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Lie Bialgebra Structures on the Lie Algebra $\mathfrak{L}$ Related to the Virasoro Algebra

Xue Chen *, Yihong Su and Jia Zheng

School of Mathematics and Statistics, Xiamen University of Technology, Xiamen 361024, China
* Correspondence: 2013111001@xmut.edu.cn

Abstract: A Lie bialgebra is a vector space endowed simultaneously with the structure of a Lie algebra and the structure of a Lie coalgebra, and some compatibility condition. Moreover, Lie brackets have skew symmetry. Because of the close relation between Lie bialgebras and quantum groups, it is interesting to consider the Lie bialgebra structures on the Lie algebra $\mathfrak{L}$ related to the Virasoro algebra. In this paper, the Lie bialgebras on $\mathfrak{L}$ are investigated by computing $\mathrm{Der}(\mathfrak{L}, \mathfrak{L}\otimes\mathfrak{L})$. It is proved that all such Lie bialgebras are triangular coboundary, and the first cohomology group $H^1(\mathfrak{L}, \mathfrak{L}\otimes\mathfrak{L})$ is trivial.

Keywords: Lie bialgebras; Yang–Baxter equation; the Lie algebra

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1. Introduction

It is well known that the Virasoro algebra plays an important role in string theory, conformal field theory, the representation theory of Kac–Moody algebras and the theory of vertex operator algebras, as well as extended affine Lie algebras (see, e.g., [1–3]). It is interesting to study various generalizations of the Virasoro algebra and other closely related algebras. The Lie algebra $\mathfrak{L}$ is an infinite-dimensional Lie algebra with a $\mathbb{C}$-basis $\{L_m, E_m, c_1, c_2 | \, m \in \mathbb{Z}\}$ and the following Lie brackets:

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}c_1,$$

$$[E_m, E_n] = \frac{m-n}{2}L_{m+n} + \delta_{m+n,0}\frac{m^3-m}{24}c_1,$$

$$[L_m, E_n] = (m-n)E_{m+n} + \delta_{m+n,0}(m^3-m)c_2,$$

for any $m, n \in \mathbb{Z}$. It is clear that $\mathfrak{L}$ contains the Virasoro algebra as its subalgebra.

Generally, a Lie algebra has a one-dimensional center, but the interesting thing about this Lie algebra is that it has a two-dimensional center. Derivations and universal central extensions of the centerless Lie algebra $\mathfrak{L}$ were studied in [4]. The automorphism group of the centerless Lie algebra $\mathfrak{L}$ was characterized in [5]. However, Lie bialgebra structures on $\mathfrak{L}$ are unknown.

In this paper, we investigate Lie bialgebra structures on $\mathfrak{L}$. The notion of Lie bialgebras was originally introduced by Drinfeld (see [6,7]) in order to search for the solutions of the Yang–Baxter quantum equation. Since then, Lie bialgebras have attracted wide attention (see, e.g., [8–19]). For instance, Lie bialgebra structures on the one-sided Witt algebra, the Witt algebra and the Virasoro algebra were proved in [8,10] to be triangular coboundary, while the generalized case was considered in [11]. Furthermore, Lie bialgebra structures in generalized Virasoro-like types were determined in [12]. Motivated by the works mentioned above, we study Lie bialgebra structures on $\mathfrak{L}$. The main result presented in the paper is Theorem 3.1, which states that every Lie bialgebra structure on $\mathfrak{L}$ is triangular coboundary.

This result makes sense since dualizing a triangular coboundary Lie bialgebra may produce new Lie algebras (see, e.g., [20]). This will be studied in a sequel.

Throughout the paper, the sets of the complex numbers, the integers, the nonzero integers, and the nonnegative integers are denoted by $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{Z}^*$, and $\mathbb{N}$, respectively.

2. Preliminaries

In this section, we recall the definitions of Lie algebras, Lie coalgebras, Lie bialgebras, and related results which will be used in Section 3.

Let $\mathfrak{g}$ be a vector space over the complex field $\mathbb{C}$. Denote by σ the *twist map* of $\mathfrak{g}\otimes\mathfrak{g}$ and ψ the *cyclic map* of $\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$, namely, $\sigma(x_1\otimes x_2)=x_2\otimes x_1$, $\psi(x_1\otimes x_2\otimes x_3)=x_2\otimes x_3\otimes x_1$, for $x_1,x_2,x_3\in\mathfrak{g}$.

Then the definitions of a Lie algebra and Lie coalgebra can be reformulated as follows. A *Lie algebra* is a pair $(\mathfrak{g},\theta)$ of a vector space $\mathfrak{g}$ and a linear map $\theta:\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$ (called the bracket of $\mathfrak{g}$) satisfying the following conditions:

$$\mathrm{Ker}(1\otimes 1-\sigma)\subset\mathrm{Ker}\theta.\,(\text{skewsymmetry}),$$

$$\theta(1\otimes\theta)(1\otimes 1\otimes 1+\psi+\psi^2)=0\,(\text{Jacobiidentity}),$$

where 1 denotes the identity map on $\mathfrak{g}$. A *Lie coalgebra* is a pair $(\mathfrak{g},\vartheta)$ of a vector space $\mathfrak{g}$ and a linear map $\vartheta:\mathfrak{g}\to\mathfrak{g}\otimes\mathfrak{g}$ (called the cobracket of $\mathfrak{g}$) satisfying the following conditions:

$$\mathrm{Im}\vartheta\subset\mathrm{Im}(1\otimes 1-\sigma).\,(\text{anti-commutativity}),$$

$$(1\otimes 1\otimes 1+\psi+\psi^2)(1\otimes\vartheta)\vartheta=0.\,(\text{Jacobiidentity}),\tag{1}$$

For a Lie algebra $\mathfrak{g}$, we shall use $[x,y]=\theta(x,y)$ to denote its Lie bracket and use the symbol "$\cdot$" to denote the *diagonal adjoint action*:

$$x\cdot(\sum_i y_i\otimes z_i)=\sum_i([x,y_i]\otimes z_i+y_i\otimes[x,z_i]),\ \text{for } x,y_i,z_i\in\mathfrak{g}$$

Definition 2.1. *A Lie bialgebra is a triple$(\mathfrak{g},\theta,\vartheta)$, where $(\mathfrak{g},\theta)$is a Lie algebra, $(\mathfrak{g},\vartheta)$is a Lie coalgebra and*

$$\vartheta\theta(x\otimes y)=x\cdot\vartheta(y)-y\cdot\vartheta(x),\textit{for any } x,y\in\mathfrak{g}\tag{2}$$

Denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$ and 1 the identity element of $U(\mathfrak{g})$. For any $r=\sum_i x_i\otimes y_i\in\mathfrak{g}\otimes\mathfrak{g}$, define $r^{ij},\ c(r),\ i,j=1,2,3$ to be the elements of $U(\mathfrak{g})\otimes U(\mathfrak{g})\otimes U(\mathfrak{g})$ by

$$r^{12}=\sum_i x_i\otimes y_i\otimes 1,\ r^{13}=\sum_i x_i\otimes 1\otimes y_i,\ r^{23}=\sum_i 1\otimes x_i\otimes y_i$$

and

$$\begin{aligned}c(r)&=[r^{12},r^{13}]+[r^{12},r^{23}]+[r^{13},r^{23}]\\&=\sum_{i,j}[x_i,x_j]\otimes y_i\otimes y_j+\sum_{i,j}x_i\otimes[y_i,x_j]\otimes y_j+\sum_{i,j}x_i\otimes x_j\otimes[y_i,y_j]\end{aligned}\tag{3}$$

Definition 2.2. (1) *A coboundary Lie bialgebra is a 4-tuple $(\mathfrak{g},\theta,\vartheta,r)$, where $(\mathfrak{g},\theta,\vartheta)$is a Lie bialgebra and $r\in\mathrm{Im}(1\otimes 1-\sigma)\subset\mathfrak{g}\otimes\mathfrak{g}$, such that $\vartheta=\vartheta_r$is a coboundary of$r$, where ϑ_r is defined by*

$$\vartheta_r(x)=x\cdot r,\ \textit{for any } x\in\mathfrak{g}$$

(2) *A coboundary Lie bialgebra* $(\mathfrak{g}, \theta, \vartheta, r)$ *is called triangular if* r *satisfies the following classical Yang–Baxter Equation (CYBE):*

$$c(r) = 0 \tag{4}$$

(3) *An element* $r \in \mathrm{Im}(1 \otimes 1 - \sigma) \subset \mathfrak{g} \otimes \mathfrak{g}$ *is said to satisfy the modified Yang–Baxter Equation (MYBE) if*

$$x \cdot c(r) = 0, \textit{ for all } x \in \mathfrak{g} \tag{5}$$

The following results come from [6,7,10].

Lemma 2.3. *Let* $\mathfrak{g}$ *be a Lie algebra and* $r \in \mathrm{Im}(1 \otimes 1 - \sigma) \subset \mathfrak{g} \otimes \mathfrak{g}$.

(1) *The triple* $(\mathfrak{g}, \ [\cdot,\cdot], \ \vartheta_r)$ *is a Lie bialgebra if, and only if,* r *satisfies MYBE.*
(2) *We have*

$$(1 \otimes 1 \otimes 1 + \psi + \psi^2)(1 \otimes \vartheta)\vartheta(x) = x \cdot c(r) \textit{ for all } x \in \mathfrak{g}$$

3. Lie Bialgebra Structures on the Lie Algebra $\mathfrak{L}$ Related to the Virasoro Algebra

In this section, the main result of this paper (Theorem 3.1) is first presented, then several lemmas and propositions are given to prove Theorem 3.1, finally Theorem 3.1 is proved.

