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Numerical Analysis of Fractional-Order Parabolic Equation Involving Atangana–Baleanu Derivative

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Abstract: In this study, the suggested q-homotopy analysis transform method is used to compute a numerical solution of a fractional parabolic equation, and the solution is obtained in a fast convergent series. The leverage and efficacy of the suggested technique are demonstrated by the test examples provided. The results that were acquired are graphically displayed. The series solution in a sizable admissible domain is handled in an extreme way by the current method. It provides us with a simple means of modifying the solution's convergence zone. The effectiveness and potential of the suggested algorithm are explicitly shown in the results using graphs.

Keywords: fractional parabolic equations; q-homotopy analysis transform method; Aboodh transform; Atangana–Baleanu operator

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1. Introduction

Fractional calculus is the study of arbitrary order differential and integral equations (FC). Newton invented FC, but it has only recently garnered the interest of a number of specialists. Within the framework of FC, the most intriguing advances in engineering and science applications have been produced over the past three decades. Due to the intrinsic complexity of heterogeneous processes, the non-integer derivative concept has been industrialized. Using fractional differential operators, one can model the behavior of multidimensional media undergoing a diffusion process [1–3]. Fractional order differential equations have shown to be a very useful tool, allowing for clearer and more precise solutions to numerous situations. As a result of the rapid rise of mathematical procedures utilizing computer software, a large number of academics began working on extended calculus in order to express their perspectives while analyzing a vast array of complex phenomena [4–6]. Consequently, symmetry analysis is a useful method for comprehending partial differential equations, particularly when examining equations generated from accounting-related mathematical concepts. Despite the notion that symmetry is the foundation of nature, "most" observations of the natural world lack symmetry. Creating unexpected events that break symmetry is a creative technique to conceal symmetry. There are two varieties of symmetry: finite and infinitesimal. Both discrete and continuous finite symmetries exist in two distinct forms. Parity and temporal inversion are instances of "discrete" natural symmetries, whereas space represents a continuous change. Mathematicians have always been fascinated by patterns. In the seventeenth century, classification of spatial and planar patterns gained off [7–10].

Distinguished researchers have provided a variety of ground-breaking recommendations for the several FC definitions that served as the basis. [11–13]. FC has also been associated with practical endeavors and is frequently employed in chaos soliton theory [14], optics [15], nanotechnology [16], human diseases [17], and other fields [18–22]. Numerous critical and nonlinear models are currently being extensively and effectively analyzed using FC. Numerous eminent scientists, including Riemann, Liouville, Caputo, and Fabrizio, have presented different definitions. However, these definitions have their own limitations.

The initial situation is not described by the Riemann–Liouville derivative; the Caputo derivation solves this issue, but does not define the phenomenon's singular core. Caputo and Fabrizio overcame the aforementioned duties in 2015 [23]. Numerous academics have utilized this derivation to analyze and solve a variety of nonlinear, challenging situations. Non-singular kernel and non-locality, which are essential to comprehending the physical behavior and nature of non-linear issues, were, nevertheless, cited as important obstacles in the CF derivative. In 2016, Atangana and Baleanu devised a unique fractional derivative and termed it AB derivative. This one of a kind derivative is defined with Mittag–Leffler functions [24]. This fractional derivative addresses all of the aforementioned issues and enables us to comprehend natural processes in a systematic and efficient manner.

The parabolic time FPDEs with varying coefficients

$$\frac{\partial^{\gamma+1}w}{\partial\theta^{\gamma+1}} + A(\eta,\xi,z)\frac{\partial^4 w}{\partial\eta^4} + B(\eta,\xi,z)\frac{\partial^4 w}{\partial\xi^4} + \frac{1}{z}C(\eta,\xi,z)\frac{\partial^4 w}{\partial z^4} = H(\eta,\xi,z,\theta),\ 0<\gamma\le 1,\ \theta>0, \quad (1)$$

where $A(\eta,\xi,z)$, $B(\eta,\xi,z)$ and $C(\eta,\xi,z)$ are positive real numbers, with initial conditions (IC's)

$$w(\eta,\xi,z,0) = f_0(\eta,\xi,z),\ w_\theta(\eta,\xi,z,0) = A_0(\eta,\xi,z), \quad (2)$$

with boundary conditions (BCs)

$$\begin{aligned}
w(\mu,\xi,z,\theta) =& h_0(\xi,z,\theta),\ w(b,\xi,z,\theta) = h_1(\xi,z,\theta),\\
w(\eta,\mu,z,\theta) =& g_0(\eta,z,\theta),\ w(\eta,b,z,\theta) = g_1(\eta,z,\theta),\\
w(\eta,\xi,\mu,\theta) =& j_0(\eta,\xi,\theta),\ w(\eta,\xi,b,\theta) = j_1(\eta,\xi,\theta),\\
w_{\eta\eta}(\mu,\xi,z,\theta) =& \bar{h}_0(\xi,z,\theta),\ w_{\eta\eta}(b,\xi,z,\theta) = \bar{h}_1(\xi,z,\theta),\\
w_{\xi\xi}(\eta,\mu,z,\theta) =& \bar{g}_0(\eta,z,\theta),\ w_{\xi\xi}(\eta,b,z,\theta) = \bar{g}_1(\eta,z,\theta),\\
w_{zz}(\eta,\xi,\mu,\theta) =& \bar{j}_0(\eta,\xi,\theta),\ w_{zz}(\eta,\xi,b,\theta) = \bar{j}_1(\eta,\xi,\theta),
\end{aligned} \quad (3)$$

where $h_i,\ g_i,\ j_i,\ \bar{h}_i,\ \bar{g}_i$and $\bar{j}_i,\ (i=0,1)$ are continuous variables and i is the beam's flexural stiffness ratio in its volume per unit mass, as is mentioned in [25–31].

