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Review

Solving the Hydrodynamical System of Equations of Inhomogeneous Fluid Flows with Thermal Diffusion: A Review

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Abstract: The present review analyzes classes of exact solutions for the convection and thermal diffusion equations in the Boussinesq approximation. The exact integration of the Oberbeck–Boussinesq equations for convection and thermal diffusion is more difficult than for the Navier–Stokes equations. It has been shown that the exact integration of the thermal diffusion equations is carried out in the Lin–Sidorov–Aristov class. This class of exact solutions is a generalization of the Ostroumov–Birikh family of exact solutions. The use of the class of exact solutions by Lin–Sidorov–Aristov makes it possible to take into account not only the inhomogeneity of the pressure field, the temperature field and the concentration field, but also the inhomogeneous velocity field. The present review shows that there is a class of exact solutions for describing the flows of incompressible fluids, taking into account the Soret and Dufour cross effects. Accurate solutions are important for modeling and simulating natural, technical and technological processes. They make it possible to find new physical mechanisms of momentum transfer for the design of new types of equipment.

Keywords: hydrodynamical system of equations; inhomogeneous fluid flows; thermal diffusion; non-stationary solution; stability of flow; exact solution

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1. Introduction

True and useful scientific research should be based on the idea that it is very important to create an adequate physical model along with a clear analytical or exact solution (which is most desired), or semi-analytical algorithm, or at least a numerical mathematical code for calculating changes in the physical system under investigation (the object of observations or physical fields reflecting their material properties). This should be undertaken while considering the results of previous observations or clearly formulated earlier hypotheses based on observations made by other researchers (e.g., Newton's laws in physics or a hydrodynamical concept of Newtonian fluid, defined as one with constant viscosity when shear rate is directly proportional to shear stress). This enables the comparison of the final results of calculations, with the help of the constructed mathematical model, with the observed data relating to them, e.g., the resulting regimes of fluid flows in hydrodynamics (which will be discussed in our review). More precisely, we will concentrate our efforts on presenting ways for solving the hydrodynamical system of equations of inhomogeneous fluid flows with thermal diffusion.

The laws of hydrodynamics lead us to the formulation of differential equations in a general form depending on the basic assumptions of fluid flows taking part in their formulation. There are two approaches to solving them: analytical methods and numerical approaches. More generally, we can formulate the classification of theoretical methods as follows:

- Exact methods;
- Approaches used in perturbation theory (asymptotical methods);
- Semi-analytical methods;
- Numerical codes combined with (or without) semi-analytical methods.

The general exact solution for the aforementioned equations of motion is unknown almost everywhere. When an analytical (or, more generally, exact) solution is sometimes impossible to obtain, numerical approaches should be used to derive at least an approximate solution of the equations to try to reduce the analytical or exact quality of a clear presentation of the solution. The method of perturbation theory allows one to construct such a solution. In hydrodynamics, this part of approximate technique is commonly used by specialists during the investigation of the stability of the obtained solutions (beyond our study).

2. The Path from the Simplest Exact Solutions to Classes of Exact Solutions

It is worth noting that vortex fluid flows are one of the most common motions of a continuous medium [1–6]. A wide range of spatial and temporal scales and the nonlinear nature of most real vortex processes make it difficult to study them [1,2,4]. An exception is the possibility to reduce the equations of motion to one-dimensional equations. In this case, the equations can be quite easily integrated and investigated for a wide range of dimensionless parameters [1,2,4,7–19].

Real Newtonian fluids are characterized by the presence of internal friction (molecular viscosity) and the constancy of density during movement [1,2,4], the latter is almost everywhere except at boiling mode at negative pressure. Therefore, to study the vortex flows of a continuous medium, the Navier–Stokes equations and the continuity equation for an incompressible fluid (incompressibility equations) are used. If a change in temperature and the presence of a dissolved substance in a liquid take place, the flows can be described by systems of Oberbeck–Boussinesq equations for describing thermal gravitational convection and thermal diffusion which are similar to the Navier–Stokes equations [1,2].

The initial (almost centenary) stage in the development of the dynamics of a viscous incompressible fluid is associated with the study of exact solutions of the Navier–Stokes equations. The field of activity expanded for hydrodynamicists when G.G. Stokes formulated a linear version of these equations [20]. The Stokes approximation for the Navier–Stokes equations is valid for Reynolds numbers much smaller than one (it is asymptotically set to zero) [4,5,7].

The asymptotic simplification of the Navier–Stokes equations, performed by L. Prandtl, led to the creation of the boundary layer theory. But even here, at first, the matter was limited to the construction of exact solutions (K. Hiemenz, K. Blasius, V.M. Falkner and S.W. Scan) [21–23].

