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On Traces of Linear Operators with Symmetrized Volterra-Type Kernels

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Abstract: A solution to the trace convergence problem, which arises in proving the mean-square convergence for the approximation of iterated Stratonovich stochastic integrals, is proposed. This approximation is based on the representation of factorized Volterra-type functions as the orthogonal series. Solving the trace convergence problem involves the theory of trace class operators for symmetrized Volterra-type kernels. The main results are primarily focused on the approximation of iterated Stratonovich stochastic integrals, which are used to implement numerical methods for solving stochastic differential equations based on the Taylor–Stratonovich expansion.

Keywords: iterated Stratonovich stochastic integral; multiple Stratonovich stochastic integral; orthogonal expansion; stochastic differential equation; symmetrization; symmetrized Volterra-type kernel; trace class operator; trace convergence problem

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1. Introduction

For the analysis of dynamical systems whose mathematical models include stochastic differential equations, numerical methods for their solution are often used. To obtain high accuracy of the approximation of output processes, it is necessary to apply numerical methods with high order of strong or mean-square convergence based on Taylor–Itô or Taylor–Stratonovich expansions [1–4]. These methods involve modeling iterated Itô or Stratonovich stochastic integrals, respectively, which can be represented as multiple stochastic integrals of functions

$$\begin{aligned}\mathbb{k}_{n_1\ldots n_k}(t_1,\ldots,t_k) &= (t_1-t_0)^{n_1}\ldots(t_k-t_0)^{n_k}\mathbf{1}(t_k-t_{k-1})\ldots\mathbf{1}(t_2-t_1)\\ &= \begin{cases}(t_1-t_0)^{n_1}\ldots(t_k-t_0)^{n_k} & \text{for } t_1<\ldots<t_k\\ 0 & \text{otherwise,}\end{cases}\end{aligned} \tag{1}$$

where $k\in\mathbb{N}=\{1,2,\ldots\}$, $t_1,\ldots,t_k\in\mathbb{T}=[t_0,T]$ and $n_1,\ldots,n_k\in\{0,1,2,\ldots\}$, $t_0\geqslant 0$, $\mathbf{1}$ is the unit step function:

$$\mathbf{1}(t)=\begin{cases}1 & \text{for } t>0\\ 0 & \text{for } t\leqslant 0.\end{cases}$$

If $n_1=\ldots=n_k=0$, then the notation $\mathbb{k}$ is used instead of $\mathbb{k}_{0\ldots 0}$ for simplicity.

In the general theory of multiple and iterated stochastic integrals, a wider class of functions is of interest:

$$\begin{aligned}\mathbb{k}_{\psi}(t_1,\ldots,t_k) &= \psi_1(t_1)\ldots\psi_k(t_k)\mathbf{1}(t_k-t_{k-1})\ldots\mathbf{1}(t_2-t_1)\\ &= \begin{cases}\psi_1(t_1)\ldots\psi_k(t_k) & \text{for } t_1<\ldots<t_k\\ 0 & \text{otherwise,}\end{cases}\end{aligned} \tag{2}$$

where $\psi_1,\ldots,\psi_k\in L_2(\mathbb{T})$. The function $\mathbb{k}_\psi$ is called the factorized Volterra-type function [5]. Here and further, $L_2(\mathbb{X})$ denotes the space of square-integrable functions $f\colon\mathbb{X}\to\mathbb{R}$, $\mathbb{R}=(-\infty,+\infty)$. Also, the space of continuous functions $f\colon\mathbb{X}\to\mathbb{R}$ is denoted by $C(\mathbb{X})$.

The multiple Itô stochastic integral introduced in [6] is defined for any function $f \in L_2(\mathbb{T}^k)$, and such an integral can be represented as a multiple random series [7]. The multiple Stratonovich stochastic integral [8,9] is more complicated, and when it is represented as a multiple random series, the trace convergence problem arises [10]. The representation of multiple stochastic integrals by multiple random series is based on the expansion of f in the orthogonal series using a basis of $L_2(\mathbb{T})$.

For clarity, we can use an analogy with the theory of linear operators. In fact, the differences between functions, for which we can define multiple Itô and Stratonovich stochastic integrals, are partly similar to the differences between Hilbert–Schmidt operators and trace class operators [11], respectively.

If we consider multiple Itô and Stratonovich stochastic integrals for numerical methods, which are used for solving stochastic differential equations, then it is enough to restrict ourselves to functions (2). In fact, we can only study functions (1) for this purpose [3]. However, multiple (or iterated) stochastic integrals should be specified with respect to all possible combinations of components of the multidimensional Wiener process [1,3,12]. In [3,13,14], the trace convergence problem in this context is studied in detail with additional smoothness conditions on the weights $\psi_1, \dots, \psi_k$ and with a restriction on both a parameter k and a basis of $L_2(\mathbb{T})$.

Note that some multiple Stratonovich stochastic integrals coincide with corresponding multiple Itô stochastic integrals, but for other Stratonovich integrals, the trace convergence problem remains. One variant of the briefly described problem is solved in this paper for the weights from $L_2(\mathbb{T})$ and without additional restrictions on both a parameter k and a basis of $L_2(\mathbb{T})$.

The motivation for this study is that the solution to the trace convergence problem provides a theoretical basis for the representation of iterated Stratonovich stochastic integrals as multiple random series and for their mean-square approximation based on partial sums of these series. This is an important component for the implementation of numerical methods for solving stochastic differential equations with high order of strong or mean-square convergence. Such methods can be used to simulate stochastic processes in different fields [15–17].

2. Preliminary Discussion and Problem Statement

Let $j_1, \dots, j_k \in \{1, 2, \dots, s\}$ and $W_1, \dots, W_s$ be independent Wiener processes defined on a probability space $(\Omega, \mathfrak{S}, \mathrm{P})$. Denote by ${}^{\mathrm{I}}\mathcal{J}_{\mathbb{T}}^{W(j_1 \dots j_k)}$ and ${}^{\mathrm{S}}\mathcal{J}_{\mathbb{T}}^{W(j_1 \dots j_k)}$ two linear operators that establish a correspondence between a function and multiple stochastic integrals for that function. The operator ${}^{\mathrm{I}}\mathcal{J}_{\mathbb{T}}^{W(j_1 \dots j_k)}$ corresponds to the multiple Itô stochastic integral, and the operator ${}^{\mathrm{S}}\mathcal{J}_{\mathbb{T}}^{W(j_1 \dots j_k)}$ corresponds to the multiple Stratonovich stochastic integral, $\mathbb{T} = [t_0, T]$:

$$
\begin{aligned}
{}^{\mathrm{I}}\mathcal{J}_{\mathbb{T}}^{W(j_1 \dots j_k)} f &= \int_{\mathbb{T}^k} f(t_1, \dots, t_k) dW_{j_1}(t_1) \dots dW_{j_k}(t_k), \\
{}^{\mathrm{S}}\mathcal{J}_{\mathbb{T}}^{W(j_1 \dots j_k)} f &= \int_{\mathbb{T}^k} f(t_1, \dots, t_k) \circ dW_{j_1}(t_1) \circ \dots \circ dW_{j_k}(t_k),
\end{aligned}
$$

where k is the integral multiplicity, which is the same as the number of arguments of f, and the symbol $\circ$ is to distinguish Itô and Stratonovich stochastic integrals.

Further, we use the following notations: $\{q_i\}_{i=0}^{\infty}$ is an orthonormal basis of $L_2(\mathbb{T})$, and $\zeta_i^{(j)}$ are independent random variables having standard normal distribution for $i = 0, 1, 2, \dots$ and $j = 1, \dots, s$. Then, according to properties of multiple stochastic integrals [18,19], we have

$$
\begin{aligned}
{}^{\mathrm{I}}\mathcal{J}_{\mathbb{T}}^{W(j_1 \dots j_k)} q_{i_1} \otimes \dots \otimes q_{i_k} &= \zeta_{i_1}^{(j_1)} * \dots * \zeta_{i_k}^{(j_k)}, \\
{}^{\mathrm{S}}\mathcal{J}_{\mathbb{T}}^{W(j_1 \dots j_k)} q_{i_1} \otimes \dots \otimes q_{i_k} &= \zeta_{i_1}^{(j_1)} \dots \zeta_{i_k}^{(j_k)},
\end{aligned}
$$

where $*$ means the Wick product defined for this case in terms of Hermite polynomials [20,21], $i_1,\ldots,i_k=0,1,2,\ldots$

If $f\in L_2(\mathbb{T}^k)$, then this function can be represented as the orthogonal series [22], i.e.,

$$f=\sum_{i_1,\ldots,i_k=0}^{\infty}F_{i_1\ldots i_k}q_{i_1}\otimes\ldots\otimes q_{i_k},$$

where

$$F_{i_1\ldots i_k}=(q_{i_1}\otimes\ldots\otimes q_{i_k},f)_{L_2(\mathbb{T}^k)},\quad i_1,\ldots,i_k=0,1,2,\ldots \tag{3}$$

Formally (without considering the convergence issues), we can write that

$$ {}^{\mathrm{I}}\mathcal{J}_{\mathbb{T}}^{W(j_1\ldots j_k)}f=\sum_{i_1,\ldots,i_k=0}^{\infty}F_{i_1\ldots i_k}\zeta_{i_1}^{(j_1)}*\ldots*\zeta_{i_k}^{(j_k)},$$

$$ {}^{\mathrm{S}}\mathcal{J}_{\mathbb{T}}^{W(j_1\ldots j_k)}f=\sum_{i_1,\ldots,i_k=0}^{\infty}F_{i_1\ldots i_k}\zeta_{i_1}^{(j_1)}\ldots\zeta_{i_k}^{(j_k)},$$

and multiple Itô and Stratonovich stochastic integrals do not generally coincide.