Theorem 3.1. *Every Lie bialgebra structure on* $\mathfrak{L}$ *is triangular coboundary.*

We introduce the grading on $\mathfrak{L}$ which will be used later. It is obvious that $\mathfrak{L} = \underset{n\in\mathbb{Z}}{\oplus} \mathfrak{L}_n$ is $\mathbb{Z}$-graded with

$$\mathfrak{L}_n = \mathrm{Span}_{\mathbb{C}}\{L_n, \, E_n \mid n \in \mathbb{Z}\} \oplus \delta_{n,0}(\mathbb{C}c_1 \oplus \mathbb{C}c_2)$$

Lemma 3.2. *Regard* $\mathfrak{L} \otimes \mathfrak{L} \otimes \mathfrak{L}$ *(the tensor product of three copies of* $\mathfrak{L}$*) as an* $\mathfrak{L}$*-module under the adjoint diagonal action of* $\mathfrak{L}$*. Suppose* $r \in \mathfrak{L} \otimes \mathfrak{L} \otimes \mathfrak{L}$ *satisfying* $a \cdot r = 0$ *for all* $a \in \mathfrak{L}$*. Then,* $r \in \mathrm{Z}(\mathfrak{L}) \otimes \mathrm{Z}(\mathfrak{L}) \otimes \mathrm{Z}(\mathfrak{L})$*, where* $\mathrm{Z}(\mathfrak{L})$ *is the center of* $\mathfrak{L}$.

Proof. Write $r = \sum_{t\in\mathbb{Z}} r_t$ as a finite sum with $r_t \in (\mathfrak{L} \otimes \mathfrak{L} \otimes \mathfrak{L})_t$. From $0 = L_0 \cdot r = -\sum_{t\in\mathbb{Z}} t r_t$, we obtain $r = r_0 \in (\mathfrak{L} \otimes \mathfrak{L} \otimes \mathfrak{L})_0$. Now we may assume that

$$\begin{aligned} r \equiv & \sum_{\substack{m,\, n \in \mathbb{Z} \\ A,\ B,\ D \in \{L,\, E\}}} \alpha_{m,n}^{A,\,B,\,D} A_m \otimes B_n \otimes D_{-(m+n)} + \sum_{\substack{k \in \mathbb{Z},\ i \in \{1,\,2\} \\ A,\ B \in \{L,\, E\}}} \beta_{k,i}^{A,\,B} A_k \otimes B_{-k} \otimes c_i \\ & + \sum_{\substack{k \in \mathbb{Z},\, i \in \{1,\,2\} \\ A,\, B \in \{L,\, E\}}} \xi_{k,i}^{A,\,B} A_k \otimes c_i \otimes B_{-k} + \sum_{\substack{k \in \mathbb{Z},\, i \in \{1,\,2\} \\ A,\, B \in \{L,\, E\}}} \rho_{k,i}^{A,\,B} c_i \otimes A_k \otimes B_{-k} + \sum_{\substack{i,\, j \in \{1,\,2\} \\ A \in \{L,\, E\}}} \lambda_{i,j}^{A} A_0 \otimes c_i \otimes c_j \\ & + \sum_{\substack{i,\, j \in \{1,\,2\} \\ A \in \{L,\, E\}}} \mu_{i,j}^{A} c_i \otimes A_0 \otimes c_j + \sum_{\substack{i,\, j \in \{1,\,2\} \\ A \in \{L,\, E\}}} \tau_{i,j}^{A} c_i \otimes c_j \otimes A_0 \ (\mathrm{mod}(\mathrm{Z}(\mathfrak{L}) \otimes \mathrm{Z}(\mathfrak{L}) \otimes \mathrm{Z}(\mathfrak{L}))) \end{aligned}$$

where all the coefficients of the tensor products are complex numbers and the sums are all finite. Fix the normal total order on $\mathbb{Z}$ compatible with its additive group structure. Define the total order on $\mathbb{Z} \times \mathbb{Z}$ by

$$(m_1, \, n_1) > (m_2, \, n_2) \Leftrightarrow m_1 > m_2, \ \text{ or } \ m_1 = m_2, \ n_1 > n_2$$

If $\alpha_{m,n}^{A,B,D} \neq 0$ for some $m, n \in \mathbb{Z}$, $A, B, D \in \{L, E\}$, let

$$(m_0, n_0) = \max\Big\{(m, n) \in \mathbb{Z} \times \mathbb{Z} \;\Big|\; \alpha_{m,n}^{A,B,D} \neq 0\Big\}.$$

Choose any $p > 0$ such that $p - m_0 \neq 0$. Then,

$$0 \neq (p - m_0)\alpha_{m_0,n_0}^{A,B,D} A_{p+m_0} \otimes B_{n_0} \otimes D_{-(m_0+n_0)}$$

is linearly independent with other terms of $L_p \cdot r$, a contradiction to the fact that $L_p \cdot r = 0$. Thus, $\alpha_{m,n}^{A,B,D} = 0$ for any $m, n \in \mathbb{Z}$, $A, B, D \in \{L, E\}$. We can similarly prove that $\beta_{k,i}^{A,B} = \xi_{k,i}^{A,B} = \rho_{k,i}^{A,B} = 0$ for any $k \in \mathbb{Z}$, $i \in \{1,2\}$, $A, B \in \{L, E\}$. Moreover, by

$$0 = L_1 \cdot r = \sum_{\substack{i,\, j \in \{1,\, 2\} \\ A \in \{L,\, E\}}} \lambda_{i,j}^{A} A_1 \otimes c_i \otimes c_j + \sum_{\substack{i,\, j \in \{1,\, 2\} \\ A \in \{L,\, E\}}} \mu_{i,j}^{A} c_i \otimes A_1 \otimes c_j + \sum_{\substack{i,\, j \in \{1,\, 2\} \\ A \in \{L,\, E\}}} \tau_{i,j}^{A} c_i \otimes c_j \otimes A_1,$$

it follows that $\lambda_{i,j}^{A} = \mu_{i,j}^{A} = \tau_{i,j}^{A} = 0$ for any $i, j \in \{1,2\}$, $A \in \{L, E\}$. This completes the proof. □

Corollary 3.3. *An element $r \in \mathrm{Im}(1 \otimes 1 - \sigma) \subset \mathfrak{L} \otimes \mathfrak{L}$ satisfies CYBE in* (4) *if, and only if, it satisfies MYBE in* (5).

Proof. It follows immediately from Lemma 3.2 and (3). □

The tensor product $V = \mathfrak{L} \otimes \mathfrak{L}$ is a $\mathbb{Z}$-graded $\mathfrak{L}$-module under the adjoint diagonal action of $\mathfrak{L}$. The gradation is given by $V = \underset{n\in\mathbb{Z}}{\oplus} V_n$, where $V_n = \sum_{\substack{p,q \in \mathbb{Z} \\ p+q=n}} \mathfrak{L}_p \otimes \mathfrak{L}_q$.

We shall discuss the derivation algebra $\mathrm{Der}(\mathfrak{L}, V)$. First, let us recall some basic definitions.

Denote by $\mathrm{Der}(\mathfrak{L}, V)$ the set of *derivations* $D : \mathfrak{L} \to V$ which are linear maps satisfying

$$D([a,\ b]) = a \cdot D(b) - b \cdot D(a) \text{ for } a, b \in L, \tag{6}$$

and $\mathrm{Inn}(\mathfrak{L}, V)$ the set of *inner derivations* u_{inn}, $u \in V$, defined by

$$u_{\mathrm{inn}} : \ a \mapsto a \cdot u \text{ for } a \in \mathfrak{L}.$$

A derivation $D \in \mathrm{Der}(\mathfrak{L}, V)$ is homogeneous of degree $\varepsilon \in \mathbb{Z}$ if $D(\mathfrak{L}_n) \in V_{\varepsilon+n}$ for all $n \in \mathbb{Z}$. Denote by $\mathrm{Der}(\mathfrak{L}, V)_\varepsilon = \{D \in \mathrm{Der}(\mathfrak{L}, V) \mid \deg D = \varepsilon\}$ for $\varepsilon \in \mathbb{Z}$. It is well known that

$$H^1(\mathfrak{L}, V) \cong \mathrm{Der}(\mathfrak{L}, V)/\mathrm{Inn}(\mathfrak{L}, V)$$

where $H^1(\mathfrak{L}, V)$ is the *first cohomology group* of the Lie algebra $\mathfrak{L}$ with coefficients in the $\mathfrak{L}$-module V.

Proposition 3.4. *Every derivation from $\mathfrak{L}$ to V is inner, i.e., $H^1(\mathfrak{L}, V) = 0$.*

Proof. We shall divide the proof of the proposition into several claims.

Claim 1. *For every $D \in \mathrm{Der}(\mathfrak{L}, V)$, we have*

$$D = \sum_{\varepsilon\in\mathbb{Z}} D_\varepsilon, \mathit{where} D_\varepsilon \in \mathrm{Der}(\mathfrak{L}, V)_\varepsilon, \tag{7}$$

which holds in the sense that for every $a \in \mathfrak{L}$*, only finitely many* $D_\varepsilon(a) \neq 0$*, and* $D(a) = \sum_{\varepsilon\in\mathbb{Z}} D_\varepsilon(a)$*(we say that (7) is summable).*

For any $\varepsilon \in \mathbb{Z}$, we define a homogeneous linear map $D_\varepsilon : \mathfrak{L} \to V$ of degree ε as follows: for any $p \in \mathbb{Z}$ and $a \in \mathfrak{L}_p$, we can write $D(a) = \sum_{m\in\mathbb{Z}} u_m$, where $u_m \in V_m$. Then, we define $D_\varepsilon(a) = u_{\varepsilon+p}$. Obviously $D_\varepsilon \in \mathrm{Der}(\mathfrak{L}, V)_\varepsilon$ and (7) hold.

Claim 2. *If* $\varepsilon \in \mathbb{Z}^*$*, then* $D_\varepsilon \in \mathrm{Inn}(\mathfrak{L}, V)$*.*

Denote $v = -\varepsilon^{-1} D_\varepsilon(L_0) \in V_\varepsilon$. Then for any $a_n \in \mathfrak{L}_n$, applying D_ε to $[\mathfrak{L}_0, a_n] = -na_n$, since $D_\varepsilon(a_n) \in V_{n+\varepsilon}$ and the action of L_0 on $V_{n+\varepsilon}$ is the scalar $-(n+\varepsilon)$, we have

$$-(n+\varepsilon)D_\varepsilon(a_n) - a_n \cdot D_\varepsilon(L_0) = -nD_\varepsilon(a_n),$$

i.e., $D_\varepsilon(a_n) = v_{\mathrm{inn}}(a_n)$. Then, $D_\varepsilon = v_{\mathrm{inn}}$ is inner.

For convenience, we shall use "$\equiv$" to denote equal modulo $\mathrm{Z}(\mathfrak{L}) \otimes \mathrm{Z}(\mathfrak{L})$ in the following.