Many scientists and mathematicians have recently developed extremely effective and exact methods for identifying and analyzing solutions to tough and nonlinear problems encountered in science and engineering. In this regard, Liao Shijun, [22] a Chinese mathematician, considers the homotopy analysis technique (HAM). Without perturbation or linearization, HAM has been used successfully and profitably to examine the behavior of nonlinear situations. For computing effort, HAM, on the other hand, necessitates a significant amount of time and computer memory. To accomplish this, an intentional procedure is combined with well-known transform techniques.

The combination of semi-analytical techniques with a suitable transformation decreases the amount of time required to examine the solutions of nonlinear problems that characterize real-world applications. In this study, we aimed to locate and explore the behavior of fractional parabolic PDE solutions derived by q-HATM. The proposed method enables us to evaluate a broader range of initial guesses and equation types in complex and nonlinear scenarios, hence allowing us to directly solve challenging NDEs. Its strength is its ability to alter two highly effective computational methods for investigating FDEs. By selecting the suitable $\hbar$, we are able to control the convergence region of solution series within a large acceptable domain. The future technique is distinguished by its use of a simple algorithm to evaluate the solution and its homotopy and axillary parameters, which enable rapid convergence in the provided solution for a nonlinear component of the given problem. In comparison to existing methods, the method under consideration is capable of maintaining high precision while minimizing computational time and effort.

2. Basic Definitions

In this section, a few definitions, theorems and property that will be useful in this article are given.

Definition 1. *The Aboodh transformation (AT) of a term $\Theta(\vartheta)$ with exponential-order*

$$\mathcal{C} = \left\{\Theta : |\Theta(\vartheta)| < Be^{p_j|\vartheta|},\ if\ \vartheta \in (-1)^i \times [0,\infty), j = 1,2; (B, p_1, p_2 > 0)\right\}$$

is written as

$$\mathcal{A}[\Theta(\vartheta)] = \mathcal{M}(\psi)$$

and expressed as

$$\mathcal{A}[\Theta(\vartheta)] = \frac{1}{\psi}\int_0^\infty \Theta(\vartheta)e^{-\psi\vartheta}d\vartheta = \mathcal{M}(\psi), \quad p_1 \le \psi \le p_2$$

Obviously, the AT is linear as the Laplace transformation (LT).

Definition 2. *The inverse AT of a term $\Theta(\vartheta)$ is expressed as*

$$\Theta(\vartheta) = \mathcal{A}^{-1}[\mathcal{M}(\psi)].$$

Definition 3. *Let $\Theta(\vartheta) \in \mathcal{E}$, then the LT is expressed as*

$$\Theta(\vartheta) = \int_0^\infty \Theta(\vartheta)e^{-s\vartheta}d\vartheta$$

The LT of $\Theta(\vartheta)$ is defined as follows.

$$\mathcal{L}[\Theta(\vartheta)] = \Theta(s).$$

Theorem 1. *If $\Theta(\vartheta) \in \mathcal{C}$ with the AT $\mathcal{A}[\Theta(\vartheta)]$ and LT $\mathcal{L}[\Theta(\vartheta)]$, then the following is the case.*

$$\mathcal{M}(\psi) = \frac{1}{\psi}\Theta(\psi)$$

Definition 4. *The Mittag–Leffler term is a special functions that often occur naturally in the result of fractional calculus, and it is expressed as*

$$E_\gamma(Z) = \sum_{\rho=0}^{\infty} \frac{Z^\rho}{\Gamma(\rho\gamma+1)}, \gamma, Z \in \mathbb{C}, \mathbb{R}e(\gamma) \ge 0,$$

In generalized type, it is given as follows:

$$E^{\xi}_{\gamma,\gamma} = \sum_{\rho=0}^{\infty} \frac{Z^\rho(\xi)_\rho}{\Gamma(\gamma+\rho\gamma)\rho!}, \gamma, \gamma, Z \in \mathbb{C}, \mathbb{R}(\gamma) \ge 0, \mathrm{Re}(\gamma) \ge 0,$$

Furthermore, we suppose $(\xi)_\rho$ to be the Pochhammer's symbols.

Definition 5. *Let $\Theta \in H^1(0,1)$ and $0 < \gamma < 1$, then the fractional AB derivative is defined as*

$${}_0^{ABC}D^\gamma_\vartheta\Theta(\vartheta) = \frac{N(\gamma)}{1-\gamma}\int_0^\vartheta \Theta'(x)E_\gamma\left(\frac{-\gamma(\vartheta-x)^\gamma}{1-\gamma}\right)dx$$

Definition 6. *Let $\Theta \in H^1(0,1)$ and $0 < \gamma < 1$, then the fractional AB derivative is expressed in the sense of Riemann–Liouville*

$$ {}_{0}^{ABR}D_{\vartheta}^{\gamma}\Theta(\vartheta) = \frac{N(\gamma)}{1-\gamma}\frac{d}{d\vartheta}\int_0^{\vartheta}\Theta(x)E_{\gamma}\left(\frac{-\gamma(\vartheta-x)^{\gamma}}{1-\gamma}\right)dx, $$

The normalization term $N(\gamma) > 0$ satisfies the conditions $N(0) = N(1) = 1$.

