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Certain Solutions of Abel's Integral Equations on Distribution Spaces via Distributional G_α-Transform

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Abstract: Abel's integral equation is an efficient singular integral equation that plays an important role in diverse fields of science. This paper aims to investigate Abel's integral equation and its solution using G_α-transform, which is a symmetric relation between Laplace and Sumudu transforms. G_α-transform, as defined via distribution space, is employed to establish a solution to Abel's integral equation, interpreted in the sense of distributions. As an application to the given theory, certain examples are given to demonstrate the efficiency and suitability of using the G_α-transform method in solving integral equations.

Keywords: Abel integral equation; distribution space; G_α-transform; Laplace-type integral transform

MSC: 44A05; 44A10; 44A35; 45E10; 46F12

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1. Introduction

Integral equations are considered to be the most effective tool for modeling various engineering topics and physical phenomena. Integral equations have been rapidly developed in recent decades and have attracted various researchers in the field of mathematics due to their enormous applications in pure and applied mathematics. Abel's equation is an integral equation developed from a real-world physical issue satisfying a differential equation. In the literature, numerous physical issues, including heat transfer, scattering theory, elasticity theory, propagation of nonlinear waves, and plasma physics are connected to Abel's integral equation [1–4]. Several methods for solving the generalized Abel's integral equation have already been built and proposed by many researchers (see, e.g., [5–12]). To solve integral equations, integral transform methods were extensively employed in [13–16] and integral transform methods are applied to find solutions to applications equations as well [17–19].

In mathematics, distributions are objects which make discontinuous functions more likely to be smooth [20,21]. They are extensively applied in physics and engineering problems. Al-Omari [22–24] extended various integral transforms—including the natural, Mellin, and double-Sumudu transforms—to a class of distributions and proposed certain applications involving certain initial values problems. However, in the sequence of such integral transforms, G_α-transform is defined for a function, f, by [25], as follows:

$$K(u) = G_\alpha\{f(\tau)\} = u^\alpha \int_0^\infty e^{-\tau/u} f(\tau)d\tau, \tag{1}$$

where $\alpha \in \mathbb{Z}$. The Laplace, Sumudu, and Elzaki transforms are special cases of the G_α-transform, obtained by decreasing the G_α-transform to their respective values of $\alpha = 0, -1$, and 1.

Kim [26] considered the value $\alpha = -2$ to provide a straightforward tool for integral transforms and introduced the G_{-2}-transform to solve the Laguerre's equation, where the G_{-2}-transform can be given as

$$G_{-2}\{f(\tau)\} = \frac{1}{u^2}\int_0^\infty e^{-\tau/u} f(\tau)d\tau.$$

Moreover, Sattaso et al. [27] discussed the properties of the G_α-transform and presented an example that is appropriate for using the G_α-transform but inappropriate for using the Sumudu and Elzaki transforms. Meanwhile, Kim et al. [28] discussed the n-th partial derivative of the G_α-transform of certain partial differential equations. On the other hand, Prasertsang et al. [29] studied the range of the G_α-transform that can be utilized to solve ordinary differential equations with variable coefficients and provided an example that corresponds to this study.

Recently, Nuruddeen et al. [30] described an essential transform coupled with Adomian's approach to studying nonlinear evolution equations endowed with non-integer derivatives. However, it is interesting to know that the strong nonlinear problems in nonlinear sciences, such as Schrodinger and nonlinear evolution equations, can be solved using this method, which is highly recommended.

Our goal in this study is to extend classes of distributions in the sense of the G_α-transform. G_α-transform has been used to solve the Abel integral equation, whereas the solution of the distributional Abel's integral equation is obtained using distributional G_α-transform.

The rest of the paper is organized as follows: Some primary definitions are briefly retrieved in Section 1. In Section 2, several properties of the classical G_α-transform and auxiliary results are derived in detail. In Section 3, a solution of Abel's integral equation is obtained by following the G_α-transform technique. In Section 4, the G_α-transform of distribution space and its application to finding a solution of Abel's integral equation are established in the sense of distributions. Illustrative examples and a conclusion are provided in Section 5 and Section 6, respectively.