Claim 3. $D_0(L_0) \equiv D_0(c_1) \equiv D_0(c_2) \equiv 0$.

For any $n \in \mathbb{Z}$ and $a_n \in \mathfrak{L}_n$, considering the action of D_0 on $[L_0, a_n] = -na_n$ and $[a_n, c_i] = 0$ $(i = 1, 2)$, respectively, we can deduce that $a_n \cdot D_0(L_0) = 0$ and $a_n \cdot D_0(c_i) = 0$ $(i = 1, 2)$. By Lemma 3.2, we have $D_0(L_0)$, $D_0(c_i) \in \mathrm{Z}(\mathfrak{L}) \otimes \mathrm{Z}(\mathfrak{L})$ for $i = 1, 2$. Thus Claim 3 is proved.

Claim 4. *By replacing* D_0 *by* $D_0 - v_{\mathrm{inn}}$ *for some* $v \in V_0$*, we can suppose* $D_0(\mathfrak{L}) \equiv 0$*.*

For any $s \in \mathbb{Z}^*$, $t \in \mathbb{Z}$, under modulo $\mathrm{Z}(\mathfrak{L}) \otimes \mathrm{Z}(\mathfrak{L})$, we can write $D_0(L_s)$ and $D_0(E_t)$ as follows.

$$\begin{aligned} D_0(L_s) \equiv & \sum_{i\in\mathbb{Z}} \alpha_{s,i} L_i \otimes L_{s-i} + \sum_{i\in\mathbb{Z}} \beta_{s,i} L_i \otimes E_{s-i} + \sum_{j=1}^{2} \alpha_s^j L_s \otimes c_j + \sum_{j=1}^{2} \beta_s^j c_j \otimes L_s \\ & + \sum_{i\in\mathbb{Z}} \xi_{s,i} E_i \otimes L_{s-i} + \sum_{i\in\mathbb{Z}} \rho_{s,i} E_i \otimes E_{s-i} + \sum_{j=1}^{2} \xi_s^j E_s \otimes c_j + \sum_{j=1}^{2} \rho_s^j c_j \otimes E_s \end{aligned} \tag{8}$$

$$\begin{aligned} D_0(E_t) \equiv & \sum_{i\in\mathbb{Z}} \lambda_{t,i} L_i \otimes L_{t-i} + \sum_{i\in\mathbb{Z}} \mu_{t,i} L_i \otimes E_{t-i} + \sum_{j=1}^{2} \lambda_t^j L_t \otimes c_j + \sum_{j=1}^{2} \mu_t^j c_j \otimes L_t \\ & + \sum_{i\in\mathbb{Z}} \tau_{t,i} E_i \otimes L_{t-i} + \sum_{i\in\mathbb{Z}} \eta_{t,i} E_i \otimes E_{t-i} + \sum_{j=1}^{2} \tau_t^j E_t \otimes c_j + \sum_{j=1}^{2} \eta_t^j c_j \otimes E_t \end{aligned} \tag{9}$$

where $\alpha_{s,i}$, $\beta_{s,i}$, $\xi_{s,i}$, $\rho_{s,i}$, $\lambda_{t,i}$, $\mu_{t,i}$, $\tau_{t,i}$, $\eta_{t,i}$, α_s^j, β_s^j, ξ_s^j, ρ_s^j, λ_t^j, μ_t^j, $\tau_t^j \eta_t^j \in \mathbb{C}$, $s \in \mathbb{Z}^*$, $t,i \in \mathbb{Z}$, $j \in \{1,2\}$, $\{i \in \mathbb{Z} \mid \alpha_{s,i} \neq 0\}$, $\{i \in \mathbb{Z} \mid \beta_{s,i} \neq 0\}$,$\{i \in \mathbb{Z} \mid \xi_{s,i} \neq 0\}$, $\{i \in \mathbb{Z} \mid \rho_{s,i} \neq 0\}$, $\{i \in \mathbb{Z} \mid \lambda_{t,i} \neq 0\}$, $\{i \in \mathbb{Z} \mid \mu_{t,i} \neq 0\}$, $\{i \in \mathbb{Z} \mid \tau_{t,i} \neq 0\}$ and $\{i \in \mathbb{Z} \mid \eta_{t,i} \neq 0\}$ are all finite sets. Note that for any $i \in \mathbb{Z}$ and $j \in \{1,2\}$, we have

$$\begin{aligned} &(L_i \otimes L_{-i})_{\mathrm{inn}}(L_1) = (1-i)L_{1+i} \otimes L_{-i} + (1+i)L_i \otimes L_{1-i}, \ (L_0 \otimes c_j)_{\mathrm{inn}}(L_1) = L_1 \otimes c_j, \\ &(L_i \otimes E_{-i})_{\mathrm{inn}}(L_1) = (1-i)L_{1+i} \otimes E_{-i} + (1+i)L_i \otimes E_{1-i}, \ (c_j \otimes L_0)_{\mathrm{inn}}(L_1) = c_j \otimes L_1, \\ &(E_i \otimes L_{-i})_{\mathrm{inn}}(L_1) = (1-i)E_{1+i} \otimes L_{-i} + (1+i)E_i \otimes L_{1-i}, \ (E_0 \otimes c_j)_{\mathrm{inn}}(L_1) = E_1 \otimes c_j, \\ &(E_i \otimes E_{-i})_{\mathrm{inn}}(L_1) = (1-i)E_{1+i} \otimes E_{-i} + (1+i)E_i \otimes E_{1-i}, \ (c_j \otimes E_0)_{\mathrm{inn}}(L_1) = c_j \otimes E_1. \end{aligned}$$

Denote

$$N_1 = \max\{\, |i| \mid \alpha_{1,i} \neq 0 \,\},\ N_2 = \max\{\, |i| \mid \beta_{1,i} \neq 0 \,\}$$
$$N_3 = \max\{\, |i| \mid \xi_{1,i} \neq 0 \,\},\ N_4 = \max\{\, |i| \mid \rho_{1,i} \neq 0 \,\}$$

Applying the induction on $\sum_{j=1}^{4} N_j$ in the above equations, by replacing D_0 by $D_0 - v_{\mathrm{inn}}$, where v is a combination of some $L_i \otimes L_{-i}$, $L_i \otimes E_{-i}$, $E_i \otimes L_{-i}$, $E_i \otimes E_{-i}$, $L_0 \otimes c_j$, $c_j \otimes L_0$, $E_0 \otimes c_j$, $c_j \otimes E_0$, we can suppose

$$\alpha_1^j = \beta_1^j = \xi_1^j = \rho_1^j = 0, \text{ for } j \in \{1,2\},\ \alpha_{1,i} = \beta_{1,i} = \xi_{1,i} = \rho_{1,i} = 0, \text{ for } i \neq -1, 2$$

Thus we have

$$D_0(L_1) \equiv \alpha_{1,-1}L_{-1} \otimes L_2 + \alpha_{1,2}L_2 \otimes L_{-1} + \beta_{1,-1}L_{-1} \otimes E_2 + \beta_{1,2}L_2 \otimes E_{-1}$$
$$+ \xi_{1,-1}E_{-1} \otimes L_2 + \xi_{1,2}E_2 \otimes L_{-1} + \rho_{1,-1}E_{-1} \otimes E_2 + \rho_{1,2}E_2 \otimes E_{-1}$$

Considering the action of D_0 on $[L_{-1}, L_1] = -2L_0$, under modulo $Z(\mathfrak{L}) \otimes Z(\mathfrak{L})$, we deduce that

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}}[(2-i)\alpha_{-1,i-1} + (2+i)\alpha_{-1,i}]L_i \otimes L_{-i} + 3\alpha_{1,-1}L_{-1} \otimes L_1 + 3\alpha_{1,2}L_1 \otimes L_{-1}\\
&+\sum_{i\in\mathbb{Z}}[(2-i)\beta_{-1,i-1} + (2+i)\beta_{-1,i}]L_i \otimes E_{-i} + 3\beta_{1,-1}L_{-1} \otimes E_1 + 3\beta_{1,2}L_1 \otimes E_{-1}\\
&+\sum_{i\in\mathbb{Z}}[(2-i)\rho_{-1,i-1} + (2+i)\rho_{-1,i}]E_i \otimes E_{-i} + 3\rho_{1,-1}E_{-1} \otimes E_1 + 3\rho_{1,2}E_1 \otimes E_{-1}\\
&+\sum_{j=1}^{2} 2\alpha^j_{-1}L_0 \otimes c_j + \sum_{j=1}^{2} 2\beta^j_{-1}c_j \otimes L_0 + \sum_{j=1}^{2} 2\xi^j_{-1}E_0 \otimes c_j + \sum_{j=1}^{2} 2\rho^j_{-1}c_j \otimes E_0 = 0
\end{aligned}$$

Comparing the coefficients of $L_0 \otimes c_j$, $c_j \otimes L_0$, $E_0 \otimes c_j$, $c_j \otimes E_0$, we obtain

$$\alpha^j_{-1} = \beta^j_{-1} = \xi^j_{-1} = \rho^j_{-1} = 0,\ j \in \{1, 2\}$$

Comparing the coefficients of $L_i \otimes L_{-i}$ for $i \in \mathbb{Z}$, we have

$$\begin{gathered}
3\alpha_{-1,-2} + \alpha_{-1,-1} + 3\alpha_{1,-1} = 0,\ \alpha_{-1,0} + 3\alpha_{-1,1} + 3\alpha_{1,2} = 0,\\
(2-i)\alpha_{-1,i-1} + (2+i)\alpha_{-1,i} = 0,\ \text{for } i \in \mathbb{Z},\ i \neq \pm1
\end{gathered}$$

Since $\{i \in \mathbb{Z} \mid \alpha_{-1,i} \neq 0\}$ is a finite set, we obtain

$$\alpha_{-1,-1} + \alpha_{-1,0} = 0,\ \alpha_{-1,i} = 0,\ \text{for } i \in \mathbb{Z},\ i \neq -2, -1, 0, 1,$$

and we have the following relations:

$$\alpha_{-1,-1} = -\alpha_{-1,0},\ \alpha_{1,-1} = \frac{1}{3}\alpha_{-1,0} - \alpha_{-1,-2},\ \alpha_{1,2} = -(\frac{1}{3}\alpha_{-1,0} + \alpha_{-1,1}).$$