Theorem 2. *The LT of AB fractional operator according to the sense of Caputo as follows:*

$$ \mathcal{L}\left[{}_{0}^{ABC}D_{\vartheta}^{\gamma}\Theta(\vartheta)\right] = \frac{N(\gamma)}{1-\gamma}\times\frac{s^{\gamma}F(s) - s^{\gamma-1}f(0)}{s^{\gamma}+\frac{\gamma}{1-\gamma}}, $$

Furthermore, the LT of AB fractional operator according to the Riemann–Liouville is defined as.

$$ \mathcal{L}\left[{}_{0}^{ABR}D_{\vartheta}^{\gamma}\Theta(\vartheta)\right] = \frac{N(\gamma)}{1-\gamma}\times\frac{s^{\gamma}F(s)}{s\gamma+\frac{\gamma}{1-\gamma}}. $$

Theorem 3. *If $\Omega, \gamma \in \mathbb{C}$, with $\mathrm{Re}(\gamma) > 0$, then the AT of $E_{\gamma}(\Omega\vartheta^{\gamma})$ is defined as:*

$$ \mathcal{M}(E_{\gamma}(\Omega\vartheta^{\gamma})) = \frac{1}{\psi^2}\left(1-\frac{\Omega}{\psi^{\gamma}}\right)^{-1} $$

where $|\Omega\psi^{-\gamma}| < 1$

Theorem 4. *Let $\gamma, \gamma \in \mathbb{C}$, with $\mathbb{R}e(\gamma) > 0, \mathbb{R}e(\gamma) > 0$, the AT of $\vartheta^{\gamma-1}E_{\gamma,\xi}^{\xi}(\Omega\vartheta^{\gamma})$ is expressed as.*

$$ \vartheta^{\gamma-1}E_{\gamma,\xi}^{\xi}(\Omega\vartheta^{\gamma}) = \frac{1}{\psi^{\gamma+1}}(1-\Omega\psi^{-\gamma})^{-\xi}, \quad |\Omega\psi^{-\gamma}| < 1 $$

Theorem 5. *If $\mathcal{M}(\psi)$ is the AT of $\Theta(\vartheta) \in \mathcal{C}$ and $\Theta(s)$ is the LT of $\Theta(\vartheta) \in \mathcal{C}$, then the AT of fractional AB derivative according to the sense of Caputo is expressed as.*

$$ \mathcal{M}\left({}_{0}^{ABC}D_{\vartheta}^{\gamma}\Theta(\vartheta)\right) = \frac{N(\gamma)\left(\mathcal{M}(\psi) - \psi^{-2}\Theta(0)\right)}{1-\gamma+\gamma\psi^{-\gamma}}. $$

Theorem 6. *Suppose that $\mathcal{M}(\psi)$ is the AT of $\Theta(\vartheta) \in \mathcal{C}$ and $\Theta(s)$ is the LT of $\Theta(\vartheta) \in \mathcal{C}$, then the AT of fractional AB derivative according to the sense Riemann–Liouville is expressed as.*

$$ \mathcal{M}\left({}_{0}^{ABR}D_{\vartheta}^{\gamma}\Theta(\vartheta)\right) = \frac{N(\gamma)\mathcal{M}(\psi)}{1-\gamma+\gamma\psi^{-\gamma}} $$

3. Methodology

In order to present the main process of q-HATM [32,33],we take a NFDE, which is written below:

$$ {}^{AB}D_{\theta}^{\gamma}\Theta(\eta,\theta) + A\Theta(\eta,\theta) + H\Theta(\eta,\theta) = B(\eta,\theta), n-1 < \gamma \leq n, \tag{4} $$

with initial condition

$$ \Theta(\eta,0) = f(\eta,0). \tag{5} $$

where ${}_{\mu}^{ABC}D_{\theta}^{\gamma}\Theta(\eta,\theta)$ symbolise the AB derivative of $\Theta(\eta,\theta)$, $B(\eta,\theta)$ signifies the source term, A and H represents the linear and nonlinear differential operator, respectively. On applying the AT to Equation (4), we obtain after simplifying

$$ \mathcal{A}_{\theta}[\Theta(\eta,\theta)] - \frac{f\eta}{s} + \left(\frac{1-\gamma+\gamma s^{-\gamma}}{N(\gamma)}\right)\mathcal{A}_{\theta}[A\Theta(\eta,\theta) + H\Theta(\eta,\theta) - B(\eta,\theta)] = 0. \tag{6} $$

The non-linear operator is defined as follows

$$\mathcal{N}[\phi(\eta,\theta;q)] - \frac{f(\eta)}{s} + \left(\frac{1-\gamma+\gamma s^{-\gamma}}{N(\gamma)}\right)\mathcal{A}_\theta[A\phi(\eta,\theta;q) + H\phi(\eta,\theta;q) - B(\eta,\theta)]. \tag{7}$$

Here, $\phi(\eta,\theta;q)$ is the real-value term with respect to η, θ and $q \in [0, \frac{1}{n}]$. Now, a homotopy is given as

$$(1-nq)\mathcal{A}_\theta[\phi(\eta,\theta;q) - \Theta_0(\eta,\theta)] = \hbar q \mathcal{N}[\phi(\eta,\theta;q)] \tag{8}$$

where $\mathcal{A}$ is signifying AT, $q \in [0, \frac{1}{n}]$ $(n \geq 1)$ is the embedding parameter and $\hbar = 0$ is an auxiliary parameters. For $q = 0$ and $q = \frac{1}{n}$, the results are defined as

$$\phi(\eta,\theta;0) = \Theta_0(\eta,\theta), \;\; \phi(\eta,\theta;\frac{1}{n}) = \Theta(\eta,\theta). \tag{9}$$