Consider a simple example for stochastic integrals of multiplicity $k=2$ under condition $j_1=j_2$. Let $f\in L_2(\mathbb{T}^2)$ and

$$f=\sum_{i_1,i_2=0}^{\infty}F_{i_1i_2}q_{i_1}\otimes q_{i_2},$$

where

$$F_{i_1i_2}=(q_{i_1}\otimes q_{i_2},f)_{L_2(\mathbb{T}^2)},\quad i_1,i_2=0,1,2,\ldots \tag{4}$$

It is known [6,7] that

$$ {}^{\mathrm{I}}\mathcal{J}_{\mathbb{T}}^{W(j_1j_1)}f=\sum_{i_1,i_2=0}^{\infty}F_{i_1i_2}\zeta_{i_1}^{(j_1)}*\zeta_{i_2}^{(j_1)},$$

where the multiple random series on the right-hand side converges in the mean-square sense (the equivalent relation is given in [3] using different notations). But the multiple random series on the right-hand side of the equality

$$ {}^{\mathrm{S}}\mathcal{J}_{\mathbb{T}}^{W(j_1j_1)}f=\sum_{i_1,i_2=0}^{\infty}F_{i_1i_2}\zeta_{i_1}^{(j_1)}\zeta_{i_2}^{(j_1)}$$

can diverge, and the equality itself may not make sense, since

$$\sum_{i_1,i_2=0}^{\infty}F_{i_1i_2}\zeta_{i_1}^{(j_1)}\zeta_{i_2}^{(j_1)}=\sum_{i_1,i_2=0}^{\infty}F_{i_1i_2}\zeta_{i_1}^{(j_1)}*\zeta_{i_2}^{(j_1)}+\sum_{i=0}^{\infty}F_{ii}\quad\left(\zeta_{i_1}^{(j_1)}\zeta_{i_2}^{(j_1)}=\zeta_{i_1}^{(j_1)}*\zeta_{i_2}^{(j_1)}+\delta_{i_1i_2}\right),$$

where $\delta_{i_1i_2}$ is the Kronecker delta, i.e.,

$$\delta_{i_1i_2}=\begin{cases}1 & \text{for } i_1=i_2\\ 0 & \text{for } i_1\neq i_2,\end{cases}$$

and the convergence of series

$$\sum_{i=0}^{\infty}F_{ii} \tag{5}$$

does not follow from condition $f\in L_2(\mathbb{T}^2)$. This series can diverge or it can converge conditionally, then its sum depends on a basis $\{q_i\}_{i=0}^{\infty}$.

Thus, the operator ${}^{\mathrm{S}}\mathcal{J}_{\mathbb{T}}^{W(j_1j_1)}$ is defined only on some linear subspace of $L_2(\mathbb{T}^2)$, although the domain of the operator ${}^{\mathrm{I}}\mathcal{J}_{\mathbb{T}}^{W(j_1j_1)}$ coincides with $L_2(\mathbb{T}^2)$. In fact, the equality

$$ {}^{\mathrm{S}}\mathcal{J}_{\mathbb{T}}^{W(j_1j_1)}f = {}^{\mathrm{I}}\mathcal{J}_{\mathbb{T}}^{W(j_1j_1)}f + \sum_{i=0}^{\infty} F_{ii} $$

makes sense if the series (5) converges absolutely, and its sum does not depend on a basis $\{q_i\}_{i=0}^{\infty}$. It this case, we can write that

$$ \sum_{i=0}^{\infty} F_{ii} = \int_{\mathbb{T}} f(t,t)dt, $$

but the integral on the right-hand side should be understood in a special way, since any function $f \in L_2(\mathbb{T}^2)$ is defined up to a set of measure zero, while the set $\{(t,t)\colon t \in \mathbb{T}\}$ has measure zero (on the plane). This integral is equal to the expectation of the multiple Stratonovich stochastic integral ${}^{\mathrm{S}}\mathcal{J}_{\mathbb{T}}^{W(j_1j_1)}f$.

If $j_1 \neq j_2$, then there is no such a problem, and the equality

$$ {}^{\mathrm{S}}\mathcal{J}_{\mathbb{T}}^{W(j_1j_2)}f = {}^{\mathrm{I}}\mathcal{J}_{\mathbb{T}}^{W(j_1j_2)}f \quad \text{a.s.} $$

holds ("a.s." means "almost surely" or "with probability 1").

For stochastic integrals of multiplicity $k = 3$, the series

$$ \sum_{i_1=0}^{\infty} F_{i_1i_1i_3}, \quad \sum_{i_1=0}^{\infty} F_{i_1i_2i_1}, \quad \sum_{i_2=0}^{\infty} F_{i_1i_2i_2} $$

can appear depending on values j_1, j_2, j_3, where $F_{i_1i_2i_3}$ are expansion coefficients (3) of $f \in L_2(\mathbb{T}^3)$, and the indices, over which the summation is not carried out, are parameters. Under certain conditions, the following equalities

$$ \sum_{i_1=0}^{\infty} F_{i_1i_1i_3} = \left(q_{i_3}, \int_{\mathbb{T}} f(t,t,\cdot)dt\right)_{L_2(\mathbb{T})}, \quad \sum_{i_1=0}^{\infty} F_{i_1i_2i_1} = \left(q_{i_2}, \int_{\mathbb{T}} f(t,\cdot,t)dt\right)_{L_2(\mathbb{T})}, $$
$$ \sum_{i_2=0}^{\infty} F_{i_1i_2i_2} = \left(q_{i_1}, \int_{\mathbb{T}} f(\cdot,t,t)dt\right)_{L_2(\mathbb{T})} $$

hold.

The number of possible variants of such series increases with the multiplicity k. For example, if $k = 4$, then it is required to consider the series

$$ \sum_{i_1=0}^{\infty} F_{i_1i_1i_3i_4}, \quad \sum_{i_1=0}^{\infty} F_{i_1i_2i_1i_4}, \quad \ldots, \quad \sum_{i_3=0}^{\infty} F_{i_1i_2i_3i_3}, \quad \sum_{i_1,i_3=0}^{\infty} F_{i_1i_1i_3i_3}, \quad \sum_{i_1,i_2=0}^{\infty} F_{i_1i_2i_1i_2}, \quad \sum_{i_1,i_2=0}^{\infty} F_{i_1i_2i_2i_1}, $$

where $F_{i_1i_2i_3i_4}$ are expansion coefficients (3) of $f \in L_2(\mathbb{T}^4)$, and the set of series is determined by values j_1, j_2, j_3, j_4. Here the indices, over which the summation is not carried out, are also parameters.

Such series appear only if some values among $j_1, \ldots, j_k$ are equal. In particular, for $k = 3$ under condition $j_1 = j_2 \neq j_3$, we need to consider only one series

$$ \sum_{i_1=0}^{\infty} F_{i_1i_1i_3}, $$

or for $k = 4$ under condition $j_1 = j_2 \neq j_3 = j_4$, we need to consider only three series

$$\sum_{i_1=0}^{\infty} F_{i_1 i_1 i_3 i_4}, \quad \sum_{i_3=0}^{\infty} F_{i_1 i_2 i_3 i_3}, \quad \sum_{i_1,i_3=0}^{\infty} F_{i_1 i_1 i_3 i_3}.$$

Now we return to stochastic integrals of multiplicity $k = 2$. To establish convergence conditions for the series (5), it is useful to apply the theory of trace class operators, but the simplest example shows that we have to be careful. Indeed, consider the Volterra integral operator $\mathcal{V}\colon L_2(\mathbb{T}) \to L_2(\mathbb{T})$ defined as

$$\mathcal{V}g(t) = \int_{t_0}^{t} g(\tau)d\tau = \int_{\mathbb{T}} \Bbbk(\tau,t)g(\tau)d\tau,$$

where $\Bbbk(t_1,t_2) = \mathbf{1}(t_2 - t_1)$ is the kernel function.

It is known [22,23] that the operator $\mathcal{V}$ is not traceable, but it can be shown that the property

$$\sum_{i=0}^{\infty} \mathbb{K}_{ii} = \frac{T - t_0}{2}$$

is satisfied for an arbitrary basis $\{q_i\}_{i=0}^{\infty}$, where

$$\mathbb{K}_{i_1 i_2} = (q_{i_1} \otimes q_{i_2}, \Bbbk)_{L_2(\mathbb{T}^2)} = \int_{\mathbb{T}} q_{i_2}(t_2) \int_{t_0}^{t_2} q_{i_1}(t_1)dt_1 dt_2, \quad i_1, i_2 = 0,1,2,\ldots$$

Note that the specified sum of that series corresponds to the well-known relation for Itô and Stratonovich stochastic integrals [3,24]:

$${}^{\mathrm{S}}\mathcal{J}_{\mathbb{T}}^{W(j_1 j_1)}\Bbbk = {}^{\mathrm{I}}\mathcal{J}_{\mathbb{T}}^{W(j_1 j_1)}\Bbbk + \frac{T - t_0}{2} \quad \text{a.s.},$$

where

$${}^{\mathrm{I}}\mathcal{J}_{\mathbb{T}}^{W(j_1 j_1)}\Bbbk = \int_{\mathbb{T}} W_{j_1}(t)dW_{j_1}(t), \quad {}^{\mathrm{S}}\mathcal{J}_{\mathbb{T}}^{W(j_1 j_1)}\Bbbk = \int_{\mathbb{T}} W_{j_1}(t) \circ dW_{j_1}(t),$$

and

$$\mathrm{E}\,{}^{\mathrm{S}}\mathcal{J}_{\mathbb{T}}^{W(j_1 j_1)}\Bbbk = \frac{T - t_0}{2},$$

where E means the expectation operator that associates a random variable with its expected value.