Comparing the coefficients of $L_i \otimes E_{-i}$ for $i \in \mathbb{Z}$, we obtain

$$\begin{gathered}
3\beta_{-1,-2} + \beta_{-1,-1} + 3\beta_{1,-1} = 0,\ \beta_{-1,0} + 3\beta_{-1,1} + 3\beta_{1,2} = 0,\\
(2-i)\beta_{-1,i-1} + (2+i)\beta_{-1,i} = 0,\ \text{for } i \in \mathbb{Z},\ i \neq \pm1
\end{gathered}$$

Since $\{i \in \mathbb{Z} \mid \beta_{-1,i} \neq 0\}$ is a finite set, we have

$$\begin{gathered}
\beta_{-1,-1} = -\beta_{-1,0},\ \beta_{-1,i} = 0,\ \text{for } i \in \mathbb{Z},\ i \neq -2, -1, 0, 1,\\
\beta_{1,-1} = \tfrac{1}{3}\beta_{-1,0} - \beta_{-1,-2},\ \beta_{1,2} = -(\tfrac{1}{3}\beta_{-1,0} + \beta_{-1,1})
\end{gathered}$$

Comparing the coefficients of $E_i \otimes L_{-i}$ for $i \in \mathbb{Z}$, we deduce that

$$\begin{gathered}
3\xi_{-1,-2} + \xi_{-1,-1} + 3\xi_{1,-1} = 0,\ \xi_{-1,0} + 3\xi_{-1,1} + 3\xi_{1,2} = 0,\\
(2-i)\xi_{-1,i-1} + (2+i)\xi_{-1,i} = 0,\ \text{for } i \in \mathbb{Z}, .i \neq \pm1.
\end{gathered}$$

Since $\{i \in \mathbb{Z} \mid \xi_{-1,i} \neq 0\}$ is a finite set, we have the following identities:

$$\begin{gathered}
\xi_{-1,-1} = -\xi_{-1,0},\ \xi_{-1,i} = 0,\ \text{for } i \in \mathbb{Z},\ i \neq -2, -1, 0, 1,\\
\xi_{1,-1} = \tfrac{1}{3}\xi_{-1,0} - \xi_{-1,-2},\ \xi_{1,2} = -(\tfrac{1}{3}\xi_{-1,0} + \xi_{-1,1})
\end{gathered}$$

Comparing the coefficients of $E_i \otimes E_{-i}$ for $i \in \mathbb{Z}$, we obtain

$$\begin{gathered}
3\rho_{-1,-2} + \rho_{-1,-1} + 3\rho_{1,-1} = 0,\ \rho_{-1,0} + 3\rho_{-1,1} + 3\rho_{1,2} = 0,\\
(2-i)\rho_{-1,i-1} + (2+i)\rho_{-1,i} = 0,\ \text{for } i \in \mathbb{Z},\ i \neq \pm1
\end{gathered}$$

Since $\{i \in \mathbb{Z} \mid \rho_{-1,i} \neq 0\}$ is a finite set, we have

$$\rho_{-1,-1} = -\rho_{-1,0},\ \rho_{-1,i} = 0,\ \text{for } i \in \mathbb{Z},\ i \neq -2,-1,0,1,$$
$$\rho_{1,-1} = \tfrac{1}{3}\rho_{-1,0} - \rho_{-1,-2},\ \rho_{1,2} = -(\tfrac{1}{3}\rho_{-1,0} + \rho_{-1,1})$$

Consequently, we can rewrite

$$\begin{aligned}
D_0(L_1) \equiv\ & (\tfrac{1}{3}\alpha_{-1,0} - \alpha_{-1,-2})L_{-1} \otimes L_2 - (\tfrac{1}{3}\alpha_{-1,0} + \alpha_{-1,1})L_2 \otimes L_{-1}\\
&+(\tfrac{1}{3}\beta_{-1,0} - \beta_{-1,-2})L_{-1} \otimes E_2 - (\tfrac{1}{3}\beta_{-1,0} + \beta_{-1,1})L_2 \otimes E_{-1}\\
&+(\tfrac{1}{3}\xi_{-1,0} - \xi_{-1,-2})E_{-1} \otimes L_2 - (\tfrac{1}{3}\xi_{-1,0} + \xi_{-1,1})E_2 \otimes L_{-1}\\
&+(\tfrac{1}{3}\rho_{-1,0} - \rho_{-1,-2})E_{-1} \otimes E_2 - (\tfrac{1}{3}\rho_{-1,0} + \rho_{-1,1})E_2 \otimes E_{-1}\\
D_0(L_{-1}) \equiv\ & \alpha_{-1,-2}L_{-2} \otimes L_1 - \alpha_{-1,0}L_{-1} \otimes L_0 + \alpha_{-1,0}L_0 \otimes L_{-1} + \alpha_{-1,1}L_1 \otimes L_{-2}\\
&+\beta_{-1,-2}L_{-2} \otimes E_1 - \beta_{-1,0}L_{-1} \otimes E_0 + \beta_{-1,0}L_0 \otimes E_{-1} + \beta_{-1,1}L_1 \otimes E_{-2}\\
&+\xi_{-1,-2}E_{-2} \otimes L_1 - \xi_{-1,0}E_{-1} \otimes L_0 + \xi_{-1,0}E_0 \otimes L_{-1} + \xi_{-1,1}E_1 \otimes L_{-2}\\
&+\rho_{-1,-2}E_{-2} \otimes E_1 - \rho_{-1,0}E_{-1} \otimes E_0 + \rho_{-1,0}E_0 \otimes E_{-1} + \rho_{-1,1}E_1 \otimes E_{-2}
\end{aligned}$$

Considering the action of D_0 on $[L_2, L_{-1}] = 3L_1$, under modulo $Z(\mathfrak{L}) \otimes Z(\mathfrak{L})$, we obtain

$$\begin{aligned}
&\alpha_{-1,-2}(4L_0 + \tfrac{1}{2}c_1) \otimes L_1 + \alpha_{-1,-2}L_{-2} \otimes L_3 - 3\alpha_{-1,0}L_1 \otimes L_0 - 2\alpha_{-1,0}L_{-1} \otimes L_2\\
&+2\alpha_{-1,0}L_2 \otimes L_{-1} + 3\alpha_{-1,0}L_0 \otimes L_1 + \alpha_{-1,1}L_3 \otimes L_{-2} + \alpha_{-1,1}L_1 \otimes (4L_0 + \tfrac{1}{2}c_1)\\
&+\beta_{-1,-2}(4L_0 + \tfrac{1}{2}c_1) \otimes E_1 + \beta_{-1,-2}L_{-2} \otimes E_3 - 3\beta_{-1,0}L_1 \otimes E_0 - 2\beta_{-1,0}L_{-1} \otimes E_2\\
&+2\beta_{-1,0}L_2 \otimes E_{-1} + 3\beta_{-1,0}L_0 \otimes E_1 + \beta_{-1,1}L_3 \otimes E_{-2} + \beta_{-1,1}L_1 \otimes (4E_0 + 6c_2)\\
&+\xi_{-1,-2}(4E_0 + 6c_2) \otimes L_1 + \xi_{-1,-2}E_{-2} \otimes L_3 - 3\xi_{-1,0}E_1 \otimes L_0 - 2\xi_{-1,0}E_{-1} \otimes L_2\\
&+2\xi_{-1,0}E_2 \otimes L_{-1} + 3\xi_{-1,0}E_0 \otimes L_1 + \xi_{-1,1}E_3 \otimes L_{-2} + \xi_{-1,1}E_1 \otimes (4L_0 + \tfrac{1}{2}c_1)\\
&+\rho_{-1,-2}(4E_0 + 6c_2) \otimes E_1 + \rho_{-1,-2}E_{-2} \otimes E_3 - 3\rho_{-1,0}E_1 \otimes E_0 - 2\rho_{-1,0}E_{-1} \otimes E_2\\
&+2\rho_{-1,0}E_2 \otimes E_{-1} + 3\rho_{-1,0}E_0 \otimes E_1 + \rho_{-1,1}E_3 \otimes E_{-2} + \rho_{-1,1}E_1 \otimes (4E_0 + 6c_2)\\
&+\sum_{i\in\mathbb{Z}}(1+i)\alpha_{2,i}L_{i-1} \otimes L_{2-i} + \sum_{i\in\mathbb{Z}}(3-i)\alpha_{2,i}L_i \otimes L_{1-i} + \sum_{i\in\mathbb{Z}}(1+i)\beta_{2,i}L_{i-1} \otimes E_{2-i}\\
&+\sum_{i\in\mathbb{Z}}(3-i)\beta_{2,i}L_i \otimes E_{1-i} + \sum_{i\in\mathbb{Z}}(1+i)\xi_{2,i}E_{i-1} \otimes L_{2-i} + \sum_{i\in\mathbb{Z}}(3-i)\xi_{2,i}E_i \otimes L_{1-i}\\
&+\sum_{i\in\mathbb{Z}}(1+i)\rho_{2,i}E_{i-1} \otimes E_{2-i} + \sum_{i\in\mathbb{Z}}(3-i)\rho_{2,i}E_i \otimes E_{1-i} + \sum_{j=1}^{2}3\alpha_2^jL_1 \otimes c_j + \sum_{j=1}^{2}3\beta_2^jc_j \otimes L_1\\
&+\sum_{j=1}^{2}3\xi_2^jE_1 \otimes c_j + \sum_{j=1}^{2}3\rho_2^jc_j \otimes E_1 - (\alpha_{-1,0} - 3\alpha_{-1,-2})L_{-1} \otimes L_2 + (\alpha_{-1,0} + 3\alpha_{-1,1})L_2 \otimes L_{-1}\\
&-(\beta_{-1,0} - 3\beta_{-1,-2})L_{-1} \otimes E_2 + (\beta_{-1,0} + 3\beta_{-1,1})L_2 \otimes E_{-1} - (\xi_{-1,0} - 3\xi_{-1,-2})E_{-1} \otimes L_2\\
&+(\xi_{-1,0} + 3\xi_{-1,1})E_2 \otimes L_{-1} - (\rho_{-1,0} - 3\rho_{-1,-2})E_{-1} \otimes E_2 + (\rho_{-1,0} + 3\rho_{-1,1})E_2 \otimes E_{-1} = 0
\end{aligned}$$

