statistics; log-extended exponential–geometric distribution; product moments; recurrence relations; single moments

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Citation: AL-Zaydi, A.M. Log-Extended Exponential–Geometric Distribution: Moments and Inference Based on Generalized Order Statistics. *Symmetry* **2023**, *15*, 1857. https://doi.org/10.3390/sym15101857

Academic Editors: Palle E.T. Jorgensen and Sergei D. Odintsov

Received: 25 August 2023
Revised: 14 September 2023
Accepted: 27 September 2023
Published: 2 October 2023

1. Introduction

The extended exponential–geometric (EEG) distribution was proposed by [1] to analyze lifetime data. If the Y random variable has the EEG distribution, then its probability density function (PDF) is given by

$$f(y) = \frac{\alpha(1+\beta)e^{-\alpha y}}{(1+\beta e^{-\alpha y})^2},$$

where $y > 0$, $\alpha > 0$ and $\beta > -1$. This distribution provides decreasing or increasing hazard functions based on the values of its parameters, providing great flexibility of fit for real applications.

The importance of models with bounded support, particularly at unit intervals, is well established for modeling rates, indices, proportions, scores, etc. A wide range of distributions have been introduced and used in numerous different fields. The log-extended exponential–geometric (LEEG) distribution was introduced by [2], applying the transformation $X = e^{-Y}$, where the random variable Y follows the EEG distribution. The LEEG distribution is an alternative to the beta distribution and has applications in the populations of cities, the sizes of power outages, and the intensities of earthquakes. Ref. [2] investigates some of the mathematical features of the LEEG distribution, such as moments and order statistics. Ref. [3] discussed the Fisher information matrix of the LEEG distribution based on simple random sample, ranked set sample, median ranked set sample, and extreme ranked set sample. Ref. [4] estimated the LEEG distribution parameters under simple random sampling and moving extremes ranked set sampling.

The PDF of the random variable X following the LEEG distribution is of the form

$$f(x) = \frac{\alpha(1+\beta)x^{\alpha-1}}{(1+\beta x^{\alpha})^2}, \tag{1}$$

and the corresponding cumulative distribution function (CDF) is

$$F(x) = \frac{(1+\beta)x^{\alpha}}{1+\beta x^{\alpha}}, \tag{2}$$

where $0 < x < 1$, $\alpha > 0$, $\beta > -1$. It is worth noting that the specific situation $\beta = 0$ corresponds to the power function, which includes the uniform distribution for $\alpha = 1$. The PDF of the LEEG distribution is flexible and has a variety of asymmetric shapes, which make it attractive for statistical purposes. For more details on the LEEG distribution and its applications, see [2,4–6].

The j-th moment of the LEEG distribution in Equation (1) is given as (see [2])

$$E[X^j] = 1 - \frac{(1+\beta)j}{\alpha}\Phi(-\beta, 1, 1+\frac{j}{\alpha}),\ j = 1, 2, \ldots, \tag{3}$$

where $\Phi(z, s, a)$ is the Lerch transcendent function of the form $\Phi(z, s, a) = \sum_{i=0}^{\infty} \frac{z^i}{(i+a)^s}$, which converges for any real number $a > 0$ if z and s are any complex numbers with either $|z| < 1$ or $|z| = 1$ and $Re(s) > 1$, see [7].

It is easy to see that (1) and (2) satisfy the following relationship:

$$\bar{F}(x) = \frac{x^{1-\alpha} + (\beta-1)x - \beta x^{1+\alpha}}{\alpha(1+\beta)} f(x), \tag{4}$$

where $\bar{F}(x) = 1 - F(x)$.

The generalized order statistics (GOS) model is a model that contains many models of ordered random variables by selecting the parameters appropriately. Ref. [8] introduced the concept of GOS as follows.

Let $n \in N$, $k \geq 1$, $\tilde{m} = (m_1, m_2, \cdots, m_{n-1}) \in \Re^{n-1}$, $M_i = \sum_{\iota=i}^{n-1} m_\iota$, $1 \leq i \leq n-1$, be the parameters such that $\gamma_i = k + n - i + M_i > 0$. Then, $X_{1:n,\tilde{m},k}, X_{2:n,\tilde{m},k}, \cdots, X_{n:n,\tilde{m},k}$ are the n GOS, and their joint PDF is of the form

$$f_{1,2,\cdots,n:n,\tilde{m},k}(x_1, x_2, \cdots, x_n) = k\left(\prod_{\nu=1}^{n-1}\gamma_\nu\right)\left(\prod_{\nu=1}^{n-1}[\bar{F}(x_\nu)]^{m_\nu} f(x_\nu)\right)[\bar{F}(x_n)]^{k-1} f(x_n), \tag{5}$$

where $F^{-1}(0) < x_1 \leq \cdots \leq x_n < F^{-1}(1)$.

Many models of ordered random variables can be viewed as special instances of GOS. If $m_i = 0$, $i = 1, \cdots, n-1$, and $k = 1$ such that $\gamma_i = n - i + 1$, then the model reduces to ordinary order statistics. For $m_i = -1$, $i = 1, \cdots, n-1$, and $k \in N$, such that $\gamma_i = k$, the model reduces to the k-th record values. Also, for $m_i = r_i$, $n = m_0 + \sum_{\nu=1}^{m_0} r_\nu$, $r_\nu \in N$, and $\gamma_i = n - \sum_{\nu=1}^{i-1} r_\nu - i + 1$, $1 \leq i \leq m_0$, where m_0 denotes the fixed number of failure of units to be observed, the model reduces to progressive Type-II censored order statistics. Here, we consider the following two cases:

Case 1: When $\gamma_i \neq \gamma_j$, $i, j = 1, \cdots, n-1$, $i \neq j$, the PDF of $X_{t:n,\tilde{m},k}$ is given by (Kamps and Cramer [9])

$$f_{t:n,\tilde{m},k}(x) = C_{t-1}\sum_{i=1}^{t} a_{i,t}[\bar{F}(x)]^{\gamma_i - 1} f(x), \tag{6}$$

and the joint PDF of $X_{t:n,\tilde{m},k}$ and $X_{s:n,\tilde{m},k}$, $1 \leq t < s \leq n$, is given by [9]

$$\begin{aligned} f_{t,s:n,\tilde{m},k}(x,y) &= C_{s-1}\left[\sum_{i=1}^{t} a_{i,t}[\bar{F}(x)]^{\gamma_i}\right]\left[\sum_{j=t+1}^{s} a_{j,s}^{(t)}\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_j}\right] \\ &\quad \times \frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)},\ x < y, \end{aligned} \tag{7}$$

where $C_{t-1} = \prod_{i=1}^{t}\gamma_i$, $a_{i,t} = \prod_{\substack{\iota=1\\ \iota\neq i}}^{t}\frac{1}{\gamma_\iota - \gamma_i}$, $1 \leq i \leq t \leq n$, and $a_{j,s}^{(t)} = \prod_{\substack{\jmath=t+1\\ \jmath\neq j}}^{s}\frac{1}{\gamma_\jmath - \gamma_j}$, $t+1 \leq j \leq s \leq n$.