The theory of hydrodynamics stability has been developed for a long time as a theory of stability of exact solutions (Couette, Hagen–Poiseuille and its analogue, the Nusselt solution) [24–27]. In a recent survey on exact methods of integrating the Navier–Stokes equations [7], all known classes of exact solutions of equations were given. By now, at least three families can be distinguished in the context of our research: the Gromeka–Beltrami–Trkal class [28–34], the Lin–Sidorov–Aristov class [7,35–37] and other solutions that, with an appropriate change in coordinates, can be reduced to these two classes. It is useful to trace the history of obtaining exact solutions from reviews and books [7,38–50].

In addition, in reviews [5,7,14,16,38–50] it was shown that most of the known exact solutions belong to the Lin–Sidorov–Aristov class [35–37]. Probably, the first described approach to integrating the Navier–Stokes equations was implemented in the work of C.C. Lin [35], published in 1957. In it, the problem of magnetohydrodynamics was studied, but the possibilities of the proposed approach turned out to be much wider. Regardless of C.C.

Lin, this family of exact solutions was used by A.F. Sidorov for problems of convection in incompressible and compressible media [36]. The most profound generalization of the ideas of Lin and Sidorov was made by S.N. Aristov. He not only repeated (independently) their results for solving problems of magnetohydrodynamics and convection, but also generalized them to an exact solution for pressure, temperature and concentration fields in the form of quadratic forms with respect to two coordinates and showed the validity of these solutions for fluid flows, taking into account internal heat release (Rayleigh's dissipative function) [37,46].

3. The First Class of Exact Solutions and Its Modifications

As applied to the Navier–Stokes equations, the Lin–Sidorov–Aristov approach can be described as follows. First and mainly, in part of the basic solution, the velocity components U, V, W depend only on terms added in the first two components that are linear in two variables. Thus, the Lin–Sidorov–Aristov velocity field is written as [7,35–47]:

$$V_x(x,y,z,t) = U(z,t) + xu_1(z,t) + yu_2(z,t),$$
$$V_y(x,y,z,t) = V(z,t) + xv_1(z,t) + yv_2(z,t)$$
$$V_z(z,t) = w(z,t)$$

A similar construction to the Lin–Sidorov–Aristov family for the equations of a three-dimensional stationary boundary layer was previously proposed in [51]. It should also be noted here that for two-dimensional (plane and axisymmetric) stationary Navier–Stokes equations, solutions linear in one of the coordinates were found even earlier by K. Hiemenz and T. von Karman [21,52]. However, the number of exact solutions in which three-dimensional effects are significant is extremely small. Meanwhile, it is precisely such solutions that are of particular interest from the point of view of applications to real problems of mechanics, atmospheric physics and oceanology, and physicochemical hydrodynamics [1,2,8,14,16,50,53–61].

Note that the Lin–Sidorov–Aristov class has recently been significantly extended [7,35–47]. This family of exact solutions is characterized by a velocity field that depends on two coordinates (horizontal or longitudinal [7,35–47]) linearly. The coefficients of linear forms depend on the third coordinate (vertical or transverse) and time [7,35–47]. It was possible to construct exact solutions for the Navier–Stokes equations, for which the field velocities are polynomials of arbitrary degree in two coordinates with a similar dependence of the coefficients of polynomials of two variables [42–45,62–65]:

$$V_x(x,y,z,t) = U(z,t) + xu_1(z,t) + yu_2(z,t) + \frac{x^2}{2}u_3(z,t) + xyu_4(z,t) + \frac{y^2}{2}u_5(z,t),$$
$$V_y(x,y,z,t) = V(z,t) + xv_1(z,t) + yv_2(z,t) + \frac{x^2}{2}v_3(z,t) + xyv_4(z,t) + \frac{y^2}{2}v_5(z,t),$$
$$V_z(z,t) = w(z,t) + xw_1(z,t) + yw_2(z,t).$$

In the article [62], for the first time, a boundary value problem for the velocity field of a shear isobaric flow of the Couette type, expressed in a quadratic form, was considered. Thus, there has been significant progress for the exact integration of the Navier–Stokes equations. The library of exact solutions of the Navier–Stokes equations has been significantly expanded over the past five years [42–45,62–65].

It is known that the most common cause that induces and maintains the movement of a fluid is convection [1,2,4,6,14,16]. The mechanisms of convective mixing are due to the inhomogeneous distribution of temperature, impurities in the form of a dissolved substance or solid inclusions, the presence of magnetic and electric fields, mechanical influences (e.g., vibration, mixing and rotation) and other reasons [1,2,4,6,14,16]. When studying convective flows, two mechanisms of stratification of the hydrodynamical fields (mainly temperature) are distinguished. The vertical stratification of the temperature field corresponds to the Rayleigh convection [1]. In Benard's classical experiments on heating the bottom layer of spermaceti (sperm of whale brain fat), the Marangoni effect, which excites horizontal convection, was not taken into account. It is obvious that the division of convection into two directions is rather arbitrary, since according to the Onsager principle, they complement each other [1].