For $j \in \{1,2\}$, comparing the coefficients of $L_1 \otimes c_j$, $c_j \otimes L_1$, $E_1 \otimes c_j$, $c_j \otimes E_1$ in the above equation, we have

$$\alpha_2^1 = -\tfrac{1}{6}\alpha_{-1,1},\ \ \alpha_2^2 = -2\beta_{-1,1},\ \ \beta_2^1 = -\tfrac{1}{6}\alpha_{-1,-2},\ \ \beta_2^2 = -2\xi_{-1,-2},$$
$$\xi_2^1 = -\tfrac{1}{6}\xi_{-1,1},\ \ \xi_2^2 = -2\rho_{-1,1},\ \ \rho_2^1 = -\tfrac{1}{6}\beta_{-1,-2},\ \ \rho_2^2 = -2\rho_{-1,-2}$$

For any $i \in \mathbb{Z}$, comparing the coefficients of $L_i \otimes L_{1-i}$, $L_i \otimes E_{1-i}$, $E_i \otimes L_{1-i}$ and $E_i \otimes E_{1-i}$, respectively, and noting that $\{i \in \mathbb{Z} \mid \alpha_{2,i} \neq 0\}$,$\{i \in \mathbb{Z} \mid \beta_{2,i} \neq 0\}$, $\{i \in \mathbb{Z} \mid \xi_{2,i} \neq 0\}$ and $\{i \in \mathbb{Z} \mid \rho_{2,i} \neq 0\}$ are finite sets, we deduce that

$$\begin{gathered}
\alpha_{-1,-2} + 5\alpha_{2,-2} = 0,\ 3\alpha_{-1,-2} - 3\alpha_{-1,0} + \alpha_{2,0} + 4\alpha_{2,-1} = 0,\ 4\alpha_{-1,-2} + 3\alpha_{-1,0} + 2\alpha_{2,1} + 3\alpha_{2,0} = 0,\\
4\alpha_{-1,1} - 3\alpha_{-1,0} + 3\alpha_{2,2} + 2\alpha_{2,1} = 0,\ 3\alpha_{-1,0} + 3\alpha_{-1,1} + 4\alpha_{2,3} + \alpha_{2,2} = 0,\ \alpha_{-1,1} + 5\alpha_{2,4} = 0,\ \alpha_{2,i} = 0\\
\beta_{-1,-2} + 5\beta_{2,-2} = 0,\ 3\beta_{-1,-2} - 3\beta_{-1,0} + \beta_{2,0} + 4\beta_{2,-1} = 0,\ 4\beta_{-1,-2} + 3\beta_{-1,0} + 2\beta_{2,1} + 3\beta_{2,0} = 0,\\
4\beta_{-1,1} - 3\beta_{-1,0} + 3\beta_{2,2} + 2\beta_{2,1} = 0,\ 3\beta_{-1,0} + 3\beta_{-1,1} + 4\beta_{2,3} + \beta_{2,2} = 0,\ \beta_{-1,1} + 5\beta_{2,4} = 0,\ \beta_{2,i} = 0\\
\xi_{-1,-2} + 5\xi_{2,-2} = 0,\ 3\xi_{-1,-2} - 3\xi_{-1,0} + \xi_{2,0} + 4\xi_{2,-1} = 0,\ 4\xi_{-1,-2} + 3\xi_{-1,0} + 2\xi_{2,1} + 3\xi_{2,0} = 0,\\
4\xi_{-1,1} - 3\xi_{-1,0} + 3\xi_{2,2} + 2\xi_{2,1} = 0,\ 3\xi_{-1,0} + 3\xi_{-1,1} + 4\xi_{2,3} + \xi_{2,2} = 0,\ \xi_{-1,1} + 5\xi_{2,4} = 0,\ \xi_{2,i} = 0\\
\rho_{-1,-2} + 5\rho_{2,-2} = 0,\ 3\rho_{-1,-2} - 3\rho_{-1,0} + \rho_{2,0} + 4\rho_{2,-1} = 0,\ 4\rho_{-1,-2} + 3\rho_{-1,0} + 2\rho_{2,1} + 3\rho_{2,0} = 0,\\
4\rho_{-1,1} - 3\rho_{-1,0} + 3\rho_{2,2} + 2\rho_{2,1} = 0,\ 3\rho_{-1,0} + 3\rho_{-1,1} + 4\rho_{2,3} + \rho_{2,2} = 0,\ \rho_{-1,1} + 5\rho_{2,4} = 0,\ \rho_{2,i} = 0
\end{gathered}$$

for $i \neq -1, 0, 1, 2, 3$. Then, we have the following identities:

$$\begin{gathered}
\alpha_{-1,-2} = -5\alpha_{2,-2} = 0,\ \alpha_{-1,1} = -5\alpha_{2,4} = 0,\ \alpha_{2,-1} = \tfrac{3}{4}\alpha_{-1,0} - \tfrac{1}{4}\alpha_{2,0}, \\
\alpha_{2,1} = -\tfrac{3}{2}\alpha_{-1,0} - \tfrac{3}{2}\alpha_{2,0},\ \alpha_{2,2} = 2\alpha_{-1,0} + \alpha_{2,0},\ \alpha_{2,3} = -\tfrac{5}{4}\alpha_{-1,0} - \tfrac{1}{4}\alpha_{2,0}, \\
\beta_{-1,-2} = -5\beta_{2,-2} = 0,\ \beta_{-1,1} = -5\beta_{2,4} = 0,\ \beta_{2,-1} = \tfrac{3}{4}\beta_{-1,0} - \tfrac{1}{4}\beta_{2,0}
\end{gathered}$$

$$\begin{gathered}
\beta_{2,1} = -\tfrac{3}{2}\beta_{-1,0} - \tfrac{3}{2}\beta_{2,0},\ \beta_{2,2} = 2\beta_{-1,0} + \beta_{2,0},\ \beta_{2,3} = -\tfrac{5}{4}\beta_{-1,0} - \tfrac{1}{4}\beta_{2,0}, \\
\xi_{-1,-2} = -5\xi_{2,-2} = 0,\ \xi_{-1,1} = -5\xi_{2,4} = 0,\ \xi_{2,-1} = \tfrac{3}{4}\xi_{-1,0} - \tfrac{1}{4}\xi_{2,0} \\
\xi_{2,1} = -\tfrac{3}{2}\xi_{-1,0} - \tfrac{3}{2}\xi_{2,0},\ \xi_{2,2} = 2\xi_{-1,0} + \xi_{2,0},\ \xi_{2,3} = -\tfrac{5}{4}\xi_{-1,0} - \tfrac{1}{4}\xi_{2,0} \\
\rho_{-1,-2} = -5\rho_{2,-2} = 0,\ \rho_{-1,1} = -5\rho_{2,4} = 0,\ \rho_{2,-1} = \tfrac{3}{4}\rho_{-1,0} - \tfrac{1}{4}\rho_{2,0}, \\
\rho_{2,1} = -\tfrac{3}{2}\rho_{-1,0} - \tfrac{3}{2}\rho_{2,0},\ \rho_{2,2} = 2\rho_{-1,0} + \rho_{2,0},\ \rho_{2,3} = -\tfrac{5}{4}\rho_{-1,0} - \tfrac{1}{4}\rho_{2,0} \\
\alpha_2^1 = \alpha_2^2 = \beta_2^1 = \beta_2^2 = \xi_2^1 = \xi_2^2 = \rho_2^1 = \rho_2^2 = 0
\end{gathered}$$

Thus we can rewrite

$$\begin{gathered}
D_0(L_1) \equiv \tfrac{1}{3}\alpha_{-1,0}L_{-1} \otimes L_2 - \tfrac{1}{3}\alpha_{-1,0}L_2 \otimes L_{-1} + \tfrac{1}{3}\beta_{-1,0}L_{-1} \otimes E_2 - \tfrac{1}{3}\beta_{-1,0}L_2 \otimes E_{-1} \\
+\tfrac{1}{3}\xi_{-1,0}E_{-1} \otimes L_2 - \tfrac{1}{3}\xi_{-1,0}E_2 \otimes L_{-1} + \tfrac{1}{3}\rho_{-1,0}E_{-1} \otimes E_2 - \tfrac{1}{3}\rho_{-1,0}E_2 \otimes E_{-1}, \\
D_0(L_{-1}) \equiv -\alpha_{-1,0}L_{-1} \otimes L_0 + \alpha_{-1,0}L_0 \otimes L_{-1} - \beta_{-1,0}L_{-1} \otimes E_0 + \beta_{-1,0}L_0 \otimes E_{-1} \\
-\xi_{-1,0}E_{-1} \otimes L_0 + \xi_{-1,0}E_0 \otimes L_{-1} - \rho_{-1,0}E_{-1} \otimes E_0 + \rho_{-1,0}E_0 \otimes E_{-1}, \\
D_0(L_2) \equiv (\tfrac{3}{4}\alpha_{-1,0} - \tfrac{1}{4}\alpha_{2,0})L_{-1} \otimes L_3 + \alpha_{2,0}L_0 \otimes L_2 - (\tfrac{3}{2}\alpha_{-1,0} + \tfrac{3}{2}\alpha_{2,0})L_1 \otimes L_1 \\
+(2\alpha_{-1,0} + \alpha_{2,0})L_2 \otimes L_0 - (\tfrac{5}{4}\alpha_{-1,0} + \tfrac{1}{4}\alpha_{2,0})L_3 \otimes L_{-1} \\
+(\tfrac{3}{4}\beta_{-1,0} - \tfrac{1}{4}\beta_{2,0})L_{-1} \otimes E_3 + \beta_{2,0}L_0 \otimes E_2 - (\tfrac{3}{2}\beta_{-1,0} + \tfrac{3}{2}\beta_{2,0})L_1 \otimes E_1 \\
+(2\beta_{-1,0} + \beta_{2,0})L_2 \otimes E_0 - (\tfrac{5}{4}\beta_{-1,0} + \tfrac{1}{4}\beta_{2,0})L_3 \otimes E_{-1} \\
+(\tfrac{3}{4}\xi_{-1,0} - \tfrac{1}{4}\xi_{2,0})E_{-1} \otimes L_3 + \xi_{2,0}E_0 \otimes L_2 - (\tfrac{3}{2}\xi_{-1,0} + \tfrac{3}{2}\xi_{2,0})E_1 \otimes L_1 \\
+(2\xi_{-1,0} + \xi_{2,0})E_2 \otimes L_0 - (\tfrac{5}{4}\xi_{-1,0} + \tfrac{1}{4}\xi_{2,0})E_3 \otimes L_{-1} \\
+(\tfrac{3}{4}\rho_{-1,0} - \tfrac{1}{4}\rho_{2,0})E_{-1} \otimes E_3 + \rho_{2,0}E_0 \otimes E_2 - (\tfrac{3}{2}\rho_{-1,0} + \tfrac{3}{2}\rho_{2,0})E_1 \otimes E_1 \\
+(2\rho_{-1,0} + \rho_{2,0})E_2 \otimes E_0 - (\tfrac{5}{4}\rho_{-1,0} + \tfrac{1}{4}\rho_{2,0})E_3 \otimes E_{-1}
\end{gathered}$$