Case 2: When $m_i = m$, $i = 1, 2, \cdots, n-1$. In this case, the PDF of $X_{t:n,\tilde{m},k}$ is given by (Kamps [8])

$$f_{t:n,\tilde{m},k}(x) \quad = \quad \frac{C_{t-1}}{(t-1)!}[\bar{F}(x)]^{\gamma_t - 1} f(x) g_m^{t-1}(F(x)), \tag{8}$$

and the joint PDF of $X_{t:n,\tilde{m},k}$ and $X_{s:n,\tilde{m},k}$, $1 \le t < s \le n$, is given by (Kamps [8])

$$\begin{aligned} f_{t,s:n,\tilde{m},k}(x,y) \quad &= \quad \frac{C_{s-1}}{(t-1)!(s-t-1)!}[\bar{F}(x)]^m g_m^{t-1}(F(x))[h_m(F(y)) - h_m(F(x))]^{s-t-1} \\ &\times [\bar{F}(y)]^{\gamma_s - 1} f(x) f(y), \ x < y, \end{aligned} \tag{9}$$

where $C_{t-1} = \prod_{i=1}^{t} \gamma_i$, $\gamma_i = k + (n-i)(m+1)$, $h_m(x) = \begin{cases} -\frac{1}{m+1}(1-x)^{m+1}, & m \neq -1, \\ -ln(1-x), & m = -1, \end{cases}$ and $g_m(x) = h_m(x) - h_m(0)$, $x \in [0,1)$.

The moments of ordered random schemes have garnered a lot of attention in recent years. Many authors studied the moments of GOS for various distributions. For example, see for some general forms of distribution [9–11] and for some specific distributions [12–16]. For more recent works, see [17–20]. However, to our knowledge, no attempt has yet been made to study the LEEG distribution based on GOS.The motivation for this paper is twofold: first, to derive the explicit expressions and recurrence relations for the single and product moments based on the GOS of the LEEG distribution; second, to obtain the best linear unbiased estimators (BLUEs) for the location and scale parameters based on progressively Type-II right censored order statistics.

The paper is organized as follows. Sections 2 and 3 provide the explicit expressions and recurrence relations for the single and product moments of GOS from the LEEG distribution. In Section 4, we compute the BLUEs for the location and scale parameters based on progressively Type-II right censored order statistics. Section 5 includes a real example that is utilized to illustrate the findings obtained in the previous section.

2. Single Moments of GOS from LEEG Distribution

The explicit expression and recurrence relation for single moments of GOS from the LEEG distribution are found in this section.

Theorem 1. *For the LEEG distribution as given in (1) and $n \in N$, $\tilde{m} \in \Re$, $k > 0$, $1 \le t \le n$, $p = 1, 2, \cdots$,*

$$\begin{aligned} &\mu_{t:n,\tilde{m},k}^{(p)} = E\left[X_{t:n,\tilde{m},k}^{p}\right] \\ &= C_{t-1} \sum_{i=1}^{t} \sum_{\nu=0}^{\gamma_i - 1} a_{i,t} (-1)^{\nu} \binom{\gamma_i - 1}{\nu} \frac{(1+\beta)^{\nu+1}}{\frac{p}{\alpha} + \nu + 1} \\ &\times {}_2F_1\left(\nu + 2, \frac{p}{\alpha} + \nu + 1; \frac{p}{\alpha} + \nu + 2; -\beta\right), \end{aligned} \tag{10}$$

where ${}_2F_1(a,\, b;\, c;\, x)$ is the Gauss hypergeometric function defined by

$${}_2F_1(a,\, b;\, c;\, z) = \sum_{u=0}^{\infty} \frac{(a)_u (b)_u}{(c)_u} \frac{z^u}{u!},$$

$|z| < 1$, and $(e)_u = e(e+1)\cdots(e+u-1)$ denotes the ascending factorial.

Proof. In view of (6), the single moment of the t-th GOS is given by

$$\mu^{(p)}_{t:n,\tilde{m},k} = E\left[X^p_{t:n,\tilde{m},k}\right]$$
$$= C_{t-1}\sum_{i=1}^{t} a_{i,t}\int_0^1 x^p[\bar{F}(x)]^{\gamma_i-1}f(x)dx. \quad (11)$$

Using (1) and (2) in (11), we have

$$\mu^{(p)}_{t:n,\tilde{m},k} = \alpha\, C_{t-1}\sum_{i=1}^{t}\sum_{\nu=0}^{\gamma_i-1} a_{i,t}(-1)^{\nu}\binom{\gamma_i-1}{\nu}(1+\beta)^{\nu+1}\int_0^1 \frac{x^{p+\alpha(\nu+1)-1}}{(1+\beta\, x^{\alpha})^{\nu+2}}dx. \quad (12)$$

Setting $x^{\alpha} = u$, we obtain

$$\mu^{(p)}_{t:n,\tilde{m},k} = C_{t-1}\sum_{i=1}^{t}\sum_{\nu=0}^{\gamma_i-1} a_{i,t}(-1)^{\nu}\binom{\gamma_i-1}{\nu}(1+\beta)^{\nu+1}\int_0^1 \frac{u^{\frac{p}{\alpha}+\nu}}{(1+\beta\, u)^{\nu+2}}du. \quad (13)$$

Now, by using the result from [21] (p. 317) given as

$$\int_0^1 x^{a-1}(1-x)^{b-1}(1-c\,x)^{-d}dx = B(a,b)\,{}_2F_1(d,\, a;\, a+b;\, c), \quad (14)$$

where $B(a,b)$ is the beta function, we obtain

$$\begin{aligned}\mu^{(p)}_{t:n,\tilde{m},k} &= C_{t-1}\sum_{i=1}^{t}\sum_{\nu=0}^{\gamma_i-1} a_{i,t}(-1)^{\nu}\binom{\gamma_i-1}{\nu}\frac{(1+\beta)^{\nu+1}}{\frac{p}{\alpha}+\nu+1}\\ &\times {}_2F_1\Big(\nu+2, \frac{p}{\alpha}+\nu+1; \frac{p}{\alpha}+\nu+2; -\beta\Big),\end{aligned} \quad (15)$$

which is (10). □

Corollary 1. *For $m_i = m \neq -1$, $i = 1,2,\cdots,n-1$, the single moment of GOS for case 2 is given by*

$$\begin{aligned}\mu^{(p)}_{t:n,m,k} &= \frac{C_{t-1}}{(t-1)!}\sum_{i=1}^{t}\sum_{\nu=0}^{\gamma_i-1}\frac{(-1)^{\nu+t-i}}{(m+1)^{t-1}}\binom{t-1}{t-i}\binom{\gamma_i-1}{\nu}\frac{(1+\beta)^{\nu+1}}{\frac{p}{\alpha}+\nu+1}\\ &\times {}_2F_1\Big(\nu+2, \frac{p}{\alpha}+\nu+1; \frac{p}{\alpha}+\nu+2; -\beta\Big).\end{aligned} \quad (16)$$

Remark 1. *Setting $m = 0$ and $k = 1$ in (16), the expression for the single moments of ordinary order statistics from the LEEG distribution can be obtained as*

$$\begin{aligned}\mu^{(p)}_{t:n} &= \frac{C_{t-1}}{(t-1)!}\sum_{i=1}^{t}\sum_{\nu=0}^{n-i}(-1)^{\nu+t-i}\binom{t-1}{t-i}\binom{n-i}{\nu}\frac{(1+\beta)^{\nu+1}}{\frac{p}{\alpha}+\nu+1}\\ &\times {}_2F_1\Big(\nu+2, \frac{p}{\alpha}+\nu+1; \frac{p}{\alpha}+\nu+2; -\beta\Big).\end{aligned} \quad (17)$$

Remark 2. *At $m_i = r_i$, $n = m_0 + \sum_{i=1}^{m_0} r_i$, and $\gamma_i = n - \sum_{\nu=1}^{i-1} r_\nu - i + 1$, $1 \leq i \leq m_0$ in (10), we obtain the expression for single moments of LEEG distribution from progressive Type-II censored order statistics.*

Theorem 2. *For the LEEG distribution as given in (1), and for $n \in N$, $\tilde{m} \in \Re$, $k > 0$, $2 \leq t \leq n$, $p = 1,2,\cdots$, the following recurrence relation for the single moments of GOS*

$$\mu^{(p)}_{t:n,\tilde{m},k} - \mu^{(p)}_{t-1:n,\tilde{m},k} = \frac{p}{\alpha(1+\beta)\gamma_t}\left[\mu^{(p-\alpha)}_{t:n,\tilde{m},k} + (\beta-1)\mu^{(p)}_{t:n,\tilde{m},k} - \beta\mu^{(p+\alpha)}_{t:n,\tilde{m},k}\right] \tag{18}$$

is satisfied.