Thus, by intensifying q from 0 to $\frac{1}{n}$, the solution $\phi(\eta,\theta;q)$ varies from $\Theta_0(\eta,\theta)$ to $\Theta(\eta,\theta)$. By using the Taylor theorem near to q, we defining $\phi(\eta,\theta;q)$ in series form and then we obtain

$$\phi(\eta,\theta;q) = \Theta_0(\eta,\theta) + \sum_{m=1}^{\infty}\Theta_m(\eta,\theta)q^m. \tag{10}$$

where

$$\Theta_m = \frac{1}{m!}\frac{\partial^m\phi(\eta,\theta;q)}{\partial q}|_{q=0}. \tag{11}$$

The series (8) converges at $q = \frac{1}{n}$ for the proper chaise of $\Theta_0(\eta,\theta)$, n and $\hbar$. Then:

$$\Theta(\eta,\theta) = \Theta_0(\eta,\theta) + \sum_{m=1}^{\infty}\Theta_m(\eta,\theta)\left(\frac{1}{n}\right)^m. \tag{12}$$

Now, m-times differentiating Equation (9) with q and later dividing by $m!$ and then putting $q = 0$, we obtain

$$\mathcal{A}_\theta[\Theta(\eta,\theta) - k_m\Theta_{m-1}(\eta,\theta)] = \hbar\Re_m(\overrightarrow{\Theta}_{m-1}), \tag{13}$$

where the vectors are defined as

$$\overrightarrow{\Theta}_m = [\Theta_0(\eta,\theta), \Theta_1(\eta,\theta), \cdots, \Theta_m(\eta,\theta)]. \tag{14}$$

On applying inverse AT on Equation (13), one can obtain

$$\Theta(\eta,\theta) = k_m\Theta_{m-1}(\eta,\theta) + \hbar\mathcal{A}_\theta^{-1}[\Re_m(\overrightarrow{\Theta}_{m-1})], \tag{15}$$

where

$$\begin{aligned}\Re_m(\overrightarrow{\Theta}_{m-1}) =& \mathcal{A}_\theta[\Theta_{m-1}(\eta,\theta)] - \left(1-\frac{k_m}{n}\right)\left(\frac{f(\eta)}{s} + \left(\frac{1-\gamma+\gamma s^{-\gamma}}{N(\gamma)}\right)(\mathcal{A}_\theta B(\eta,\theta))\right)\\ &+ \left(\frac{1-\gamma+\gamma s^{-\gamma}}{N(\gamma)}\right)\mathcal{A}_\theta[A\Theta_{m-1} + H_{m-1}],\end{aligned} \tag{16}$$

and

$$k_m = \begin{cases} 0, & m \leq 1, \\ n, & m > 1. \end{cases} \tag{17}$$

In Equation (16), H_m signifies homotopy polynomials is given as

$$H_m = \frac{1}{m!}\left[\frac{\partial^m\phi(\eta,\theta;q)}{\partial q^m}\right]_{q=0} \text{ and } \phi(\eta,\theta;q) = \phi_0 + q\phi_1 + q^2\phi_2 + \cdots. \tag{18}$$

By the aid of Equations (15) and (16), one can obtain

$$\begin{aligned}\Theta_m(\eta,\theta) =&(k_m+\hbar)\Theta_{m-1}(\eta,\theta) - \left(1-\frac{k_m}{n}\right)\left(\frac{f(\eta)}{s} + \left(\frac{1-\gamma+\gamma s^{-\gamma}}{N(\gamma)}\right)(\mathcal{A}_\theta B(\eta,\theta))\right) \\ &+ \left(\frac{1-\gamma+\gamma s^{-\gamma}}{N(\gamma)}\right)\mathcal{A}_\theta[A\Theta_{m-1}+H_{m-1}],\end{aligned} \tag{19}$$

Using Equation (19), one can obtain the series of $\Theta_m(\eta,\theta)$. Lastly, the series q-HATM solution is defined as

$$\Theta(\eta,\theta) = \sum_{m=0}^{\infty}\Theta_m(\eta,\theta). \tag{20}$$

4. Numerical Problems

In order to present the solution procedure and efficiency of the future scheme, in this subsection of the paper, a few numerical problems are presented.

4.1. Problem

Consider the fractional order one dimensional parabolic equation [34]:

$${}^{ABC}D_\theta^{\gamma+1} + \left(\frac{1}{\eta}+\frac{\eta^4}{120}\right)\frac{\partial^4\Theta}{\partial\eta^4} = 0,\ \ 0<\alpha\leq 1,\ \ \theta>0, \tag{21}$$

with initial conditions

$$\Theta(\eta,0)=0,\ \ \Theta_\theta(\eta,0) = 1+\frac{\eta^5}{120}, \tag{22}$$

with boundary conditions

$$\begin{aligned}&\Theta(\frac{1}{2},\theta) = \left(1+\frac{(1/2)^5}{120}\right)\sin(\theta),\ \Theta(1,\theta) = \frac{121}{120}\sin(\theta),\\ &\Theta_{\eta\eta}(\frac{1}{2},\theta) = \frac{1}{6}\left(\frac{1}{2}\right)^3\sin(\theta),\ \Theta_{\eta\eta}(1,\theta) = \frac{1}{6}\sin(\theta).\end{aligned} \tag{23}$$