Considering the action of D_0 on $[L_1, L_{-2}] = 3L_{-1}$, under modulo $\mathrm{Z}(\mathfrak{L}) \otimes \mathrm{Z}(\mathfrak{L})$, we have

$$\begin{gathered}
\sum_{i\in\mathbb{Z}}(1-i)\alpha_{-2,i}L_{1+i} \otimes L_{-2-i} + \sum_{i\in\mathbb{Z}}(3+i)\alpha_{-2,i}L_i \otimes L_{-1-i} + \sum_{i\in\mathbb{Z}}(1-i)\beta_{-2,i}L_{1+i} \otimes E_{-2-i} \\
+\sum_{i\in\mathbb{Z}}(3+i)\beta_{-2,i}L_i \otimes E_{-1-i} + \sum_{i\in\mathbb{Z}}(1-i)\xi_{-2,i}E_{1+i} \otimes L_{-2-i} + \sum_{i\in\mathbb{Z}}(3+i)\xi_{-2,i}E_i \otimes L_{-1-i}
\end{gathered}$$

$$\begin{gathered}
+\sum_{i\in\mathbb{Z}}(1-i)\rho_{-2,i}E_{1+i} \otimes E_{-2-i} + \sum_{i\in\mathbb{Z}}(3+i)\rho_{-2,i}E_i \otimes E_{-1-i} + \sum_{j=1}^{2}3\alpha^j_{-2}L_{-1} \otimes c_j \\
+\sum_{j=1}^{2}3\beta^j_{-2}c_j \otimes L_{-1} + \sum_{j=1}^{2}3\xi^j_{-2}E_{-1} \otimes c_j + \sum_{j=1}^{2}3\rho^j_{-2}c_j \otimes E_{-1} + \tfrac{1}{3}\alpha_{-1,0}L_{-3} \otimes L_2
\end{gathered}$$

$$\begin{gathered}
+\tfrac{4}{3}\alpha_{-1,0}L_{-1} \otimes L_0 + \tfrac{1}{6}\alpha_{-1,0}L_{-1} \otimes c_1 - \tfrac{4}{3}\alpha_{-1,0}L_0 \otimes L_{-1} - \tfrac{1}{6}\alpha_{-1,0}c_1 \otimes L_{-1} - \tfrac{1}{3}\alpha_{-1,0}L_2 \otimes L_{-3} \\
+\tfrac{1}{3}\beta_{-1,0}L_{-3} \otimes E_2 + \tfrac{4}{3}\beta_{-1,0}L_{-1} \otimes E_0 + 2\beta_{-1,0}L_{-1} \otimes c_2 - \tfrac{4}{3}\beta_{-1,0}L_0 \otimes E_{-1} - \tfrac{1}{6}\beta_{-1,0}c_1 \otimes E_{-1} \\
-\tfrac{1}{3}\beta_{-1,0}L_2 \otimes E_{-3} + \tfrac{1}{3}\xi_{-1,0}E_{-3} \otimes L_2 + \tfrac{4}{3}\xi_{-1,0}E_{-1} \otimes L_0 + \tfrac{1}{6}\xi_{-1,0}E_{-1} \otimes c_1 - \tfrac{4}{3}\xi_{-1,0}E_0 \otimes L_{-1} \\
-2\xi_{-1,0}c_2 \otimes L_{-1} - \tfrac{1}{3}\xi_{-1,0}E_2 \otimes L_{-3} + \tfrac{1}{3}\rho_{-1,0}E_{-3} \otimes E_2 + \tfrac{4}{3}\rho_{-1,0}E_{-1} \otimes E_0 + 2\rho_{-1,0}E_{-1} \otimes c_2 \\
-\tfrac{4}{3}\rho_{-1,0}E_0 \otimes E_{-1} - 2\rho_{-1,0}c_2 \otimes E_{-1} - \tfrac{1}{3}\rho_{-1,0}E_2 \otimes E_{-3} + 3\alpha_{-1,0}L_{-1} \otimes L_0 - 3\alpha_{-1,0}L_0 \otimes L_{-1} \\
+3\beta_{-1,0}L_{-1} \otimes E_0 - 3\beta_{-1,0}L_0 \otimes E_{-1} + 3\xi_{-1,0}E_{-1} \otimes L_0 - 3\xi_{-1,0}E_0 \otimes L_{-1} + 3\rho_{-1,0}E_{-1} \otimes E_0 \\
-3\rho_{-1,0}E_0 \otimes E_{-1} = 0
\end{gathered}$$

For any $i \in \mathbb{Z}$ and $j \in \{1, 2\}$, comparing the coefficients of $L_i \otimes L_{-1-i}$, $L_i \otimes E_{-1-i}$, $E_i \otimes L_{-1-i}$, $E_i \otimes E_{-1-i}$, $L_{-1} \otimes c_j$, $c_j \otimes L_{-1}$, $E_{-1} \otimes c_j$ and $c_j \otimes E_{-1}$ in the above equation, and noting that $\{i \in \mathbb{Z} \mid \alpha_{-2,i} \neq 0\}$, $\{i \in \mathbb{Z} \mid \beta_{-2,i} \neq 0\}$, $\{i \in \mathbb{Z} \mid \xi_{-2,i} \neq 0\}$ and $\{i \in \mathbb{Z} \mid \rho_{-2,i} \neq 0\}$ are finite sets, we obtain

$$\alpha_{-1,0}=0,\ \alpha_{-2,-3}=-\tfrac{1}{4}\alpha_{-2,0},\ \alpha_{-2,-2}=\alpha_{-2,0},\ \alpha_{-2,-1}=-\tfrac{3}{2}\alpha_{-2,0},\ \alpha_{-2,1}=-\tfrac{1}{4}\alpha_{-2,0},\ \alpha_{-2,i}=0,$$
$$\beta_{-1,0}=0,\ \beta_{-2,-3}=-\tfrac{1}{4}\beta_{-2,0},\ \beta_{-2,-2}=\beta_{-2,0},\ \beta_{-2,-1}=-\tfrac{3}{2}\beta_{-2,0},\ \beta_{-2,1}=-\tfrac{1}{4}\beta_{-2,0},\ \beta_{-2,i}=0,$$
$$\xi_{-1,0}=0,\ \xi_{-2,-3}=-\tfrac{1}{4}\xi_{-2,0},\ \xi_{-2,-2}=\xi_{-2,0},\ \xi_{-2,-1}=-\tfrac{3}{2}\xi_{-2,0},\ \xi_{-2,1}=-\tfrac{1}{4}\xi_{-2,0},\ \xi_{-2,i}=0,\ \rho_{-1,0}=0,$$
$$\rho_{-2,-3}=-\tfrac{1}{4}\rho_{-2,0},\ \rho_{-2,-2}=\rho_{-2,0},\ \rho_{-2,-1}=-\tfrac{3}{2}\rho_{-2,0},\ \rho_{-2,1}=-\tfrac{1}{4}\rho_{-2,0},\ \rho_{-2,i}=0$$
$$\alpha^1_{-2}=\ -\tfrac{1}{18}\alpha_{-1,0}=0,\ \alpha^2_{-2}=\ -\tfrac{2}{3}\beta_{-1,0}=0,\ \beta^1_{-2}=\ \tfrac{1}{18}\alpha_{-1,0}=0,\ \beta^2_{-2}=\ \tfrac{2}{3}\xi_{-1,0}=0$$
$$\xi^1_{-2}=\ -\tfrac{1}{18}\xi_{-1,0}=0,\ \xi^2_{-2}=\ -\tfrac{2}{3}\rho_{-1,0}=0,\ \rho^1_{-2}=\ \tfrac{1}{18}\beta_{-1,0}=0,$$
$$\rho^2_{-2}=\ \tfrac{2}{3}\rho_{-1,0}=0$$

