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$L(2,1)$-Labeling Halin Graphs with Maximum Degree Eight

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Abstract: Suppose that T is a plane tree without vertices of degree 2 and with at least one vertex of at least degree 3, and C is the cycle obtained by connecting the leaves of T in a cyclic order. Set $G = T \cup C$, which is called a Halin graph. A k-$L(2,1)$-labeling of a graph $G = (V, E)$ is a mapping $f : V(G) \to \{0, 1, \ldots, k\}$ such that, for any $x_1, x_2 \in V(G)$, it holds that $|f(x_1) - f(x_2)| \geq 2$ if $x_1x_2 \in E(G)$, and $|f(x_1) - f(x_2)| \geq 1$ if the distance between x_1 and x_2 is 2 in G. The $L(2,1)$-labeling number, denoted $\lambda(G)$, of G is the least k for which G is k-$L(2,1)$-labelable. In this paper, we prove that every Halin graph G with $\Delta = 8$ has $\lambda(G) \leq 10$. This improves a known result, which states that every Halin graph G with $\Delta \geq 9$ satisfies $\lambda(G) \leq \Delta + 2$. This result, together with some known results, shows that every Halin graph G satisfies $\lambda(G) \leq \Delta + 6$.

Keywords: Halin graph; $L(2,1)$-labeling; maximum degree

1. Introduction

Graph coloring and labeling play significant roles in graph theory and combinatorial optimization, for example, in the famous Four-Color Problem stimulating the rapid development of graph theory and network theory, where many symmetric properties are widely investigated and used, such as, symmetric graphs generated from automorphism groups, symmetric embedding, and drawings of graphs in the surface. Stanley [1] introduced a homogeneous symmetric function generalization of the chromatic polynomial of a graph to investigate the graph coloring problems. In 2018, Gross et al. [2] explored the relation between graph symmetry and colorings.

This paper focuses on simple graphs. Given a graph G, the notation $V(G)$, $E(G)$, $|G|$, and $\Delta(G)$ (or simply, Δ) are used to denote the vertex set, the edge set, the vertex number, and the maximum degree of G, respectively. For a vertex v of G, let $N_G(v)$ (or simply, $N(v)$) denote the set of vertices that are adjacent to v in G. We say that v is a *d-vertex*, a *d^+-vertex*, and a *d^--vertex* if the degree of v is d, at least d, and at most d, respectively. The *distance*, denoted $d_G(y_1, y_2)$, between two vertices y_1 and y_2, is defined as the length of a shortest path from y_1 to y_2 in G.

Assume that $k \geq 2$ is an integer. A *k-$L(2,1)$-labeling* of a graph G is a mapping $f : V(G) \to \{0, 1, \ldots, k\}$ such that, for any $x_1, x_2 \in V(G)$, it holds that $|f(x_1) - f(x_2)| \geq 2$ if $x_1x_2 \in E(G)$, and $|f(x_1) - f(x_2)| \geq 1$ if $d_G(x_1, x_2) = 2$. The *$L(2,1)$-labeling number*, denoted $\lambda(G)$, of G is the least k for which G is k-$L(2,1)$-labelable.

The $L(2,1)$-labeling of graphs stems from the famous frequency channel assignment problem, due to Hale [3]. By the definition, it holds trivially that $\lambda(G) \geq \Delta + 1$ for any graph G. Griggs and Yeh [4] put forward the following conjecture.

Conjecture 1. *For a graph G with $\Delta \geq 2$, $\lambda(G) \leq \Delta^2$.*

Conjecture 1 remains open. In 1996, Chang and Kuo [5] first proved that $\lambda(G) \leq \Delta^2 + \Delta$ for any graph G. Later, this result was improved to $\lambda(G) \leq \Delta^2 + \Delta - 1$ in [6], and furthermore to $\lambda(G) \leq \Delta^2 + \Delta - 2$ in [7]. By means of probabilistic analysis, Havet et al. [8]

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showed that there is a constant Δ_0 so that every graph G with $\Delta \geq \Delta_0$ has $\lambda(G) \leq \Delta^2$. It was shown in [9] that $\lambda(G) \leq 2\Delta + 35$ for a planar graph G. Molloy and Salavatipour [10] decreased this bound to $\lambda(G) \leq \lceil 5\Delta/3 \rceil + 95$. Wang and Lih [11] proved that if a planar graph G does not contain a cycle of length three or four, then $\lambda(G) \leq \Delta + 21$. Zhu et al. [12] reinforced this result by demonstrating that every planar graph G having no cycles of length four satisfies $\lambda(G) \leq \Delta + 19$. Wang [13] confirmed that a $\Delta \geq 3$ tree T has $\lambda(T) = \Delta + 1$, provided no two Δ-vertices x and y in T satisfies $d_T(x, y) \in \{1, 2, 4\}$.

Suppose that T is a plane tree with $\Delta \geq 3$ and without 2-vertices. Let $v \in (T)$. We say that v is a *leaf* if $d(v) = 1$, and a *handle* if $d(v) \geq 2$ and v is adjacent to at most one 2^+-vertex. A *d-handle* is a handle that is of degree d. Let C be the cycle obtained by connecting the leaves of T in a cyclic order. Define the graph $G = T \cup C$, which is called a *Halin graph*. The vertices in $V(C)$ and in $V(G) \setminus V(C)$ are called the *outer vertices* and *inner vertices* of G, respectively. As a special case, we call G a *wheel* if $|V(G) \setminus V(C)| = 1$.

Halin graphs are a class of important planar graphs as they possess many interesting structural properties. It is well known that Halin graphs are minimal 3-connected graphs. Namely, every Halin graph is 3-connected, whereas each of its subgraphs is not. In an earlier paper, Bondy and Lovász [14] showed that Halin graphs are almost pancyclic with the possible exception of an even cycle. Stadler [15] proved that Halin graphs other than necklaces have a unique minimum cycle basis. Chandran et al. [16] showed that the boxicity of a Halin graph is 2. For other results on Halin graphs, the reader is referred to [17–22].

Suppose that G is a Halin graph. The third author of this paper proved in [23] that: (a) $\lambda(G) \leq \Delta + 7$; (b) $\lambda(G) \leq \Delta + 2$ for $\Delta \geq 9$; (c) $\lambda(G) \leq 9$ for $\Delta = 3$. Chen and Wang [24] showed that if $\Delta \leq 7$, then $\lambda(G) \leq 10$. The goal of this paper is to extend these results by showing the following consequences:

(1) If G is a Halin graph with $\Delta = 8$, then $\lambda(G) \leq 10$;
(2) For every Halin graph G, it holds that $\lambda(G) \leq \Delta + 6$.

2. Structural Analysis

The proof of the main result in this paper is by induction on the vertex number of graphs. To do this, we need to find some special structures in graphs under consideration that can be reduced in the induction process. Such special structures may consist of 14 configurations, as described in the following lemma.

Lemma 1. *Let $G = T \cup C$ be a Halin graph with $\Delta = 8$ that is not a wheel. Then C contains a path $P_k = x_1 x_2 \cdots x_k$, satisfying one of the following conditions, as shown in Figure 1:*

(C1) *$k = 4$, and there exists a vertex v adjacent to two 3-handles u_1 and u_2 such that $N(u_1) = \{v, x_1, x_2\}$ and $N(u_2) = \{v, x_3, x_4\}$.*

(C2) *$k = 5$, and there exists a vertex v adjacent to a 3-handle u_1 and a 4-handle u_2 such that $N(u_1) = \{v, x_1, x_2\}$ and $N(u_2) = \{v, x_3, x_4, x_5\}$.*

(C3) *$k = 6$, and there exists a vertex v adjacent to two 4-handles u_1 and u_2 such that $N(u_1) = \{v, x_1, x_2, x_3\}$ and $N(u_2) = \{v, x_4, x_5, x_6\}$.*

(C4) *$k = 4$, and there exists a vertex v adjacent to x_4 and a 4-handle u such that $N(u) = \{v, x_1, x_2, x_3\}$.*

(C5) *$k = 3$, and there exists a 5^--vertex v adjacent to x_3 and a 3-handle u such that $N(u) = \{v, x_1, x_2\}$.*

(C6) *$k = 4$, and there exists a 6^--vertex v adjacent to x_1, x_4 and a 3-handle u such that $N(u) = \{v, x_2, x_3\}$.*

(C7) *$k = 5$, and there exists a 7-vertex v adjacent to x_1, x_4, x_5 and a 3-handle u such that $N(u) = \{v, x_2, x_3\}$.*

(C8) *$k = 6$, and there exists a 8-vertex v adjacent to x_1, x_4, x_5, x_6 and a 3-handle u such that $N(u) = \{v, x_2, x_3\}$.*

(C9) *$k = 7$, and there exists a vertex v adjacent to x_1, x_4, x_7 and two 3-handles u_1 and u_2 such that $N(u_1) = \{v, x_2, x_3\}$ and $N(u_2) = \{v, x_5, x_6\}$.*

(C10) $k = 8$, *and there exists a vertex* v *adjacent to* x_1, x_4, x_5, x_8 *and two* 3*-handles* u_1 *and* u_2 *such that* $N(u_1) = \{v, x_2, x_3\}$ *and* $N_G(u_2) = \{v, x_6, x_7\}$.

(C11) $k \geq 4$, *and there exists a* $(k+1)$*-handle* u *such that* $N(u) = \{v, x_1, x_2, \ldots, x_k\}$.

(C12) $k \geq 7$, *and there exists a vertex* v *adjacent to* $x_3, x_4, \ldots, x_k, w$ *and a* 3*-handle* u *such that* $N(u) = \{v, x_1, x_2\}$ *and* $N(v) = \{u, x_3, \ldots, x_k, w\}$.

(C13) $k \geq 8$, *and there exists a vertex* v *adjacent to* $x_3, x_4, \ldots, x_{k-2}, w$ *and two* 3*-handles* u_1 *and* u_2 *such that* $N(u_1) = \{v, x_1, x_2\}$, $N(u_2) = \{v, x_{k-1}, x_k\}$, *and* $N(v) = \{u_1, u_2, x_3, \ldots, x_{k-2}, w\}$.

(C14) $k = 10$, *and there exists a vertex* v *adjacent to* x_3, x_4, x_7, x_8, w *and three* 3*-handles* u_1, u_2, u_3 *such that* $N(u_1) = \{v, x_1, x_2\}$, $N(u_2) = \{v, x_5, x_6\}$, $N(u_3) = \{v, x_9, x_{10}\}$, *and* $N(v) = \{u_1, u_2, u_3, x_3, x_4, x_7, x_8, w\}$.