2. Preliminaries

In this section, we present the necessary details and properties of G_α-transform and establish several theorems related to G_α-transform, which are useful in the sequel.

Given that the power series function $f(\tau) = \sum_{k=0}^{\infty} a_k\tau^k$ is piecewise continuous on $[0,\infty)$ and of exponential order at infinity, then the discrete analog of the G_α-transform (1) is given by

$$G_\alpha\{f(\tau)\} = \sum_{k=0}^{\infty} k! a_k u^{\alpha+k+1}.$$

The G_α-transform and Laplace transform duality relationships are expressed as

$$K(u) = u^\alpha F\left(\frac{1}{u}\right); \quad F(s) = s^\alpha K\left(\frac{1}{s}\right), \tag{2}$$

where F denotes the Laplace transform and K denotes the G_α-transform. The G_α-transform of the n-th order derivative of a function f is defined by

$$G_\alpha\{f^{(n)}(\tau)\} = \frac{1}{u^n}G_\alpha\{f\} - \frac{1}{u^{n-1}}f(0)u^\alpha - \cdots - \frac{1}{u}f^{(n-2)}(0)u^\alpha - f^{(n-1)}(0)u^\alpha.$$

The convolution theorem of the G_α-transform is given by

$$G_\alpha\{f * g\}(\tau) = \frac{1}{u^\alpha}G_\alpha\{f(\tau)\}G_\alpha\{g(\tau)\}. \tag{3}$$

If $f^{(n)}$ is the n-th order derivative of f , then we have

$$
\begin{aligned}
G_\alpha\{\tau^m f^{(n)}(\tau)\} = {} & u^{2m}\frac{d^m G_\alpha\{f^{(n)}(\tau)\}}{du^m} - \binom{m}{1}[\alpha-(m-1)]u^{2m-1}\frac{d^{m-1}G_\alpha\{f^{(n)}(\tau)\}}{du^{m-1}} \\
& + \cdots - \binom{m}{m-1}[\alpha-(m-1)][\alpha-(m-2)]\cdots(\alpha-1)u^{m+1}\frac{dG_\alpha\{f^{(n)}(\tau)\}}{du} \\
& + [\alpha-(m-1)][\alpha-(m-2)]\cdots\alpha u^m G_\alpha\{f^{(n)}(\tau)\}.
\end{aligned}
$$

Here, we present some useful properties of the G_α-transform pertinent to the present paper.

Theorem 1 (Inversion formula of G_α-transform)**.** *Let K be the G_α-transform of f and*

(i) The meromorphic function $s^\alpha K(1/s)$ contains singularities with $Re(s) < a$;

(ii) There is a circular region Γ of radius R, M and N are positive constants, such that

$$|s^\alpha T(1/s)| < M/R^N.$$

Then, the function f is defined as

$$G_\alpha^{-1}\{K(s)\} = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{s\tau}s^\alpha K\left(\frac{1}{s}\right)ds.$$

That is,

$$f(\tau) = \sum \text{residues of } \left[e^{s\tau}s^\alpha K\left(\frac{1}{s}\right)\right]. \tag{4}$$

Moreover, by utilizing (2), (4) can be written as

$$f(\tau) = \sum \text{residues of } \left[\frac{1}{u^\alpha}e^{\tau/u}K(u)\right]. \tag{5}$$

Proof. If $F(s) = \mathcal{L}\{f(\tau)\}$ and $K(u) = G_\alpha\{f(\tau)\}$ are, respectively, the Laplace transform and the G_α-transform of $f(\tau)$; then, for $\tau > 0$, $f(\tau)$ is the complex inversion formula of the Laplace transform given as

$$f(\tau) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{s\tau}F(s)ds,$$

where $s = x + iy$ is a complex variable. The branch points, fundamental singularities, and poles are included. The residue theorem leads to the following result:

$$f(\tau) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{s\tau}F(s)ds = \sum \text{res}[e^{s\tau}F(s)].$$