The study of horizontal convection began later than the study of the Rayleigh mechanism in the transfer of angular momentum in a liquid [1]. However, the inhomogeneity of the perturbing force field is often found in nature [1,14,16]. The flows of water and air masses, the astrophysical motions of the interstellar medium, the growth of crystals, the flows of biological fluids and other processes are due to the presence of horizontal (longitudinal) density gradients. These gradients may be due to the dependence of density on temperature, the concentration of dissolved substances, pressure, and the magnetic and electrical properties of the liquid [1,14,16].

If we compare the system of equations for describing the flows of isothermal liquids, then the system of equations for describing convective flows contains an additional scalar equation (heat equation). Thus, finding exact solutions to the Oberbeck–Boussinesq equations is a more difficult problem compared to the exact integration of the Navier–Stokes equations for Newtonian fluids [1]. At the same time, it is useful to recall that the Oberbeck–Boussinesq equations are obviously approximate. First, the change in density is taken into account only for the Archimedes force. In the terms for acceleration, the derivative is assumed to be equal to the constant value [1]. Secondly, the continuity equation is replaced by the incompressibility equation, that is, the compressibility of the liquid due to thermal effects is not taken into account [1].

The first researcher to draw attention to the importance of convection in the study of motion was Thomson [1]. Benard began to systematically study convection through the methods of experimental hydrodynamics. In his experiments, Benard considered Rayleigh convection (the influence of gravitational instability on the structure of a hydrodynamic flow) to be dominant [66–68]. Later studies revealed the influence of not only the body force of Archimedes, but also the surface force of Marangoni [1,14,16,66–68].

The exact solution of the equations describing thermocapillary convection was first found by Birikh [9]. The history of the development and generalization of the Birikh solution and the interpretation of the physical meaning of the mechanisms of thermocapillary convection can be traced in [10,13–18,41,43–45]. Note that the question of describing thermocapillary flows near extreme temperatures, in which the horizontal temperature gradient is equal to zero, still remains open. The known solutions do not allow an exhaustive study of the announced class of flows. Thus, the goal of this work is to construct an exact solution of the Oberbeck–Boussinesq thermocapillary convection system that describes an extreme temperature field [69–77].

In order to understand the mechanisms of convection, it is important to have a large library of exact Oberbeck–Boussinesq solutions. Theoretical research in this area has been carried out by the pioneering scientific publications of Ostroumov and Birikh [8,9]. The papers [10–19,78] lay the foundation for unidirectional convective flows. To date, several classes of exact solutions of the three-dimensional system of Oberbeck–Boussinesq equations describing the flow of a viscous incompressible fluid have been obtained [78–84]. The main idea in constructing classes of exact solutions of the Navier–Stokes equations is related to the modification of the velocity field, which depends linearly on the spatial acceleration. A generalization of the exact Ostroumov–Birikh solution for layered and shear flows has been carried out in [78–84].

A significant gap in the study of horizontal convection is observed in the study of binary fluids. In this case, the cross-dissipative effects of Soret and Dufour play certain roles [85,86]. In most studies, when the Dufour effect in a liquid is neglected, we can explain this by the fact that it is small [87–94]. However, the Dufour effect must be taken into account when moving gases, even if the mechanical flow rates of the gas are small (the gas is incompressible), but the characteristic rates of diffusion and thermal diffusion are significantly higher than in a liquid [87–95].

4. Equations of Motion to Describe Thermal Diffusion

The description of the hydrodynamics of a binary incompressible mixture is based on the use of the Navier–Stokes and incompressibility equations, supplemented by the energy equation and the conservation equation for the light component of the mixture [1]:

$$
\begin{gathered}
\rho \frac{d\mathbf{V}}{dt} = -\nabla p + \nu \Delta \mathbf{V}, \\
\nabla \cdot \mathbf{V} = 0, \\
\rho T \frac{dS}{dt} = -\nabla \cdot \mathbf{q} + \mu \nabla \cdot \mathbf{j}, \\
\rho \frac{dC}{dt} = -\nabla \cdot \mathbf{j}.
\end{gathered} \tag{1}
$$

Here in Equation (1), $\mathbf{V}(t, x, y, z) = \left(V_x, V_y, V_z\right)$ is the velocity vector; p is the pressure; ρ is the density; and ν is the kinematic viscosity of the mixture, $\nabla = \mathbf{i}_1 \frac{\partial}{\partial x} + \mathbf{i}_2 \frac{\partial}{\partial y} + \mathbf{i}_3 \frac{\partial}{\partial z}$ is the Hamilton operator; $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ are the orts of the Cartesian rectangular coordinate system; $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplace operator; $\frac{d}{dt} = \frac{\partial}{\partial t} + \nabla \cdot \mathbf{V}$ is the total derivative (individual derivative, material derivative, particle derivative or derivative along the trajectory), consisting of the sum of the local derivative (local derivative) and the convective derivative [1]; C is the concentration of the light component; T is the absolute temperature; S is the entropy of a mass unit; μ is the chemical potential of the mixture; $\mathbf{q}$ is the heat flux density; and $\mathbf{j}$ is the density of the diffusion flux of the matter of the light component [1].