Proof. Since G is not a wheel, $|V(G) \setminus V(C)| \geq 2$. If $|V(G) \setminus V(C)| = 2$, then (C4), (C5), or (C11) holds clearly. Thus, assume that $|V(G) \setminus V(C)| \geq 3$. Let $P = y_1 y_2 \ldots y_n$ be the longest path in $G - V(C)$. Then $n \geq 3$ and y_1, y_n are handles in T. Let $y_3, z_1, z_2, \ldots, z_m$ denote the neighbors of y_2 in T in clockwise order, where $2 \leq m \leq 7$, and $y_1 = z_l$ for some $1 \leq l \leq m$. Thus each z_i is either a handle or a leaf in T by the choice of P.

If y_2 is adjacent to a 5^+-handle, then (C11) holds. If y_2 is adjacent to two consecutive 4^--handles in $N(y_2)$, then either (C1), (C2), or (C3) holds. So suppose that neither 5^+-handles nor two consecutive 4^--handles are contained in $N(y_2)$. If y_2 is adjacent to a 4-handle, say z_j, then at least one of z_{j-1} and z_{j+1} is a leaf in T, where the indices are taken modulo m. Thus, (C4) occurs in G. Otherwise, all handles in $N(y_2)$ are 3-handles. Let β denote the number of 3-handles in $\{z_1, z_2, \ldots, z_m\}$. Then $1 \leq \beta \leq 4$ since $d(y_2) \leq \Delta \leq 8$. If $d(y_2) \leq 5$, then (C5) holds obviously. Hence, assume that $6 \leq d(y_2) \leq 8$. □

Case 1. $\beta = 1$.

If z_1 or z_m is 3-handle, then (C12) holds. Otherwise, z_i is a 3-handle for some $2 \leq i \leq m-1$. If $d(y_2) = 6$, then (C6) holds. If $d(y_2) \geq 7$, then (C8) holds.

Case 2. $\beta = 2$.

Suppose that z_i and z_j are 3-handles in $\{z_1, z_2, \ldots, z_m\}$ with $1 \leq i < j \leq m$. Assume that $|\{z_1, z_2, \ldots, z_{i-1}\}| \leq |\{z_{j+1}, z_{j+2}, \ldots, z_m\}|$. It suffices to consider the following three possibilities by symmetry.

- $i = 1$ and $j = m$. Then (C13) holds;
- $i = 1$ and $j \leq m-1$. Then z_{j-1} and z_{j+1} are leaves in T. If $d(y_2) \leq 6$, then (C6) holds. If $d(y_2) = 7$, then (C7) holds. If $d(y_2) = 8$, then $m = 7$, we have $|\{z_2, z_3, \ldots, z_{s-1}\}| \geq 3$ or $|\{z_{s+1}, z_{s+2}, \ldots, z_7\}| \geq 3$. Thus, (C8) holds;
- $i \geq 2$ and $j \leq m-1$. If $d(y_2) \leq 6$, then (C6) holds. If $d(y_2) = 7$, then (C7) holds. Otherwise, $d(y_2) = 8$. Note that $2 \leq j - i \leq 4$. If $j - i = 2$, then (C9) holds. If $j - i = 3$, then (C10) holds. If $j - i = 4$, that is $i = 2$ and $j = 6$, then (C8) holds.

Case 3. $\beta = 3$.

Suppose that z_p, z_q, z_r are 3-handles adjacent to y_2 with $1 \leq p < q < r \leq m$. Assume that $|\{z_1, z_2, \ldots, z_{p-1}\}| \leq |\{z_{r+1}, z_{r+2}, \ldots, z_m\}|$, say. If $d(y_2) = 6$, then (C6) holds. Assume that $d(y_2) = 7$. Then z_{q-1} and z_{q+1} are leaves in T. If both z_1 and z_6 are 3-handles, then (C7) holds. Otherwise, we have $p = 1$, $q = 3$ and $r = 5$, and hence (C9) holds. Now assume that $d(y_2) = 8$. If $p \neq 1$ and $r \neq 7$, then it is easy to get that $p = 2$, $q = 4$, and $r = 6$, and hence (C9) holds. Otherwise, we assume that $p = 1$. If $r = 7$, then it follows that $q \in \{3, 4\}$ or $q \in \{4, 5\}$, say the former holds. If $q = 3$, then (C8) holds. If $q = 4$, then (C14) holds. Assume that $r \leq 6$. It is easy to see that $r \in \{5, 6\}$. If $r = 5$, then (C9) holds. If $r = 6$, then (C9) or (C10) holds.

Case 4. $\beta = 4$.

It is immediate to derive that z_1, z_3, z_5, z_7 are 3-handles, and hence (C9) holds.

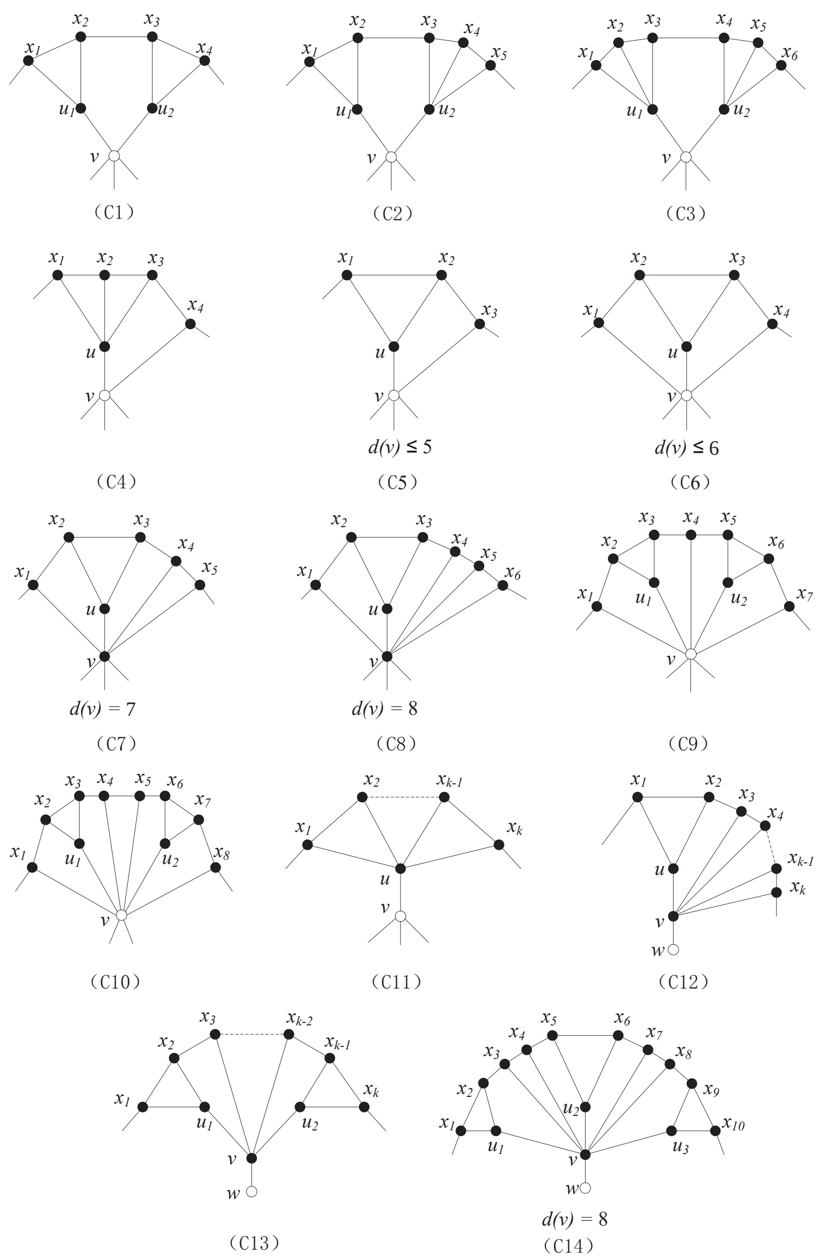

Figure 1. Configurations (C1)–(C14) in Lemma 1.

3. Preliminary Results

An $L^*(2,1)$-*labeling* of a graph G is defined to be a one-to-one $L(2,1)$-labeling. A function L is said to be an *assignment* for the graph G if it assigns a list $L(v)$ of possible labels to each vertex v of G. If G has an $L(2,1)$-labeling (or $L^*(2,1)$-labeling, respectively) f such that $f(v) \in L(v)$ for all vertices v, then we say that f is an L-$L(2,1)$-*labeling* (or L-$L^*(2,1)$-*labeling*, respectively) of G. Given an integer $n \geq 1$, we use $\underline{n}$ to denote three consecutive integers $n-1, n, n+1$.

Lemma 2 below is an easy observation and hence we omit its proof.

Lemma 2. *Let L be a list assignment for an edge xy such that $|L(x)|, |L(y)| \geq 2$. Then xy has an L-$L(2,1)$-labeling unless $L(x) = L(y) = \{p, p+1\}$ for some integer p.*

Lemma 3. *Let $P = x_1x_2x_3$ be a path. Let L be a list assignment for $V(P)$ such that $|L(x_1)| \geq 2$, $|L(x_2)| \geq 4$, and $|L(x_3)| \geq 3$. Then P has an L-$L^*(2,1)$-labeling.*

Proof. Without loss of generality, assume that $|L(x_1)| = 2$, $|L(x_2)| = 4$, and $|L(x_3)| = 3$. Furthermore, let $L(x_2) = \{a, b, c, d\}$ with $a < b < c < d$.

First suppose that there exists $p \in L(x_1) \setminus L(x_2)$. Label x_1 with p and then define a list assignment L' for x_2 and x_3: $L'(x_2) = L(x_2) \setminus \{p-1, p+1\}$ and $L'(x_3) = L(x_3) \setminus \{p\}$. Then $|L'(x_2)| \geq 4 - 2 = 2$ and $|L'(x_3)| \geq 3 - 1 = 2$. By Lemma 2, x_2 and x_3 are not L'-$L(2,1)$-labelable only if $L'(x_2) = L'(x_3) = \{q, q+1\}$ for some integer q, that is, $L(x_2) = \{p-1, p+1, q, q+1\}$ and $L(x_3) = \{p, q, q+1\}$. Since $p-1, p+1, q, q+1$ are mutually distinct, we may assume that $p+1 < q$. Let $r \in L(x_1) \setminus \{p\}$. If $r < q$, then we label x_1, x_2, x_3 with $r, q+1, p$, respectively. Otherwise, $r \geq q$, we label x_1 with r, x_2 with $p-1$, and x_3 with a label in $\{q, q+1\} \setminus \{r\}$.