Proof. We use the result of [11] for the LEEG distribution to obtain the recurrence relation for the single moments of GOS:

$$E[\xi(X_{t:n,\tilde{m},k})] - E[\xi(X_{t-1:n,\tilde{m},k})] = C_{t-2}\int_{-\infty}^{\infty}\xi'(x)\sum_{i=1}^{t}a_{i,t}[\bar{F}(x)]^{\gamma_i}dx.$$

Let $\xi(x) = x^p$, then

$$\mu^{(p)}_{t:n,\tilde{m},k} - \mu^{(p)}_{t-1:n,\tilde{m},k} = pC_{t-2}\int_{-\infty}^{\infty}x^{p-1}\sum_{i=1}^{t}a_{i,t}[\bar{F}(x)]^{\gamma_i}dx. \tag{19}$$

Now, using (4) in (19), we obtain

$$\begin{aligned}\mu^{(p)}_{t:n,\tilde{m},k} - \mu^{(p)}_{t-1:n,\tilde{m},k} &= \frac{pC_{t-1}}{\alpha(1+\beta)\gamma_t}\int_0^1 (x^{p-\alpha} + (\beta-1)x^p - \beta x^{p+\alpha}) \\ &\quad\times\sum_{i=1}^{t}a_{i,t}[\bar{F}(x)]^{\gamma_i-1}f(x)dx.\end{aligned} \tag{20}$$

By simplifying the previous expression, we obtain the required result. □

Corollary 2. *If $m_i = m$, $i = 1,2,\cdots,n-1$, then the recurrence relation for the single moments of GOS for case 2 is as follows:*

$$\begin{aligned}&\mu^{(p)}_{t:n,m,k} - \mu^{(p)}_{t-1:n,m,k} \\ &= \frac{p}{\alpha(1+\beta)(k+(n-t)(m+1))}\left[\mu^{(p-\alpha)}_{t:n,m,k} + (\beta-1)\mu^{(p)}_{t:n,m,k} - \beta\mu^{(p+\alpha)}_{t:n,m,k}\right].\end{aligned} \tag{21}$$

Remark 3. *Setting $m = 0$ and $k = 1$ in (21), we get the recurrence relation for the single moments of ordinary order statistics for the LEEG distribution as*

$$\mu^{(p)}_{t:n} - \mu^{(p)}_{t-1:n} = \frac{p}{\alpha(1+\beta)(n-t+1)}\left[\mu^{(p-\alpha)}_{t:n} + (\beta-1)\mu^{(p)}_{t:n} - \beta\mu^{(p+\alpha)}_{t:n}\right]. \tag{22}$$

Remark 4. *Setting $m_i = r_i$, $n = m_0 + \sum_{i=1}^{m_0} r_i$, and $\gamma_i = n - \sum_{\nu=1}^{i-1} r_\nu - i + 1$, $1 \le i \le m_0$ in (18), then the recurrence relation for the single moments of progressive Type-II censored order statistics of the LEEG distribution is given by*

$$\begin{aligned}&\mu^{(r_1,\cdots,r_{m_0})^{(p)}}_{t:m_0:n} - \mu^{(r_1,\cdots,r_{m_0})^{(p)}}_{t-1:m_0:n} \\ &= \frac{p}{\alpha(1+\beta)\gamma_t}\left[\mu^{(r_1,\cdots,r_{m_0})^{(p-\alpha)}}_{t:m_0:n} + (\beta-1)\mu^{(r_1,\cdots,r_{m_0})^{(p)}}_{t:m_0:n} - \beta\mu^{(r_1,\cdots,r_{m_0})^{(p+\alpha)}}_{t:m_0:n}\right].\end{aligned} \tag{23}$$

3. Product Moments of GOS from LEEG Distribution

In this part, the explicit expression and recurrence relation for product moments of GOS are considered.

Theorem 3. *For $n \in N$, $\tilde{m} \in \Re$, $k > 0$, $1 \le t < s \le n$, $p, q = 1,2,\cdots$, the product moment of tth and sth GOS from the LEEG distribution is given by*

$$\mu_{t,s:n,\tilde{m},k}^{(p,q)} = E\left[X_{t:n,\tilde{m},k}^{p} X_{s:n,\tilde{m},k}^{q}\right]$$
$$= C_{s-1} \sum_{i=1}^{t} \sum_{j=t+1}^{s} \sum_{\nu_1=0}^{\gamma_i-\gamma_j-1} \sum_{\nu_2=0}^{\gamma_j-1} \sum_{\nu_3=0}^{\infty} a_{i,t} a_{j,s}^{(t)} (-1)^{\nu_1+\nu_2+\nu_3} \binom{\gamma_i-\gamma_j-1}{\nu_1}$$
$$\times \binom{\gamma_j-1}{\nu_2} \binom{\nu_1+\nu_3+1}{\nu_1+1} \frac{\beta^{\nu_3}(1+\beta)^{\nu_1+\nu_2+2}}{(\frac{p}{\alpha}+\nu_1+\nu_3+1)(\frac{p+q}{\alpha}+\nu_1+\nu_2+\nu_3+2)}$$
$$\times {}_2F_1\left(\nu_2+2, \frac{p+q}{\alpha}+\nu_1+\nu_2+\nu_3+2; \frac{p+q}{\alpha}+\nu_1+\nu_2+\nu_3+3; -\beta\right). \quad (24)$$

Proof. The product moment of GOS is given by

$$\mu_{t,s:n,\tilde{m},k}^{(p,q)} = C_{s-1} \sum_{i=1}^{t} \sum_{j=t+1}^{s} a_{i,t}\, a_{j,s}^{(t)} \int_0^1 \int_0^y x^p y^q [\bar{F}(x)]^{\gamma_i-\gamma_j-1}$$
$$\times [\bar{F}(y)]^{\gamma_j-1} f(x) f(y) dx dy. \quad (25)$$