Nevertheless, it is possible to apply the theory of trace class operators by symmetrization of $f \in L_2(\mathbb{T}^2)$. Linear operators ${}^{\mathrm{I}}\mathcal{J}_{\mathbb{T}}^{W(j_1 j_1)}$ and ${}^{\mathrm{S}}\mathcal{J}_{\mathbb{T}}^{W(j_1 j_1)}$ have one useful property: they can be considered on the set of equivalence classes constructed by symmetrization. In particular,

$$\begin{aligned}
{}^{\mathrm{I}}\mathcal{J}_{\mathbb{T}}^{W(j_1 j_1)} f &= {}^{\mathrm{I}}\mathcal{J}_{\mathbb{T}}^{W(j_1 j_1)} f^* = {}^{\mathrm{I}}\mathcal{J}_{\mathbb{T}}^{W(j_1 j_1)} \langle f \rangle, \\
{}^{\mathrm{S}}\mathcal{J}_{\mathbb{T}}^{W(j_1 j_1)} f &= {}^{\mathrm{S}}\mathcal{J}_{\mathbb{T}}^{W(j_1 j_1)} f^* = {}^{\mathrm{I}}\mathcal{J}_{\mathbb{T}}^{W(j_1 j_1)} \langle f \rangle,
\end{aligned}$$

where

$$f^*(t_1,t_2) = f(t_2,t_1), \quad \langle f(t_1,t_2) \rangle = \frac{f(t_1,t_2) + f^*(t_1,t_2)}{2} = \frac{f(t_1,t_2) + f(t_2,t_1)}{2},$$

and expansion coefficients (4) do not change under condition $i_1 = i_2$ for such functions, $\langle\cdot\rangle$ is the symmetrization operator.

In fact, if $\langle f \rangle = \langle g \rangle$, then

$${}^{\mathrm{I}}\mathcal{J}_{\mathbb{T}}^{W(j_1 j_1)} f = {}^{\mathrm{I}}\mathcal{J}_{\mathbb{T}}^{W(j_1 j_1)} g, \quad {}^{\mathrm{S}}\mathcal{J}_{\mathbb{T}}^{W(j_1 j_1)} f = {}^{\mathrm{S}}\mathcal{J}_{\mathbb{T}}^{W(j_1 j_1)} g,$$

hence

$${}^{\mathrm{I}}\mathcal{J}_{\mathbb{T}}^{W(j_1 j_1)}\Bbbk = {}^{\mathrm{I}}\mathcal{J}_{\mathbb{T}}^{W(j_1 j_1)} c, \quad {}^{\mathrm{S}}\mathcal{J}_{\mathbb{T}}^{W(j_1 j_1)}\Bbbk = {}^{\mathrm{S}}\mathcal{J}_{\mathbb{T}}^{W(j_1 j_1)} c,$$

where $c(t_1, t_2) = \langle \Bbbk(t_1, t_2) \rangle = 1/2$, and the linear operator with trivial kernel c is traceable.

Iterated stochastic integrals can be represented by solutions to corresponding systems of stochastic differential equations. For the integral

$$ {}^{\mathrm{I}}\mathcal{J}_{\mathbb{T}}^{W(j_1 \ldots j_k)} \Bbbk_{\psi} = \int_{\mathbb{T}} \ldots \int_{t_0}^{t_3} \int_{t_0}^{t_2} \psi_1(t_1)\psi_2(t_2) \ldots \psi_k(t_k) dW_{j_1}(t_1) dW_{j_2}(t_2) \ldots dW_{j_k}(t_k), $$

we have ${}^{\mathrm{I}}\mathcal{J}_{\mathbb{T}}^{W(j_1 \ldots j_k)} \Bbbk_{\psi} = {}^{\mathrm{I}}X_k(T)$, where ${}^{\mathrm{I}}X_k$ is the component of the solution to the system of Itô stochastic differential equations

$$ d\,{}^{\mathrm{I}}X_l(t) = \psi_l(t) {}^{\mathrm{I}}X_{l-1}(t) dW_{j_l}(t), \quad {}^{\mathrm{I}}X_l(t_0) = 0, \quad l = 1, \ldots, k, \tag{6} $$

and ${}^{\mathrm{I}}X_0(t) = 1$.

Similarly, for the integral

$$ {}^{\mathrm{S}}\mathcal{J}_{\mathbb{T}}^{W(j_1 \ldots j_k)} \Bbbk_{\psi} = \int_{\mathbb{T}} \ldots \int_{t_0}^{t_3} \int_{t_0}^{t_2} \psi_1(t_1)\psi_2(t_2) \ldots \psi_k(t_k) \circ dW_{j_1}(t_1) \circ dW_{j_2}(t_2) \circ \ldots \circ dW_{j_k}(t_k), $$

we have ${}^{\mathrm{S}}\mathcal{J}_{\mathbb{T}}^{W(j_1 \ldots j_k)} \Bbbk_{\psi} = {}^{\mathrm{S}}X_k(T)$, where ${}^{\mathrm{S}}X_k$ is the component of the solution to the system of Stratonovich stochastic differential equations

$$ d\,{}^{\mathrm{S}}X_l(t) = \psi_l(t) {}^{\mathrm{S}}X_{l-1}(t) \circ dW_{j_l}(t), \quad {}^{\mathrm{S}}X_l(t_0) = 0, \quad l = 1, \ldots, k, \tag{7} $$

and ${}^{\mathrm{S}}X_0(t) = 1$.

Note that for the system (6) we can derive the equivalent system of Stratonovich stochastic differential equations

$$ d\,{}^{\mathrm{I}}X_l(t) = -\frac{1}{2}\, \delta_{j_{l-1} j_l} \psi_{l-1}(t) \psi_l(t)\, {}^{\mathrm{I}}X_{l-2}(t) dt + \psi_l(t)\, {}^{\mathrm{I}}X_{l-1}(t) \circ dW_{j_l}(t), \quad {}^{\mathrm{I}}X_l(t_0) = 0, $$

and the system (7) can be transformed into the equivalent system of Itô stochastic differential equations

$$ d\,{}^{\mathrm{S}}X_l(t) = \frac{1}{2}\, \delta_{j_{l-1} j_l} \psi_{l-1}(t) \psi_l(t)\, {}^{\mathrm{S}}X_{l-2}(t) dt + \psi_l(t)\, {}^{\mathrm{S}}X_{l-1}(t) dW_{j_l}(t), \quad {}^{\mathrm{S}}X_l(t_0) = 0. $$

Here, in addition to above notations, we use the following: ${}^{\mathrm{I}}X_{-1}(t) = {}^{\mathrm{S}}X_{-1}(t) = 0$, $j_0 = 0$, $\delta_{j_{l-1} j_l}$ is the Kronecker delta.

The structure of obtained equations shows that the differences between Itô and Stratonovich stochastic differential equations exist only when some values $j_1, \ldots, j_k$ with neighboring indices are equal. And these differences form traces, which are sums of expansion coefficients $\mathbb{K}^{\psi}_{i_1 \ldots i_k}$ of the function $\Bbbk_{\psi}$ given by the formula (2):

$$ \begin{aligned} \mathbb{K}^{\psi}_{i_1 \ldots i_k} &= (q_{i_1} \otimes \ldots \otimes q_{i_k}, \Bbbk_{\psi})_{L_2(\mathbb{T}^k)} \\ &= \int_{\mathbb{T}} q_{i_k}(t_k) \psi_k(t_k) \ldots \int_{t_0}^{t_3} q_{i_2}(t_2) \psi_2(t_2) \int_{t_0}^{t_2} q_{i_1}(t_1) \psi_1(t_1) dt_1 \ldots dt_k, \\ & i_1, \ldots, i_k = 0, 1, 2, \ldots \end{aligned} \tag{8} $$

Traces are formed only by summing over neighboring pairs of indices. If we assume that possible parameters are fixed (all indices, over which the summation is not carried out, are parameters), then it suffices to consider the following series:

$$\sum_{i=0}^{\infty} \mathbb{K}_{ii}^{\psi} \quad (k=2), \quad \sum_{i_1,i_3=0}^{\infty} \mathbb{K}_{i_1 i_1 i_3 i_3}^{\psi} \quad (k=4), \quad \dots,$$
$$\sum_{i_1,i_3,\dots,i_{k-1}=0}^{\infty} \mathbb{K}_{i_1 i_1 i_3 i_3 \dots i_{k-1} i_{k-1}}^{\psi} \quad (k=2\gamma,\ \gamma \in \mathbb{N}).$$

The problem statement is to prove the absolute convergence of these series and to express their sums as a functional depending on the weights $\psi_1, \dots, \psi_k \in L_2(\mathbb{T})$.

Next sections present proofs of the absolute convergence of these series regardless of a basis $\{q_i\}_{i=0}^{\infty}$. The main results are formulated separately for cases $k=2$ and $k=2\gamma$, $\gamma \in \mathbb{N}$.

3. Main Result for the Case $k = 2$

Consider the Hilbert–Schmidt operator $\mathcal{F}\colon L_2(\mathbb{T}) \to L_2(\mathbb{T})$ with the kernel $f \in L_2(\mathbb{T}^2)$ given by the relation ("a.e." means "almost everywhere")

$$x = \mathcal{F}g \iff x(t) = \int_{\mathbb{T}} f(\tau,t) g(\tau) d\tau \quad \text{a.e. on} \quad \mathbb{T} = [t_0, T].$$

This operator is the trace class operator [23,25] if there exist functions $\check{f}, \hat{f} \in L_2(\mathbb{T}^2)$ such that

$$f(t_1,t_2) = \int_{\mathbb{T}} \check{f}(t_1,\tau) \hat{f}(\tau,t_2) d\tau, \quad \check{f}, \hat{f} \in L_2(\mathbb{T}^2). \tag{9}$$

Conversely, if $\mathcal{F}$ is the trace class operator, then there exists a (nonunique) representation (9) for its kernel.