The intended result of this theorem is obtained by referring to the prior relationship, $s^\alpha K(1/s)$, between the transforms F and K. □

Theorem 2. *Let $f(\tau)$ and $g(\tau)$ be continuous functions and $F(u)$ and $G(u)$ be their respective G_α-transforms for $\tau \geq 0$ and a complex number u. Then, $f(\tau) = g(\tau)$ if $F(u) = G(u)$ everywhere.*

Proof. Let $F(u) = G(u)$; then, we have

$$G_\alpha\{f(\tau), u\} = G_\alpha\{g(\tau), u\}.$$

In view of the relation between the G_α-transform and the Laplace transform, we obtain

$$\mathcal{L}\left\{f(\tau), \frac{1}{u}\right\} = \mathcal{L}\left\{g(\tau), \frac{1}{u}\right\}.$$

By the uniqueness property of the Laplace transform, we have

$$f(\tau) = g(\tau).$$

Therefore, the proof is completed. □

Theorem 3 ([27] The existence of the G_α-transform). *Let $|f(\tau)| \leq Me^{k\tau}$ for $M \geq 0, \tau > C$ and k, C are constants; f is a piecewise continuous function on $[0, \infty)$ of exponential order at infinity. If F is the G_α-transform of f, then F exists for $u < 1/k$.*

The existence of the G_α-transform for derivatives and higher derivatives is demonstrated in the following theorems:

Theorem 4 (The G_α-transform of derivatives)**.**

(i) *If f is differentiable on $(0, \infty)$, $f(\tau) = 0$ for $\tau < 0$ and $f' \in L_{loc}$, then $dom(G_\alpha\{f\}) \subset dom(f')$ and*

$$G_\alpha\{f'\} = \frac{1}{u} G_\alpha\{f\} - u^\alpha f(0+) \quad \text{for } u \in dom(G_\alpha\{f\});$$

(ii) *In a broader sense, if f is differentiable on (h, ∞), $f(\tau) = 0$ for $\tau < 0$ and $f' \in L_{loc}$ then*

$$G_\alpha\{f'\} = \frac{1}{u} G_\alpha\{f\} - u^\alpha e^{-h/u} f(h+) \quad \text{for } u \in dom(G_\alpha\{f\}).$$

Proof. We begin by stating (1) as follows: The local integrability entails the existence of $f(h+)$, because if $x > h$, then

$$f(x) = f(h+1) - \int_x^{h+1} f'(\tau) d\tau \to f(h+1) - \int_h^{h+1} f'(\tau) d\tau \ \text{ as } x \to h^+.$$

Let $u \in dom(G_\alpha\{f\})$. If $z \in D_0$ (z : z differentiable and $z(0) = 0$), then—by integrating by parts—we have

$$\begin{aligned}
&u^\alpha \int z\left(\frac{\tau}{\gamma}\right) e^{-\tau/u} f'(\tau) d\tau \\
&= u^\alpha \int_h^\infty z\left(\frac{\tau}{\gamma}\right) e^{-\tau/u} f'(\tau) d\tau \\
&= \lim_{x \to h+} \left[-u^\gamma z\left(\frac{x}{\gamma}\right) e^{-x/u} f(x)\right] - u^\alpha \int_h^\infty e^{-\tau/u} \left[\frac{1}{\gamma} z'\left(\frac{\tau}{\gamma}\right) - \frac{1}{u} z\left(\frac{\tau}{\gamma}\right)\right] f(\tau) d\tau.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&- u^\alpha z\left(\frac{h}{\gamma}\right) e^{-h/u} f(h+) \to -u^\alpha z(0) e^{-h/u} f(h+) \quad \text{as } \gamma \to \infty, \\
&- \frac{u^\alpha}{\gamma} \int z'\left(\frac{\tau}{\gamma}\right) e^{-\tau/u} f(\tau) d\tau + \frac{u^\alpha}{u} \int z\left(\frac{\tau}{\gamma}\right) e^{-\tau/u} f(\tau) d\tau \to 0 + \frac{1}{u} z(0) G_\alpha\{f\}
\end{aligned}$$

as $\gamma \to \infty$. Thus, for any $z \in D_0$, we have

$$\lim_{\gamma \to \infty} \left[u^\alpha \int z\left(\frac{\tau}{\gamma}\right) e^{-\tau/u} f'(\tau) d\tau\right] = \frac{z(0)}{u} G_\alpha\{f\} - u^\alpha z(0) e^{-h/u} f(h+).$$