Using Aboodh transform on Equation (21) and then applying Equation (22), we obtain

$$\mathcal{A}_\theta[\Theta(\eta,\theta)] - \frac{\left(1+\frac{\eta^5}{120}\right)}{s^2} + \left(\frac{1-\gamma+\gamma s^{-\gamma}}{N(\gamma)}\right)\mathcal{A}_\theta\left[\left(\frac{1}{\eta}+\frac{\eta^4}{120}\right)\frac{\partial^4\Theta}{\partial\eta^4}\right] = 0, \tag{24}$$

The nonlinear term $\mathcal{N}$ is represented with the aid of the given method, as below

$$\mathcal{N}[\phi(\eta,\theta;q)] = \mathcal{A}_\theta[\phi(\eta,\theta;q)] - \frac{\left(1+\frac{\eta^5}{120}\right)}{s^2} + \left(\frac{1-\gamma+\gamma s^{-\gamma}}{N(\gamma)}\right)\mathcal{A}_\theta\left[\left(\frac{1}{\eta}+\frac{\eta^4}{120}\right)\frac{\partial^4\phi(\eta,\theta;q)}{\partial\eta^4}\right]. \tag{25}$$

The deformation equation of m-th order with the aid of $q-HATM$ at $\mathbb{H}(\eta,\theta)=1$, is defined by

$$\mathcal{A}_\theta[\Theta_m(\eta,\theta) - k_m\Theta_{m-1}(\eta,\theta)] = \hbar\Re_m[\overrightarrow{\Theta}_{m-1}], \tag{26}$$

where

$$\Re_m[\overrightarrow{\Theta}_{m-1}] = \mathcal{A}_\theta[\Theta(\eta,\theta)] - (1-\frac{k_m}{n})\frac{\left(1+\frac{\eta^5}{120}\right)}{s^2} + \left(\frac{1-\gamma+\gamma s^{-\gamma}}{N(\gamma)}\right)\mathcal{A}_\theta\left[\left(\frac{1}{\eta}+\frac{\eta^4}{120}\right)\frac{\partial^4\Theta}{\partial\eta^4}\right]. \tag{27}$$

Using inverse Aboodh transform on Equation (26), it reduces to

$$\Theta_m(\eta,\theta) = k_m\Theta_{m-1}(\eta,\theta) + \mathcal{A}_\theta^{-1}\left[\hbar\Re_m[\overrightarrow{\Theta}_{m-1}]\right]. \tag{28}$$

Using initial conditions to simplify the above equation, we can find the terms of the series solution as

$$
\begin{aligned}
\Theta_0(\eta,\theta) =& 1+\frac{\eta^5}{120},\\
\Theta_1(\eta,\theta) =& \frac{\hbar\Big(\gamma\Big(-120\theta+\frac{(\eta^5+120)\theta^{\gamma+2}}{\Gamma(\gamma+3)}\Big)+(-\gamma\eta^5+\eta^5+120)\theta\Big)}{120\mathcal{N}[\gamma]},\\
\Theta_2(\eta,\theta) =& \frac{n\hbar\Big(\gamma\Big(-120\theta+\frac{(\eta^5+120)\theta^{\gamma+2}}{\Gamma(\gamma+3)}\Big)+(-\gamma\eta^5+\eta^5+120)\theta\Big)}{120\mathcal{N}[\gamma]}\\
&+\frac{\hbar^2\Big(\gamma^2\Big(\frac{\theta^{2\alpha+3}}{\Gamma(2\alpha+4)}+\theta\Big)+\gamma\Big(-2\theta+\frac{(\mathcal{N}[\gamma]-2\gamma+2)\theta^{\alpha+2}}{\Gamma(\alpha+3)}\Big)+(-(\alpha-1)\mathcal{N}[\gamma]+1)\theta\Big)(\eta^5+120)}{120\mathcal{N}[\gamma]^2},\\
\Theta_3(\eta,\theta) =& n\Big(\frac{n\hbar\Big(\gamma\Big(-120\theta+\frac{(\eta^5+120)\theta^{\gamma+2}}{\Gamma(\gamma+3)}\Big)+(-\gamma\eta^5+\eta^5+120)\theta\Big)}{120\mathcal{N}[\gamma]}\\
&+\frac{\hbar^2\Big(\gamma^2\Big(\frac{\theta^{2\alpha+3}}{\Gamma(2\alpha+4)}+\theta\Big)+\gamma\Big(-2\theta+\frac{(\mathcal{N}[\gamma]-2\gamma+2)\theta^{\alpha+2}}{\Gamma(\alpha+3)}\Big)+(-(\alpha-1)\mathcal{N}[\gamma]+1)\theta\Big)(\eta^5+120)}{120\mathcal{N}[\gamma]^2}\Big)\\
&+\hbar\Big(\frac{1}{120\mathcal{N}[\gamma]^2}\Big(\hbar\Big((\eta^5+120)\gamma\Big(\frac{\theta^{2\gamma+3}\gamma(2\hbar+n)}{\Gamma(2\alpha+4)}+\frac{(4\hbar+2n+\mathcal{N}[\gamma](\hbar+n)-2\gamma(2\hbar+n))\theta^{\gamma+2}}{\Gamma(\alpha+3)}\Big)\\
&+\theta(\eta^5+120)(\alpha-1^2)(2\hbar+n)\Big)\Big)+\frac{1}{120\mathcal{N}[\gamma]^3}\Big(\Big(-(\alpha-1)\Big(\mathcal{N}[\gamma]^2\theta(\hbar+n)+\frac{3\hbar\gamma^2\theta^{2\alpha+3}}{\Gamma(2\alpha+4)}\Big)\\
&+\hbar\Big(\frac{3\theta^{\alpha+2}\gamma(\gamma-1)^2}{\Gamma(\alpha+3)}+\gamma^3\Big(\frac{\theta^{3\alpha+4}}{\Gamma(3\alpha+5)}-\theta\Big)+\theta(3\gamma^2-3\gamma+1)\Big)\Big)(\eta^5+120)\hbar\Big)\Big),\\
&\vdots
\end{aligned}
\tag{29}
$$