Using binomial expansion, (25) can be rewritten as

$$\mu_{t,s:n,\tilde{m},k}^{(p,q)} = C_{s-1} \sum_{i=1}^{t} \sum_{j=t+1}^{s} \sum_{\nu_1=0}^{\gamma_i-\gamma_j-1} \sum_{\nu_2=0}^{\gamma_j-1} a_{i,t}\, a_{j,s}^{(t)} (-1)^{\nu_1+\nu_2}$$
$$\times \binom{\gamma_i-\gamma_j-1}{\nu_1} \binom{\gamma_j-1}{\nu_2} \int_0^1 y^q [F(y)]^{\nu_2} f(y) I(y) dy, \quad (26)$$

where

$$I(y) = \int_0^y x^p [F(x)]^{\nu_1} f(x) dx.$$

Substituting the values of $f(x)$ and $F(x)$ from (1) and (2), we obtain

$$I(y) = (1+\beta)^{\nu_1+1} \int_0^{y^\alpha} \frac{u^{\frac{p}{\alpha}+\nu_1}}{(1+\beta u)^{\nu_1+2}} du,$$

where $u = x^\alpha$. Now using Equation (3) 194.1 from [21] (p. 315), we obtain

$$I(y) = (1+\beta)^{\nu_1+1} \frac{y^{p+\alpha(\nu_1+1)}}{\frac{p}{\alpha}+\nu_1+1} {}_2F_1\left(\nu_1+2, \frac{p}{\alpha}+\nu_1+1; \frac{p}{\alpha}+\nu_1+2; -\beta y^\alpha\right)$$
$$= (1+\beta)^{\nu_1+1} \sum_{\nu_3=0}^{\infty} (-1)^{\nu_3} \beta^{\nu_3} \binom{\nu_1+\nu_3+1}{\nu_1+1} \frac{y^{p+\alpha(\nu_1+\nu_3+1)}}{\frac{p}{\alpha}+\nu_1+\nu_3+1}. \quad (27)$$

Substituting (27) into (26), we obtain

$$\mu_{t,s:n,\tilde{m},k}^{(p,q)} = C_{s-1} \sum_{i=1}^{t} \sum_{j=t+1}^{s} \sum_{\nu_1=0}^{\gamma_i-\gamma_j-1} \sum_{\nu_2=0}^{\gamma_j-1} \sum_{\nu_3=0}^{\infty} a_{i,t}\, a_{j,s}^{(t)} (-1)^{\nu_1+\nu_2+\nu_3} \binom{\gamma_i-\gamma_j-1}{\nu_1}$$
$$\times \binom{\gamma_j-1}{\nu_2} \binom{\nu_1+\nu_3+1}{\nu_1+1} \frac{\beta^{\nu_3}(1+\beta)^{\nu_1+\nu_2+2}}{\frac{p}{\alpha}+\nu_1+\nu_3+1} \int_0^1 \frac{w^{\frac{p+q}{\alpha}+\nu_1+\nu_2+\nu_3+1}}{(1+\beta w)^{\nu_2+2}} dw, \quad (28)$$

where $w = y^\alpha$. By appling (14) in (28), with $a = \frac{p+q}{\alpha} + \nu_1 + \nu_2 + \nu_3 + 2$, $b = 1$, $d = \nu_2 + 2$ and $c = -\beta$, we obtain the required result. □

Corollary 3. *When $m_1 = m_2 = \cdots = m_{n-1} = m \neq -1$, the product moment of GOS for case 2 is given as*

$$\mu^{(p,q)}_{t,s:n,m,k} = C_{s-1} \sum_{i=1}^{t} \sum_{j=t+1}^{s} \sum_{\nu_1=0}^{\gamma_i-\gamma_j-1} \sum_{\nu_2=0}^{\gamma_j-1} \sum_{\nu_3=0}^{\infty} \frac{(-1)^{\nu_1+\nu_2+\nu_3+t-i+s-j}}{(m+1)^{s-2}(t-1)!(s-t-1)!}$$
$$\times \binom{t-1}{t-i} \binom{s-t-1}{s-j} \binom{\gamma_i-\gamma_j-1}{\nu_1} \binom{\gamma_j-1}{\nu_2} \binom{\nu_1+\nu_3+1}{\nu_1+1}$$
$$\times \frac{\beta^{\nu_3}(1+\beta)^{\nu_1+\nu_2+2}}{(\frac{p}{\alpha}+\nu_1+\nu_3+1)(\frac{p+q}{\alpha}+\nu_1+\nu_2+\nu_3+2)}$$
$$\times {}_2F_1\left(\nu_2+2, \frac{p+q}{\alpha}+\nu_1+\nu_2+\nu_3+2; \frac{p+q}{\alpha}+\nu_1+\nu_2+\nu_3+3; -\beta\right), \quad (29)$$

where $\gamma_i = k + (n-i)(m+1)$.

Remark 5. *Putting $m = 0$ and $k = 1$ in (29), we obtain the product moment of ordinary order statistics as*

$$\mu^{(p,q)}_{t,s:n} = \frac{n!}{(t-1)!(s-t-1)!(n-s)!} \sum_{i=1}^{t} \sum_{j=t+1}^{s} \sum_{\nu_1=0}^{j-i-1} \sum_{\nu_2=0}^{n-j} \sum_{\nu_3=0}^{\infty} (-1)^{\nu_1+\nu_2+\nu_3+t-i+s-j}$$
$$\times \binom{t-1}{t-i} \binom{s-t-1}{s-j} \binom{j-i-1}{\nu_1} \binom{n-j}{\nu_2} \binom{\nu_1+\nu_3+1}{\nu_1+1}$$
$$\times \frac{\beta^{\nu_3}(1+\beta)^{\nu_1+\nu_2+2}}{(\frac{p}{\alpha}+\nu_1+\nu_3+1)(\frac{p+q}{\alpha}+\nu_1+\nu_2+\nu_3+2)}$$
$$\times {}_2F_1\left(\nu_2+2, \frac{p+q}{\alpha}+\nu_1+\nu_2+\nu_3+2; \frac{p+q}{\alpha}+\nu_1+\nu_2+\nu_3+3; -\beta\right). \quad (30)$$

Remark 6. *At $m_i = r_i$, $n = m_0 + \sum_{i=1}^{m_0} r_i$, and $\gamma_i = n - \sum_{\nu=1}^{i-1} r_\nu - i + 1$, $1 \leq i \leq m_0$ in (24), the explicit expression for the product moment of progressive Type-II censored order statistics of the LEEG distribution can be obtained.*

Theorem 4. *For $n \in N$, $\tilde{m} \in \Re$, $k > 0$, $1 \leq t < s \leq n$, $p, q = 1, 2, \cdots$, the recurrence relation for product moments of GOS from the LEEG distribution is given by*

$$\mu^{(p,q)}_{t,s:n,\tilde{m},k} - \mu^{(p,q)}_{t,s-1:n,\tilde{m},k} = \frac{q}{\alpha(1+\beta)\,\gamma_s}\Big[\mu^{(p,q-\alpha)}_{t,s:n,\tilde{m},k} + (\beta-1)\mu^{(p,q)}_{t,s:n,\tilde{m},k} - \beta\mu^{(p,q+\alpha)}_{t,s:n,\tilde{m},k}\Big]. \quad (31)$$

Proof. From [11], we have

$$E[\xi(X_{t:n,\tilde{m},k}, X_{s:n,\tilde{m},k})] - E[\xi(X_{t:n,\tilde{m},k}, X_{s-1:n,\tilde{m},k})] =$$
$$C_{s-2} \int_{-\infty}^{\infty} \int_{x}^{\infty} \frac{d}{dy}\xi(x,y) \sum_{i=1}^{t} \sum_{j=t+1}^{s} a_{i,t}\, a^{(t)}_{j,s} [\bar{F}(x)]^{\gamma_i-\gamma_j-1} [\bar{F}(y)]^{\gamma_j-1} f(x) f(y)\, dy\, dx.$$