Next suppose that $L(x_1) \subset L(x_2)$. By symmetry, we only need to deal with the following two cases.

Case 1. $L(x_1) = \{a, t\}$ where $t \in \{b, c, d\}$.

If there exists a label $r \in L(x_3) \setminus \{a\}$ such that $r < c$, then we label x_1 with a, x_2 with d, and x_3 with r. Otherwise, there exist $r_1, r_2 \in L(x_3) \setminus \{a\}$ such that $r_2 > r_1 \geq c$. There are three subcases to be considered, as follows.

• $t = b$. If $r_1 > c$, or $r_2 > d$, then we label x_1 with a, x_2 with c, and x_3 with r_2. Otherwise, $r_1 = c$ and $r_2 \leq d$. If $a \in L(x_3)$, then we label x_1 with b, x_2 with d, and x_3 with a. So assume that $a \notin L(x_3)$. Thus, there is $r_3 \in L(x_3) \setminus \{c, r_2\}$ with $r_3 > c$. If $r_3 > r_2$, then we label x_1 with a, x_2 with c, and x_3 with r_3. If $c < r_3 < r_2$, then we label x_1 with a, x_2 with c, and x_3 with r_2;

• $t = c$. If $r_1 > c$, then we label x_1 with a, x_2 with c, and x_3 with r_2. Otherwise, $r_1 = c$. If $r_2 \geq d$, then we label x_1 with c, x_2 with a, and x_3 with r_2. Otherwise, $c < r_2 < d$, we label x_1 with a, x_2 with d, and x_3 with c;

• $t = d$. If $r_2 > d$, then we label x_1 with a, x_2 with c, and x_3 with r_2. Otherwise, $r_2 \leq d$ and henceforth $c \leq r_1 < d$. If $r_1 = c$, then we label x_1 with d, x_2 with a, and x_3 with c. Otherwise, $r_1 > c$, we label x_1 with a, x_2 with c, and x_3 with r_2.

Case 2. $L(x_1) = \{b, c\}$.

Let $L(x_3) = \{q_1, q_2, q_3\}$ with $q_1 < q_2 < q_3$. If $q_3 \geq d$, then we label x_1 with c, x_2 with a, and x_3 with q_3. If $q_1 \leq a$, then we label x_1 with b, x_2 with d, and x_3 with q_1. Otherwise, $a < q_1 < q_2 < q_3 < d$, we label x_1 with b, x_2 with d, and x_3 with some label in $\{q_1, q_2\} \setminus \{b\}$. $\square$

Lemma 4. *Let $P = x_1x_2x_3x_4$ be a path. Let L be a list assignment for $V(P)$ such that $|L(x_1)| \geq 2$, $|L(x_2)|, |L(x_3)| \geq 5$, and $|L(x_4)| \geq 3$. Then P has an L-$L^*(2,1)$-labeling.*

Proof. Assume that $|L(x_1)| = 2$, $|L(x_2)| = |L(x_3)| = 5$, and $|L(x_4)| = 3$. If there is a label $a \in L(x_1)$ such that $|L(x_2) \cap \{\underline{a}\}| \leq 2$, then we label x_1 with a and then define a list assignment L' for x_2, x_3, x_4 as follows: $L'(x_2) = L(x_2) \setminus \{\underline{a}\}$, $L'(x_i) = L(x_i) \setminus \{a\}$ for $i = 3, 4$. It is easy to calculate that $|L'(x_2)| \geq 3$, $|L'(x_3)| \geq 4$, and $|L'(x_4)| \geq 2$. By Lemma 3, x_2, x_3, x_4 are L'-$L(2,1)$-labelable.

If there is a label $b \in L(x_1) \setminus L(x_4)$, then we label x_1 with b and then define a list assignment L' for x_2, x_3, x_4 as follows: $L'(x_2) = L(x_2) \setminus \{\underline{b}\}$, $L'(x_3) = L(x_3) \setminus \{b\}$, and $L'(x_4) = L(x_4)$. Then $|L'(x_2)| \geq 2$, $|L'(x_3)| \geq 4$, and $|L'(x_4)| = 3$. By Lemma 3, x_2, x_3, x_4 are L'-$L(2,1)$-labelable.

Otherwise, we have $L(x_1) \subset L(x_2) \cap L(x_4)$, and for each $a \in L(x_1)$, we have $|L(x_2) \cap \{\underline{a}\}| = 3$. Let $L(x_1) = \{p, q\}$ with $p < q$. Then $p-1, p, p+1, q-1, q, q+1 \in L(x_2)$. Since $|L(x_2)| = 5$, we obtain that $q - p \leq 2$.

Case 1. $q = p + 1$, say $p = 5$ and $q = 6$.

Since $|L(x_3)| = 5$, there must exist a label $r \in L(x_3)$ such that $r \leq 4$ or $r \geq 9$, so that we can label x_1, x_2, x_3, x_4 with $5, 7, r, 6$, respectively.

Case 2. $q = p + 2$, say $p = 5$ and $q = 7$.

It follows that $L(x_1) = \{5, 7\}$, $L(x_2) = \{4, 5, 6, 7, 8\}$, and $5, 7 \in L(x_4)$. If there is $r \in L(x_3)$ such that $r \leq 3$, then we label x_1, x_2, x_3, x_4 with $5, 8, r, 7$, respectively. If there is

$r \in L(x_3)$ such that $r \geq 9$, then we label x_1, x_2, x_3, x_4 with $7, 4, r, 5$, respectively. Otherwise, $L(x_3) = \{4, 5, 6, 7, 8\}$. Let $b \in L(x_4) \setminus \{5, 7\}$. If $b \leq 4$, then we label x_1, x_2, x_3, x_4 with $5, 8, 6, b$, respectively. If $b \geq 8$, then we label x_1, x_2, x_3, x_4 with $5, 7, 4, b$, respectively. If $b = 6$, then we label x_1, x_2, x_3, x_4 with $5, 7, 4, 6$, respectively. □

Lemma 5. *Let $P = x_1x_2x_3x_4$ be a path. Let L be a list assignment for $V(P)$ such that $|L(x_1)|, |L(x_4)| \geq 2$ and $|L(x_2)|, |L(x_3)| \geq 5$. Then P has an L-$L(2,1)$-labeling.*

Proof. Assume that $|L(x_1)| = |L(x_4)| = 2$ and $|L(x_2)| = |L(x_3)| = 5$. If there is a label $a \in L(x_1)$ such that $|L(x_2) \cap \{\underline{a}\}| \leq 2$, then we label x_1 with a and then define a list assignment L' for x_2, x_3, x_4 as follows: $L'(x_2) = L(x_2) \setminus \{\underline{a}\}$, $L'(x_3) = L(x_3) \setminus \{a\}$, and $L'(x_4) = L(x_4)$. Then $|L'(x_2)| \geq 3$, $|L'(x_3)| \geq 4$, and $|L'(x_4)| = 2$. By Lemma 3, x_2, x_3, x_4 are L'-$L(2,1)$-labelable. So suppose that $L(x_1) \subset L(x_2)$ and for each $a \in L(x_1)$, it holds that $|L(x_2) \cap \{\underline{a}\}| = 3$. Similarly, $L(x_4) \subset L(x_3)$ and for each $b \in L(x_4)$, we have $|L(x_3) \cap \{\underline{b}\}| = 3$. Thus, each of $L(x_2)$ and $L(x_3)$ contains at least four consecutive labels, and each of $L(x_1)$ and $L(x_4)$ consists of two labels whose difference is exactly 1 or 2.

Analogous to the proof of Lemma 4, we give the following discussion by symmetry.

Case 1. $L(x_1) = \{5, 7\}$ and $L(x_2) = \{4, 5, 6, 7, 8\}$.

First assume that $L(x_3) = \{b, b+1, b+2, b+3, b+4\}$ and $L(x_4) = \{b+1, b+3\}$ for some integer b. If $b = 4$, i.e., $L(x_3) = \{4, 5, 6, 7, 8\}$ and $L(x_4) = \{5, 7\}$, then we label x_1, x_2, x_3, x_4 with $5, 8, 4, 7$, respectively. If $b \geq 5$, then we label x_1, x_2, x_3, x_4 with $7, 4, b+3, b+1$, respectively. If $b \leq 3$, then we label x_1, x_2, x_3, x_4 with $5, 8, b, b+3$, respectively.

Next assume that $\{b, b+1, b+2, b+3\} \subset L(x_3)$ and $L(x_4) = \{b+1, b+2\}$. If $b = 4$, i.e., $L(x_4) = \{5, 6\}$, then we label x_1, x_2, x_3, x_4 with $5, 8, 4, 6$, respectively. If $b \geq 5$, then we label x_1, x_2, x_3, x_4 with $7, 4, b+3, b+1$, respectively. If $b \leq 3$, then we label x_1, x_2, x_3, x_4 with $5, 8, b, b+2$, respectively.

Case 2. $L(x_1) = \{5, 6\}$, $\{4, 5, 6, 7\} \subset L(x_2)$, $L(x_4) = \{b+1, b+2\}$, and $\{b, b+1, b+2, b+3\} \subset L(x_3)$.

If $b = 4$, i.e., $L(x_4) = \{5, 6\}$, then we label x_1, x_2, x_3, x_4 with $5, 7, 4, 6$, respectively. If $b \geq 5$, then we label x_1, x_2, x_3, x_4 with $6, 4, b+3, b+1$, respectively. If $b \leq 3$, then we label x_1, x_2, x_3, x_4 with $5, 7, b, b+2$, respectively. □

Lemma 6. *Let $P = x_1x_2x_3x_4$ be a path. Let L be a list assignment for $V(P)$ such that $|L(x_1)|, |L(x_4)| \geq 2$ and $|L(x_2)|, |L(x_3)| \geq 6$. Then P has an L-$L^*(2,1)$-labeling.*

Proof. Let $|L(x_1)| = |L(x_4)| = 2$ and $|L(x_2)| = |L(x_3)| = 6$. If there is a label $a \in L(x_1)$ such that $|L(x_2) \cap \{\underline{a}\}| \leq 2$, then we label x_1 with a, x_4 with $b \in L(x_4) \setminus \{a\}$, and then define a list assignment L' for x_2, x_3 as follows: $L'(x_2) = L(x_2) \setminus \{\underline{a}, b\}$, and $L'(x_3) = L(x_3) \setminus \{\underline{b}, a\}$. Then $|L'(x_2)| \geq 3$ and $|L'(x_3)| \geq 2$. By Lemma 2, x_2, x_3 are L'-$L(2,1)$-labelable.