The series solution is given by

$$
\Theta(\eta,\theta)=\Theta_0(\eta,\theta)+\frac{1}{n}\Theta_1(\eta,\theta)+\frac{1}{n^2}\Theta_2(\eta,\theta)+\frac{1}{n^3}\Theta(\eta,\theta)+\cdots. \tag{30}
$$

Putting the values of $\Theta_m(\eta,\theta)$ $(m=0,1,2,\cdots)$ in Equation (30), we have

$$
\begin{aligned}
\Theta(\eta,\theta) =&1+\frac{\eta^5}{120}+\left(\frac{1}{n}\right)\frac{\hbar\Big(\gamma\Big(-120\theta+\frac{(\eta^5+120)\theta^{\gamma+2}}{\Gamma(\gamma+3)}\Big)+(-\gamma\eta^5+\eta^5+120)\theta\Big)}{120\mathcal{N}[\gamma]}\\
&+\left(\frac{1}{n^2}\right)\frac{n\hbar\Big(\gamma\Big(-120\theta+\frac{(\eta^5+120)\theta^{\gamma+2}}{\Gamma(\gamma+3)}\Big)+(-\gamma\eta^5+\eta^5+120)\theta\Big)}{120\mathcal{N}[\gamma]}\\
&+\left(\frac{1}{n^2}\right)\frac{\hbar^2\Big(\gamma^2\Big(\frac{\theta^{2\alpha+3}}{\Gamma(2\alpha+4)}+\theta\Big)+\gamma\Big(-2\theta+\frac{(\mathcal{N}[\gamma]-2\gamma+2)\theta^{\alpha+2}}{\Gamma(\alpha+3)}\Big)+(-(\alpha-1)\mathcal{N}[\gamma]+1)\theta\Big)(\eta^5+120)}{120\mathcal{N}[\gamma]^2}\\
&+\left(\frac{1}{n^3}\right)\Big(n\Big(\frac{n\hbar\Big(\gamma\Big(-120\theta+\frac{(\eta^5+120)\theta^{\gamma+2}}{\Gamma(\gamma+3)}\Big)+(-\gamma\eta^5+\eta^5+120)\theta\Big)}{120\mathcal{N}[\gamma]}\\
&+\frac{\hbar^2\Big(\gamma^2\Big(\frac{\theta^{2\alpha+3}}{\Gamma(2\alpha+4)}+\theta\Big)+\gamma\Big(-2\theta+\frac{(\mathcal{N}[\gamma]-2\gamma+2)\theta^{\alpha+2}}{\Gamma(\alpha+3)}\Big)+(-(\alpha-1)\mathcal{N}[\gamma]+1)\theta\Big)(\eta^5+120)}{120\mathcal{N}[\gamma]^2}\Big)\\
&+\hbar\Big(\frac{1}{120\mathcal{N}[\gamma]^2}\Big(\hbar\Big((\eta^5+120)\gamma\Big(\frac{\theta^{2\gamma+3}\gamma(2\hbar+n)}{\Gamma(2\alpha+4)}+\frac{(4\hbar+2n+\mathcal{N}[\gamma](\hbar+n)-2\gamma(2\hbar+n))\theta^{\gamma+2}}{\Gamma(\alpha+3)}\Big)\\
&+\theta(\eta^5+120)(\alpha-1^2)(2\hbar+n)\Big)\Big)+\frac{1}{120\mathcal{N}[\gamma]^3}\Big(\Big(-(\alpha-1)\Big(\mathcal{N}[\gamma]^2\theta(\hbar+n)+\frac{3\hbar\gamma^2\theta^{2\alpha+3}}{\Gamma(2\alpha+4)}\Big)\\
&+\hbar\Big(\frac{3\theta^{\alpha+2}\gamma(\gamma-1)^2}{\Gamma(\alpha+3)}+\gamma^3\Big(\frac{\theta^{3\alpha+4}}{\Gamma(3\alpha+5)}-\theta\Big)+\theta(3\gamma^2-3\gamma+1)\Big)\Big)(\eta^5+120)\hbar\Big)\Big)\Big)
\end{aligned}
\tag{31}
$$

Substituting $n=1, \hbar=-1$ and $\gamma=1$, we obtain

$$
\begin{aligned}
\Theta(\eta,\theta) =&(\frac{1+\eta^5}{120})\theta-\frac{(\frac{1}{6}\theta^3+2\theta)(\eta^5+120)}{30}+\frac{29\theta(\eta^5+120)}{30}+\frac{(\eta^5+120)(\frac{1}{120}\theta^5+\frac{1}{6}\theta^3+4\theta)}{40}\\
&-\frac{(\frac{1}{5040}\theta^7+60\theta)(\eta^5+120)}{60}-\frac{(\eta^5+120)\theta^5}{7200}+\frac{\theta^9}{362880}+\frac{\eta^5\theta^9}{43545600}+\frac{\theta^7(\eta^5+120)}{604800}+\cdots,\\
=&(1+\frac{\eta^5}{120})\theta+(\frac{-\eta^5}{720}-\frac{1}{6})\theta^3+(\frac{1}{120}+\frac{\eta^5}{14400})\theta^5+(\frac{-\eta^5}{604800}-\frac{1}{5040})\theta^7+(\frac{1}{362880}+\frac{\eta^5}{43545600})\theta^9+\cdots,\\
=&\left(1+\frac{\eta^5}{120}\right)\left(\theta-\frac{1}{6}\theta^3+\frac{1}{120}\theta^5-\frac{1}{5040}\theta^7+\frac{1}{362880}\theta^9+\cdots\right),
\end{aligned}
\tag{32}
$$

$$
\Theta(\eta,\theta)=\left(1+\frac{\eta^5}{120}\right)\sin(\theta).
\tag{33}
$$

which is the exact solution [34] of Equation (21).