For further derivation of the equations of hydrodynamics of an inhomogeneous fluid, we will use the Boussinesq approximation [1]. When studying various fluid flows, it is necessary, in the vast majority of cases, to take into account the influence of temperature and the concentration of substances dissolved in the fluid on the structure of hydrodynamic field profiles. Convective currents are the most common movements in the universe [1]. When the temperature changes, a variety of processes occur in nature, space and industry.

Accounting for temperature stratification leads to significant complication of the equations of motion [1]. It is established that in order to achieve an equilibrium state of an unevenly heated liquid in a conservative gravity field, a necessary and sufficient condition is the existence of one vertical component of the temperature gradient in the absence of horizontal components [1]. The description of thermal diffusion, as well as the study of convection, is a consequence of the integration of the Oberbeck–Boussinesq equations. The assumptions and errors of mathematical modeling were indicated above, but these equations have proven themselves in solving real problems and have been repeatedly verified experimentally [1,87–95]. They contain the simplest dependence of density on temperature, and this dependence is taken into account only in the Archimedes force. However, these equations have proven to be extremely useful tools for modeling thermal processes in fluids. A detailed historical and bibliographic review on this topic can be found in [1]. Note that [1,87–95] shows the possibility of a linear approximation of the dependence of density on the concentration of a dissolved substance, similar to thermal convection:

$$\rho = \rho_0 (1 - \beta_1 T - \beta_2 C).$$

Here, ρ_0 is the density of a mixture of liquids at average values of temperature and concentration, β_1 and β_2 are the coefficients of thermal and concentration expansion of the liquid, respectively. Note that the inequality $\beta_2 > 0$ is true, since the light component of the liquid mixture is considered.

5. The Possibility of Constructing a New Exact Solution Based on a Known Solution

By analogy with the derivation of the Oberbeck–Boussinesq equations, we present the equations describing the thermal diffusion flows of a viscous incompressible fluid [1]:

$$\begin{gathered}\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} = -\nabla P + \nu \Delta \mathbf{V} + g(\beta_1 T + \beta_2 C)\mathbf{i}_3,\\ \frac{\partial T}{\partial t} + (\mathbf{V} \cdot \nabla)T = \left(\chi + \alpha^2 dn\right)\Delta T + \alpha dn \Delta C,\\ \frac{\partial C}{\partial t} + (\mathbf{V} \cdot \nabla)C = d\Delta C + \alpha d \Delta T,\\ \nabla \cdot \mathbf{V} = 0.\end{gathered} \tag{2}$$

In the system of Equation (2), the following notations are introduced: P is the pressure deviation from hydrostatic pressure divided by the constant average fluid density; g is acceleration due to gravity; χ, d, α are the coefficients of thermal diffusivity, diffusion and thermal diffusion; and $n = \left[\frac{T}{c_p}\left(\frac{\partial \mu}{\partial C}\right)_{T,P}\right]_0$ is the thermodynamic parameter [1].

The system of Equation (2) is the object of further study. Note that when finding exact solutions to the Oberbeck–Boussinesq system, group analysis is often used. The method of finding symmetries in the equations of mathematical physics was presented for the first time in the works of the Norwegian mathematician Lee. Further development of this methodology for equations was carried out. This theory was significantly developed in the works of L. V. Ovsyannikov [96–98], N. Kh. Ibragimov [98–101], V. Fushchich [102,103], P. Olver [104], J. Bluman [105,106], B. Cantwell [107] and others [108–112]. For the mechanics of a viscous incompressible fluid, the methods of group analysis were systematically used by Pukhnachev [7,113]. The articles in which the methods of local and nonlocal symmetries of the motion of liquids and gases in various force fields were developed [96–113] should be noted. These methods can be used to multiply the exact solution by the previously calculated exact solution [96–113].

Let us present the main invariant (group) properties of system (2), which help find exact solutions for a previously calculated solution [7]:

1. Invariance of equations with respect to shift in all independent variables;
2. Invariance of equations with respect to consistent stretching in independent and dependent variables;
3. Invariance of rotation around one of the coordinate axes of the coordinate system;
4. Invariance of the Galilean transformation.

6. Probably the Most General Class of Exact Solutions for Describing Thermal Diffusion Flows

By projecting the system of Equation (2), written in an invariant form, onto the axes of a rectangular Cartesian coordinate system, we can obtain the following non-stationary system of nonlinear partial differential equations:

$$\begin{gathered}
\frac{\partial V_x}{\partial t} + V_x\frac{\partial V_x}{\partial x} + V_y\frac{\partial V_x}{\partial y} + V_z\frac{\partial V_x}{\partial z} = -\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right),\\
\frac{\partial V_y}{\partial t} + V_x\frac{\partial V_y}{\partial x} + V_y\frac{\partial V_y}{\partial y} + V_z\frac{\partial V_y}{\partial z} = -\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_y}{\partial z^2}\right),\\
\frac{\partial V_z}{\partial t} + V_x\frac{\partial V_z}{\partial x} + V_y\frac{\partial V_z}{\partial y} + V_z\frac{\partial V_z}{\partial z}\\
= -\frac{\partial P}{\partial z} + \nu\left(\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2}\right) + g(\beta_1 T + \beta_2 c),\\
\frac{\partial T}{\partial t} + V_x\frac{\partial T}{\partial x} + V_y\frac{\partial T}{\partial y} + V_z\frac{\partial T}{\partial z}\\
= \left(\chi + \alpha^2 dn\right)\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \alpha dn\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2}\right),\\
\frac{\partial C}{\partial t} + V_x\frac{\partial C}{\partial x} + V_y\frac{\partial C}{\partial y} + V_z\frac{\partial C}{\partial z}\\
= \alpha\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2}\right) + \alpha d\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2}\right),\\
\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 0.
\end{gathered} \tag{3}$$

In article [41], a class of exact solutions of the Navier–Stokes system for an incompressible fluid in the Boussinesq approximation (Oberbeck–Boussinesq systems for thermal diffusion equations) was proposed, written in the form (2) or (3):

$$\begin{gathered}
V_x(x,y,z,t) = U(z,t) + xu_1(z,t) + yu_2(z,t),\\
V_y(x,y,z,t) = V(z,t) + xv_1(z,t) + yv_2(z,t),\\
V_z(z,t) = w(z,t),\\
P(x,y,z,t) = P_0(z,t) + xP_1(z,t) + yP_2(z,t)\\
+ \frac{x^2}{2}P_{11}(z,t) + \frac{y^2}{2}P_{22}(z,t) + xyP_{12}(z,t),\\
T(x,y,z,t) = T_0(z,t) + xT_1(z,t) + yT_2(z,t)\\
+ \frac{x^2}{2}T_{11}(z,t) + \frac{y^2}{2}T_{22}(z,t) + xyT_{12}(z,t),\\
C(x,y,z,t) = C_0(z,t) + xC_1(z,t) + yC_2(z,t)\\
+ \frac{x^2}{2}C_{11}(z,t) + \frac{y^2}{2}C_{22}(z,t) + xyC_{12}(z,t).
\end{gathered} \tag{4}$$

Expressions (4) for hydrodynamic fields, which are a generalization of the method of separation of variables [7,114–116], represent a new class of exact solutions of the Navier–Stokes equations in the Boussinesq approximation, which is the most general and contains all the exact solutions known so far, except for some cases. Substituting the representations of hydrodynamic fields (4) into the system of Equation (3), we obtain the following system:

$$\begin{gathered}
\frac{\partial U}{\partial t} + x\frac{\partial u_1}{\partial t} + y\frac{\partial u_2}{\partial t} + (U + xu_1 + yu_2)u_1 + (V + xv_1 + yv_2)u_2 +\\
w\left(\frac{\partial U}{\partial z} + x\frac{\partial u_1}{\partial z} + y\frac{\partial u_2}{\partial z}\right) = -(P_1 + xP_{11} + yP_{12}) + \nu\left(\frac{\partial^2 U}{\partial z^2} + x\frac{\partial^2 u_1}{\partial z^2} + y\frac{\partial^2 u_2}{\partial z^2}\right),\\
\frac{\partial V}{\partial t} + x\frac{\partial v_1}{\partial t} + y\frac{\partial v_2}{\partial t} + (U + xu_1 + yu_2)v_1 + (V + xv_1 + yv_2)v_2 +\\
w\left(\frac{\partial V}{\partial z} + x\frac{\partial v_1}{\partial z} + y\frac{\partial v_2}{\partial z}\right) = -(P_2 + yP_{22} + xP_{12}) + \nu\left(\frac{\partial^2 V}{\partial z^2} + x\frac{\partial^2 v_1}{\partial z^2} + y\frac{\partial^2 v_2}{\partial z^2}\right),
\end{gathered}$$