Let $\xi(x,y) = x^p y^q$, and in view of (4), we obtain

$$
\begin{aligned}
&\mu^{(p,q)}_{t,s:n,\tilde{m},k} - \mu^{(p,q)}_{t,s-1:n,\tilde{m},k} \\
&= \frac{q\,C_{s-2}}{\alpha(1+\beta)} \int_0^1 \int_x^1 x^p y^{q-1} (y^{1-\alpha} + (\beta-1)y - \beta y^{1+\alpha}) \\
&\times \sum_{i=1}^{t} \sum_{j=t+1}^{s} a_{i,t}\, a^{(t)}_{j,s} [\bar{F}(x)]^{\gamma_i-\gamma_j-1} [\bar{F}(y)]^{\gamma_j-1} f(x) f(y) dy dx, \\
&= \frac{q\,C_{s-2}}{\alpha(1+\beta)} \int_0^1 \int_x^1 x^p (y^{q-\alpha} + (\beta-1)y^q - \beta y^{q+\alpha}) \\
&\times \sum_{i=1}^{t} \sum_{j=t+1}^{s} a_{i,t}\, a^{(t)}_{j,s} [\bar{F}(x)]^{\gamma_i-\gamma_j-1} [\bar{F}(y)]^{\gamma_j-1} f(x) f(y) dy dx,
\end{aligned} \tag{32}
$$

which, after simplification, yields (31). □

Corollary 4. *When $m_1 = m_2 = \cdots = m_{n-1} = m$, the recurrence relation for the product moments of GOS for case 2 is given by*

$$
\begin{aligned}
&\mu^{(p,q)}_{t,s:n,m,k} - \mu^{(p,q)}_{t,s-1:n,m,k} \\
&= \frac{q}{\alpha(1+\beta)\,(k+(n-s)(m+1))} \left[\mu^{(p,q-\alpha)}_{t,s:n,m,k} + (\beta-1)\mu^{(p,q)}_{t,s:n,m,k} - \beta\mu^{(p,q+\alpha)}_{t,s:n,m,k}\right].
\end{aligned} \tag{33}
$$

Remark 7. *For $m_i = 0$, $i = 1, 2, \cdots, n-1$, and $k = 1$ in (33), we obtain the recurrence relation for product moments of ordinary order statistics for the LEEG distribution as*

$$
\begin{aligned}
&\mu^{(p,q)}_{t,s:n} - \mu^{(p,q)}_{t,s-1:n} \\
&= \frac{q}{\alpha(1+\beta)\,(n-s+1)} \left[\mu^{(p,q-\alpha)}_{t,s:n} + (\beta-1)\mu^{(p,q)}_{t,s:n} - \beta\mu^{(p,q+\alpha)}_{t,s:n}\right].
\end{aligned} \tag{34}
$$

Remark 8. *If $m_i = r_i$, $n = m_0 + \sum_{i=1}^{m_0} r_i$, and $\gamma_i = n - \sum_{v=1}^{i-1} r_v - i + 1$, $1 \le i \le m_0$ in (31), then the recurrence relation for the product moments of progressive Type-II censored order statistics of the LEEG distribution is given by*

$$
\begin{aligned}
&\mu^{(r_1,\cdots,r_{m_0})^{(p,q)}}_{t,s:m_0:n} - \mu^{(r_1,\cdots,r_{m_0})^{(p,q)}}_{t,s-1:m_0:n} \\
&= \frac{q}{\alpha(1+\beta)\,\gamma_s} \left[\mu^{(r_1,\cdots,r_{m_0})^{(p,q-\alpha)}}_{t,s:m_0:n} + (\beta-1)\mu^{(r_1,\cdots,r_{m_0})^{(p,q)}}_{t,s:m_0:n} - \beta\mu^{(r_1,\cdots,r_{m_0})^{(p,q+\alpha)}}_{t,s:m_0:n}\right].
\end{aligned} \tag{35}
$$

Utilizing the explicit expressions for the single and product moments given in Theorems (1) and (3), we calculated the means and variances of the LEEG distribution based on ordinary order statistics and progressively Type-II censored order statistics as shown in Tables 1–4. For simplicity, the notation $6, 3*0$, for example, is used in Tables 3–7 to signify the censoring scheme $(6, 0, 0, 0)$. Figure 1 shows the behavior of the means and variances of order statistics.

In the case of moments of order statistics, the following identities are utilized to validate the computations of means, variances, and covariances:

$$
\sum_{t=1}^{n} \mu^{(p)}_{t:n} = n\,\mu^{(p)}_{1:1},\ p = 1, 2,
$$

$$
\sum_{t=1}^{n} \sum_{s=1}^{n} \sigma_{t,s:n} = n\,\sigma_{1,1:1},
$$

where $\sigma_{t,s:n} = \mu_{t,s:n} - \mu_{t:n}\,\mu_{s:n}$; see [22].

Table 1. Means and variances of order statistics from the LEEG distribution with $\alpha = 1$ and $\beta = 0.60$.

n	t	Mean	Variance	n	t	Mean	Variance
1	1	0.422238	0.080920	5	5	0.770882	0.030253
2	1	0.259205	0.044246	6	1	0.098893	0.008677
2	2	0.585271	0.064435	6	2	0.208859	0.017581
3	1	0.185320	0.026220	6	3	0.331650	0.025809
3	2	0.406977	0.047544	6	4	0.469360	0.031781
3	3	0.674418	0.049039	6	5	0.624512	0.032833
4	1	0.143734	0.017019	6	6	0.800156	0.024595
4	2	0.310076	0.033070	7	1	0.085487	0.006617
4	3	0.503879	0.043237	7	2	0.179328	0.013493
4	4	0.731265	0.038047	7	3	0.282688	0.020170
5	1	0.117220	0.011841	7	4	0.396933	0.025870
5	2	0.249790	0.023674	7	5	0.523680	0.029329
5	3	0.400505	0.033536	7	6	0.664845	0.028542
5	4	0.572795	0.037832	7	7	0.822708	0.020377

Table 2. Means and variances of order statistics from the LEEG distribution with $\alpha = 1$ and $\beta = -0.60$.

n	t	Mean	Variance	n	t	Mean	Variance
1	1	0.648566	0.074582	5	5	0.918337	0.006691
2	1	0.495220	0.068969	6	1	0.265884	0.035198
2	2	0.801912	0.033165	6	2	0.468232	0.037207
3	1	0.404715	0.058211	6	3	0.625414	0.029112
3	2	0.676228	0.041340	6	4	0.749834	0.019481
3	3	0.864754	0.017230	6	5	0.850033	0.011130
4	1	0.343812	0.048791	6	6	0.931997	0.004683
4	2	0.587425	0.041959	7	1	0.239218	0.030376
4	3	0.765030	0.024948	7	2	0.425876	0.034262
4	4	0.897996	0.010237	7	3	0.574124	0.028874
5	1	0.299609	0.041219	7	4	0.693800	0.021245
5	2	0.520626	0.039999	7	5	0.791860	0.014037
5	3	0.687624	0.028167	7	6	0.873302	0.008073
5	4	0.816634	0.016145	7	7	0.941780	0.003449