If $L(x_1) \neq L(x_4)$, then we label x_1 with a label $a \in L(x_1) \setminus L(x_4)$ and then define a list assignment L' for x_2, x_3, x_4 as follows: $L'(x_2) = L(x_2) \setminus \{\underline{a}\}$, $L'(x_3) = L(x_3) \setminus \{a\}$, and $L'(x_4) = L(x_4)$. Noting that $|L'(x_2)| \geq 3$, $|L'(x_3)| \geq 5$, and $|L'(x_4)| = 2$, x_2, x_3, x_4 are L'-$L(2,1)$-labelable by Lemma 3.

Otherwise, we may assume that $L(x_1) = L(x_4) = \{a, b\}$ with $a < b$, and furthermore $L(x_2) = L(x_3) = \{a-1, a, a+1, b-1, b, b+1\}$. Label x_1, x_2, x_3, x_4 with $a, b+1, a+1, b$, respectively. □

Lemma 7. *Let $P = x_1x_2x_3x_4x_5$ be a path. If L is a list assignment for $V(P)$ satisfying the following conditions* (1) *and* (2)*, then P has an L-$L^*(2,1)$-labeling.*

1. *$|L(x_1)| \geq 2$, $|L(x_2)|, |L(x_3)|, |L(x_4)| \geq 5$, and $|L(x_5)| \geq 3$.*
2. *$L(x_2) = L(x_3) = L(x_4) = S$ and $L(x_1), L(x_5) \subset S$.*

Proof. Assume that $|L(x_1)| = 2$, $|L(x_5)| = 3$, $|S| = 5$, and $S = \{a_1, a_2, a_3, a_4, a_5\}$ with $a_1 < a_2 < a_3 < a_4 < a_5$. Then the proof splits into the following two cases.

Case 1. $L(x_5)$ contains a_1 or a_5, say $a_1 \in L(x_5)$.

If $a_3 \in L(x_1)$, then we label x_1, x_2, x_3, x_4, x_5 with a_3, a_5, a_2, a_4, a_1, respectively. If $a_4 \in L(x_1)$, then we label x_1, x_2, x_3, x_4, x_5 with a_4, a_2, a_5, a_3, a_1, respectively. Otherwise, $L(x_1) \subset \{a_1, a_2, a_5\}$.

- $L(x_1) = \{a_1, a_2\}$. If $a_3 \in L(x_5)$ or $a_4 \in L(x_5)$, we have a similar proof. Otherwise, $L(x_5) = \{a_1, a_2, a_5\}$. Label x_1, x_2, x_3, x_4, x_5 with a_2, a_4, a_1, a_3, a_5, respectively;
- $L(x_1) = \{a_1, a_5\}$. With the similar reasoning, we have $L(x_5) = \{a_1, a_2, a_5\}$, and x_1, x_2, x_3, x_4, x_5 can be labeled with a_5, a_3, a_1, a_4, a_2, respectively;
- $L(x_1) = \{a_2, a_5\}$. In view of the above discussion, we may assume that $L(x_5) = \{a_1, a_3, a_4\}$. It suffices to label x_1, x_2, x_3, x_4, x_5 with a_2, a_4, a_1, a_5, a_3, respectively.

Case 2. $L(x_5) = \{a_2, a_3, a_4\}$.

Note that at least one of a_1, a_2, a_3, a_5 is in $L(x_1)$. For each possible situation, we can construct a proper labeling for $V(P)$, similarly to the foregoing argument. □

Lemma 8. *Let $P = x_1x_2 \dots x_7$ be a path. If L is a list assignment for $V(P)$ satisfying the following conditions* (1) *and* (2)*, then P has an L-$L^*(2,1)$-labeling.*

(1) $L(x_3) = L(x_4) = L(x_5) = S$, $L(x_1) \subset L(x_2) \subset S$, *and* $L(x_7) \subset L(x_6) \subset S$;

(2) $|L(x_1)|, |L(x_7)| \geq 2$, $|L(x_2)|, |L(x_6)| \geq 6$, *and* $|S| \geq 7$.

Proof. Assume that $|L(x_1)| = |L(x_7)| = 2$, $|L(x_2)| = |L(x_6)| = 6$, and $|S| = 7$. In addition, let $S = \{a_1, a_2, \dots, a_7\}$ with $a_1 < a_2 < \cdots < a_7$. If there is a label $p \in L(x_1)$ such that $|L(x_2) \cap \{\underline{p}\}| \leq 2$, then we label x_1 with p and x_7 with $q \in L(x_7) \setminus \{p\}$, and then define a list assignment L' for x_2, x_3, x_4, x_5, x_6 as follows:

$L'(x_2) = L(x_2) \setminus \{\underline{p}, q\}$,

$L'(x_6) = L(x_6) \setminus \{\underline{q}, p\}$,

$L'(x_i) = L(x_i) \setminus \{\underline{p}, q\}$ for $i = 3, 4, 5$.

It follows that $|L'(x_2)| \geq 3$, $|L'(x_6)| \geq 2$, and $|L(x_i)| \geq 5$ for $i = 3, 4, 5$. Observing that $L'(x_3) = L'(x_4) = L'(x_5)$ and $L'(x_2), L'(x_6) \subset L'(x_3)$, Lemma 4 asserts that x_2, x_3, x_4, x_5, x_6 are L'-$L^*(2,1)$-labelable. Otherwise, for any $p \in L(x_1)$ we have $|L(x_2) \cap \{\underline{p}\}| = 3$, and for any $q \in L(x_7)$ we have $|L(x_6) \cap \{\underline{q}\}| = 3$. Assume that $L(x_1) = \{\overline{a, b}\}$ with $a < b$, and $L(x_7) = \{c, d\}$ with $c < d$. Then $L(x_2) = \{a-1, a, a+1, b-1, b, b+1\}$ and $L(x_6) = \{c-1, c, c+1, d-1, d, d+1\}$. By symmetry, we have to consider the following three cases.

Case 1. $L(x_1) = \{a_2, a_5\}$ and $L(x_2) = S \setminus \{a_7\}$.

If $L(x_7) = \{a_2, a_5\}$ and $L(x_6) = S \setminus \{a_7\}$, then we label $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ with $a_2, a_6, a_4, a_7, a_1, a_3, a_5$, respectively. If $L(x_7) = \{a_2, a_6\}$ and $L(x_6) = S \setminus \{a_4\}$, then we label $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ with $a_5, a_2, a_4, a_1, a_7, a_3, a_6$, respectively. If $L(x_7) = \{a_3, a_6\}$ and $L(x_6) = S \setminus \{a_1\}$, then we label $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ with $a_2, a_6, a_4, a_7, a_1, a_5, a_3$, respectively.

Case 2. $L(x_1) = \{a_2, a_6\}$ and $L(x_2) = S \setminus \{a_4\}$.

If $L(x_7) = \{a_2, a_6\}$ and $L(x_6) = S \setminus \{a_4\}$, then we label $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ with $a_2, a_5, a_1, a_4, a_7, a_3, a_6$, respectively. If $L(x_7) = \{a_3, a_6\}$ and $L(x_6) = S \setminus \{a_1\}$, then we label $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ with $a_2, a_6, a_4, a_1, a_7, a_5, a_3$, respectively.

Case 3. $L(x_1) = L(x_7) = \{a_3, a_6\}$ and $L(x_2) = L(x_6) = S \setminus \{a_1\}$.

It suffices to label $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ with $a_3, a_5, a_2, a_7, a_1, a_4, a_6$, respectively. □

Lemma 9. *Let $C = xyzx$ be a* 3*-cycle. Let L be a list assignment for $V(C)$ such that $|L(x)| \geq 3$, $|L(y)| \geq 4$, and $|L(z)| \geq 5$. Then C has an L-$L(2,1)$-labeling.*

Proof. Let a denote the minimum integer in the set $L(x) \cup L(y) \cup L(z)$. The proof is split into the following three cases.

- $a \in L(x)$. Labeling x with a, we define a list assignment L' for y, z as follows: $L'(y) = L(y) \setminus \{a, a+1\}$ and $L'(z) = L(z) \setminus \{a, a+1\}$. Then $|L'(y)| \geq 4 - 2 = 2$ and $|L'(z)| \geq 5 - 2 = 3$. By Lemma 2, y, z are L'-$L(2,1)$-labelable;

- $a \in L(y)$ and $a \notin L(x)$. Labeling y with a, we define a list assignment L' for x, z as follows: $L'(x) = L(x) \setminus \{a+1\}$ and $L'(z) = L(z) \setminus \{a, a+1\}$. Then $|L'(x)| \geq 3-1 = 2$ and $|L'(z)| \geq 5-2 = 3$. By Lemma 2, x, z are L'-$L(2,1)$-labelable;
- $a \in L(z)$ and $a \notin L(x) \cup L(y)$. Labeling z with a, we define a list assignment L' for x, y as follows: $L'(x) = L(x) \setminus \{a+1\}$ and $L'(y) = L(y) \setminus \{a+1\}$. Then $|L'(x)| \geq 3-1 = 2$ and $|L'(y)| \geq 4-1 = 3$. By Lemma 2, x, y are L'-$L(2,1)$-labelable.

□

Wheels are special Halin graphs, which contain only one inner vertex. The $L(2,1)$-labeling number of a wheel has been determined in [23]:

Lemma 10. *Let W_n be a wheel with n vertices. Then $\lambda(W_n) = 6$ if $4 \leq n \leq 5$, and $\lambda(W_n) = \Delta(W_n) + 1$ if $n \geq 6$.*

4. $L(2,1)$-Labeling

Now we give the main result of this paper, i.e., Theorem 1, whose proof depends on the structural lemma in Section 2 and auxiliary lemmas in Section 3.

Theorem 1. *Let G be a Halin graph with $\Delta = 8$. Then $\lambda(G) \leq 10$.*

Proof. Let $B = \{0, 1, \ldots, 10\}$ denote a set of 11 labels. The proof is proceeded by induction on the vertex number $|G|$. Since $\Delta = 8$, we see that $|G| \geq 9$. If $|G| = 9$, then G is a wheel of nine vertices and hence the result holds from Lemma 10. So suppose that G is a Halin graph with $\Delta = 8$ and $|G| \geq 10$. Then G is clearly not a wheel. By Lemma 1, there exists a path $x_1 x_2 \cdots x_k$ in C such that one of the conditions (C1)–(C14) holds. □

In the sequel, let $y \in N_C(x_1) \setminus \{x_2\}$, $z \in N_C(x_k) \setminus \{x_{k-1}\}$, $N_G(y) = \{x_1, y_1, y_2\}$, and $N_G(z) = \{x_k, z_1, z_2\}$. We will reduce these 14 configurations one by one.