for $i\neq-3,-2,-1,0,1$. Consequently, we can rewrite

$$\begin{aligned}
&D_0(L_1)\equiv D_0(L_{-1})\equiv 0,\\
&D_0(L_2)\equiv -\tfrac{1}{4}\alpha_{2,0}L_{-1}\otimes L_3+\alpha_{2,0}L_0\otimes L_2-\tfrac{3}{2}\alpha_{2,0}L_1\otimes L_1+\alpha_{2,0}L_2\otimes L_0\\
&\quad-\tfrac{1}{4}\alpha_{2,0}L_3\otimes L_{-1}-\tfrac{1}{4}\beta_{2,0}L_{-1}\otimes E_3+\beta_{2,0}L_0\otimes E_2-\tfrac{3}{2}\beta_{2,0}L_1\otimes E_1\\
&\quad+\beta_{2,0}L_2\otimes E_0-\tfrac{1}{4}\beta_{2,0}L_3\otimes E_{-1}-\tfrac{1}{4}\xi_{2,0}E_{-1}\otimes L_3+\xi_{2,0}E_0\otimes L_2\\
&\quad-\tfrac{3}{2}\xi_{2,0}E_1\otimes L_1+\xi_{2,0}E_2\otimes L_0-\tfrac{1}{4}\xi_{2,0}E_3\otimes L_{-1}-\tfrac{1}{4}\rho_{2,0}E_{-1}\otimes E_3\\
&\quad+\rho_{2,0}E_0\otimes E_2-\tfrac{3}{2}\rho_{2,0}E_1\otimes E_1+\rho_{2,0}E_2\otimes E_0-\tfrac{1}{4}\rho_{2,0}E_3\otimes E_{-1},\\
&D_0(L_{-2})\equiv -\tfrac{1}{4}\alpha_{-2,0}L_{-3}\otimes L_1+\alpha_{-2,0}L_{-2}\otimes L_0-\tfrac{3}{2}\alpha_{-2,0}L_{-1}\otimes L_{-1}+\alpha_{-2,0}L_0\otimes L_{-2}\\
&\quad-\tfrac{1}{4}\alpha_{-2,0}L_1\otimes L_{-3}-\tfrac{1}{4}\beta_{-2,0}L_{-3}\otimes E_1+\beta_{-2,0}L_{-2}\otimes E_0-\tfrac{3}{2}\beta_{-2,0}L_{-1}\otimes E_{-1}\\
&\quad+\beta_{-2,0}L_0\otimes E_{-2}-\tfrac{1}{4}\beta_{-2,0}L_1\otimes E_{-3}-\tfrac{1}{4}\xi_{-2,0}E_{-3}\otimes L_1+\xi_{-2,0}E_{-2}\otimes L_0\\
&\quad-\tfrac{3}{2}\xi_{-2,0}E_{-1}\otimes L_{-1}+\xi_{-2,0}E_0\otimes L_{-2}-\tfrac{1}{4}\xi_{-2,0}E_1\otimes L_{-3}-\tfrac{1}{4}\rho_{-2,0}E_{-3}\otimes E_1\\
&\quad+\rho_{-2,0}E_{-2}\otimes E_0-\tfrac{3}{2}\rho_{-2,0}E_{-1}\otimes E_{-1}+\rho_{-2,0}E_0\otimes E_{-2}-\tfrac{1}{4}\rho_{-2,0}E_1\otimes E_{-3}
\end{aligned}$$

Applying D_0 to $[L_2,L_{-2}]=4L_0+\frac{1}{2}c_1$, we have

$$\alpha_{-2,0}=-\alpha_{2,0},\ \beta_{-2,0}=-\beta_{2,0},\ \xi_{-2,0}=-\xi_{2,0},\ \rho_{-2,0}=-\rho_{2,0}$$

Denote

$$\begin{aligned}
v=&-\frac{1}{4}\alpha_{2,0}(L_{-1}\otimes L_1-2L_0\otimes L_0+L_1\otimes L_{-1})\\
&-\tfrac{1}{4}\beta_{2,0}(L_{-1}\otimes E_1-2L_0\otimes E_0+L_1\otimes E_{-1})\\
&-\tfrac{1}{4}\xi_{2,0}(E_{-1}\otimes L_1-2E_0\otimes L_0+E_1\otimes L_{-1})\\
&-\frac{1}{4}\rho_{2,0}(E_{-1}\otimes E_1-2E_0\otimes E_0+E_1\otimes E_{-1})
\end{aligned}$$

Replacing D_0 by D_0-v_{inn}, we obtain

$$D_0(L_{\pm1})\equiv D_0(L_{\pm2})\equiv 0$$

Since $\mathfrak{L}$ has a Virasoro subalgebra $\mathfrak{L}':=\mathrm{Span}_{\mathbb{C}}\{L_m\,|\,m\in\mathbb{Z}\}$, which can be generated by the set $\{L_{-2},L_{-1},L_1,L_2\}$, then we have

$$D_0(L_m)\equiv 0,\ \text{for any } m\in\mathbb{Z} \tag{10}$$

Considering the action of D_0 on $[L_1,E_1]=0$ and $[L_{-1},E_{-1}]=0$, respectively, under modulo $\mathrm{Z}(\mathfrak{L})\otimes\mathrm{Z}(\mathfrak{L})$, we obtain

$$\lambda_{1,i}=\mu_{1,i}=\tau_{1,i}=\eta_{1,i}=0,\ \text{for } i\neq 0,1$$
$$\lambda_{1,0}+\lambda_{1,1}=\mu_{1,0}+\mu_{1,1}=\tau_{1,0}+\tau_{1,1}=\eta_{1,0}+\eta_{1,1}=0,$$
$$\lambda_{-1,i}=\mu_{-1,i}=\tau_{-1,i}=\eta_{-1,i}=0,\ \text{for } i\neq 0,-1,$$
$$\lambda_{-1,0}+\lambda_{-1,-1}=\mu_{-1,0}+\mu_{-1,-1}=\tau_{-1,0}+\tau_{-1,-1}=\eta_{-1,0}+\eta_{-1,-1}=0$$

Then, we can write

$$\begin{aligned} D_0(E_1) \equiv{}& \lambda_{1,0}L_0 \otimes L_1 - \lambda_{1,0}L_1 \otimes L_0 + \mu_{1,0}L_0 \otimes E_1 - \mu_{1,0}L_1 \otimes E_0 \\ & + \tau_{1,0}E_0 \otimes L_1 - \tau_{1,0}E_1 \otimes L_0 + \eta_{1,0}E_0 \otimes E_1 - \eta_{1,0}E_1 \otimes E_0 \\ & + \sum_{j=1}^{2} \lambda_1^j L_1 \otimes c_j + \sum_{j=1}^{2} \mu_1^j c_j \otimes L_1 + \sum_{j=1}^{2} \tau_1^j E_1 \otimes c_j + \sum_{j=1}^{2} \eta_1^j c_j \otimes E_1, \\ D_0(E_{-1}) \equiv{}& -\lambda_{-1,0}L_{-1} \otimes L_0 + \lambda_{-1,0}L_0 \otimes L_{-1} - \mu_{-1,0}L_{-1} \otimes E_0 + \mu_{-1,0}L_0 \otimes E_{-1} \\ & -\tau_{-1,0}E_{-1} \otimes L_0 + \tau_{-1,0}E_0 \otimes L_{-1} - \eta_{-1,0}E_{-1} \otimes E_0 + \eta_{-1,0}E_0 \otimes E_{-1} \\ & + \sum_{j=1}^{2} \lambda_{-1}^j L_{-1} \otimes c_j + \sum_{j=1}^{2} \mu_{-1}^j c_j \otimes L_{-1} + \sum_{j=1}^{2} \tau_{-1}^j E_{-1} \otimes c_j + \sum_{j=1}^{2} \eta_{-1}^j c_j \otimes E_{-1} \end{aligned}$$

Applying D_0 to $[L_2, E_{-1}] = 3E_1$, under modulo $\mathrm{Z}(\mathfrak{L}) \otimes \mathrm{Z}(\mathfrak{L})$, we obtain

$$\begin{gathered} \lambda_{1,0} = \lambda_{-1,0} = \mu_{1,0} = \mu_{-1,0} = \tau_{1,0} = \tau_{-1,0} = \eta_{1,0} = \eta_{-1,0} = 0, \\ \lambda_{-1}^j = \lambda_1^j,\ \mu_{-1}^j = \mu_1^j,\ \tau_{-1}^j = \tau_1^j,\ \eta_{-1}^j = \eta_1^j, \text{ for } j \in \{1,2\} \end{gathered}$$

Thus, we can rewrite

$$\begin{aligned} D_0(E_1) &\equiv \sum_{j=1}^{2} \lambda_1^j L_1 \otimes c_j + \sum_{j=1}^{2} \mu_1^j c_j \otimes L_1 + \sum_{j=1}^{2} \tau_1^j E_1 \otimes c_j + \sum_{j=1}^{2} \eta_1^j c_j \otimes E_1, \\ D_0(E_{-1}) &\equiv \sum_{j=1}^{2} \lambda_1^j L_{-1} \otimes c_j + \sum_{j=1}^{2} \mu_1^j c_j \otimes L_{-1} + \sum_{j=1}^{2} \tau_1^j E_{-1} \otimes c_j + \sum_{j=1}^{2} \eta_1^j c_j \otimes E_{-1} \end{aligned}$$

Applying D_0 to $[E_1, E_{-1}] = L_0$, under modulo $\mathrm{Z}(\mathfrak{L}) \otimes \mathrm{Z}(\mathfrak{L})$, we have

$$\lambda_1^j = \mu_1^j = \tau_1^j = \eta_1^j = 0, \text{ for } j \in \{1,2\}.$$

Then,

$$D_0(E_1) \equiv D_0(E_{-1}) \equiv 0 \tag{11}$$

Since the Lie algebra $\mathfrak{L}$ is generated by the set $\{L_{-2}, L_{-1}, L_1, L_2, E_1\}$, using (10) and (11), we obtain $D_0(\mathfrak{L}) \equiv 0$. Then, Claim 4 is proved.

Claim 5. $\mathfrak{L}$ *is perfect,* i.e., $[\mathfrak{L}, \mathfrak{L}] = \mathfrak{L}$.

By Lie brackets of $\mathfrak{L}$, we have

$$\begin{gathered} L_m = \tfrac{1}{m}[L_m, L_0] \in [\mathfrak{L}, \mathfrak{L}] \text{ for } m \neq 0,\ L_0 = \tfrac{1}{2}[L_1, L_{-1}] \in [\mathfrak{L}, \mathfrak{L}], \\ c_1 = 2[L_2, L_{-2}] - 8L_0 \in [\mathfrak{L}, \mathfrak{L}],\ E_m = \tfrac{1}{m}[E_m, L_0] \in [\mathfrak{L}, \mathfrak{L}] \text{ for } m \neq 0, \\ E_0 = \tfrac{1}{2}[L_1, E_{-1}] \in [\mathfrak{L}, \mathfrak{L}],\ c_2 = \tfrac{1}{6}[L_2, E_{-2}] - \tfrac{2}{3}E_0 \in [\mathfrak{L}, \mathfrak{L}] \end{gathered}$$

Note that $\{L_m, E_m, c_1, c_2 | \, m \in \mathbb{Z}\}$ is a $\mathbb{C}$-basis of the Lie algebra $\mathfrak{L}$. Thus $\mathfrak{L}$ is perfect. Claim 5 is proved.

Claim 6. $D_0 = 0$.

It is proved that $D_0(\mathfrak{L}) \subseteq \mathrm{Z}(\mathfrak{L}) \otimes \mathrm{Z}(\mathfrak{L})$ in Claim 4. Because $\mathfrak{L} = [\mathfrak{L},\ \mathfrak{L}]$ by Claim 5, we have $D_0(\mathfrak{L}) \subseteq \mathfrak{L} \cdot D_0(\mathfrak{L}) = 0$. Then, Claim 6 is proved.