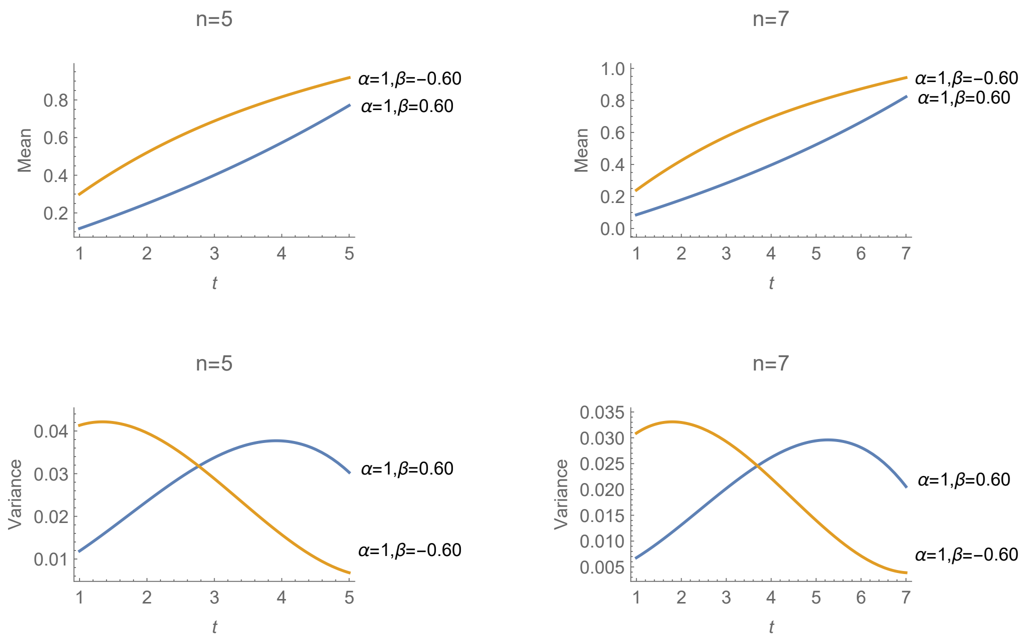

Figure 1. Behavior of the means and variances of order statistics.

Table 3. Means of progressively Type-II right censored order statistics from the LEEG distribution with $\alpha = 1$ and $\beta = -0.60, 0.60$.

β	m_0, n	Scheme	Mean				
−0.60	2, 10	8, 0	0.184425	0.700137			
−0.60	2, 10	0, 8	0.184425	0.336070			
0.60	2, 10	8, 0	0.060714	0.462408			
0.60	2, 10	0, 8	0.060714	0.125713			
−0.60	3, 10	7, 0, 0	0.184425	0.572918	0.827356		
−0.60	3, 10	0, 0, 7	0.184425	0.336070	0.462289		
0.60	3, 10	7, 0, 0	0.060714	0.308828	0.615987		
0.60	3, 10	0, 0, 7	0.060714	0.125713	0.195435		
−0.60	4, 10	6, 3 ∗ 0	0.184425	0.499126	0.720503	0.880782	
−0.60	4, 10	3 ∗ 0, 6	0.184425	0.336070	0.462289	0.568530	
0.60	4, 10	6, 3 ∗ 0	0.060714	0.238722	0.449041	0.699460	
0.60	4, 10	3 ∗ 0, 6	0.060714	0.125713	0.195435	0.270376	
−0.60	5, 10	5, 4 ∗ 0	0.184425	0.450070	0.646292	0.794714	0.909472
−0.60	5, 10	4 ∗ 0, 5	0.184425	0.336070	0.462289	0.568530	0.658882
0.60	5, 10	5, 4 ∗ 0	0.060714	0.199081	0.357645	0.540437	0.752468
0.60	5, 10	4 ∗ 0, 5	0.060714	0.125713	0.195435	0.270376	0.351098

Table 4. Variances of progressively Type-II right censored order statistics from the LEEG distribution with $\alpha = 1$ and $\beta = -0.60, 0.60$.

β	m_0, n	Scheme	Variance				
−0.60	2, 10	8, 0	0.020622	0.053982			
−0.60	2, 10	0, 8	0.020622	0.026442			
0.60	2, 10	8, 0	0.003455	0.073392			
0.60	2, 10	0, 8	0.003455	0.007080			
−0.60	3, 10	7, 0, 0	0.020622	0.050871	0.024723		
−0.60	3, 10	0, 0, 7	0.020622	0.026442	0.025562		
0.60	3, 10	7, 0, 0	0.003455	0.042132	0.057478		
0.60	3, 10	0, 0, 7	0.003455	0.007080	0.010784		
−0.60	4, 10	6, 3 ∗ 0	0.020622	0.044609	0.030720	0.013162	
−0.60	4, 10	3 ∗ 0, 6	0.020622	0.026442	0.025562	0.021982	
0.60	4, 10	6, 3 ∗ 0	0.003455	0.026471	0.043965	0.043332	
0.60	4, 10	3 ∗ 0, 6	0.003455	0.007080	0.010784	0.014420	
−0.60	5, 10	5, 4 ∗ 0	0.020622	0.039344	0.031529	0.018897	0.007958
−0.60	5, 10	4 ∗ 0, 5	0.020622	0.026442	0.025562	0.021982	0.017631
0.60	5, 10	5, 4 ∗ 0	0.003455	0.018404	0.031813	0.039411	0.033399
0.60	5, 10	4 ∗ 0, 5	0.003455	0.007080	0.010784	0.014420	0.017772

4. BLUEs for Location-Scale LEEG Distribution

Suppose $\mathbf{Y}' = (Y_{1:m_0:n}, Y_{2:m_0:n}, \cdots, Y_{m_0:m_0:n})$ is a progressively Type-II right censored sample from a location-scale LEEG distribution with location parameter θ_1 and scale parameter θ_2. Let $X_{i:m_0:n} = (Y_{i:m_0:n} - \theta_1)/\theta_2$, $i = 1, 2, \cdots, m_0$, denote the corresponding progressively Type-II right censored sample from the standard distribution, $E(X_{i:m_0:n}) = \mu_i$ and $Cov(X_{i:m_0:n}, X_{j:m_0:n}) = \sigma_{ij}$. Denote $\boldsymbol{\theta} = (\theta_1, \theta_2)'$, $\boldsymbol{\mu}' = (\mu_1, \mu_2, \cdots, \mu_{m_0})$ as the mean vector, and $\boldsymbol{\Sigma} = ((\sigma_{ij}))$, $1 \le i,\ j \le m_0$ as the variance–covariance matrix.

Then, the BLUE of the parameter $\boldsymbol{\theta}$ is given by

$$\boldsymbol{\theta}^* = \frac{1}{\Delta}\begin{pmatrix} \boldsymbol{\mu}'\,\Gamma\,\mathbf{Y} \\ -\mathbf{1}'\Gamma\,\mathbf{Y} \end{pmatrix},$$

where $\Gamma = \boldsymbol{\Sigma}^{-1}(\ \boldsymbol{\mu}\ \mathbf{1}'\ - \mathbf{1}\ \ \boldsymbol{\mu}'\ )\ \boldsymbol{\Sigma}^{-1}$, $\Delta = (\boldsymbol{\mu}'\ \boldsymbol{\Sigma}^{-1}\ \boldsymbol{\mu})(\mathbf{1}'\ \boldsymbol{\Sigma}^{-1}\ \mathbf{1}) - (\boldsymbol{\mu}'\ \boldsymbol{\Sigma}^{-1}\ \mathbf{1})^2$ and $\mathbf{1}' = (1, 1, \cdots, 1)_{m_0 \times 1}$.