(C1) Let $H = G - \{x_1, x_2, x_3, x_4\} + \{yu_1, u_1u_2, u_2z\}$. Then H is a Halin graph with $\Delta(H) = 8$ and $|H| < |G|$. By the induction hypothesis, H has an $L(2,1)$-labeling f with the label set B. Define a list assignment L for x_1, x_2, x_3, x_4 as follows:

$L(x_1) = B \setminus \{\underline{f(u_1)}, \underline{f(y)}, f(v), f(y_1), f(y_2)\}$,
$L(x_2) = B \setminus \{\overline{f(u_1)}, \overline{f(y)}, f(v), f(u_2)\}$,
$L(x_3) = B \setminus \{\underline{f(u_2)}, f(z), f(v), f(u_1)\}$,
$L(x_4) = B \setminus \{\overline{f(u_2)}, f(z), f(v), f(z_1), f(z_2)\}$.

Since $|B| \geq 11$, it follows that $|L(x_1)| \geq 11-3-3-3 \geq 2$ and $|L(x_2)| \geq 11-3-3 \geq 5$. Similarly, $|L(x_4)| \geq 2$ and $|L(x_3)| \geq 5$. By Lemma 5, x_1, x_2, x_3, x_4 are L'-$L(2,1)$-labelable.

(C2) Let $H = G - x_4 + x_3x_5$. Then H is a Halin graph with $\Delta(H) = 8$ and $|H| < |G|$. By the induction hypothesis, H has an $L(2,1)$-labeling f with the label set B. Define a list assignment L for x_2, x_3, x_4 as follows:

$L(x_2) = B \setminus \{\underline{f(x_1)}, \underline{f(u_1)}, f(y), f(v), f(u_2)\}$,
$L(x_3) = B \setminus \{\underline{f(u_2)}, \overline{f(x_1)}, f(u_1), f(v), f(x_5)\}$,
$L(x_4) = B \setminus \{\overline{f(u_2)}, f(x_5), f(v), f(z)\}$.

Then $|L(x_2)| \geq 11-3-3-3 \geq 2$, $|L(x_3)| \geq 11-3-4 \geq 4$, and $|L(x_4)| \geq 11-3-3-2 \geq 5$. By Lemma 3, x_2, x_3, x_4 can be labeled properly.

(C3) Let $H = G - x_2 + x_1x_3$. Then H is a Halin graph with $\Delta(H) = 8$ and $|H| < |G|$. By the induction hypothesis, H has an $L(2,1)$-labeling f with the label set B. Define a list assignment L for x_2, x_3, x_4 as follows:

$L(x_2) = B \setminus \{\underline{f(x_1)}, f(u_1), f(y), f(v)\}$,
$L(x_3) = B \setminus \{\underline{f(u_1)}, \overline{f(x_1)}, f(u_2), f(v), f(x_5)\}$,
$L(x_4) = B \setminus \{\overline{f(u_2)}, f(x_5), f(u_1), f(v), f(x_6)\}$.

Then $|L(x_2)| \geq 3$, $|L(x_3)| \geq 4$, and $|L(x_4)| \geq 2$. By Lemma 3, x_2, x_3, x_4 are L-$L(2,1)$-labelable.

(C4) Let $H = G - x_2 + x_1x_3$. Then H is a Halin graph with $\Delta(H) = 8$ and $|H| < |G|$. By the induction hypothesis, H has an $L(2,1)$-labeling f with the label set B. Define a list assignment L for x_1, x_2, x_3 as follows:

$L(x_1) = B \setminus \{\underline{f(y), f(u)}, f(y_1), f(y_2), f(v)\}$,
$L(x_2) = B \setminus \{\underline{f(u), f(y)}, f(v), f(x_4)\}$,
$L(x_3) = B \setminus \{\underline{f(u), f(x_4)}, f(z), f(v)\}$.

Then $|L(x_1)| \geq 2$, $|L(x_2)| \geq 5$, and $|L(x_3)| \geq 3$. By Lemma 3, x_1, x_2, x_3 are L-$L(2,1)$-labelable.

(C5) Let $N(v) = \{u, x_3, t_1, \ldots, t_l\}$. Since $d(v) \leq 5$, we see that $l \leq 3$. Let $H = G - \{x_1, x_2\} + \{x_3u, uy\}$. Then H is a Halin graph with $\Delta(H) = 8$ and $|H| < |G|$. By the induction hypothesis, H has an $L(2,1)$-labeling f using B. Erasing the label of u, we define a list assignment L for x_1, x_2, u as follows:

$L(x_1) = B \setminus \{\underline{f(y)}, f(y_1), f(y_2), f(v), f(x_3)\}$,
$L(x_2) = B \setminus \{\underline{f(x_3)}, f(z), f(y), f(v)\}$,
$L(u) = B \setminus \{\underline{f(v)}, f(y), f(x_3), f(t_1), \ldots, f(t_l)\}$.

Then $|L(x_1)| \geq 4$, $|L(x_2)| \geq 5$, and $|L(u)| \geq 3$. By Lemma 9, u, x_1, x_2 are L-$L(2,1)$-labelable.

(C6) Let $N(v) = \{u, x_1, x_4, t_1, \ldots, t_m\}$. Since $d(v) \leq 6$, we see that $m \leq 3$. Let $H = G - \{x_2, x_3\} + \{x_1u, ux_4\}$. Then H is a Halin graph with $\Delta(H) = 8$ and $|H| < |G|$. By the induction hypothesis, H admits an $L(2,1)$-labeling f using B. Erasing the label of u, we define a list assignment L for x_2, x_3, u as follows:

$L(x_2) = B \setminus \{\underline{f(x_1)}, f(y), f(v), f(x_4)\}$,
$L(x_3) = B \setminus \{\underline{f(x_4)}, f(z), f(v), f(x_1)\}$,
$L(u) = B \setminus \{\underline{f(v)}, f(x_1), f(x_4), f(t_1), \ldots, f(t_m)\}$.

Since $m \leq 3$, we have that $|L(u)| \geq 3$ and $|L(x_2)|, |L(x_3)| \geq 5$. By Lemma 9, u, x_2, x_3 are L-$L(2,1)$-labelable.

(C7) Set $N(v) = \{u, x_1, x_4, x_5, t_1, t_2, t_3\}$ because $d(v) = 7$. Let $H = G - \{x_2, x_3\} + \{x_1u, ux_4\}$. Then H is a Halin graph with $\Delta(H) = 8$ and $|H| < |G|$. By the induction hypothesis, H admits an $L(2,1)$-labeling f using B. Let $f(x_1) = a$, $f(x_4) = b$, and $f(x_5) = c$. Deleting the label of u, we define a list assignment L for x_2, x_3, u as follows:

$L(x_2) = B \setminus \{\underline{a}, b, f(v), f(y)\}$,
$L(x_3) = B \setminus \{\underline{b}, a, c, f(v)\}$,
$L(u) = B \setminus \{\underline{f(v)}, a, b, c, f(t_1), f(t_2), f(t_3)\}$.

Then $|L(u)| \geq 2$ and $|L(x_2)|, |L(x_3)| \geq 5$. It is easy to show by Lemmas 2 and 9 that u, x_2, x_3 cannot be labeled only if $L(x_2) = L(x_3) = S = \{i, i+1, i+2, i+3, i+4\}$ and $L(u) = \{i+1, i+3\}$ for some $i \in B$. This implies that $a-1, a, a+1, b-1, b, b+1, c \notin S$, and thus $i+1, i+3 \notin \{c-1, c, c+1\}$. Let s denote the neighbor of x_5 other than v and x_4. Relabel x_4 with a label $p \in \{i+1, i+3\} \setminus \{f(s)\}$. If $p = i+1$, then we label u with b, x_3 with $i+3$, and x_2 with i. If $p = i+3$, then we label u with b, x_3 with $i+1$, and x_2 with $i+4$.

(C8) Set $N(v) = \{u, x_1, x_4, x_5, x_6, t_1, t_2, t_3\}$, and let $H = G - \{x_2, x_3\} + \{x_1u, ux_4\}$. Then H is a Halin graph with $\Delta(H) = 8$ and $|H| < |G|$. By the induction hypothesis, H admits an $L(2,1)$-labeling f using B such that $x_1, x_4, u, x_5, x_6, y, v$ are labeled by a, b, c, d, e, g, h, respectively. Define a list assignment L for x_2 and x_3 as follows:

$L(x_2) = B \setminus \{\underline{a}, \underline{c}, h, g, b\}$,
$L(x_3) = B \setminus \{\underline{b}, \underline{c}, h, d, a\}$.

Then $|L(x_2)| \geq 2$ and $|L(x_3)| \geq 2$. By Lemma 2, x_2 and x_3 cannot be labeled only if $L(x_2) = L(x_3) = \{i, i+1\}$ for some $i \in B$. It follows that

$$\{a-1, a+1, g\} = \{b-1, b+1, d\} \tag{1}$$

Note that a, b, c, d, e are distinct and $d \notin \{c-1, c, c+1\}$. Switch the labels of u and x_4 and then define a new list assignment L' as follows:

$L'(x_2) = B \setminus \{\underline{a}, \underline{b}, h, g, c\}$,
$L'(x_3) = L(x_3)$.

We assert that x_2, x_3 are L'-$L(2,1)$-labelable. If not, we have that $L'(x_2) = L'(x_3) = L(x_3) = \{i, i+1\}$ for some $i \in B$. Thus,

$$\{a-1, a+1, g\} = \{c-1, c+1, d\} \tag{2}$$

Combining (1) and (2), we get that $\{b-1, b+1, d\} = \{c-1, c+1, d\}$, i.e., $b = c$, which contradicts the fact that $b \neq c$.