Figure 1, three dimensional graphs of Section 4.1 at (a) $\gamma=0.5$, (b) $\gamma=0.6$, (c) $\gamma=0.7$, (d) $\gamma=0.9$, (e) $\gamma=1$, and (f) Exact. Figure 2, two dimensional plots of the Section 4.1 for various values of α. Table 1, absolute error (AE) at $n=1, \hbar=-1$ and various fractional order absolute error (AE) at $n=1, \hbar=-1$ and various fractional order.

Table 1. Absolute error (AE) at $n=1, \hbar=-1$ and various fractional orders.

θ	η	AE ($\gamma=0.6$)	AE ($\gamma=0.7$)	AE ($\gamma=0.9$)	AE ($\gamma=1$)
0.1	1	0.01012218412	0.03251045834	0.01102023356	1.0×10^{-10}
	2	0.0127154723	0.04083958402	0.0138435992	1.0×10^{-10}
	3	0.0303665632	0.0975313750	0.0330607006	1.0×10^{-10}
	4	0.0957006955	0.3073716060	0.1041912988	3.0×10^{-10}
	5	0.271458714	0.871871383	0.295542627	1.0×10^{-9}

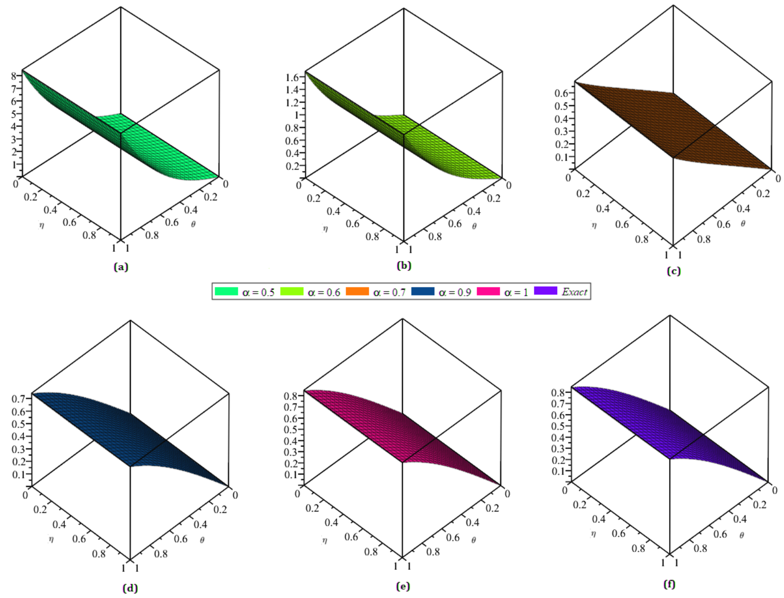

Figure 1. 3D graphs of Section 4.1 at (**a**) $\gamma = 0.5$, (**b**) $\gamma = 0.6$, (**c**) $\gamma = 0.7$, (**d**) $\gamma = 0.9$, (**e**) $\gamma = 1$, and (**f**) Exact.

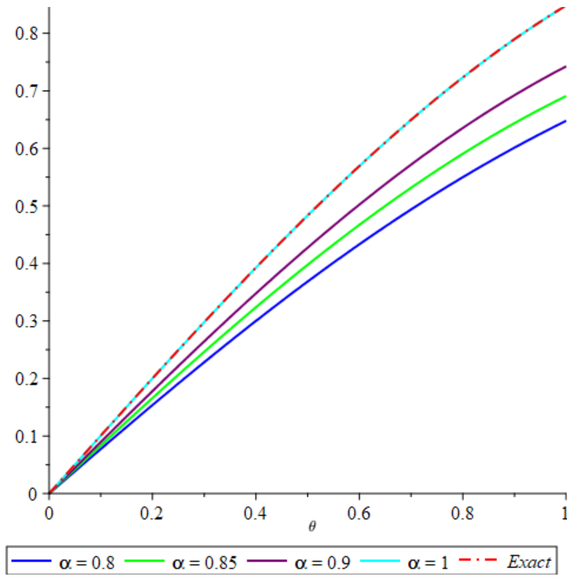

Figure 2. Tow dimensional plots of the Section 4.1 for various values of α.

4.2. Problem

Consider the fractional-order two-dimensional parabolic equation [34]:

$$ {}^{ABC}D_{\theta}^{\gamma+1}\Theta + 2\left(\frac{1}{\eta^2} + \frac{\eta^4}{6!}\right)\frac{\partial^4\Theta}{\partial\eta^4} + 2\left(\frac{1}{\xi^2} + \frac{\xi^4}{6!}\right)\frac{\partial^4\Theta}{\partial\xi^4} = 0, \quad 0 < \alpha \leq 1, \quad \theta > 0, \tag{34} $$

with initial conditions

$$ \Theta(\eta,\xi,0) = 0, \quad \Theta_{\theta}(\eta,\xi,0) = 2 + \frac{\eta^6}{6!} + \frac{\xi^6}{6!}, \tag{35} $$