One of the simplest examples of trace class operators is the operator with the kernel

$$f(t_1,t_2) = \varphi(t_1)\psi(t_2), \quad \varphi, \psi \in L_2(\mathbb{T}). \tag{10}$$

Here, it suffices to show that f can be represented by the equality (9). For this, we assume that $\check{f}(t_1,t_2) = \varphi(t_1)/\sqrt{T-t_0}$ and $\hat{f}(t_1,t_2) = \psi(t_2)/\sqrt{T-t_0}$. Then

$$f(t_1,t_2) = \frac{\varphi(t_1)\psi(t_2)}{T-t_0} \int_{t_0}^{T} d\tau = \varphi(t_1)\psi(t_2).$$

Let $\mathcal{S}^{\varepsilon}\colon L_2(\mathbb{T}^2) \to C(\mathbb{T}^2)$ be the averaging operator [23]:

$$\mathcal{S}^{\varepsilon} f(t_1,t_2) = \frac{1}{4\varepsilon^2} \int_{D_{\varepsilon}(t_1,t_2)} f(\tau_1,\tau_2) d\tau_1 d\tau_2, \quad \varepsilon > 0,$$

where $D_{\varepsilon}(t_1,t_2) = \{(\tau_1,\tau_2) \in \mathbb{T}^2\colon \max\{|t_1-\tau_1|, |t_2-\tau_2|\} < \varepsilon\}$, i.e., $\mathcal{S}^{\varepsilon}$ is a linear operator, which associates a function f with a continuous function that has well-defined value at each point (t_1,t_2) as the average value of f on the square $D_{\varepsilon}(t_1,t_2) \subset \mathbb{R}^2$ centered at this point (f should be defined by zero outside the square $\mathbb{T}^2$). Then $\bar{f}(t_1,t_2) = f(t_1,t_2)$ a.e. on $\mathbb{T}^2$, where

$$\bar{f} = \lim_{\varepsilon \to 0} \mathcal{S}^{\varepsilon} f.$$

Theorem 1 ([23,25])**.** *Let $\mathcal{F}\colon L_2(\mathbb{T}) \to L_2(\mathbb{T})$ be the trace class operator with the kernel $f \in L_2(\mathbb{T}^2)$ and let $\{q_i\}_{i=0}^{\infty}$ be a basis of $L_2(\mathbb{T})$. Then*

$$\operatorname{tr} \mathcal{F} = \sum_{i=0}^{\infty} F_{ii} = \operatorname{tr} f, \tag{11}$$

where $F_{i_1 i_2}$ are expansion coefficients (4) of f relative to the basis $\{q_{i_1} \otimes q_{i_2}\}_{i_1,i_2=0}^{\infty}$, and

$$\operatorname{tr} f = \lim_{\varepsilon \to 0} \int_{\mathbb{T}} \mathcal{S}^{\varepsilon} f(t,t) dt.$$

Remark 1. *The series in the relation* (11) *is called the trace of the operator* $\mathcal{F}$. *It converges absolutely, and its sum does not depend on a basis* $\{q_i\}_{i=0}^{\infty}$.

Next, we prove two technical lemmas, after that we can formulate and prove one of the main results.

Lemma 1. *Operators with symmetric kernels*

$$f(t_1,t_2) = t_1^{m+n} t_2^n \mathbf{1}(t_2 - t_1) + t_2^{m+n} t_1^n \mathbf{1}(t_1 - t_2), \tag{12}$$
$$f(t_1,t_2) = t_1^n t_2^{m+n} \mathbf{1}(t_2 - t_1) + t_2^n t_1^{m+n} \mathbf{1}(t_1 - t_2), \tag{13}$$

where $m \in \mathbb{N}$ *and* $n \in \{0,1,2,\dots\}$, *are trace class operators.*

Proof of Lemma 1. Let $\check{f}(t_1,t_2) = m t_1^n t_2^{m-1} \mathbf{1}(t_1 - t_2)$ and $\hat{f}(t_1,t_2) = t_2^l \mathbf{1}(t_2 - t_1)$ with $m \geqslant 1$ and $l, n > -1/2$. According to the representation (9), we have

$$f(t_1,t_2) = m t_1^n t_2^l \int_{\mathbb{T}} \tau^{m-1} \mathbf{1}(t_1 - \tau) \mathbf{1}(t_2 - \tau) d\tau = m t_1^n t_2^l \int_{t_0}^{\min\{t_1,t_2\}} \tau^{m-1} d\tau$$
$$= t_1^n t_2^l \min\{t_1^m, t_2^m\} - t_0^m t_1^n t_2^l = t_1^{m+n} t_2^l \mathbf{1}(t_2 - t_1) + t_1^n t_2^{l+m} \mathbf{1}(t_1 - t_2) - t_0^m t_1^n t_2^l.$$

The term $t_0^m t_1^n t_2^l$ defines the trace class operator with the kernel (10). In addition, we can restrict ourselves to conditions $m \in \mathbb{N}$ and $l = n \in \{0,1,2,\dots\}$, so the function (12) defines the trace class operator.

Let $\check{f}(t_1,t_2) = t_1^n \mathbf{1}(t_2 - t_1)$ and $\hat{f}(t_1,t_2) = m t_1^{m-1} t_2^l \mathbf{1}(t_1 - t_2)$, $m \geqslant 1$ and $l, n > -1/2$. Using the representation (9), we obtain

$$f(t_1,t_2) = m t_1^n t_2^l \int_{\mathbb{T}} \tau^{m-1} \mathbf{1}(\tau - t_1) \mathbf{1}(\tau - t_2) d\tau = m t_1^n t_2^l \int_{\max\{t_1,t_2\}}^{T} \tau^{m-1} d\tau$$
$$= T^m t_1^n t_2^l - t_1^n t_2^l \max\{t_1^m, t_2^m\} = T^m t_1^n t_2^l - t_1^n t_2^{l+m} \mathbf{1}(t_2 - t_1) - t_1^{m+n} t_2^l \mathbf{1}(t_1 - t_2).$$

Similarly, the function (13) also defines the trace class operator, since the term $T^m t_1^n t_2^l$ defines the trace class operator with the kernel (10). □

Lemma 2. *Let* λ, μ *be polynomials. Then the operator* $\mathcal{F}$ *with the symmetric kernel*

$$f(t_1,t_2) = \lambda(t_1)\mu(t_2)\mathbf{1}(t_2 - t_1) + \lambda(t_2)\mu(t_1)\mathbf{1}(t_1 - t_2)$$

is the trace class operator.

Proof of Lemma 2. The operator $\mathcal{G}$ with the symmetric kernel

$$g(t_1,t_2) = t_1^{n_1} t_2^{n_2} \mathbf{1}(t_2 - t_1) + t_2^{n_1} t_1^{n_2} \mathbf{1}(t_1 - t_2), \quad n_1, n_2 = 0,1,2,\dots,$$

is the trace class operator.

Indeed, for $n_1 = n_2$ we have the kernel $g(t_1,t_2) = (t_1 t_2)^{n_1}$, which satisfies the condition (10). For $n_1 \neq n_2$, the required result follows from Lemma 1. For functions (12) and (13), conditions $m + n = n_1 > n_2 = n$ and $n = n_1 < n_2 = m + n$, respectively, should be satisfied.

The function f is represented as a linear combination of functions g, and it defines the trace class operator $\mathcal{F}$, since the space of trace class operators is linear [26]. □

Theorem 2. *Let* $\varphi, \psi \in L_2(\mathbb{T})$ *and let* $\{q_i\}_{i=0}^{\infty}$ *be a basis of* $L_2(\mathbb{T})$. *Then*

$$\sum_{i=0}^{\infty} \int_{\mathbb{T}} \varphi(t_2) q_i(t_2) \int_{t_0}^{t_2} \psi(t_1) q_i(t_1) dt_1 dt_2 = \frac{1}{2} (\varphi, \psi)_{L_2(\mathbb{T})}. \tag{14}$$

Proof of Theorem 2. Define functions $g, g^* \in L_2(\mathbb{T}^2)$ as follows:

$$g(t_1,t_2) = \lambda(t_1)\mu(t_2)\mathbf{1}(t_2-t_1), \quad g^*(t_1,t_2) = \mu(t_1)\lambda(t_2)\mathbf{1}(t_1-t_2) = g(t_2,t_1).$$

Then their expansion coefficients $G_{i_1i_2}$ and $G^*_{i_1i_2}$ relative to the basis $\{q_{i_1}\otimes q_{i_2}\}_{i_1,i_2=0}^{\infty}$ are defined by the formula (4):

$$G_{i_1i_2} = (q_{i_1}\otimes q_{i_2}, g)_{L_2(\mathbb{T}^2)} = \int_{\mathbb{T}} \mu(t_2)q_{i_1}(t_2)\int_{t_0}^{t_2}\lambda(t_1)q_{i_2}(t_1)dt_1dt_2,$$
$$G^*_{i_1i_2} = (q_{i_1}\otimes q_{i_2}, g^*)_{L_2(\mathbb{T}^2)} = \int_{\mathbb{T}} \lambda(t_2)q_{i_1}(t_2)\int_{t_2}^{T}\mu(t_1)q_{i_2}(t_1)dt_1dt_2,$$

and for them the condition $G_{i_1i_2} = G^*_{i_2i_1}$ holds due to the symmetry, $i_1,i_2 = 0,1,2,\ldots$

Let

$$f(t_1,t_2) = g(t_1,t_2) + g^*(t_1,t_2) = g(t_1,t_2) + g(t_2,t_1) = f(t_2,t_1), \quad f \in L_2(\mathbb{T}^2),$$

i.e., $f = \langle g\rangle$, where $\langle\cdot\rangle$ is the symmetrization operator. Then expansion coefficients $F_{i_1i_2}$ of f are determined as

$$F_{i_1i_2} = G_{i_1i_2} + G^*_{i_1i_2} = G_{i_1i_2} + G_{i_2i_1} = F_{i_2i_1} \quad\text{and}\quad F_{ii} = 2G_{ii} \quad\text{for}\quad i = 0,1,2,\ldots$$

Moreover, we can write that

$$\lim_{\varepsilon\to 0}\mathcal{S}^{\varepsilon}f(t,t) = \lambda(t)\mu(t) \quad\text{and}\quad \operatorname{tr} f = \int_{\mathbb{T}}\lambda(\tau)\mu(\tau)d\tau = (\lambda,\mu)_{L_2(\mathbb{T})},$$

and this means that the equality (14) is equivalent to the relation (11), which holds for the trace class operator $\mathcal{F}$ with some kernel f according to Theorem 1.