(C9) Let $H = G - \{x_2, x_3, x_5, x_6\} + \{x_1u_1, u_1x_4, x_4u_2, u_2x_7\}$. Then H is a Halin graph with $\Delta(H) = 8$ and $|H| < |G|$. By the induction hypothesis, H admits an $L(2,1)$-labeling f using B such that $u_1, x_4, u_2, x_1, x_7, v, y, z$ are labeled by a, b, c, d, e, g, h, i, respectively. Define a list assignment L for x_2, x_3, x_5, x_6 as follows:
$L(x_2) = B \setminus \{\underline{a}, \underline{d}, b, g, h\}$,
$L(x_3) = B \setminus \{\underline{a}, \underline{b}, d, g\}$,
$L(x_5) = B \setminus \{\underline{c}, \underline{b}, e, g\}$,
$L(x_6) = B \setminus \{\underline{c}, \underline{e}, b, g, i\}$.

Then $|L(x_2)|, |L(x_6)| \geq 2$ and $|L(x_3)|, |L(x_5)| \geq 3$. By Lemma 2, we can show that x_2, x_3, x_5, x_6 cannot be labeled only if $L(x_2) = \{i, i+1\}$, $L(x_3) = \{i, i+1, p\}$, $L(x_5) = \{j, j+1, p\}$, and $L(x_6) = \{j, j+1\}$ for some $i, j \in B$. Note that i may be equal to j. It implies that any two labels in $\{c-1, c, c+1, b-1, b, b+1, e, g\}$ are distinct, and any two labels in $\{c-1, c, c+1, e-1, e, e+1, b, g, i\}$ are distinct. A similar conclusion holds for the sets $\{a-1, a, a+1, b-1, b, b+1, d, g\}$ and $\{a-1, a, a+1, d-1, d, d+1, g, h\}$. Now we switch the labels of x_4 and u_2 and then define a new list assignment L' for x_2, x_3, x_5, x_6 as follows:
$L'(x_2) = B \setminus \{\underline{a}, \underline{d}, c, g, h\}$,
$L'(x_3) = B \setminus \{\underline{a}, \underline{c}, d, g\}$,
$L'(x_5) = L(x_5)$,
$L'(x_6) = B \setminus \{\underline{b}, \underline{e}, c, g, i\}$.

It is easy to confirm that $L'(x_2) = (L(x_2) \cup \{b\}) \setminus \{c\} = \{i, i+1, b\} \setminus \{c\}$. If $|L'(x_2)| \geq 3$, the proof can be reduced to the previous case. Otherwise, since $b \neq c$, we get that $c = i$ or $c = i+1$, that is, $L'(x_2) = \{b, i+1\}$ or $L'(x_2) = \{b, i\}$. Since $i, i+1 \in L(x_3)$, we see that $|b-i| \geq 2$ and $|b-(i+1)| \geq 2$. This shows that two labels in $L'(x_2)$ are not consecutive. Thus, x_2, x_3, x_5, x_6 admit an L'-$L(2,1)$-labeling.

(C10) Without loss of generality, assume that $d(v) = 8$ and $N(v) = \{u_1, u_2, x_1, x_4, x_5, x_8, t_1, t_2\}$. Let $H = G - \{u_1, u_2, x_2, x_3, \ldots, x_7\} + x_1x_8$. Then H is a Halin graph with $\Delta(H) \leq 8$ and $|H| < |G|$. If $\Delta(H) \leq 7$, then H is 10-$L(2,1)$-labelable by the result in [24]. If $\Delta(H) = 8$, then H is also 10-$L(2,1)$-labelable by the induction hypothesis. Thus, H always admits an $L(2,1)$-labeling f using B such that $v, x_1, x_8, y, z, t_1, t_2$ are labeled by $i_1, i_2, i_3, i_4, i_5, i_6, i_7$, respectively. Define a list assignment L as follows:
$L(u_1) = L(u_2) = L(x_4) = L(x_5) = S = B \setminus \{\underline{i_1}, i_2, i_3, i_6, i_7\}$,
$L(x_2) = B \setminus \{\underline{i_2}, i_1, i_4\}$,
$L(x_3) = B \setminus \{i_1, i_2\}$,
$L(x_6) = B \setminus \{i_1, i_3\}$,
$L(x_7) = B \setminus \{\underline{i_3}, i_1, i_5\}$.

It is easy to deduce that $|L(x_2)|, |L(x_7)| \geq 6$, $|L(x_3)|, |L(x_6)| \geq 9$, and $|S| \geq 4$. Assume that $|S| = 4$ and $S = \{a, b, c, d\}$ with $0 \leq a < b < c < d \leq 10$. To complete the proof, we consider the following three subcases.

(C10.1) $a = 0$. (If $d = 10$, we have a similar argument.)

Label x_4 with a, u_1 with b, x_5 with c, and u_2 with d. Define a list assignment L' for x_2, x_3, x_6, x_7 as follows:
$L'(x_2) = L(x_2) \setminus \{\underline{b}, 0\}$,
$L'(x_3) = L(x_3) \setminus \{\underline{b}, 0, 1, c\}$,
$L'(x_6) = L(x_6) \setminus \{\underline{c}, \underline{d}, 0\}$,
$L'(x_7) = L(x_7) \setminus \{\underline{d}, c\}$.

It is easy to show that $|L'(x_3)| \geq 3$ and $|L'(x_2)|, |L'(x_6)|, |L'(x_7)| \geq 2$. By Lemma 2, x_2 and x_3 are L'-$L(2,1)$-labelable. If x_6, x_7 are also L'-$L(2,1)$-labelable, we are done. Otherwise, by Lemma 2, we have $L'(x_6) = L'(x_7) = \{j, j+1\}$ for some $j \in B$. This implies that $d \geq c+3$, as otherwise we derive that $|L'(x_6)| \geq 3$, which is impossible. Now we switch the labels of x_5 and u_2 to induce a new list assignment L'' from L'. On the one hand, it still holds that $|L''(x_2)| \geq 2$ and $|L''(x_3)| \geq 3$, and hence x_2, x_3 are L''-$L(2,1)$-labelable. On the other hand, $L''(x_6) = L'(x_6)$, and $L''(x_7) = (L'(x_7) \cup \{d-1, d+1\}) \setminus \{c-1, c+1\} = \{j, j+1, d-1, d+1\} \setminus \{c-1, c+1\}$. Since $j, j+1 \notin \{c-1, c+1, d-1, d+1\}$ and $d \geq c+3$, it follows that either $|L''(x_7)| \geq 3$ or $L''(x_7) = \{j, k\}$ with $|j-k| \geq 2$. By Lemma 2, x_6 and x_7 are L''-$L(2,1)$-labelable.

(C10.2) $b \leq a+2$. (If $d \leq c+2$, we have a similar discussion).

After x_4, u_1, x_5, u_2 with a, b, c, d, respectively, we define a list assignment L' for x_2, x_3, x_6, x_7 as follows:

$L'(x_2) = L(x_2) \setminus \{\underline{b}, a\}$,
$L'(x_3) = L(x_3) \setminus \{\underline{b}, \underline{a}, c\}$,
$L'(x_6) = L(x_6) \setminus \{\underline{c}, \underline{d}, a\}$,
$L'(x_7) = L(x_7) \setminus \{\underline{d}, c\}$.

Since $b \leq a+2$, it follows that $|\{a-1, a, a+1, b-1, b, b+1\}| \leq 5$. Thus, $|L'(x_3)| \geq 3$ and $|L'(x_2)|, |L'(x_6)|, |L'(x_7)| \geq 2$. The remaining discussion is analogous to (C10.1).

(C10.3) $b \geq a+3$ and $c \leq b+2$.

Label x_5, x_4, u_1, u_2 with a, b, c, d, respectively, and define a list assignment L' as follows:

$L'(x_2) = L(x_2) \setminus \{\underline{c}, b\}$,
$L'(x_3) = L(x_3) \setminus \{\underline{c}, \underline{b}, a\}$,
$L'(x_6) = L(x_6) \setminus \{\underline{a}, \underline{d}, b\}$,
$L'(x_7) = L(x_7) \setminus \{\underline{d}, a\}$.

Since $c \leq b+2$, we derive that $|L'(x_3)| \geq 3$ and $|L'(x_2)|, |L'(x_6)|, |L'(x_7)| \geq 2$. The remaining discussion is analogous to (C10.1).

(C11) Note that $4 \leq k \leq 7$. Let $H = G - \{x_1, x_2, \ldots, x_k\} + \{yu, uz\}$. By the induction hypothesis or the result in [24], H has an $L(2,1)$-labeling f using B such that u, y, z are labeled with p, q, r, respectively. Define a list assignment L for $x_1, x_2, \ldots, x_k$ as follows:

$L(x_1) = B \setminus \{\underline{p}, \underline{q}, f(v), f(y_1), f(y_2)\}$,
$L(x_k) = B \setminus \{\underline{p}, \underline{r}, f(v), f(z_1), f(z_2)\}$,
$L(x_2) = B \setminus \{\underline{p}, f(v), f(y)\}$,
$L(x_{k-1}) = B \setminus \{\underline{p}, f(v), f(z)\}$,
$L(x_i) = B \setminus \{p, \underline{f}(v)\}$ for $i = 3, 4, \ldots, k-2$.

Then $|L(x_1)|, |L(x_k)| \geq 2$, $|L(x_2)|, |L(x_{k-1})| \geq 6$, and $|L(x_i)| \geq 7$ for $i = 3, 4, \ldots, k-2$. According to the size of k, we have to deal with the following subcases.

(C11.1) $k = 4$.

Since $|L(x_1)|, |L(x_4)| \geq 2$ and $|L(x_2)|, |L(x_3)| \geq 6$, x_1, x_2, x_3, x_4 are L-$L(2,1)$-labelable by Lemma 6.

(C11.2) $k = 5$.

We note that $L(x_1) \subset L(x_2) \subset L(x_3)$, $L(x_5) \subset L(x_4) \subset L(x_3)$, $|L(x_1)|, |L(x_5)| \geq 2$, $|L(x_2)|, |L(x_4)| \geq 6$, and $|L(x_3)| \geq 7$. Assume, w.l.o.g., that $|L(x_1)| = |L(x_5)| = 2$, $|L(x_2)| = |L(x_4)| = 6$, and $|L(x_3)| = 7$. If $L(x_1) \neq L(x_5)$, then we label x_5 with a label $a \in L(x_5) \setminus L(x_1)$ and then define a list assignment L' for x_1, x_2, x_3, x_4 as follows:

$L'(x_1) = L(x_1)$,
$L'(x_4) = L(x_4) \setminus \{\underline{a}\}$,
$L'(x_i) = L(x_i) \setminus \{a\}$ for $i = 2, 3$.