This suggests that $e^{-\tau/u} f'(\tau)$ converges, i.e., $u \in \text{dom}(G_\alpha\{f\})$ and

$$G_\alpha\{f'\} = \frac{1}{u} G_\alpha\{f\} - u^\alpha e^{-h/u} f(h+).$$

□

Next, we will consider the G_α-transform of higher derivatives and their matrix form. First, we introduce the following notation. For $P(x) = \sum_{k=0}^{n}(a_k/x^k)$ is a polynomial in $1/x$, where $n \geq 0$ and $a_n \neq 0$, we define M_p to be the $1 \times n$ matrix as follows:

$$M_p(x) = \left(\frac{1}{x}\,\frac{1}{x^2}\,\frac{1}{x^3}\cdots\frac{1}{x^n}\right)\begin{pmatrix} a_1 & a_2 & \cdots & \cdots & \cdots & a_n \\ a_2 & a_3 & \cdots & \cdots & a_n & 0 \\ a_3 & \cdots & \cdots & a_n & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_n & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}. \tag{6}$$

Indeed, M_p establishes a linear mapping of $\mathbb{C}^n$ into $\mathbb{C}$. We write vectors y in $\mathbb{C}^n$ as either the row vectors or the column vectors interchangeably, depending on which is more convenient when $M_p(x)y$ is to be computed and the matrix representation (6) of $M_p(x)$ is used. Of course, y must be written as a column vector, as follows:

$$M_p(x)y = \sum_{i=1}^{n}\frac{1}{x^i}\sum_{k=0}^{n-i} a_{i+k}y_k \tag{7}$$

for any $y = (y_0, y_1, \ldots, y_{n-1}) \in \mathbb{C}^n$. M_p is a unique linear mapping of $\{0\} = \mathbb{C}^0$ into $\mathbb{C}$ (empty matrix), if $n = 0$. Generally, if $n > 0$ and f is $n-1$ times differentiable on an interval (a, b), we will write

$$\rho(f;a;n) = \Big(f(a+), f'(a+), \ldots, f^{(n-1)}(a+)\Big) \in \mathbb{C}^n$$

and

$$\theta(f;b;n) = \Big(f(b-), f'(b-), \ldots, f^{(n-1)}(b-)\Big) \in \mathbb{C}^n.$$

If $a = 0$, $\rho(f;n)$ is written for $\rho(f;0;n)$. If $n = 0$, then we define

$$\rho(f;a;0) = \theta(f;b;0) = 0 \in \mathbb{C}^0.$$

Theorem 5 (G_α-transform for higher derivatives)**.** *If f is n times differentiable on $(0,\infty)$, $f(t) = 0$ for $t < 0$ and $f^{(n)} \in L_{loc}$, then $f^{(k)} \in L_{loc}$ for $0 \leq k \leq n-1$, $dom(G_\alpha\{f\}) \subset dom(f^{(n)})$ and, for any polynomial P of degree n, we have*

$$P(u)G_\alpha\{y\}(u) = G_\alpha\{P(D)f\}(u) + u^{\alpha+1}M_p(u)\rho(y;n)$$

for $u \in dom(G_\alpha\{f\})$. In particular,

$$G_\alpha\{f^{(n)}\}(u) = \frac{1}{u^n}G_\alpha\{f\}(u) - \left(\frac{1}{u^n}, \frac{1}{u^{n-1}}, \ldots, \frac{1}{u}\right)u^{\alpha+1}\rho(f;n).$$

(In this case, $\rho(f;n)$ is written as a column vector). For $n = 2$, we have

$$G_\alpha\{f''\}(u) = \frac{1}{u^2}G_\alpha\{f\}(u) - u^{\alpha-1}f(0+) - u^{\alpha}f'(0+).$$