Moreover, the variances and covariances of these BLUEs are given by

$$Var(\theta_1^*)/\theta_2^2 = \frac{\boldsymbol{\mu}'\,\boldsymbol{\Sigma}^{-1}\,\boldsymbol{\mu}}{\Delta},$$

$$Var(\theta_2^*)/\theta_2^2 = \frac{\mathbf{1}'\,\boldsymbol{\Sigma}^{-1}\,\mathbf{1}}{\Delta},$$

and

$$Cov(\theta_1^*, \theta_2^*)/\theta_2^2 = \frac{(-\boldsymbol{\mu}'\,\boldsymbol{\Sigma}^{-1}\,\mathbf{1})}{\Delta}.$$

See [23,24] for more details.

The coefficients of the BLUEs for θ_1 and θ_2 and the values $\mathrm{Var}(\theta_1^*)/\theta_2^2$, $\mathrm{Var}(\theta_2^*)/\theta_2^2$, and $\mathrm{Cov}(\theta_1^*, \theta_2^*)/\theta_2^2$ are displayed in Tables 5–7 for sample size $n = 10$ and for different choices of m_0 and progressive censoring schemes. The coefficients of the BLUEs θ_1^* and θ_2^* in Tables 5 and 6 can be checked by the conditions $\sum_{i=1}^{m_0} a_i = 1$ and $\sum_{i=1}^{m_0} b_i = 0$.

Table 5. Coefficients for the BLUEs of θ_1 for some selected progressive censoring schemes for the location-scale LEEG distribution with $\alpha = 1$ and $\beta = -0.60, 0.60$.

β	m_0	n	Scheme	Coefficients (a_i)				
−0.60	2	10	8, 0	1.35761	−0.35761			
−0.60	2	10	0, 8	2.21617	−1.21617			
0.60	2	10	8, 0	1.15115	−0.15115			
0.60	2	10	0, 8	1.93408	−0.93408			
−0.60	3	10	7, 0, 0	1.267510	0.048873	−0.316382		
−0.60	3	10	0, 0, 7	1.622630	0.090463	−0.713096		
0.60	3	10	7, 0, 0	1.084910	0.044168	−0.129077		
0.60	3	10	0, 0, 7	1.465590	−0.028833	−0.436757		
−0.60	4	10	6, 3 ∗ 0	1.233440	0.035801	0.051191	−0.32043	
−0.60	4	10	3 ∗ 0, 6	1.424650	0.058870	0.071805	−0.555328	
0.60	4	10	6, 3 ∗ 0	1.079010	−0.057814	0.147298	−0.168491	
0.60	4	10	3 ∗ 0, 6	1.309520	−0.019550	−0.018046	−0.271927	
−0.60	5	10	5, 4 ∗ 0	1.212280	0.029705	0.040361	0.054430	−0.336772
−0.60	5	10	4 ∗ 0, 5	1.324870	0.043549	0.053086	0.064140	−0.485642
0.60	5	10	5, 4 ∗ 0	1.154350	−0.249614	0.081079	0.283290	−0.269101
0.60	5	10	4 ∗ 0, 5	1.231460	−0.014845	−0.013701	−0.012619	−0.190292

Table 6. Coefficients for the BLUEs of θ_2 for some selected progressive censoring schemes for the location-scale LEEG distribution with $\alpha = 1$ and $\beta = -0.60, 0.60$.

β	m_0	n	Scheme	Coefficients (b_i)			
−0.60	2	10	8, 0	−1.93907	1.93907		
−0.60	2	10	0, 8	−6.59438	6.59438		
0.60	2	10	8, 0	−2.48946	2.48946		
0.60	2	10	0, 8	−15.3849	15.3849		
−0.60	3	10	7, 0, 0	−1.50756	−0.12082	1.62838	
−0.60	3	10	0, 0, 7	−3.43490	−0.361011	3.79591	
0.60	3	10	7, 0, 0	−1.88178	0.146175	1.73560	
0.60	3	10	0, 0, 7	−7.63449	0.409117	7.22538	
−0.60	4	10	6, 3 ∗ 0	−1.37515	−0.069004	−0.100254	1.54441
−0.60	4	10	3 ∗ 0, 6	−2.41641	−0.19848,	−0.241947	2.85684
0.60	4	10	6, 3 ∗ 0	−1.86443	0.583966	−0.312108	1.592570
0.60	4	10	3 ∗ 0, 6	−5.02877	0.254115	0.234554	4.54010

Table 6. *Cont.*

β	m_0	n	Scheme	Coefficients (b_i)				
−0.60	5	10	$5, 4*0$	−1.310220	−0.048427	−0.066312	−0.089994	1.514960
−0.60	5	10	$4*0, 5$	−1.927040	−0.123346	−0.150142	−0.181194	2.381720
0.60	5	10	$5, 4*0$	−2.310470	1.680380	−0.497668	−0.637334	1.765090
0.60	5	10	$4*0, 5$	−3.706610	0.174442	0.160968	0.148230	3.222970

Table 7. The values of $\mathrm{Var}(\theta_1^*)/\theta_2^2$, $\mathrm{Var}(\theta_2^*)/\theta_2^2$ and $\mathrm{Cov}(\theta_1^*, \theta_2^*)/\theta_2^2$ for some selected progressive censoring schemes for the location-scale LEEG distribution with $\alpha = 1$ and $\beta = -0.60, 0.60$.

β	m_0	n	Scheme	$\frac{Var(\theta_1^*)}{\theta_2^2}$	$\frac{Var(\theta_2^*)}{\theta_2^2}$	$\frac{Cov(\theta_1^*,\theta_2^*)}{\theta_2^2}$
−0.60	2	10	$8, 0$	0.039000	0.234720	−0.071469
−0.60	2	10	$0, 8$	0.056385	0.691219	−0.160697
0.60	2	10	$8, 0$	0.004760	0.423011	−0.023590
0.60	2	10	$0, 8$	0.007123	0.924341	−0.058274
−0.60	3	10	$7, 0, 0$	0.033814	0.094203	−0.044520
−0.60	3	10	$0, 0, 7$	0.042320	0.292673	−0.085827
0.60	3	10	$7, 0, 0$	0.003466	0.159245	−0.005680
0.60	3	10	$0, 0, 7$	0.005344	0.437374	−0.028838
−0.60	4	10	$6, 3*0$	0.032441	0.060941	−0.037824
−0.60	4	10	$3*0, 6$	0.037611	0.168037	−0.061599
0.60	4	10	$6, 3*0$	0.002489	0.072280	0.002225
0.60	4	10	$3*0, 6$	0.004751	0.272057	−0.018936
−0.60	5	10	$5, 4*0$	0.031844	0.048360	−0.035150
−0.60	5	10	$4*0, 5$	0.035224	0.110639	−0.049896
0.60	5	10	$5, 4*0$	0.000822	0.012654	0.010497
0.60	5	10	$4*0, 5$	0.004454	0.186895	−0.013908

5. Real Data Application

In this part, we consider a real data set as an application of the estimating method provided in this article. The data set is available in [25] and consists of seventy-three observations on seven variables. Many authors analyzed these data; see [26–28]. The data set's description may be found in [27].

Here, attention is on the variable FIRMCOST (divided by 100), which is a measure of the cost effectiveness of the firm's risk management practices. A random sample of size ten is selected as follows: 0.002, 0.0036, 0.004, 0.0122, 0.0407, 0.0571, 0.0849, 0.1357, 0.1833, 0.2222. From the above data, we generate a progressively Type-II censored sample with $n = 10$, $m_0 = 5$, and the censoring scheme $r = (5, 0, 0, 0, 0)$. The generated progressively Type-II censored sample is shown in Table 8.