If λ, μ are polynomials, then the operator $\mathcal{F}\colon L_2(\mathbb{T}) \to L_2(\mathbb{T})$ with the kernel f is the trace class operator according to Lemma 2, and an arbitrary function from $L_2(\mathbb{T})$ can be approximated using polynomials (polynomials are dense in $L_2(\mathbb{T})$).

Further, let $\varphi, \psi \in L_2(\mathbb{T})$ and

$$\varphi = \lim_{n\to\infty}\varphi_n, \quad \psi = \lim_{m\to\infty}\psi_m,$$

where

$$\varphi_n = \sum_{i=0}^{n}\Phi_i\hat{P}_i \quad\text{and}\quad \psi_m = \sum_{i=0}^{m}\Psi_i\hat{P}_i, \quad n,m \in \{0,1,2,\ldots\},$$

and Φ_i, Ψ_i are expansion coefficients of φ, ψ relative to orthonormal Legendre polynomials $\{\hat{P}_i\}_{i=0}^{\infty}$, i.e.,

$$\Phi_i = (\hat{P}_i, \varphi)_{L_2(\mathbb{T})}, \quad \Psi_i = (\hat{P}_i, \psi)_{L_2(\mathbb{T})}.$$

Then we can establish the following equality for arbitrary n and m:

$$\sum_{i=0}^{\infty}\left[\int_{\mathbb{T}}\varphi_n(t_2)q_i(t_2)\int_{t_0}^{t_2}\psi_m(t_1)q_i(t_1)dt_1dt_2 + \int_{\mathbb{T}}\psi_m(t_2)q_i(t_2)\int_{t_2}^{T}\varphi_n(t_1)q_i(t_1)dt_1dt_2\right] = (\varphi_n,\psi_m)_{L_2(\mathbb{T})}, \tag{15}$$

where the series converges absolutely, and its sum does not depend on a basis $\{q_i\}_{i=0}^{\infty}$.

If we fix n in the equality (15), then it defines a bounded linear functional in $L_2(\mathbb{T})$, which is given by the function φ_n (the trace of operator is also a bounded linear functional but in the space of trace class operators [26]). Letting $m \to \infty$, we obtain a bounded linear functional given by the function ψ:

$$\sum_{i=0}^{\infty} \Big[\int_{\mathbb{T}} \varphi_n(t_2) q_i(t_2) \int_{t_0}^{t_2} \psi(t_1) q_i(t_1) dt_1 dt_2 + \int_{\mathbb{T}} \psi(t_2) q_i(t_2) \int_{t_2}^{T} \varphi_n(t_1) q_i(t_1) dt_1 dt_2 \Big] = (\varphi_n, \psi)_{L_2(\mathbb{T})}.$$

Similarly, letting $n \to \infty$, we obtain the relation

$$\sum_{i=0}^{\infty} \Big[\int_{\mathbb{T}} \varphi(t_2) q_i(t_2) \int_{t_0}^{t_2} \psi(t_1) q_i(t_1) dt_1 dt_2 + \int_{\mathbb{T}} \psi(t_2) q_i(t_2) \int_{t_2}^{T} \varphi(t_1) q_i(t_1) dt_1 dt_2 \Big] = 2 \sum_{i=0}^{\infty} \int_{\mathbb{T}} \varphi(t_2) q_i(t_2) \int_{t_0}^{t_2} \psi(t_1) q_i(t_1) dt_1 dt_2 = (\varphi, \psi)_{L_2(\mathbb{T})},$$

which proves the theorem. □

4. Main Result for the Case $k = 2\gamma$ for $\gamma \in \mathbb{N}$

Define the Hilbert–Schmidt operator $\mathcal{F}\colon L_2(\mathbb{T}^\gamma) \to L_2(\mathbb{T}^\gamma)$ with the kernel $f \in L_2(\mathbb{T}^k)$, $k = 2\gamma$ for $\gamma \in \mathbb{N}$:

$$x = \mathcal{F}g \iff$$
$$x(t_1, t_2, \ldots, t_\gamma) = \int_{\mathbb{T}^\gamma} f(\tau_1, t_1, \ldots, \tau_\gamma, t_\gamma) g(\tau_1, \tau_2, \ldots, \tau_\gamma) d\tau_1 \ldots d\tau_\gamma \quad \text{a.e. on} \quad \mathbb{T}^\gamma = [t_0, T]^\gamma.$$

It is known [23,25] that if the function f is represented as

$$f(t_1, \ldots, t_k) = \int_{\mathbb{T}^\gamma} \check{f}(t_1, \tau_1, t_3, \tau_2, \ldots, t_{k-1}, \tau_\gamma) \hat{f}(\tau_1, t_2, \tau_2, t_4, \ldots, \tau_\gamma, t_k) d\tau_1 \ldots d\tau_\gamma, \tag{16}$$

where $\check{f}, \hat{f} \in L_2(\mathbb{T}^k)$, then $\mathcal{F}$ is the trace class operator.

Let $\mathcal{S}^\varepsilon\colon L_2(\mathbb{T}^k) \to C(\mathbb{T}^k)$ be the averaging operator [23]:

$$\mathcal{S}^\varepsilon f(t_1, \ldots, t_k) = \frac{1}{(2\varepsilon)^k} \int_{D_\varepsilon(t_1, \ldots, t_k)} f(\tau_1, \ldots, \tau_k) d\tau_1 \ldots d\tau_k, \quad \varepsilon > 0,$$

where $D_\varepsilon(t_1, \ldots, t_k) = \{(\tau_1, \ldots, \tau_k) \in \mathbb{T}^k\colon \max_{l=1,\ldots,k} |t_l - \tau_l| < \varepsilon\}$, i.e., $\mathcal{S}^\varepsilon$ is a linear operator, which associates a function f with a continuous function that has well-defined value at each point $(t_1, \ldots, t_k)$ as the average value of f on the hypercube $D_\varepsilon(t_1, \ldots, t_k) \subset \mathbb{R}^k$ centered at this point (f should be defined by zero outside the hypercube $\mathbb{T}^k$). In this case, we have $\bar{f}(t_1, \ldots, t_k) = f(t_1, \ldots, t_k)$ a.e. on $\mathbb{T}^k$, where

$$\bar{f} = \lim_{\varepsilon \to 0} \mathcal{S}^\varepsilon f.$$

Theorem 3 ([23,25])**.** *Let $\mathcal{F}\colon L_2(\mathbb{T}^\gamma) \to L_2(\mathbb{T}^\gamma)$ be the trace class operator with the kernel $f \in L_2(\mathbb{T}^k)$ and let $\{q_i\}_{i=0}^\infty$ be a basis of $L_2(\mathbb{T})$. Then*

$$\operatorname{tr} \mathcal{F} = \sum_{i_1, i_3, \ldots, i_{k-1}=0}^{\infty} F_{i_1 i_1 i_3 i_3 \ldots i_{k-1} i_{k-1}} = \operatorname{tr} f, \tag{17}$$

where $F_{i_1 \ldots i_k}$ are expansion coefficients (3) of f relative to the basis $\{q_{i_1} \otimes \ldots \otimes q_{i_k}\}_{i_1, \ldots, i_k=0}^\infty$, and

$$\operatorname{tr} f = \lim_{\varepsilon \to 0} \int_{\mathbb{T}^\gamma} \mathcal{S}^\varepsilon f(t_1, t_1, t_3, t_3, \ldots, t_{k-1}, t_{k-1}) dt_1 dt_3 \ldots dt_{k-1}.$$

Remark 2.

1. *The series in the relation (17) is the trace of the operator $\mathcal{F}$. It converges absolutely, and its sum does not depend on a basis $\{q_i\}_{i=0}^{\infty}$.*
2. *Obviously, Theorem 1 is the particular case of Theorem 3.*
3. *There is the trace-oriented definition of the averaging operator $\mathcal{S}^{\varepsilon}$ [25]. However, the above definition naturally agrees with the definition of the multiple Stratonovich stochastic integral from [9,19], and the main results presented in this paper are directly related to such integrals.*

Now we give the technical lemma and then formulate and prove a more general result compared to Theorem 2.

Lemma 3. *If the function $f_r \in L_2(\mathbb{T}^2)$ defines the trace class operator $\mathcal{F}_r \colon L_2(\mathbb{T}) \to L_2(\mathbb{T})$, $r = 1, \ldots, \gamma$, then the operator $\mathcal{F} \colon L_2(\mathbb{T}^\gamma) \to L_2(\mathbb{T}^\gamma)$ with the kernel*

$$f(t_1, \ldots, t_k) = \prod_{r=1}^{\gamma} f_r(t_{2r-1}, t_{2r}) \tag{18}$$

is the trace class operator.