Proof. We employ induction on n. If $n = 0$, then the conclusion is obviously true. If $n = 1$, then the conclusion is equivalent to Theorem 4. Assume that the conclusion is true for some value of $n \geq 1$. Let $P(x) = \sum_{k=0}^{n+1}(a_k/x^k)$ have degree $n+1$ and take the form $P(x) = a_0 + (1/x)W(x)$, where $W(x) = \sum_{k=0}^{n}(a_{k+1}/x^k)$. It follows that $P(D)f = a_0 f + W(D)z$. Hence, by using Theorem 3, we obtain

$$G_\alpha\{P(D)f\}(u) = a_0 G_\alpha\{f\}(u) + W(u)\left[\frac{1}{u}G_\alpha\{f\}(u) - u^\alpha f(0+)\right]$$
$$- u^{\alpha+1}\sum_{i=1}^{n}\frac{1}{u^i}\sum_{k=0}^{n-i}a_{i+k+1}f^{(k+1)}(0+),$$

using (7) and setting $z^{(k)} = f^{(k+1)}$. The summation can, therefore, be expressed as follows:

$$\sum_{i=1}^{n}\frac{1}{u^i}\sum_{k=1}^{n-i+1}a_{i+k}f^{(k)}(0+) = \sum_{i=1}^{n+1}\frac{1}{u^i}\sum_{k=0}^{n-i+1}a_{i+k}f^{(k)}(0+) - \frac{1}{u}\left[\frac{1}{u^n}a_{n+1}f(0+) + \sum_{i=1}^{n}\frac{1}{u^{i-1}}a_i f(0+)\right]$$
$$= M_p(u)\rho(f;n) - \frac{1}{u}W(u)f(0+).$$

As a result, we obtain

$$G_\alpha\{P(D)f\}(u) = \left[a_0 + W(u)\frac{1}{u}\right]G_\alpha\{f\}(u) - u^\alpha W(u)f(0+)$$
$$- u^{\alpha+1}M_p(u)\rho(f;n) + u^\alpha W(u)f(0+)$$
$$= P(u)G_\alpha\{f\}(u) - u^{\alpha+1}M_p(u)\rho(f;n).$$

□

The Abel integral equation is provided by [31]

$$f(s) = \int_a^s \frac{g(\tau)}{(s-\tau)^\gamma}d\tau, \quad s > a,\ 0 < \gamma < 1 \tag{8}$$

and its solution is

$$g(\tau) = \frac{\sin\gamma\pi}{\pi}\frac{d}{d\tau}\left[\int_a^\tau \frac{f(s)}{(\tau-s)^{1-\gamma}}ds\right]. \tag{9}$$

There are two ways to find the solution, which are illustrated in [31].

3. The G_α-Transform of Abel's Integral Equation

In this section, we use the G_α-transform to prove Abel's integral equation. Abel's integral equation is expressed as

$$f(\tau) = \int_0^\tau \frac{g(s)}{(\tau-s)^\beta}ds, \quad 0 < \beta < 1, \tag{10}$$

which is sometimes written as

$$f(\tau) = g(\tau) * \tau_+^{-\beta}, \tag{11}$$

where $\tau_+^{-\beta} = \tau^{-\beta}H(\tau)$ and $H(\tau)$ is a Heavside's unit step function. By applying the G_α-transform on both sides of (11), and using the convolution theorem of the G_α-transform (3) in (11), we obtain

$$G_\alpha\{f(\tau)\} = u^{-\alpha}G_\alpha\{g(\tau)\}G_\alpha\{\tau_+^{-\beta}\}, \tag{12}$$

where $f(\tau) = \tau^{n-1}/(n-1)!$ and the G_α-transform is $F(u) = u^{n+\alpha}$. Similarly, we prove that the G_α-transform is $F(u) = \Gamma(1-\beta)u^{-\beta+\alpha+1}$, if $f(\tau) = \tau^{-\beta}$. Putting the value of $G_\alpha\{\tau^{-\beta}\}$ in (12), we obtain