$$
\begin{gathered}
\frac{\partial w}{\partial t}+w\frac{\partial w}{\partial z}=\frac{\partial P_0}{\partial z}+x\frac{\partial P_1}{\partial z}+y\frac{\partial P_2}{\partial z}+\frac{x^2}{2}\frac{\partial P_{11}}{\partial z}+\frac{y^2}{2}\frac{\partial P_{22}}{\partial z}+xy\frac{\partial P_{12}}{\partial z}+\nu\frac{\partial^2 w}{\partial z^2} \\
+g\beta_1\left(T_0+xT_1+yT_2+\frac{x^2}{2}T_{11}+\frac{y^2}{2}T_{22}+xyT_{12}\right) \\
+g\beta_2\left(C_0+xC_1+yC_2+\frac{x^2}{2}C_{11}+\frac{y^2}{2}C_{22}+xyC_{12}\right), \\
\frac{\partial T_0}{\partial t}+x\frac{\partial T_1}{\partial t}+y\frac{\partial T_2}{\partial t}+\frac{x^2}{2}\frac{\partial T_{11}}{\partial t}+\frac{y^2}{2}\frac{\partial T_{22}}{\partial t}+xy\frac{\partial T_{12}}{\partial t} \\
+(U+xu_1+yu_2)(T_1+xT_{11}+yT_{12})+(V+xv_1+yv_2)(T_2+yT_{22}+xT_{12}) \\
+w\left(\frac{\partial T_0}{\partial z}+x\frac{\partial T_1}{\partial z}+y\frac{\partial T_2}{\partial z}+\frac{x^2}{2}\frac{\partial T_{11}}{\partial z}+\frac{y^2}{2}\frac{\partial T_{22}}{\partial z}+xy\frac{\partial T_{12}}{\partial z}\right) \\
=\left(\chi+\alpha^2 dn\right)\left(T_{11}+T_{22}+\frac{\partial^2 T_0}{\partial z^2}+x\frac{\partial^2 T_1}{\partial z^2}+y\frac{\partial^2 T_2}{\partial z^2}+\frac{x^2}{2}\frac{\partial^2 T_{11}}{\partial z^2}+\frac{y^2}{2}\frac{\partial^2 T_{22}}{\partial z^2}+xy\frac{\partial^2 T_{12}}{\partial z^2}\right) \\
+\alpha dn\left(C_{11}+C_{22}+\frac{\partial^2 C_0}{\partial z^2}+x\frac{\partial^2 C_1}{\partial z^2}+y\frac{\partial^2 C_2}{\partial z^2}+\frac{x^2}{2}\frac{\partial^2 C_{11}}{\partial z^2}+\frac{y^2}{2}\frac{\partial^2 C_{22}}{\partial z^2}+xy\frac{\partial^2 C_{12}}{\partial z^2}\right), \\
\frac{\partial C_0}{\partial t}+x\frac{\partial C_1}{\partial t}+y\frac{\partial C_2}{\partial t}+\frac{x^2}{2}\frac{\partial C_{11}}{\partial t}+\frac{y^2}{2}\frac{\partial C_{22}}{\partial t}+xy\frac{\partial C_{12}}{\partial t} \\
+(U+xu_1+yu_2)(C_1+C_{11}x+C_{12}y)+(V+xv_1+yv_2)(C_2+C_{12}x+C_{22}y) \\
+w\left(\frac{\partial c_0}{\partial z}+x\frac{\partial c_1}{\partial z}+y\frac{\partial c_2}{\partial z}+\frac{x^2}{2}\frac{\partial c_{11}}{\partial z}+\frac{y^2}{2}\frac{\partial c_{22}}{\partial z}+xy\frac{\partial c_{12}}{\partial z}\right) \\
=\alpha\left(T_{11}+T_{22}+\frac{\partial^2 T_0}{\partial z^2}+x\frac{\partial^2 T_1}{\partial z^2}+y\frac{\partial^2 T_2}{\partial z^2}+\frac{x^2}{2}\frac{\partial^2 T_{11}}{\partial z^2}+\frac{y^2}{2}\frac{\partial^2 T_{22}}{\partial z^2}+xy\frac{\partial^2 T_{12}}{\partial z^2}\right) \\
+\alpha d\left(C_{11}+C_{22}+\frac{\partial^2 C_0}{\partial z^2}+x\frac{\partial^2 C_1}{\partial z^2}+y\frac{\partial^2 C_2}{\partial z^2}+\frac{x^2}{2}\frac{\partial^2 C_{11}}{\partial z^2}+\frac{y^2}{2}\frac{\partial^2 C_{22}}{\partial z^2}+xy\frac{\partial^2 C_{12}}{\partial z^2}\right), \\
\frac{\partial w}{\partial z}+u_1+v_2=0.
\end{gathered}
$$

Equating the coefficients in polynomial expressions to zero, we obtain the following system, consisting of twenty-five non-stationary nonlinear partial differential equations, to determine the twenty-five unknown functions:

$$\overset{\frown}{L}U+Uu_1+Vu_2+P_1=0,\ \overset{\frown}{L}u_1+u_1^2+v_1u_2+P_{11}=0,$$

$$\overset{\frown}{L}u_2+u_2u_1+v_2u_2+P_{12}=0,$$

$$\overset{\frown}{L}V+Uv_1+Vv_2+P_2=0,$$

$$\overset{\frown}{L}v_1+v_1u_1+v_1v_2+P_{12}=0,$$

$$\overset{\frown}{L}v_2+u_2v_1+v_2^2+P_{22}=0,$$

$$\overset{\frown}{L}w+\frac{\partial P_0}{\partial z}-g(\beta_1T_0+\beta_2C_0)=0,$$

$$\frac{\partial P_1}{\partial z}=g(\beta_1T_1+\beta_2C_1),\ \frac{\partial P_2}{\partial z}=g(\beta_1T_2+\beta_2C_2),$$

$$\frac{\partial P_{11}}{\partial z}=g(\beta_1T_{11}+\beta_2C_{11}),\ \frac{\partial P_{12}}{\partial z}=g\beta(\beta_1T_{12}+\beta_2C_{12}),$$