Then $|L'(x_1)| \geq 2$, $|L'(x_2)| \geq 5$, $|L'(x_3)| \geq 6$, and $|L'(x_4)| \geq 3$. By Lemma 4, x_1, x_2, x_3, x_4 are L-$L(2,1)$-labelable. Otherwise, $L(x_1) = L(x_5) = \{p, q\}$ with $p < q$. If $|L(x_2) \cap \{\underline{p}\}| \leq 2$, then we label x_1 with p and x_5 with q and then define a list assignment L' for x_2, x_3, x_4 as follows:

$L'(x_2) = L(x_2) \setminus \{\underline{p}, q\}$,
$L'(x_3) = L(x_3) \setminus \{p, q\}$,
$L'(x_4) = L(x_4) \setminus \{\underline{q}, p\}$.

Then $|L'(x_2)| \geq 3$, $|L'(x_3)| \geq 5$, and $|L'(x_4)| \geq 2$. By Lemma 3, x_2, x_3, x_4 are L'-$L(2,1)$-labelable. Otherwise, $|L(x_2) \cap \{\underline{p}\}| = 3$. Similarly, $|L(x_2) \cap \{\underline{q}\}| = 3$, $|L(x_4) \cap \{\underline{p}\}| = 3$, and $|L(x_4) \cap \{\underline{q}\}| = 3$. This implies that $L(x_2) = L(x_4) = \{p-1, p, p+1, q-1, q, q+1\}$, and $L(x_3) = \{p-1, p, p+1, q-1, q, q+1, r\}$. Since $p < q$, we have $p+1 < q-1$. If $r > q+1$, we label x_1 with q, x_2 with $p-1$, x_3 with r, x_4 with $q-1$, and x_5 with p. If $r < p-1$, we have a similar labeling. If $p+1 < r < q-1$, we label x_1 with q, x_2 with $p-1$, x_3 with r, x_4 with $q+1$, and x_5 with p.

(C11.3) $k = 6$.

Assume that $|L(x_1)| = |L(x_6)| = 2$, $|L(x_2)| = |L(x_5)| = 6$, and $|L(x_3)| = |L(x_4)| = 7$. Note that $L(x_3) = L(x_4)$, $L(x_1) \subset L(x_2) \subset L(x_3)$ and $L(x_6) \subset L(x_5) \subset L(x_4)$. If there is $a \in L(x_1)$ such that $|L(x_2) \cap \{\underline{a}\}| \leq 2$, then we label x_1 with a, x_6 with $b \in L(x_6) \setminus \{a\}$, and then define a list assignment L' for x_2, x_3, x_4, x_5 as follows:

$L'(x_2) = L(x_2) \setminus \{\underline{a}, b\}$,

$L'(x_i) = L(x_4) \setminus \{a, b\}$ for $i = 3, 4$,

$L'(x_5) = L(x_5) \setminus \{\underline{b}, a\}$,

Then $|L'(x_2)| \geq 3$, $|L'(x_3)|, |L'(x_4)| \geq 5$, and $|L'(x_5)| \geq 2$. By Lemma 4, x_2, x_3, x_4, x_5 are L'-$L(2,1)$-labelable. Otherwise, for each $a \in L(x_1)$ we have $|L(x_2) \cap \{\underline{a}\}| = 3$ and for each $b \in L(x_6)$ we have $|L(x_5) \cap \{\underline{b}\}| = 3$. Let $L(x_3) = L(x_4) = S = \{b_1, b_2, \ldots, b_7\}$ with $b_1 < b_2 < \cdots < b_7$. Since $|L(x_2)| = |L(x_5)| = 6$ and $L(x_2), L(x_5) \subset S$, we only need to consider the following cases by symmetry.

- $L(x_1) = \{b_2, b_5\}$. Then $L(x_2) = S \setminus \{b_7\}$. If $L(x_6) = \{b_2, b_5\}$, then $L(x_5) = S \setminus \{b_7\}$, we label $x_1, x_2, x_3, x_4, x_5, x_6$ with $b_2, b_6, b_4, b_1, b_3, b_5$, respectively. If $L(x_6) = \{b_2, b_6\}$, or $L(x_6) = \{b_3, b_6\}$, then $L(x_5) = S \setminus \{b_4\}$, or $L(x_5) = S \setminus \{b_1\}$, we label $x_1, x_2, x_3, x_4, x_5, x_6$ with $b_5, b_1, b_4, b_7, b_3, b_6$, respectively;
- $L(x_1) = \{b_2, b_6\}$. Then $L(x_2) = S \setminus \{b_4\}$. If $L(x_6) = \{b_2, b_6\}$, then $L(x_5) = S \setminus \{b_4\}$, we label $x_1, x_2, x_3, x_4, x_5, x_6$ with $b_2, b_7, b_3, b_5, b_1, b_6$, respectively. If $L(x_6) = \{b_3, b_6\}$, then $L(x_5) = S \setminus \{b_1\}$, we label $x_1, x_2, x_3, x_4, x_5, x_6$ with $b_2, b_7, b_4, b_1, b_5, b_3$, respectively;
- $L(x_1) = L(x_6) = \{b_3, b_6\}$. Then $L(x_2) = L(x_5) = S \setminus \{b_1\}$. We label $x_1, x_2, x_3, x_4, x_5, x_6$ with $b_3, b_7, b_5, b_1, b_4, b_6$, respectively.

(C11.4) $k = 7$.

Since $|L(x_1)|, |L(x_7)| \geq 2$, $|L(x_2)|, |L(x_6)| \geq 6$, $|L(x_3)|, |L(x_4)|, |L(x_5)| \geq 7$, $L(x_3) = L(x_4) = L(x_5)$, $L(x_1) \subset L(x_2) \subset L(x_3)$, and $L(x_7) \subset L(x_6) \subset L(x_5)$, Lemma 8 guarantees that $x_1, x_2, \ldots, x_7$ are L-$L(2,1)$-labelable.

(C12) Note that $6 \leq k \leq 8$. Let $H = G - \{u, x_1, x_2, \ldots, x_k\} + \{yv, vz\}$. By the induction hypothesis or the result in [24], H has an $L(2,1)$-labeling f using B such that $v, w, y, z, y_1, y_2, z_1, z_2$ are labeled with $a, b, c, d, c_1, c_2, d_1, d_2$, respectively. Label x_2 with some label in $\{a-1, a+1\} \setminus \{c\}$, say $a+1$. Then we label x_k with $e \in B \setminus \{\underline{a}, \underline{d}, b, d_1, d_2\}$, x_1 with $g \in B \setminus \{\underline{c}, a, a+1, c_1, c_2\}$, and u with $h \in B \setminus \{\underline{g}, \underline{a}, a+2, b, c, e\}$. Now we define a list assignment L for $x_3, x_4, \ldots, x_{k-1}$ as follows:

$L(x_3) = B \setminus \{\underline{a}, a+2, b, e, g, h\}$,

$L(x_i) = B \setminus \{\underline{a}, b, e, h\}$ for $i = 4, 5, \ldots, k-2$,

$L(x_{k-1}) = B \setminus \{\underline{a}, \underline{e}, b, d, h\}$.

It follows that $|L(x_3)| \geq 3$, $|L(x_{k-1})| \geq 2$, and $|L(x_i)| \geq 5$ for $i = 4, 5, \ldots, k-2$. If $k = 6$, then x_3, x_4, x_5 are L-$L(2,1)$-labelable by Lemma 3. If $k = 7$, then x_3, x_4, x_5, x_6 are L-$L(2,1)$-labelable by Lemma 4. If $k = 8$, then x_3, x_4, x_5, x_6, x_7 are L-$L(2,1)$-labelable by Lemma 7.

(C13) Note that $7 \leq k \leq 9$. Let $H = G - \{u_1, u_2, x_1, x_2, \ldots, x_k\} + \{yv, vz\}$. By the induction hypothesis or the result in [24], H has an $L(2,1)$-labeling f using B such that $v, w, y, z, y_1, y_2, z_1, z_2$ are labeled with $a, b, c, d, c_1, c_2, d_1, d_2$, respectively. Label x_2 with some label in $\{a-1, a+1\} \setminus \{c\}$, say $a+1$, and x_{k-1} with some label in $\{a-1, a+1\} \setminus \{c\}$, say $a-1$, Then we label x_1 with $g \in B \setminus \{\underline{c}, \underline{a+1}, c_1, c_2\}$, x_k with $h \in B \setminus \{\underline{d}, \underline{a-1}, d_1, d_2\}$, u_1 with $e' \in B \setminus \{\underline{e}, \underline{a+1}, a-1, b, c\}$, and u_2 with $h' \in B \setminus \{\underline{h}, \underline{a-1}, a+1, b, d, e'\}$.

Afterwards we define a list assignment L for $x_3, x_4, \ldots, x_{k-2}$ as follows:

$L(x_3) = B \setminus \{\underline{a}, a+2, b, e, e', h'\}$,

$L(x_{k-2}) = B \setminus \{\underline{a}, a-2, b, e', h, h'\}$,
$L(x_i) = B \setminus \{\underline{a}, b, e', h'\}$ for $i = 4, 5, \ldots, k-2$.

Then $|L(x_3)|, |L(x_{k-2})| \geq 3$, and $|L(x_i)| \geq 5$ for $i = 4, 5, \ldots, k-3$. If $k = 7$, then x_3, x_4, x_5 can be L-$L(2,1)$-labeled by Lemma 3. If $k = 8$, then x_3, x_4, x_5, x_6 can be L-$L(2,1)$-labeled by Lemma 4. If $k = 9$, then x_3, x_4, x_5, x_6, x_7 can be L-$L(2,1)$-labeled by Lemma 7.

(C14) Let $H = G - \{u_1, u_2, u_3, x_1, x_2, \ldots, x_k\} + \{yv, vz\}$. By the induction hypothesis or the result in [24], H has an $L(2,1)$-labeling f using B such that $v, w, y, z, y_1, y_2, z_1, z_2$ are labeled with $a, b, c, d, c_1, c_2, d_1, d_2$, respectively. Similarly to the proof of the previous cases, we label, w.l.o.g., x_2, x_5 with $a+1$, x_6, x_9 with $a-1$, x_1 with g, x_{10} with h, u_1 with g' and u_3 with h'. Define a list assignment L for x_3, x_4, x_7, x_8 as follows:
$L(x_3) = B \setminus \{\underline{a}, a+2, b, g, g', h'\}$,
$L(x_4) = B \setminus \{\underline{a}, a+2, b, g', h'\}$,
$L(x_7) = B \setminus \{\underline{a}, a-2, b, g', h'\}$,
$L(x_8) = B \setminus \{\underline{a}, a-2, b, g', h, h'\}$.