$$F(u) = u^{-\alpha}G_\alpha\{g(\tau)\}\cdot\Gamma(1-\beta)u^{-\beta+\alpha+1}$$

and

$$\begin{aligned} G_\alpha\{g(\tau)\} &= \frac{F(u)}{\Gamma(1-\beta)} \cdot \frac{u^{\beta-\alpha-1}}{u^{-\alpha}} \\ &= \frac{F(u)\Gamma(\beta)}{\Gamma(\beta)\Gamma(1-\beta)} \cdot \frac{u^{\beta-\alpha-1}}{u^{-\alpha}} \\ &= \frac{\Gamma(\beta)\sin\pi\beta}{\pi} \cdot \frac{F(u)u^{\beta-\alpha-1}}{u^{-\alpha}} \\ &= \frac{\Gamma(\beta)\sin\pi\beta}{\pi} \cdot F(u)u^{\beta-1}, \end{aligned}$$

where $\Gamma(\beta)\Gamma(1-\beta) = \frac{\pi}{\sin\pi\beta}$. Hence, it yields

$$\begin{aligned} G_\alpha\{g(\tau)\} &= \frac{\sin\pi\beta}{\pi}\Big[\Gamma(\beta)F(u)u^{\beta-1}\Big], \\ uG_\alpha\{g(\tau)\} &= \frac{\sin\pi\beta}{\pi}G_\alpha\{f(\tau)*\tau^{\beta-1}\}; \quad G_\alpha\{\tau^{n-1}\} = \Gamma(n)u^{n+\alpha}. \end{aligned}$$

From $G_\alpha\{\int_0^\infty f(\tau)d\tau\} = uF(u)$ (see [27]) and the definition of the convolution, we obtain

$$\begin{aligned} G_\alpha\{\int_0^\tau g(x)dx\} &= \frac{\sin\pi\beta}{\pi}G_\alpha\{\int_0^\tau f(x)(\tau-x)^{\beta-1}dx\} \\ &= \frac{\sin\pi\beta}{\pi}G_\alpha\{k(\tau)\}, \end{aligned}$$

where $k(\tau) = \int_0^\tau f(x)(\tau-x)^{\beta-1}dx,\ k(0) = 0$. We have

$$G_\alpha\{k'(\tau)\} = \frac{K(u)}{u}.$$

Therefore, we obtain

$$\begin{aligned} uG_\alpha\{g(\tau)\} &= \frac{\sin\pi\beta}{\pi}uG_\alpha\{k'(\tau)\}, \\ G_\alpha\{g(\tau)\} &= G_\alpha\{\frac{\sin\pi\beta}{\pi}k'(\tau)\}. \end{aligned}$$

By the uniqueness theorem, we obtain

$$g(\tau) = \frac{\sin\pi\beta}{\pi}\frac{d}{d\tau}\left[\int_0^\tau f(x)(\tau-x)^{\beta-1}dx\right].$$

Thus, we have obtained the solution of the Abel's integral equation.

4. The G_α-Transform and Abel's Integral Equation on Distribution Spaces

The G_α-transform of certain distribution spaces is discussed in this section. Moreover, a relationship is established to solve Abel's integral equation using the distributional G_α-transform.

Suppose that f is a locally integrable function, then we have

$$\langle f,\phi\rangle = \langle f(\tau),\phi(\tau)\rangle \triangleq \int_{-\infty}^{\infty} f(\tau)\phi(\tau)d\tau \tag{13}$$

is the distribution f through the convergent integral (i.e., f in $\mathcal{D}'$).

According to Theorem 5, the G_α-transform of the function f generates a distribution. In other words, f is in $\mathcal{D}'$ and ϕ belongs to $\mathcal{D}$, where $\mathcal{D}$ and $\mathcal{D}'$ stand for the testing function space and it is dual space, respectively. The linearity property, as defined in [27] is

$$G_\alpha\{af(\tau) + bg(\tau)\} = aG_\alpha\{f(\tau)\} + bG_\alpha\{g(\tau)\}, \tag{14}$$

where a and b are any numbers.