Claim 7. *For every* $D \in \mathrm{Der}(\mathfrak{L},\ V)$,$D = \sum_{\varepsilon \in \mathbb{Z}} D_\varepsilon$ *is a finite sum, where* $D_\varepsilon \in \mathrm{Der}(\mathfrak{L},\ V)_\varepsilon$.

According to the above claims, for any $\varepsilon \in \mathbb{Z}$, we can suppose $D_\varepsilon \in (u_\varepsilon)_{\mathrm{inn}}$ for some $u_\varepsilon \in V_\varepsilon$. If $\{\varepsilon \in \mathbb{Z}^* \mid u_\varepsilon \neq 0\}$ is an infinite set, then we have $D(L_0) = -\sum_{\varepsilon \in \mathbb{Z}^*} \varepsilon u_\varepsilon$ is an infinite sum, a contradiction with the fact that $D \in \mathrm{Der}(\mathfrak{L},\ V)$. This proves Claim 7 and Proposition 3.4. □

Lemma 3.5. *Suppose* $\omega \in V$ *such that* $a \cdot \omega \in \mathrm{Im}(1 \otimes 1 - \sigma)$ *for all* $a \in \mathfrak{L}$. *Then*$\omega \in \mathrm{Im}(1 \otimes 1 - \sigma)$.

Proof. It is easy to see that $\mathfrak{L} \cdot \mathrm{Im}(1 \otimes 1 - \sigma) \subset \mathrm{Im}(1 \otimes 1 - \sigma)$. After a few of steps in each of which ω is replaced by $\omega - \gamma$ for some $\gamma \in \mathrm{Im}(1 \otimes 1 - \sigma)$, we shall prove that $\omega = 0$ and thus $\omega \in \mathrm{Im}(1 \otimes 1 - \sigma)$. We can write $\omega = \sum_{k \in \mathbb{Z}} \omega_k$, where $\omega_k \in V_k$. Clearly,

$$\omega \in \mathrm{Im}(1 \otimes 1 - \sigma) \Leftrightarrow \omega_k \in \mathrm{Im}(1 \otimes 1 - \sigma) \text{ for all } k \in \mathbb{Z}$$

Then, without loss of generality, we may assume that $\omega = \omega_k$ is homogeneous. For any $k \neq 0$, $\omega_k = -\frac{1}{k} L_0 \cdot \omega_k \in \mathrm{Im}(1 \otimes 1 - \sigma)$. Thus, we can suppose $\omega = \omega_0 \in V_0$. Now ω can be written as

$$\begin{aligned} \omega = & \sum_{i \in \mathbb{Z}} \alpha_i L_i \otimes L_{-i} + \sum_{i \in \mathbb{Z}} \beta_i L_i \otimes E_{-i} + \sum_{i \in \mathbb{Z}} \xi_i E_i \otimes L_{-i} + \sum_{i \in \mathbb{Z}} \rho_i E_i \otimes E_{-i} \\ & + \sum_{j=1}^{2} \alpha_0^j L_0 \otimes c_j + \sum_{j=1}^{2} \beta_0^j c_j \otimes L_0 + \sum_{j=1}^{2} \xi_0^j E_0 \otimes c_j + \sum_{j=1}^{2} \rho_0^j c_j \otimes E_0 \end{aligned}$$

where $\alpha_i, \beta_i, \xi_i, \rho_i, \alpha_0^j, \beta_0^j, \xi_0^j, \rho_0^j \in \mathbb{C}$, for $i \in \mathbb{Z}$, $j \in \{1, 2\}$; $\{i \in \mathbb{Z} \mid \alpha_i \neq 0\}$, $\{i \in \mathbb{Z} \mid \beta_i \neq 0\}$, $\{i \in \mathbb{Z} \mid \xi_i \neq 0\}$ and $\{i \in \mathbb{Z} \mid \rho_i \neq 0\}$ are finite sets. For any $i \in \mathbb{Z}$, since $\gamma_{1,i} := L_i \otimes L_{-i} - L_{-i} \otimes L_i$, $\gamma_{2,i} := L_i \otimes E_{-i} - E_{-i} \otimes L_i$, $\gamma_{3,i} := E_i \otimes E_{-i} - E_{-i} \otimes E_i$,$\gamma_1 := L_0 \otimes c_1 - c_1 \otimes L_0$, $\gamma_2 := L_0 \otimes c_2 - c_2 \otimes L_0$, $\gamma_3 := E_0 \otimes c_1 - c_1 \otimes E_0$, $\gamma_4 := E_0 \otimes c_2 - c_2 \otimes E_0$ are all in $\mathrm{Im}(1 \otimes 1 - \sigma)$, by replacing ω by $\omega - \gamma$, where γ is a combination of some $\gamma_{1,i}$, $\gamma_{2,i}$, $\gamma_{3,i}$, γ_1, γ_2, γ_3 and γ_4, one can suppose

$$\xi_i = \beta_0^j = \rho_0^j = 0, \text{ for any } i \in \mathbb{Z},\ j \in \{1, 2\} \tag{12}$$

$$\alpha_i\,,\, \rho_i \neq 0 \Rightarrow i \geq 0 \tag{13}$$

Thus ω has the following form

$$\omega = \sum_{i \in \mathbb{N}} \alpha_i L_i \otimes L_{-i} + \sum_{i \in \mathbb{Z}} \beta_i L_i \otimes E_{-i} + \sum_{i \in \mathbb{N}} \rho_i E_i \otimes E_{-i} + \sum_{j=1}^{2} \alpha_0^j L_0 \otimes c_j + \sum_{j=1}^{2} \xi_0^j E_0 \otimes c_j$$

Suppose that there exists $i > 0$ such that $\alpha_i \neq 0$. Let $j > 0$ be such that $j \neq i$. It is easy to see that the term $L_{i+j} \otimes L_{-i}$ appears in $L_j \cdot \omega$. However the term $L_{-i} \otimes L_{i+j}$ cannot appear in $L_j \cdot \omega$ by (13), a contradiction with the fact that $L_j \cdot \omega \in \mathrm{Im}(1 \otimes 1 - \sigma)$. Thus, we can further suppose that $\alpha_i = 0$ for any $i \in \mathbb{Z}^*$. Similarly, we also can suppose that $\rho_i = 0$ for any $i \in \mathbb{Z}^*$. Then, ω can be written as

$$\omega = \alpha_0 L_0 \otimes L_0 + \sum_{i \in \mathbb{Z}} \beta_i L_i \otimes E_{-i} + \rho_0 E_0 \otimes E_0 + \sum_{j=1}^{2} \alpha_0^j L_0 \otimes c_j + \sum_{j=1}^{2} \xi_0^j E_0 \otimes c_j$$

Noting that $\mathrm{Im}(1 \otimes 1 - \sigma) \subset \mathrm{Ker}(1 \otimes 1 + \sigma)$ and using that $\mathfrak{L} \cdot \omega \in \mathrm{Im}(1 \otimes 1 - \sigma)$, we have

$$\begin{aligned} 0 &= (1 \otimes 1 + \sigma) L_1 \cdot \omega \\ &= 2\alpha_0 (L_1 \otimes L_0 + L_0 \otimes L_1) + 2\rho_0 (E_1 \otimes E_0 + E_0 \otimes E_1) \\ &+ \sum_{j=1}^{2} \alpha_0^j (L_1 \otimes c_j + c_j \otimes L_1) + \sum_{j=1}^{2} \xi_0^j (E_1 \otimes c_j + c_j \otimes E_1) \\ &+ \sum_{i \in \mathbb{Z}} [(2-i)\beta_{i-1} + (1+i)\beta_i] L_i \otimes E_{1-i} + \sum_{i \in \mathbb{Z}} [(2-i)\beta_{i-1} + (1+i)\beta_i] E_{1-i} \otimes L_i \end{aligned}$$

Since $\{i \in \mathbb{Z} \mid \beta_i \neq 0\}$ is a finite set, comparing the coefficient of the tensor products in the above equation, it follows that

$$\begin{gathered} \alpha_0 = \rho_0 = \alpha_0^j = \xi_0^j = \beta_i = 0, \text{ for } j \in \{1, 2\},\ i \in \mathbb{Z}, \\ i \neq -1,\, 0,\, 1 \\ \beta_0 = -2\beta_{-1} = -2\beta_1 \end{gathered}$$

Thus ω has the following form

$$\omega = \beta_1(L_{-1} \otimes E_1 - 2L_0 \otimes E_0 + L_1 \otimes E_{-1})$$

Considering the computation

$$0 = (1 \otimes 1 + \sigma)L_2 \cdot \omega.$$

$$= \beta_1(1 \otimes 1 + \sigma)[6L_1 \otimes E_1 + L_{-1} \otimes E_3 - 4L_2 \otimes E_0 - 4L_0 \otimes E_2 + L_3 \otimes E_{-1}]$$

it follows immediately that $\beta_1 = 0$. Thus $\omega = 0$. This completes the proof. □

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Let $(\mathfrak{L}, [\cdot, \cdot], \vartheta)$ be a Lie bialgebra structure on $\mathfrak{L}$. By (2) and (6), $\vartheta \in \mathrm{Der}(\mathfrak{L}, V)$. By Proposition 3.4, $\mathrm{Der}(\mathfrak{L}, V) = \mathrm{Inn}(\mathfrak{L}, V)$. Thus, there exists $r \in V$ such that $\vartheta = \vartheta_r$, where ϑ_r is defined by (1) of Definition 2.2. Namely, $\vartheta(a) = a \cdot r$ for any $a \in \mathfrak{L}$. By (1), $\mathrm{Im}\vartheta \subset \mathrm{Im}(1 \otimes 1 - \sigma)$. Hence, by Lemma 3.5, $r \in \mathrm{Im}(1 \otimes 1 - \sigma)$. By Lemma 2.3, $a \cdot c(r) = 0$, for all $a \in \mathfrak{L}$. By Corollary 3.3, $c(r) = 0$. Therefore, (1) and (2) of Definition 2.2 imply that $(\mathfrak{L}, [\cdot, \cdot], \vartheta)$ is a triangular coboundary Lie bialgebra. □

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