A simple plot of these censored values against the expected values given in Table 3 for $\alpha = 1$ and $\beta = 0.60$ shows a quite high correlation (i.e., a correlation coefficient of the order of 0.9743). Hence, we can assume that these data comes from the LEEG distribution with $\alpha = 1$ and $\beta = 0.60$.

Table 8. Generated progressively Type-II censored sample from the data of [25].

i	1	2	3	4	5
$y_{i:5:10}$	0.002	0.004	0.0849	0.1833	0.2222
r_i	5	0	0	0	0

Based on the progressively Type-II censored sample presented above, we find the BLUEs of θ_1 and θ_2 to be $\theta_1^* = 0.00033$ and $\theta_2^* = 0.23523$, respectively, and their standard errors to be $SE(\theta_1^*) = 0.00674$ and $SE(\theta_2^*) = 0.02646$.

6. Conclusions

In this study, we obtain explicit expressions and recurrence relations for the single and product moments of GOS from the LEEG distribution. Based on these results, the means and variances of ordinary order statistics and progressively Type-II censored order statistics are calculated using Mathematica software. Further, based on progressively Type-II censored order statistics, we obtain the BLUEs of the location and scale parameters of the LEEG distribution. In addition, a real data set is used for illustration. The findings of this study can be utilized to obtain the best linear invariant estimators for the location and scale parameters of the LEEG distribution, as well as the best linear unbiased predictors and best linear invariant predictors for future unobserved GOS. It is also worth noting that the recurrence relations presented here can be used to characterize the LEEG distribution.

Funding: This research received no external funding.

Data Availability Statement: This study relied on previously published data. These earlier studies are listed as references in the text at appropriate places.

Acknowledgments: The author would like to acknowledge the Deanship of Scientific Research at Taif University for supporting this work.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Adamidis, K.; Dimitrakopoulou, T.; Loukas, S. On an extension of the exponential-geometric distribution. *Stat. Probab. Lett.* **2005,** *73*, 259–269. [CrossRef]
2. Jodra, P.; Jimenez-Gamero, M.D. A quantile regression model for bounded responses based on the exponential-geometric distribution. *Revstat-Stat. J.* **2020**, *18*, 415–436.
3. Yang, R.; Chen, W.; Yao, D.; Long, C.; Dong, Y.; Shen, B. The efficiency of ranked set sampling design for parameter estimation for the log-extended exponential–geometric distribution. *Iran. J. Sci. Technol.-Trans. A Sci.* **2020**, *44*, 497–507. [CrossRef]
4. Yang, R.; Chen, W.; Dong, Y. Log-extended exponential-geometric parameters estimation using simple random sampling and moving extremes ranked set sampling. *Commun. Stat. Simul. Comput.* **2023**, *52*, 267–277. [CrossRef]
5. Mitzenmacher, M. A brief history of generative models for power law and lognormal distributions. *Internet Math.* **2004**, *1*, 226–251. [CrossRef]
6. Clauset, A.; Shalizi, C.R.; Newman, M.E. Power-law distributions in empirical data. *SIAM Rev.* **2009**, *51*, 661–703. [CrossRef]
7. Apostol, T.M. *Zeta and Related Functions*; Cambridge University Press: Cambridge, UK, 2010; Chapter 25, pp. 601–616.
8. Kamps, U. *A Concept of Generalized Order Statistics*; Teubner: Stuttgart, Germany, 1995; p. 210.
9. Kamps, U.; Cramer, E. On distributions of generalized order statistics. *Statistics* **2001**, *35*, 269–280. [CrossRef]
10. Ahmad, A.E.B.A.; Fawzy, M.A. Recurrence relations for single moments of generalized order statistics from doubly truncated distributions. *J. Stat. Plan. Inference* **2003**, *117*, 241–249. [CrossRef]
11. Athar, H.; Islam, H.M.U. Recurrence relations for single and product moments of generalized order statistics from a general class of distribution. *Metron* **2004**, *62*, 327–337.
12. Khan, R.U.; Kumar, D.; Athar, H. Moments of generalized order statistics from Erlang-truncated exponential distribution and its characterization. *Int. J. Stat. Syst.* **2010**, *5*, 455–464.
13. Kumar, D. Exact moments of generalized order statistics from type II exponentiated log-logistic distribution. *Hacet. J. Math. Stat.* **2015**, *44*, 715–733. [CrossRef]
14. Zarrin, S.; Athar, H.; Abdel-Aty, Y. Relations for moments of generalized order statistics from power Lomax distribution. *J. Stat. Appl. Probab. Lett.* **2019**, *6*, 29–36. [CrossRef]
15. Athar, H.; Zarrin, S.; Noor, Z. Moment properties of generalized order statistics from Weibull-geometric distribution. *Appl. Math. E-Notes* **2019**, *19*, 199–209.
16. Gupta, N.; Anwar, Z. Relations for single and product moments of odds generalized exponential-Pareto distribution based on generalized order statistics and its characterization. *Stat. Optim. Inf. Comput.* **2019**, *7*, 160–170. [CrossRef]
17. Khan, M.J. S.; Sharma, A.; Iqrar, S. On moments of Lindley distribution based on generalized order statistics. *Am. J. Math. Manag. Sci.* **2020**, *39*, 214–233. [CrossRef]
18. Athar, H.; Noor, Z.; Zarrin, S.; Almutairi, H.N. Expectation Properties of Generalized Order Statistics from Poisson Lomax Distribution. *Stat. Optim. Inf. Comput.* **2021**, *9*, 735–747. [CrossRef]
19. Alharbi, Y.F. A Study on Moments of Generalized Order Statistics from Lomax-Exponential Distribution and Characterization. *Int. J. Math. Comput. Sci.* **2021**, *16*, 1179–1202.
20. Athar, H.; Fawzy, M.A.; Alharbi, Y.F. Moment Properties of Generalized Order Statistics Based on Modified Weighted Rayleigh Distribution. *Eur. J. Stat.* **2023**, *3*, 1. [CrossRef]

21. Gradshteyn, I.S.; Ryzhik, I.M. *Table of Integrals, Series, and Products*, 7th ed.; Elsevier/Academic Press: Amsterdam, The Netherlands, 2007; Volume 48, p. 1171.
22. Joshi, P.C.; Balakrishnan, N. Recurrence relations and identities for the product moments of order statistics. *Sankhyā Ser. B* **1982**, *44* , 39–49.
23. Balakrishnan, N.; Cohen, A.C. *Order Statistics and Inference: Estimation Methods*; Elsevier: Amsterdam, The Netherlands, 2014.
24. Arnold, B.C.; Balakrishnan, N.; Nagaraja, H.N. *A First Course in Order Statistics*; Society for Industrial and Applied Mathematics: Delhi, India, 2008.
25. Frees, E.W. *Regression Modeling with Actuarial and Financial Applications*; Cambridge University Press: Cambridge, UK, 2010.
26. Schmit, J.T.; Roth, K. Cost effectiveness of risk management practices. *J. Risk Insur.* **1990**, *57*, 455–470. [CrossRef]
27. Gómez-Déniz, E.; Sordo, M.A.; Calderín-Ojeda, E. The Log–Lindley distribution as an alternative to the beta regression model with applications in insurance. *Insur. Math. Econ.* **2014**, *54*, 49–57. [CrossRef]
28. Jodrá, P.; Jimenez-Gamero, M.D. A note on the Log-Lindley distribution. *Insur. Math. Econ.* **2016**, *71*, 189–194. [CrossRef]