Proof of Lemma 3. Since $\mathcal{F}_r$ is the trace class operator, for its kernel f_r there exists a representation (9), i.e.,

$$f_r(\theta_1, \theta_2) = \int_{\mathbb{T}} \check{f}_r(\theta_1, \tau)\hat{f}_r(\tau, \theta_2) d\tau, \quad \check{f}_r, \hat{f}_r \in L_2(\mathbb{T}^2), \quad r = 1, \ldots, \gamma,$$

but then the representation (16) holds for the function f if

$$\check{f}(t_1, \ldots, t_k) = \prod_{r=1}^{\gamma} \check{f}_r(t_{2r-1}, t_{2r}), \quad \hat{f}(t_1, \ldots, t_k) = \prod_{r=1}^{\gamma} \hat{f}_r(t_{2r-1}, t_{2r}),$$

where $\check{f}, \hat{f} \in L_2(\mathbb{T}^k)$. Hence, $\mathcal{F}$ is the trace class operator. □

Theorem 4. *Let $\psi_l \in L_2(\mathbb{T})$, $l = 1, \ldots, k$, and let $\{q_i\}_{i=0}^{\infty}$ be a basis of $L_2(\mathbb{T})$. Then*

$$\sum_{i_1, i_3, \ldots, i_{k-1}=0}^{\infty} \int_{\mathbb{T}} q_{i_k}(t_k)\psi_k(t_k) \ldots \int_{t_0}^{t_3} q_{i_2}(t_2)\psi_2(t_2) \int_{t_0}^{t_2} q_{i_1}(t_1)\psi_1(t_1) dt_1 \ldots dt_k \Big|_{i_1=i_2, \ldots, i_{k-1}=i_k}$$
$$= \frac{1}{2^\gamma}(\psi_1 \otimes \psi_3 \otimes \ldots \otimes \psi_{k-1}, \psi_2 \otimes \psi_4 \otimes \ldots \otimes \psi_k)_{L_2(\Delta\mathbb{T}^\gamma)}, \tag{19}$$

where $\Delta\mathbb{T}^\gamma = \{(t_1, \ldots, t_\gamma) \in \mathbb{T}^\gamma \colon t_1 < \ldots < t_\gamma\}$.

Proof of Theorem 4. Define functions $g_r, g_r^* \in L_2(\mathbb{T}^2)$ as follows:

$$g_r(\theta_1, \theta_2) = \lambda_r(\theta_1)\mu_r(\theta_2)\mathbf{1}(\theta_2 - \theta_1), \quad g_r^*(\theta_1, \theta_2) = \mu_r(\theta_1)\lambda_r(\theta_2)\mathbf{1}(\theta_1 - \theta_2) = g_r(\theta_2, \theta_1),$$

and let

$$f_r(\theta_1, \theta_2) = g_r(\theta_1, \theta_2) + g_r^*(\theta_1, \theta_2) = g_r(\theta_1, \theta_2) + g_r(\theta_2, \theta_1) = f_r(\theta_2, \theta_1), \quad r = 1, \ldots, \gamma.$$

If λ_r, μ_r are polynomials, then the operator $\mathcal{F}_r \colon L_2(\mathbb{T}) \to L_2(\mathbb{T})$ with the kernel f_r is the trace class operator according to Lemma 2. Therefore, the operator $\mathcal{F} \colon L_2(\mathbb{T}^\gamma) \to L_2(\mathbb{T}^\gamma)$ with the kernel

$$
\begin{aligned}
f(t_1,\dots,t_k) &= \prod_{r=1}^{\gamma} f_r(t_{2r-1},t_{2r}) = \prod_{r=1}^{\gamma}\big(g_r(t_{2r-1},t_{2r}) + g_r^*(t_{2r-1},t_{2r})\big) \\
&= \prod_{r=1}^{\gamma}\big(g_r(t_{2r-1},t_{2r}) + g_r(t_{2r},t_{2r-1})\big) = \sum_{p=0}^{2^\gamma-1} h_p(t_1,\dots,t_k)
\end{aligned}
$$

is the trace class operator according to Lemma 3, where

$$
h_0(t_1,\dots,t_k) = \prod_{r=1}^{\gamma} g_r(t_{2r-1},t_{2r}) = \prod_{r=1}^{\gamma} \lambda_r(t_{2r-1})\mu_r(t_{2r})\mathbf{1}(t_{2r}-t_{2r-1}), \tag{20}
$$

and for $p = 1,\dots,2^\gamma - 1$ the function h_p is obtained from h_0 by permutation of variables in pairs (t_{2r-1},t_{2r}) if the binary representation of p is $(p_\gamma\dots p_1)_2$ and $p_r = 1$. So, values of f are not change by permutation of variables t_{2r-1} and t_{2r}, $r = 1,\dots,\gamma$, i.e., f is the symmetrized function relatively pairs $(t_1,t_2),(t_3,t_4),\dots,(t_{k-1},t_k)$:

$$
f = 2^\gamma\langle h_0\rangle = 2^\gamma\langle h_p\rangle \quad \forall\, p\in\{0,\dots,2^\gamma-1\},
$$

where $\langle\cdot\rangle$ is the corresponding symmetrization operator.

Functions of type (20) allow to obtain an approximation to the function

$$
x(t_1,\dots,t_k) = y(t_1,\dots,t_k)\prod_{r=1}^{\gamma}\mathbf{1}(t_{2r}-t_{2r-1}), \quad y\in L_2(\mathbb{T}^k),
$$

where, for example,

$$
\begin{aligned}
y(t_1,\dots,t_k) &= \psi_1(t_1)\dots\psi_k(t_k)\prod_{r=1}^{\gamma-1}\mathbf{1}(t_{2r+1}-t_{2r}) \\
&= \psi_1(t_1)\dots\psi_k(t_k)\mathbf{1}(t_3-t_2)\mathbf{1}(t_5-t_4)\dots\mathbf{1}(t_{k-1}-t_{k-2}),
\end{aligned}
$$

and then x coincides with the function $\mathbb{k}_\psi$ given by the formula (2), and its expansion coefficients $\mathbb{K}^{\psi}_{i_1\dots i_k}$ satisfy the relation (8).

We can represent y as a product of two functions of γ arguments:

$$
\varphi(t_1,t_4,t_5,\dots)\psi(t_2,t_3,t_6,\dots), \quad \varphi,\psi\in L_2(\mathbb{T}^\gamma),
$$

where the first function depends on variables t_l for $l = 4\beta-3$ and $l = 4\beta$, and the second one depends on variables t_l for $l = 4\beta-1$ and $l = 4\beta-2$, where $\beta\in\mathbb{N}$ under condition $l\in\{1,\dots,k\}$. Thus, we separate variables so that the list of arguments of each function does not include both variables that form any pair $(t_1,t_2),(t_3,t_4),\dots,(t_{k-1},t_k)$:

$$
\varphi(t_1,t_4,t_5,\dots) = \begin{cases}
\psi_1(t_1)\psi_4(t_4)\psi_5(t_5)\mathbf{1}(t_5-t_4)\times\dots & \\
\quad\times\,\psi_{k-4}(t_{k-4})\psi_{k-3}(t_{k-3})\mathbf{1}(t_{k-3}-t_{k-4})\psi_k(t_k) & \text{for even } \gamma \\
\psi_1(t_1)\psi_4(t_4)\psi_5(t_5)\mathbf{1}(t_5-t_4)\times\dots & \\
\quad\times\,\psi_{k-2}(t_{k-2})\psi_{k-1}(t_{k-1})\mathbf{1}(t_{k-1}-t_{k-2}) & \text{for odd } \gamma,
\end{cases}
$$

$$
\psi(t_2,t_3,t_6,\dots) = \begin{cases}
\psi_2(t_2)\psi_3(t_3)\mathbf{1}(t_3-t_2)\times\dots & \\
\quad\times\,\psi_{k-2}(t_{k-2})\psi_{k-1}(t_{k-1})\mathbf{1}(t_{k-1}-t_{k-2}) & \text{for even } \gamma \\
\psi_2(t_2)\psi_3(t_3)\mathbf{1}(t_3-t_2)\times\dots & \\
\quad\times\,\psi_{k-2}(t_{k-2})\psi_{k-1}(t_{k-1})\mathbf{1}(t_{k-1}-t_{k-2})\psi_k(t_k) & \text{for odd } \gamma.
\end{cases}
$$

Further, represent function $\varphi,\psi\in L_2(\mathbb{T})$ as

$$
\varphi = \lim_{n\to\infty}\varphi_n, \quad \psi = \lim_{m\to\infty}\psi_m,
$$

where

$$\varphi_n = \sum_{i_1,\ldots,i_\gamma=0}^{n} \Phi_{i_1,\ldots,i_\gamma} \hat{P}_{i_1} \otimes \ldots \otimes \hat{P}_{i_\gamma}, \quad \psi_m = \sum_{i_1,\ldots,i_\gamma=0}^{m} \Psi_{i_1,\ldots,i_\gamma} \hat{P}_{i_1} \otimes \ldots \otimes \hat{P}_{i_\gamma},$$
$$n, m \in \{0, 1, 2, \ldots\},$$

and $\Phi_{i_1,\ldots,i_\gamma}, \Psi_{i_1,\ldots,i_\gamma}$ are expansion coefficients of φ, ψ relative to orthonormal Legendre polynomials $\{\hat{P}_i\}_{i=0}^{\infty}$, i.e.,

$$\Phi_{i_1,\ldots,i_\gamma} = (\hat{P}_{i_1} \otimes \ldots \otimes \hat{P}_{i_\gamma}, \varphi)_{L_2(\mathbb{T}^\gamma)}, \quad \Psi_{i_1,\ldots,i_\gamma} = (\hat{P}_{i_1} \otimes \ldots \otimes \hat{P}_{i_\gamma}, \psi)_{L_2(\mathbb{T}^\gamma)}.$$

This implies that

$$y = \lim_{n\to\infty} \lim_{m\to\infty} y_{nm} = \lim_{n\to\infty} y_n, \quad x = \lim_{n\to\infty} \lim_{m\to\infty} x_{nm} = \lim_{n\to\infty} x_n,$$

where

$$\begin{aligned} y_{nm}(t_1,\ldots,t_k) &= \varphi_n(t_1,t_4,t_5,\ldots)\psi_m(t_2,t_3,t_6,\ldots), \\ y_n(t_1,\ldots,t_k) &= \varphi_n(t_1,t_4,t_5,\ldots)\psi(t_2,t_3,t_6,\ldots), \\ x_{nm}(t_1,\ldots,t_k) &= y_{nm}(t_1,\ldots,t_k)\prod_{r=1}^{\gamma} \mathbf{1}(t_{2r}-t_{2r-1}), \\ x_n(t_1,\ldots,t_k) &= y_n(t_1,\ldots,t_k)\prod_{r=1}^{\gamma} \mathbf{1}(t_{2r}-t_{2r-1}). \end{aligned}$$

Moreover, for functions

$$f_{nm} = 2^\gamma \langle x_{nm} \rangle, \quad f_n = 2^\gamma \langle x_n \rangle, \quad f = 2^\gamma \langle x \rangle,$$

we have

$$f = \lim_{n\to\infty} \lim_{m\to\infty} f_{nm} = \lim_{n\to\infty} f_n,$$

since $\langle \cdot \rangle$ is a linear bounded operator [23].