According to (13) and (14), we can conclude that $f = g$ almost everywhere if the locally integrable functions $f(\tau)$ and $g(\tau)$ are absolutely integrable over $0 < \tau < \infty$, and their G_α-transforms $F(u)$ and $G(u)$ are equal everywhere.

The proof of the Parseval equation for the distributional G_α-transform is provided below, which will be used to analyze the problem of this paper.

Theorem 6. *If the locally integrable functions $f(\tau)$ and $g(\tau)$ are absolutely integrable over $0 < \tau < \infty$, then*

$$\int_0^\infty F(u)G(u)du = \int_0^\infty u^{\alpha+1} f(\tau)g(-\tau)d\tau. \tag{15}$$

Proof. Both sides of Equation (15) converge because, as demonstrated in Section 2, the transforms $F(u)$ and $G(u)$ are bounded and continuous for all u. Moreover, from (5) we have

$$\begin{aligned}\int_0^\infty F(u)G(u)du &= \int_0^\infty f(\tau)d\tau \int_0^\infty u^\alpha G(u)e^{-(\tau/u)}du \\ &= \int_0^\infty u^{2\alpha} f(\tau)d\tau \int_0^\infty u^{-\alpha} G(u)e^{-(\tau/u)}du \\ &= \int_0^\infty u^{2\alpha} f(\tau)g(-\tau)d\tau.\end{aligned}$$

The aforementioned integral is absolutely integrable. Hence, it follows

$$\int_0^\infty F(u)G(u)du = \int_0^\infty u^{2\alpha} f(\tau)g(-\tau)d\tau.$$

Furthermore, we take into account that $g(\tau) = f^*(-\tau)$, such that

$$\int_0^\infty F(u)F^*(u)du = \int_0^\infty u^{2\alpha} f(\tau)f^*(\tau)d\tau.$$

Consequently,

$$\int_0^\infty |F(u)|^2 du = \int_0^\infty u^{2\alpha} |f(\tau)|^2 d\tau.$$

As a result, the Parseval relation of the G_α-transform is

$$\|F\| = u^{2\alpha}\|f\|.$$

Therefore, the proof is completed. □

It is easy to see that (8) can be obtained by a convolution of a distribution, we refer to [31]. Consider the convolution as a bilinear operation $* : \mathcal{D}'[a,\infty) \times \mathcal{D}'[b,\infty) \to \mathcal{D}'[a+b,\infty)$. If $u \in \mathcal{D}'[a,\infty)$ and $v \in \mathcal{D}'[b,\infty)$ are locally integrable functions, then

$$(u * v)(\tau) = \int_a^{\tau-b} u(\omega)v(\tau-\omega)d\omega, \quad \tau > a+b.$$

When $b = 0$ and $v \in \mathcal{D}'[0,\infty)$, we have $u * vs. \in \mathcal{D}'[a,\infty)$. As a result, the operator of the space $\mathcal{D}'[a,\infty)$, defined by the convolution with v, is given by

$$(u * v)(\tau) = \int_a^\tau u(\omega)v(\tau-\omega)d\omega, \quad \tau > a,$$

where u and v are locally integrable functions.

The convolution of (8) results in

$$f = g * \tau_+^{-\beta}, \tag{16}$$

where $\tau_+^{-\beta} = \tau^{-\beta}H(\tau)$ and $\tau_+^{-\beta}$ is locally integrable, since $0 < \beta < 1$.

Abel's integral equation can be interpreted in the sense of distributions, according to Equation (16), and the functions f and g can be thought of as components of $\mathcal{D}'[a,\infty)$. Similarly, (9) can also be interpreted in the sense of distributions, as shown by

$$g(\tau) = \frac{\sin\beta\pi}{\pi}\frac{d}{d\tau}\left(f * \tau^{\beta-1}\right). \tag{17}$$

It may be noted that the G_α-transform is identified as having an affinity for the mixed spaces since $\mathcal{D}'[a,\infty)$, as well as the previously mentioned others, is one of the mixed distribution spaces that are identified with the space of distribution $\mathcal{D}'(R)$; the support of which is contained in $[a,\infty)$.