$$\frac{\partial P_{22}}{\partial z}=g(\beta_1T_{22}+\beta_2C_{22}),$$

$$\overset{\frown}{M}T_0+UT_1+VT_2-\left(\chi+\alpha^2dn\right)(T_{11}+T_{22})-\alpha dn\left(C_{11}+C_{22}+\frac{\partial^2 C_0}{\partial z^2}\right)=0,$$

$$\overset{\frown}{M}T_1 + UT_{11} + VT_{12} + u_1T_1 + v_1T_2 - \alpha dn\frac{\partial^2 C_1}{\partial z^2} = 0,$$

$$\overset{\frown}{M}T_2 + UT_{12} + VT_{22} + u_2T_1 + v_2T_2 - \alpha dn\frac{\partial^2 C_2}{\partial z^2} = 0,$$

$$\overset{\frown}{M}T_{11} + 2u_1T_{11} + 2v_1T_{12} - \alpha dn\frac{\partial^2 C_{11}}{\partial z^2} = 0,$$

$$\overset{\frown}{M}T_{22} + 2u_2T_{12} + 2v_2T_{22} - \alpha dn\frac{\partial^2 C_{22}}{\partial z^2} = 0,$$

$$\overset{\frown}{M}T_{12} + u_1T_{12} + u_2T_{11} + v_1T_{22} + v_2T_{12} - \alpha dn\frac{\partial^2 C_{12}}{\partial z^2} = 0,$$

$$\hat{N}C_0 + UC_1 + VC_2 - d(C_{11} + C_{22}) - \alpha d\left(T_{11} + T_{22} + \frac{\partial^2 T_0}{\partial z^2}\right) = 0,$$

$$\hat{N}C_1 + UC_{11} + VC_{12} + u_1C_1 + v_1C_2 - \alpha d\frac{\partial^2 T_1}{\partial z^2} = 0,$$

$$\hat{N}C_2 + UC_{12} + VC_{22} + u_2C_1 + v_2C_2 - \alpha d\frac{\partial^2 T_2}{\partial z^2} = 0,$$

$$\hat{N}c_{11} + 2u_1c_{11} + 2v_1c_{12} - \alpha d\frac{\partial^2 T_{11}}{\partial z^2} = 0,$$

$$\hat{N}c_{22} + 2u_2c_{12} + 2v_2c_{22} - \alpha d\frac{\partial^2 T_{22}}{\partial z^2} = 0,$$

$$\hat{N}c_{12} + u_1c_{12} + u_2c_{11} + v_1c_{22} + v_2c_{12} - \alpha d\frac{\partial^2 T_{12}}{\partial z^2} = 0,$$

$$\frac{\partial w}{\partial z} + u_1 + v_2 = 0. \tag{5}$$

To shorten the notations and clarify their physical meaning, the resulting system of quasilinear partial differential Equation (5), which inherits the nonlinear properties of the Oberbeck–Boussinesq system (1), is written in operator form. Parabolic partial differential operators of heat conduction type with a convective term are written as follows:

$$\begin{gathered}\overset{\frown}{L} = \tfrac{\partial}{\partial t} + w\tfrac{\partial}{\partial z} - \nu\tfrac{\partial^2}{\partial z^2}\\ \overset{\frown}{M} = \tfrac{\partial}{\partial t} + w\tfrac{\partial}{\partial z} - \left(\chi + \alpha^2 dn\right)\tfrac{\partial^2}{\partial z^2}\\ \hat{N} = \tfrac{\partial}{\partial t} + w\tfrac{\partial}{\partial z} - \alpha\tfrac{\partial^2}{\partial z^2}\end{gathered} \tag{6}$$

A characteristic feature of the system of Equation (5) is the presence, in the heat equation of system (5), of a term containing the concentration. Systems (1) and (2), as a consequence of Equation (4), allow one to describe the fluid motions while considering the cross-dissipative Soret and Dufour effects [1,2,87–107].

When solving system (4), it is necessary to formulate the initial and boundary conditions for solving boundary and initial-boundary problems of fluid mechanics within the class of exact solutions (4). For this, it is necessary and sufficient to require the representation of the boundary conditions in the class (4). Boundary conditions can be written for class (4) in the form of specifying a non-uniform distribution of velocities and stresses at the boundaries, by setting quadratic forms of pressure and concentration at the boundaries, using the thermocapillary effect in describing thermal diffusion flows and setting many other types of boundary conditions.

Many papers investigate large-scale flow (fluid flow in a thin layer), where one of the geometric variables is negligibly small compared to the others. Then, as a hydrodynamic model, one can use a flat horizontal layer, the fluid motion of which is a class of Couette flows. Earlier in [8,9], exact solutions were constructed for various boundary value problems while taking into account the convection effect. In papers [10–19], the class of exact solutions published in [8,9] was taken as a basis. At present, the family of exact solutions announced in [14,16,36,37,41] is the widest known polynomial exact solution of the equations of hydrodynamics. A bibliographic overview of obtaining this class of solutions is contained in [14,16,36,37,41] and in the bibliographic references to the article.