The function f_{nm} defines the trace class operator because x_{nm} is the function of type (20). Its expansion coefficients $F^{nm}_{i_1\ldots i_k}$ relative to the basis $\{q_{i_1} \otimes \ldots \otimes q_{i_k}\}_{i_1,\ldots,i_k=0}^{\infty}$ are given by the formula (3):

$$F^{nm}_{i_1\ldots i_k} = (q_{i_1} \otimes \ldots \otimes q_{i_k}, f_{nm})_{L_2(\mathbb{T}^k)}, \quad i_1,\ldots,i_k = 0,1,2,\ldots$$

Expansion coefficients $X^{nm}_{i_1\ldots i_k}$ of x_{nm} can be similarly determined. According to both the linearity and the symmetry, they are related to $F^{nm}_{i_1\ldots i_k}$ by

$$F^{nm}_{i_1 i_1 i_3 i_3 \ldots i_{k-1} i_{k-1}} = 2^\gamma X^{nm}_{i_1 i_1 i_3 i_3 \ldots i_{k-1} i_{k-1}}.$$

Since the convergence in the norm implies the weak convergence, expansion coefficients for limit functions can be defined as follows:

$$F_{i_1\ldots i_k} = \lim_{n\to\infty} \lim_{m\to\infty} F^{nm}_{i_1\ldots i_k} = \lim_{n\to\infty} F^{n}_{i_1\ldots i_k}, \quad X_{i_1\ldots i_k} = \lim_{n\to\infty} \lim_{m\to\infty} X^{nm}_{i_1\ldots i_k} = \lim_{n\to\infty} X^{n}_{i_1\ldots i_k}.$$

Further, we can write that

$$\lim_{\varepsilon\to 0} \mathcal{S}^\varepsilon f_{nm}(t_1,t_1,t_3,t_3,\ldots,t_{k-1},t_{k-1}) = \varphi_n(t_1,t_3,\ldots,t_{k-1})\psi_m(t_1,t_3,\ldots,t_{k-1}),$$
$$\operatorname{tr} f_{nm} = \int_{\mathbb{T}^\gamma} \varphi_n(\tau_1,\ldots,\tau_\gamma)\psi_m(\tau_1,\ldots,\tau_\gamma)d\tau_1\ldots d\tau_\gamma = (\varphi_n,\psi_m)_{L_2(\mathbb{T}^\gamma)},$$

and in accordance with Theorem 3 we have the following equality for arbitrary n and m:

$$\sum_{i_1,i_3,\ldots,i_{k-1}=0}^{\infty} F^{nm}_{i_1 i_1 i_3 i_3 \ldots i_{k-1} i_{k-1}} = 2^{\gamma} \sum_{i_1,i_3,\ldots,i_{k-1}=0}^{\infty} X^{nm}_{i_1 i_1 i_3 i_3 \ldots i_{k-1} i_{k-1}} = (\varphi_n, \psi_m)_{L_2(\mathbb{T}^{\gamma})}, \tag{21}$$

where the series converge absolutely, and their sums do not depend on a basis $\{q_i\}_{i=0}^{\infty}$. Similar relations hold for functions f_n and f.

We can fix n in the equality (21). Then it defines a bounded linear functional in $L_2(\mathbb{T}^{\gamma})$, which is given by the function φ_n (the trace of operator is also a bounded linear functional but in the space of trace class operators [26]). Letting $m \to \infty$, we obtain a bounded linear functional given by the function ψ:

$$\sum_{i_1,i_3,\ldots,i_{k-1}=0}^{\infty} F^{n}_{i_1 i_1 i_3 i_3 \ldots i_{k-1} i_{k-1}} = 2^{\gamma} \sum_{i_1,i_3,\ldots,i_{k-1}=0}^{\infty} X^{n}_{i_1 i_1 i_3 i_3 \ldots i_{k-1} i_{k-1}} = (\varphi_n, \psi)_{L_2(\mathbb{T}^{\gamma})}.$$

Letting $n \to \infty$, we obtain the following result:

$$\sum_{i_1,i_3,\ldots,i_{k-1}=0}^{\infty} F_{i_1 i_1 i_3 i_3 \ldots i_{k-1} i_{k-1}} = 2^{\gamma} \sum_{i_1,i_3,\ldots,i_{k-1}=0}^{\infty} X_{i_1 i_1 i_3 i_3 \ldots i_{k-1} i_{k-1}} = (\varphi, \psi)_{L_2(\mathbb{T}^{\gamma})},$$

hence

$$\begin{aligned}
\sum_{i_1,i_3,\ldots,i_{k-1}=0}^{\infty} \mathbb{K}^{\psi}_{i_1 i_1 i_3 i_3 \ldots i_{k-1} i_{k-1}} &= \frac{1}{2^{\gamma}} \int_{\mathbb{T}^{\gamma}} \varphi(\tau_1, \ldots, \tau_{\gamma}) \psi(\tau_1, \ldots, \tau_{\gamma}) d\tau_1 \ldots d\tau_{\gamma} \\
&= \frac{1}{2^{\gamma}} \int_{\mathbb{T}^{\gamma}} \psi_1(\tau_1)\psi_2(\tau_1) \ldots \psi_{k-1}(\tau_{\gamma})\psi_k(\tau_{\gamma}) \\
&\qquad \times \mathbf{1}(\tau_2 - \tau_1)\mathbf{1}(\tau_3 - \tau_2) \ldots \mathbf{1}(\tau_{\gamma} - \tau_{\gamma-1}) d\tau_1 \ldots d\tau_{\gamma} \\
&= \frac{1}{2^{\gamma}} (\psi_1 \otimes \psi_3 \otimes \ldots \otimes \psi_{k-1}, \psi_2 \otimes \psi_4 \otimes \ldots \otimes \psi_k)_{L_2(\Delta \mathbb{T}^{\gamma})},
\end{aligned}$$

i.e., the theorem is proved. □

Remark 3.

1. *A transition to symmetrized Volterra-type kernels is made in proving Theorems 2 and 4. The series in relations (14) and (19) converge regardless of a basis $\{q_i\}_{i=0}^{\infty}$, and as a consequence they converge absolutely.*
2 *Certainly, Theorem 2 is the particular case of Theorem 4. However, it is useful to formulate and prove Theorem 2 separately for two reasons. Firstly, for the case $k = 2$, the proof is more simple, and it gives an idea of the proof in the general case. Secondly, this theorem is sufficient for solving the trace convergence problem if we consider not multiple series but iterated ones, for example,*

$$\sum_{i_1=0}^{\infty} \sum_{i_3=0}^{\infty} \cdots \sum_{i_{k-1}=0}^{\infty} \mathbb{K}^{\psi}_{i_1 i_1 i_3 i_3 \ldots i_{k-1} i_{k-1}} \quad \textit{or} \quad \sum_{i_{k-1}=0}^{\infty} \cdots \sum_{i_3=0}^{\infty} \sum_{i_1=0}^{\infty} \mathbb{K}^{\psi}_{i_1 i_1 i_3 i_3 \ldots i_{k-1} i_{k-1}}$$

instead of

$$\sum_{i_1,i_3,\ldots,i_{k-1}=0}^{\infty} \mathbb{K}^{\psi}_{i_1 i_1 i_3 i_3 \ldots i_{k-1} i_{k-1}}.$$

For solving the trace convergence problem with iterated series, it suffices to apply Theorem 2 iteratively. This approach seems appropriate for applications to iterated Stratonovich stochastic integrals.

3. *How the trace convergence problem is related to the definition of the multiple Stratonovich stochastic integral is shown in [27].*

As an example, we can find the sum of series

$$\sum_{i_1,i_3,\ldots,i_{k-1}=0}^{\infty} \mathbb{K}_{i_1 i_1 i_3 i_3 \ldots i_{k-1} i_{k-1}}^{n_1 n_2 n_3 n_4 \ldots n_{k-1} n_k},$$

where $\mathbb{K}_{i_1 i_1 i_3 i_3 \ldots i_{k-1} i_{k-1}}^{n_1 n_2 n_3 n_4 \ldots n_{k-1} n_k}$ are expansion coefficients (8) of the function $\Bbbk_{n_1 \ldots n_k}$ given by the formula (1) for $k = 2\gamma$ and $\gamma \in \mathbb{N}$, $\mathbb{T} = [0,1]$.