It is not difficult to see that $|L(x_3)|, |L(x_8)| \geq 3$, $|L(x_4)|, |L(x_7)| \geq 4$, $L(x_3) \subset L(x_4)$, $L(x_8) \subset L(x_7)$, and $3 \leq |L(x_4) \cap L(x_7)| \leq 4$. Assume that $|L(x_3)| = |L(x_8)| = 3$, and $|L(x_4)| = |L(x_7)| = 4$.

Claim 1 x_3, x_4, x_7, x_8 are L-$L^*(2,1)$-labelable.

Proof. First, assume that $L(x_3) = L(x_8) = \{\alpha, \beta, \gamma\}$ with $\alpha < \beta < \gamma$. Then $\alpha, \beta, \gamma \in L(x_4) \cap L(x_7)$. Furthermore, assume that $L(x_4) = \{\alpha, \beta, \gamma, \phi\}$. If $\phi < \alpha$, then we label x_3, x_4, x_7, x_8 with $\beta, \phi, \alpha, \gamma$, respectively. If $\alpha < \phi < \beta$, then we label x_3, x_4, x_7, x_8 with $\gamma, \phi, \alpha, \beta$, respectively. If $\beta < \phi < \gamma$, then we label x_3, x_4, x_7, x_8 with $\alpha, \phi, \beta, \gamma$, respectively. If $\gamma < \phi$, then we label x_3, x_4, x_7, x_8 with $\beta, \phi, \alpha, \gamma$, respectively.

Next assume that $L(x_3) \neq L(x_8)$. Label x_3 with $\alpha \in L(x_3) \setminus L(x_8)$ and x_4 with $\beta \in L(x_4) \setminus \{\underline{\alpha}\}$. Define a list assignment L' for x_7, x_8 as follows:
$L'(x_7) = L(x_7) \setminus \{\alpha, \beta\}$,
$L'(x_8) = L(x_8) \setminus \{\beta\}$.

Then $|L'(x_7)| \geq 2$ and $|L'(x_8)| \geq 2$. By Lemma 2, x_7, x_8 are not L'-$L(2,1)$-labelable only if $L'(x_7) = L'(x_8) = \{p, p+1\}$ for some $p \in B$. It turns out that $L(x_7) = \{\alpha, \beta, p, p+1\}$, $L(x_8) = \{\beta, p, p+1\}$, $\alpha, \beta \in L(x_4)$, and $\alpha \in L(x_3)$. Since $|L(x_4) \cap L(x_7)| \geq 3$ by their definitions, at least one of p and $p+1$ belongs to $L(x_4)$.

Case I. $\beta < p$.

- $\alpha < \beta$. If $p \in L(x_4)$, we label x_3, x_4, x_7, x_8 with $\alpha, p, p+1, \beta$, respectively. Otherwise, $p \notin L(x_4)$. Then $p \notin L(x_3)$ since $L(x_3) \subset L(x_4)$, and $p+1 \in L(x_4)$. This implies that at least one of β and $p+1$ belongs to $L(x_3)$ since $L(x_3) \subset L(x_4)$ and $|L(x_4) \cap L(x_7)| \geq 3$. Label x_7 with α, x_8 with p, x_3 with some label $\gamma \in L(x_3) \cap \{\beta, p+1\}$, and x_4 with some label in $\{\beta, p+1\} \setminus \{\gamma\}$;
- $\beta < \alpha < p$. If $p+1 \in L(x_4)$, we label x_3, x_4, x_7, x_8 with $\alpha, p+1, p, \beta$, respectively. Otherwise, $p+1 \notin L(x_4)$ and hence $p+1 \notin L(x_3)$, and $p \in L(x_4)$. If $\beta \in L(x_3)$, we label x_3, x_4, x_7, x_8 with $\beta, p, \alpha, p+1$, respectively. Otherwise, $L(x_3) = \{p, \alpha, \gamma\}$ and $L(x_4) = \{p, \alpha, \beta, \gamma\}$ for some $\gamma \in B$. If $\gamma < \alpha$, then we label x_3, x_4, x_7, x_8 with $p, \gamma, \beta, p+1$, respectively. If $\gamma > p$, then we label x_3, x_4, x_7, x_8 with $\alpha, p, \beta, p+1$, respectively;
- $\alpha > p+1$. If $p \in L(x_4)$, we label x_3, x_4, x_7, x_8 with $\alpha, p, p+1, \beta$, respectively. Otherwise, $p \notin L(x_4)$ and hence $p \notin L(x_3)$, and $p+1 \in L(x_4)$. If $p+1 \in L(x_3)$, we label x_3, x_4, x_7, x_8 with $p+1, \beta, \alpha, p$, respectively. Otherwise, $\beta \in L(x_3)$, we label x_3, x_4, x_7, x_8 with $\beta, p+1, \alpha, p$, respectively.

Case II. $\beta > p+1$.

- $\alpha < p$. If $p+1 \in L(x_4)$, then we label x_3, x_4, x_7, x_8 with $\alpha, p+1, \beta, p$, respectively. Otherwise, $p+1 \notin L(x_4)$, and $p \in L(x_4)$. If $\beta \in L(x_3)$, then we label x_3, x_4, x_7, x_8 with $\beta, p, \alpha, p+1$, respectively. Otherwise, $p \in L(x_3)$, then we label x_3, x_4, x_7, x_8 with $p, \beta, \alpha, p+1$, respectively;

- $p+1 < \alpha < \beta$. If $p \in L(x_4)$, then we label x_3, x_4, x_7, x_8 with $\alpha, p, \beta, p+1$, respectively. Otherwise, $p \notin L(x_4)$, and $p+1 \in L(x_4)$. If $\beta \in L(x_3)$, then we label x_3, x_4, x_7, x_8 with $\beta, p+1, \alpha, p$, respectively. Otherwise, $p+1 \in L(x_3)$, we label x_3, x_4, x_7, x_8 with $p+1, \beta, \alpha, p$, respectively;
- $\alpha > \beta$. If $p+1 \in L(x_4)$, then we label x_3, x_4, x_7, x_8 with $\alpha, p+1, \beta, p$, respectively. Otherwise, $p+1 \notin L(x_4)$, and $p \in L(x_4)$. If $\beta \in L(x_3)$, then we label x_3, x_4, x_7, x_8 with $\beta, p, \alpha, p+1$, respectively. Otherwise, $p \in L(x_3)$, we label x_3, x_4, x_7, x_8 with $p, \beta, \alpha, p+1$, respectively. The completes the proof of Claim 1.

□

By Claim 1, we label x_3, x_4, x_7, x_8 with p_1, p_2, p_3, p_4, respectively. Finally, we label u_2 with some label in $B \setminus \{\underline{a}, b, g', h', p_1, p_2, p_3, p_4\}$.

Combining Theorem 1 and the results in [23], we obtain easily the following:

Corollary 1. *For every Halin graph G, it holds that $\lambda(G) \le \Delta + 6$.*

5. Concluding Remarks

Halin graphs are important and interesting planar graphs. The research of structures and parameters for Halin graphs has attracted considerable attention in the recent decades. The $L(2,1)$-labeling of graphs can be thought of as the generalization of the proper vertex coloring of graphs, which are of wide applications in frequency channel assignment, traffic phasing, task assignment, and other practical problems. This paper has contributed with the $L(2,1)$-labeling of Halin graphs.

We first proved that the $L(2,1)$-labeling number of each Halin graph of maximum degree 8 is at most 10. To explain that the upper bound 10 is sharp, we observed the graph H^*, depicted in Figure 2. Note that H^* is a Halin graph consisting of three 8-vertices and twenty 3-vertices, which implies that $\Delta(H^*) = 8$. It was shown in [4] that if a graph G contains a Δ-vertex that is adjacent to at least two Δ-vertices, then $\lambda(G) \ge \Delta + 2$. This fact immediately implies that $\lambda(H^*) \ge \Delta(H^*) + 2 = 8 + 2 = 10$. On the other hand, a 10-$L(2,1)$-labeling of H^* using the labels $0, 1, \ldots, 10$ is constructed in Figure 2, which gives that $\lambda(H^*) \le 10$. Consequently, $\lambda(H^*) = 10$.

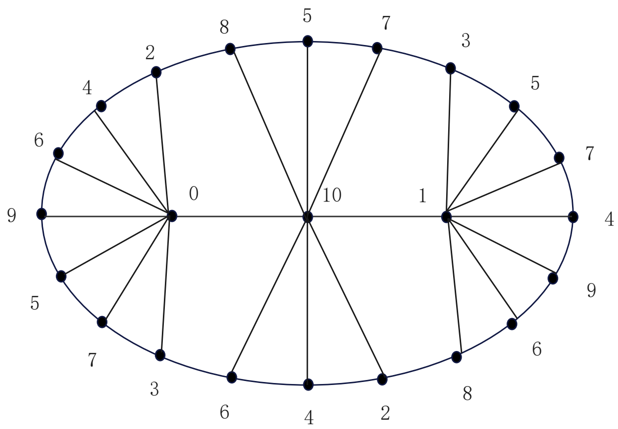

Figure 2. A Hain graph H^* with $\Delta = 8$ and $\lambda(H^*) = 10$.

Our Theorem 1 and the result in [23] confirm that every Halin graph G with $\Delta \ge 8$ has $\lambda(G) \le \Delta + 2$. Here, the lower bound 8 for Δ does not seem like the best possibility. Thus, we would like to propose the following problem:

Problem 1. *Determine the least integer Δ_0 such that every Halin graph G with $\Delta(G) \ge \Delta_0$ has $\lambda(G) \le \Delta + 2$.*

It is easy to check that the complete graph K_4 is a Halin graph with $\lambda(K_4) = 6 = \Delta(K_4) + 3$. This fact and Theorem 1 imply that $4 \leq \Delta_0 \leq 8$.

The second result we establish in this paper is that every Halin graph G has $\lambda(G) \leq \Delta + 6$. We feel that the constant 6 in the expression can be further improved.

Problem 2. *Determine the least constant C such that every Halin graph G has $\lambda(G) \leq \Delta + C$.*

The above discussion clearly implies that $3 \leq C \leq 6$.

To obtain the main contributions of this paper, we first analyzed the structures of Halin graphs with the maximum degree 8, i.e., that is, 14 inevitable configurations (C1)–(C14) were found in the graph under consideration. These structural characterizations could perhaps be applied to the study of other problems.

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