Studies within the class of exact solutions for studying the flows of binary fluids with allowance for the Soret effect and neglecting the cross-Dufour effect were published in [41]. In the development of the considered class of exact solutions [44,59,65], diffusion processes were added, making it possible to fully and more accurately describe fluid flows and evaluate their influence on the formation of countercurrents in a swirling fluid.

Let us just mention some of the main works in this area. The works in refs. [1–5] are noteworthy, where it is shown that taking into account weak compressibility leads to effects that cannot be satisfactorily described by the Boussinesq approximation. This is also discussed in [1], where experimental data are presented that allow us to assert the possibility of using the Boussinesq approximation for compressible fluids. Another area of research on convection can be considered regarding the movement of a fluid under conditions of reduced gravity (weightlessness). Studies on this topic are displayed in [1–5]. The study of exact solutions of the equations of natural convection began much later than the properties of the isothermal Navier–Stokes equations. Until now, each new exact solution of the Oberbeck–Boussinesq equations can be regarded as an event that allows researchers a fresh look at the qualitative and quantitative characteristics of a moving stream. The complexity of calculating the exact solutions of the Oberbeck–Boussinesq system is due to the non-linearity of the convective temperature derivative in the heat equation.

The first class of exact solutions of the equations of natural convection can be considered to be the Ostroumov–Birikh (Birikh–Ostroumov) class [8,9]. In his book, G. A. Ostroumov [8] focused on what the unpublished solution of N. K. Guk represented. Despite this, the name of the class of solutions indicated above has stuck in the literature. G.A. Ostroumov described the motion of a fluid in a cylindrical pipe induced by a longitudinal temperature gradient [8].

The Birikh flow [9] describes the stationary convective flow of a viscous incompressible fluid in an infinite vertical or horizontal layer. Recall that the motions of a viscous incompressible Ostroumov–Birikh (Birikh–Ostroumov) fluid belong to the class of velocities linearly dependent on part of the coordinates for convective motions (the Lin–Sidorov–Aristov class) [35–37]. The properties of the Ostroumov–Birikh (Birikh–Ostroumov) flow [8,9] are presently being studied in great detail [35–37]. The group nature of solutions was studied in [87–94], and in the same article, non-stationary analogs of these flows are given. In the articles [41,87–94], generalizations of the Ostroumov–Birikh flow for binary fluids, which are unidirectional flows, are constructed. In [15,94], the possibility of using this class in modeling evaporation processes is shown. Thermocapillary flows of a viscous incompressible fluid were exhaustively reflected in the reviews by Napolitano [10], O.N. Goncharova and O.A. Kabov [15]. Articles [15,94] contain information that supplements the material about the nature of such flows. This class of exact solutions has been repeatedly studied for stability with respect to various types of perturbations. There is nothing surprising in this, since the onset of convection can be understood as the loss of stability of the isothermal motion of a fluid when a thermal disturbance is introduced. Information of this kind can be found in articles that reflect the main areas of research, not only in purely convective motion, but also in electric, vibrational, magnetic and other fields [14,16]. Here, it is appropriate to single out a series of works by V.B. Bekezhanova, who used the study of the stability of the Ostroumov–Birikh flow in modeling convection in Lake Baikal [94]. These works are surprising in that the author shows that in certain cases, the Soret effect can be neglected, while the Dufour effect is decisive in the thermal diffusion flow for liquids. Close to the Ostroumov–Birikh class is the Shliomis class [94]. Generalizations of these solutions were carried out by S.N. Aristov and A.F. Sidorov [36,37]. Note that the structure of velocities, temperature and pressure has found application in geophysical hydrodynamics. Interesting works in this area are by S.N. Aristov and K.G. Schwartz [14,16], A.G. Kirdyashkin [117], Kuo [118,119] and others.

7. Conclusions

A review has been made of exact solutions for describing the thermodiffusion of a viscous incompressible fluid in the articles given. The authors of this review have obtained the most general class of exact solutions to date. This class includes almost all known exact solutions. They are based on special cases (the Ostroumov–Birikh and Gershuni–Betchelov families), announced, as a rule, first for Marangoni convection. An important component of the Lin–Sidorov–Aristov class is generalizations of the Hiemenz–Ryabushinsky and von Karman families. The exact solutions obtained in various articles are important for geophysical fluid dynamics, microfluidics and nanofluidics. The exact solutions announced in this review will be useful when testing new boundary conditions and computer programs. And, most importantly, they will find their use in describing the processes of growing crystals for microelectronics and simulating the processes of the rotation of liquid matter where direct measurements are impossible. Exact solutions will help to understand the problem of describing topologically complex flows, which are now called turbulent. Let us also mention works such as [120–125], where these references are within the framework of the semi-analytical and analytical approaches to the study of mathematical hydrodynamical models.

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