If (16) and (17) express the solution of Abel's integral equation on certain distribution spaces, the distributional G_α-transform is used in a similar way (as in Section 3) to derive the solution of Abel's integral equation. As a result, the analysis has been explicitly justified and explained.

5. Application

In this section, we provide two examples to illustrate our application of the G_α-transform to integral equations.

Example 1. *Consider the following integral equation by the G_α-transform method:*

$$\int_0^\tau \frac{f(x)}{(\tau-x)^{\frac{1}{2}}}dx = \tau^{\frac{5}{2}}. \tag{18}$$

In terms of convolution, Equation (18) can be expressed in the following form:

$$f(\tau) * \tau^{-\frac{1}{2}} = \tau^{\frac{5}{2}}. \tag{19}$$

By applying the G_α-transform on both sides of (19), we have

$$G_\alpha[f(\tau) * \tau^{-\frac{1}{2}}] = G_\alpha[\tau^{\frac{5}{2}}].$$

using the convolution of the G_α-transform from (3) given in the tables of the G_α-transform [27], the previous equation reveals that

$$\begin{aligned}
\frac{1}{u^\alpha}G_\alpha[f(\tau)]\cdot\Gamma\left(\frac{1}{2}\right)u^{\alpha+\frac{1}{2}} &= \left(\frac{5}{2}\right)!u^{\alpha+\frac{7}{2}},\\
G_\alpha[f(\tau)]\cdot\Gamma\left(\frac{1}{2}\right) &= \left(\frac{5}{2}\right)!u^{\alpha+3},\\
G_\alpha[f(\tau)] &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{5}{2}\right)!u^{\alpha+3}.
\end{aligned}$$

By allowing the inverse G_2-transform to act on the preceding equation, we write

$$\begin{aligned} f(\tau) &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{5}{2}\right)!\left(\frac{\tau^2}{2}\right) \\ &= \frac{1}{\sqrt{\pi}}\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi}\left(\frac{\tau^2}{2}\right) \\ &= \frac{15}{16}\tau^2. \end{aligned}$$

Example 2. *Consider the following integral equation by the G_α-transform method*

$$\frac{1}{\sqrt{\tau}} = \int_0^\tau \frac{f(x)}{\sqrt{\tau - x}}dx. \tag{20}$$

In view of the concept of convolution, we write (20) into the form

$$\frac{1}{\sqrt{\tau}} = f(\tau) * \tau^{-\frac{1}{2}}. \tag{21}$$

Applying the G_α-transform on both sides of (21) reveals

$$G_\alpha[\frac{1}{\sqrt{\tau}}] = G_\alpha[f(\tau) * \tau^{-\frac{1}{2}}].$$

using the convolution of the G_α-transform from (3) presented in the tables of the G_α-transform [27], the previous equation becomes

$$\begin{aligned} \left(-\frac{1}{2}\right)u^{\alpha+\frac{1}{2}} &= \frac{1}{u^\alpha}G_\alpha[f(\tau)] \cdot \left(-\frac{1}{2}\right)u^{\alpha+\frac{1}{2}}, \\ G_\alpha[f(\tau)] &= u^\alpha. \end{aligned}$$

Therefore, applying the inverse G_α-transform to both sides implies

$$f(\tau) = \delta(\tau).$$

6. Conclusions

Even though our paper investigates a solution to Abel's integral equation by following a G_α-transform method and further expressing Abel's integral equation on certain distribution space through the distributional G_α-transform, it has believed that the rest of the integral equations may be defined on different distribution spaces by employing the said distributional transform to obtain a solution of Abel's integral equation, as noted, a solution to the Abel integral equation has been received by applying G_α-transform. Moreover, a solution of certain integral equations is given in distributions by applying G_α-transform. Some examples to illustrate the solution of integral equations using the G_α-transform is provided in the distributions space.

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