Applying Theorem 4, we obtain

$$\begin{aligned}
&\sum_{i_1,i_3,\ldots,i_{k-1}=0}^{\infty} \mathbb{K}_{i_1 i_1 i_3 i_3 \ldots i_{k-1} i_{k-1}}^{n_1 n_2 n_3 n_4 \ldots n_{k-1} n_k} \\
&= \frac{1}{2^\gamma} \int_{[0,1]^\gamma} \tau_1^{n_1} \tau_1^{n_2} \tau_2^{n_3} \tau_2^{n_4} \ldots \tau_\gamma^{n_{k-1}} \tau_\gamma^{n_k} \mathbf{1}(\tau_2 - \tau_1) \mathbf{1}(\tau_3 - \tau_2) \ldots \mathbf{1}(\tau_\gamma - \tau_{\gamma-1}) d\tau_1 d\tau_2 \ldots d\tau_\gamma \\
&= \frac{1}{2^\gamma} \int_0^1 \tau_\gamma^{n_{k-1}+n_k} \ldots \int_0^{\tau_3} \tau_2^{n_3+n_4} \int_0^{\tau_2} \tau_1^{n_1+n_2} d\tau_1 d\tau_2 \ldots d\tau_\gamma,
\end{aligned}$$

and then we should integrate sequentially, i.e.,

$$\begin{aligned}
&\sum_{i_1,i_3,\ldots,i_{k-1}=0}^{\infty} \mathbb{K}_{i_1 i_1 i_3 i_3 \ldots i_{k-1} i_{k-1}}^{n_1 n_2 n_3 n_4 \ldots n_{k-1} n_k} \\
&= \frac{1}{2^\gamma (n_1 + n_2 + 1)} \int_0^1 \tau_\gamma^{n_{k-1}+n_k} \ldots \int_0^{\tau_4} \tau_3^{n_5+n_6} \int_0^{\tau_3} \tau_2^{n_1+n_2+n_3+n_4+1} d\tau_2 d\tau_3 \ldots d\tau_\gamma \\
&= \frac{1}{2^\gamma (n_1 + n_2 + 1)(n_1 + n_2 + n_3 + n_4 + 2)} \\
&\quad \times \int_0^1 \tau_\gamma^{n_{k-1}+n_k} \ldots \int_0^{\tau_4} \tau_3^{n_1+n_2+n_3+n_4+n_5+n_6+2} d\tau_3 d\tau_4 \ldots d\tau_\gamma = \ldots \\
&= \frac{1}{2^\gamma} \prod_{j=1}^{\gamma} \left(\sum_{l=1}^{j} (n_{2l-1} + n_{2l} + 1) \right)^{-1}.
\end{aligned}$$

For the particular case $n_1 = \ldots = n_k = 0$ when $\Bbbk_{n_1 \ldots n_k} = \Bbbk$, we conclude that

$$\sum_{i_1,i_3,\ldots,i_{k-1}=0}^{\infty} \mathbb{K}_{i_1 i_1 i_3 i_3 \ldots i_{k-1} i_{k-1}} = \frac{1}{2^\gamma \gamma!},$$

where expansion coefficients $\mathbb{K}_{i_1 i_1 i_3 i_3 \ldots i_{k-1} i_{k-1}} = \mathbb{K}_{i_1 i_1 i_3 i_3 \ldots i_{k-1} i_{k-1}}^{0 \ldots 0}$ corresponds to the function $\Bbbk = \Bbbk_{0 \ldots 0}$, i.e.,

$$\sum_{i=0}^{\infty} \mathbb{K}_{ii} = \frac{1}{2} \quad (k = 2, \gamma = 1), \qquad \sum_{i_1,i_3=0}^{\infty} \mathbb{K}_{i_1 i_1 i_3 i_3} = \frac{1}{8} \quad (k = 4, \gamma = 2), \quad \ldots$$

These values correspond to expectations of the simplest iterated Stratonovich stochastic integrals of even multiplicities:

$$\begin{aligned}
\mathrm{E}\, {}^{\mathrm{S}}\mathcal{J}_{[0,1]}^{W(j_1 j_1)} \Bbbk &= \mathrm{E} \int_0^1 W_{j_1}(t) \circ dW_{j_1}(t) = \frac{1}{2} \quad (k = 2), \\
\mathrm{E}\, {}^{\mathrm{S}}\mathcal{J}_{[0,1]}^{W(j_1 j_1 j_3 j_3)} \Bbbk &= \mathrm{E} \int_0^1 \int_0^t \int_0^\tau W_{j_1}(\theta) \circ dW_{j_1}(\theta) \circ dW_{j_3}(\tau) \circ dW_{j_3}(t) = \frac{1}{8} \quad (k = 4), \quad \ldots,
\end{aligned}$$

where $j_1, j_3, \ldots \in \{1, 2, \ldots, s\}$ $(j_1 = j_2, j_3 = j_4, \ldots)$, E means the expectation operator.

Next, we present some numerical results. In Table 1, partial sums of series

$$\sum_{i=0}^{L-1} \mathbb{K}_{ii} \quad (k=2), \qquad \sum_{i_1,i_3=0}^{L-1} \mathbb{K}_{i_1 i_1 i_3 i_3} \quad (k=4)$$

are given under conditions that $L = 4, 8, \ldots, 64$ and the basis $\{q_i\}_{i=0}^{\infty}$ is chosen as follows:

$$q_i(t) = \begin{cases} 1 & \text{for } i=0 \\ \sqrt{2}\cos i\pi t & \text{for even } i>0 \\ \sqrt{2}\sin(i+1)\pi t & \text{for odd } i, \end{cases} \qquad (F)$$

$$q_i(t) = \begin{cases} 1 & \text{for } i=0 \\ \sqrt{2}\cos i\pi t & \text{for } i>0, \end{cases} \qquad (C)$$

$$q_i(t) = \sqrt{2}\sin(i+1)\pi t, \qquad (S)$$

i.e., we consider three cases: the Fourier basis, cosines (for expansion of even functions in Fourier series), and sines (for expansion of odd functions in Fourier series). For $k = 2$ and for both bases (F) and (C), partial sums coincide with the exact value, since in this case q_0 is the constant function. Otherwise, partial sums approximate the corresponding exact values.

Table 1. Partial sums of series.

Basis	$L=4$	$L=8$	$L=16$	$L=32$	$L=64$	**Exact**
$k=2$						
(F)	0.500000	0.500000	0.500000	0.500000	0.500000	1/2
(C)	0.500000	0.500000	0.500000	0.500000	0.500000	1/2
(S)	0.450316	0.474799	0.487351	0.493669	0.496834	1/2
$k=4$						
(F)	0.101826	0.112996	0.118849	0.121881	0.123429	1/8
(C)	0.110621	0.118255	0.121733	0.123392	0.124202	1/8
(S)	0.093084	0.107566	0.115893	0.120348	0.122650	1/8

Table 2 gives the estimates of the expectation of iterated Stratonovich stochastic integrals ${}^{\mathrm{S}}\mathcal{J}^{W(j_1 j_1)}_{[0,1]}\Bbbk$ and ${}^{\mathrm{S}}\mathcal{J}^{W(j_1 j_1 j_3 j_3)}_{[0,1]}\Bbbk$ under the same conditions. We use the following partial sums of multiple random series for their simulation:

$${}^{\mathrm{S}}\mathcal{J}^{W(j_1 j_1)}_{[0,1]}\Bbbk \approx \sum_{i_1,i_2=0}^{L-1} \mathbb{K}_{i_1 i_2} \zeta^{(j_1)}_{i_1} \zeta^{(j_1)}_{i_2},$$

$${}^{\mathrm{S}}\mathcal{J}^{W(j_1 j_1 j_3 j_3)}_{[0,1]}\Bbbk \approx \sum_{i_1,i_2,i_3,i_4=0}^{L-1} \mathbb{K}_{i_1 i_2 i_3 i_4} \zeta^{(j_1)}_{i_1} \zeta^{(j_1)}_{i_2} \zeta^{(j_3)}_{i_3} \zeta^{(j_3)}_{i_4},$$

where $\zeta^{(j)}_i$ are independent random variables having standard normal distribution for $i = 0, 1, 2, \ldots$ and $j = j_1, j_3$ (we assume that $j_1 \neq j_3$). These estimates are obtained from 10^6 realizations of iterated Stratonovich stochastic integrals. Obviously, they correspond to partial sums from Table 1.

Moreover, these numerical results together with data from [4,21] show that the Fourier basis, which used for approximation of iterated Stratonovich stochastic integrals in the Milstein method and then in the strong 1.5 order method [1,2] does not provide high accuracy. This is noted in [3] based on a comparison the Fourier basis with Legendre polynomials. However, the presented result indicates that trigonometric functions can be effectively used for the approximation of iterated Stratonovich stochastic integrals, but it is only necessary to restrict ourselves to cosines.

Table 2. The estimates of the expectation of iterated Stratonovich stochastic integrals.

Basis	$L = 4$	$L = 8$	$L = 16$	$L = 32$	$L = 64$	Exact
$k = 2$						
(F)	0.499867	0.499904	0.499622	0.500227	0.499657	1/2
(C)	0.499242	0.499384	0.499971	0.499813	0.500034	1/2
(S)	0.449581	0.474902	0.486190	0.494549	0.496971	1/2
$k = 4$						
(F)	0.101850	0.113372	0.118709	0.121523	0.123627	1/8
(C)	0.110726	0.118727	0.121476	0.123099	0.124747	1/8
(S)	0.093502	0.107255	0.115730	0.120210	0.122978	1/8

5. Conclusions

In this paper, one variant of the trace convergence problem is solved. This problem is to prove the absolute convergence of traces that are formed by summing the expansion coefficients of factorized Volterra-type functions. Here we restrict ourselves to summing over neighboring pairs of indices only, assuming that this is sufficient. Solving the trace convergence problem involves the theory of trace class operators for symmetrized Volterra-type kernels. In general, i.e., for all square-integrable functions, this problem has no solution. Therefore, it is required to reduce the class of functions.

The main application of the presented results is related to the mean-square approximation of iterated Stratonovich stochastic integrals, which are used to implement numerical methods for solving stochastic differential equations based on the Taylor–Stratonovich expansion. In addition, these results can be relevant to other stochastic integrals with similar properties. For example, the obtained results can be applied to iterated Ogawa stochastic integrals [28], since they also require the representation of factorized Volterra-type functions as the orthogonal series.

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