with boundary conditions

$$\begin{aligned}
&\Theta(\frac{1}{2},\xi,\theta) = \left(2+\frac{(1/2)^6}{6!}+\frac{\xi^6}{6!}\right)\sin(\theta),\ \Theta(1,\xi,\theta) = \left(2+\frac{1}{6!}+\frac{\xi^6}{6!}\right)\sin(\theta),\\
&\Theta(\eta,\frac{1}{2},\theta) = \left(2+\frac{\eta^6}{6!}+\frac{(1/2)^6}{6!}\right)\sin(\theta),\ \Theta(\eta,1,\theta) = \left(2+\frac{\eta^6}{6!}+\frac{1}{6!}\right)\sin(\theta),\\
&\Theta_{\eta\eta}(\frac{1}{2},\xi,\theta) = \frac{(\frac{1}{2})^4}{4!}\sin(\theta),\ \Theta_{\eta\eta}(1,\xi,\theta) = \frac{1}{24}\sin(\theta),\\
&\Theta_{\xi\xi}(\eta,\frac{1}{2},\theta) = \frac{(\frac{1}{2})^4}{4!}\sin(\theta),\ \Theta_{\xi\xi}(\eta,1,\theta) = \frac{1}{24}\sin(\theta).
\end{aligned} \tag{36}$$

Taking AT on Equation (34) and then using Equation (35), we obtain

$$\mathcal{A}_\theta[\Theta(\eta,\xi,\theta)] - \frac{\left(2+\frac{\eta^6}{6!}+\frac{\xi^6}{6!}\right)}{s^2} + \left(\frac{1-\gamma+\gamma s^{-\gamma}}{N(\gamma)}\right)\mathcal{A}_\theta\left[2\left(\frac{1}{\eta^2}+\frac{\eta^4}{6!}\right)\frac{\partial^4\Theta}{\partial\eta^4} + 2\left(\frac{1}{\xi^2}+\frac{\xi^4}{6!}\right)\frac{\partial^4\Theta}{\partial\xi^4}\right] = 0, \tag{37}$$

The non-linear operator $\mathcal{N}$ is presented with the help of future algorithm as below

$$\begin{aligned}
\mathcal{N}[\phi(\eta,\xi,\theta;q)] =& \mathcal{A}_\theta[\phi(\eta,\xi,\theta;q)] - \frac{\left(2+\frac{\eta^6}{6!}+\frac{\xi^6}{6!}\right)}{s^2}\\
&+ \left(\frac{1-\gamma+\gamma s^{-\gamma}}{N(\gamma)}\right)\mathcal{A}_\theta\left[2\left(\frac{1}{\eta^2}+\frac{\eta^4}{6!}\right)\frac{\partial^4\phi(\eta,\xi,\theta;q)}{\partial\eta^4} + 2\left(\frac{1}{\xi^2}+\frac{\xi^4}{6!}\right)\frac{\partial^4\phi(\eta,\xi,\theta;q)}{\partial\xi^4}\right].
\end{aligned} \tag{38}$$

The deformation equation of m-th order with the help of $q-HATM$ at $\mathbb{H}(\eta,\theta) = 1$, is given by

$$\mathcal{A}_\theta[\Theta_m(\eta,\xi,\theta) - k_m\Theta_{m-1}(\eta,\xi,\theta)] = \hbar\Re_m[\overrightarrow{\Theta}_{m-1}], \tag{39}$$

where

$$\begin{aligned}
\Re_m[\overrightarrow{\Theta}_{m-1}] =& \mathcal{A}_\theta[\Theta(\eta,\xi,\theta)] - (1-\frac{k_m}{n})\frac{\left(2+\frac{\eta^6}{6!}+\frac{\xi^6}{6!}\right)}{s^2}\\
&+ \left(\frac{1-\gamma+\gamma s^{-\gamma}}{N(\gamma)}\right)\mathcal{A}_\theta\left[2\left(\frac{1}{\eta^2}+\frac{\eta^4}{6!}\right)\frac{\partial^4\Theta}{\partial\eta^4} + 2\left(\frac{1}{\xi^2}+\frac{\xi^4}{6!}\right)\frac{\partial^4\Theta}{\partial\xi^4}\right].
\end{aligned} \tag{40}$$

Using inverse Aboodh transform on Equation (39), it reduces to

$$\Theta_m(\eta,\xi,\theta) = k_m\Theta_{m-1}(\eta,\xi,\theta) + \mathcal{A}_\theta^{-1}\left[\hbar\Re_m[\overrightarrow{\Theta}_{m-1}]\right]. \tag{41}$$

Using initial conditions to simplify the above equation, we can find the terms of the series solution as

$$
\begin{aligned}
\Theta_0(\eta,\xi,\theta) =& 2+\frac{\eta^6}{6!}+\frac{\xi^6}{6!},\\
\Theta_1(\eta,\xi,\theta) =& \frac{\hbar\Big(-\gamma\Big(3\theta+\frac{(-2+\gamma)\theta^{\gamma+2}}{\Gamma(\gamma+3)}+\theta(\gamma^2+2)\Big)(\eta^6+\xi^6+1440)\Big)}{(720\gamma)},\\
\Theta_2(\eta,\xi,\theta) =& \frac{n\hbar\Big(-\gamma\Big(3\theta+\frac{(-2+\gamma)\theta^{\gamma+2}}{\Gamma(\gamma+3)}+\theta(\gamma^2+2)\Big)(\eta^6+\xi^6+1440)\Big)}{(720\gamma)}\\
&+\frac{\hbar^2}{720}\Big(\frac{1}{\gamma^2}\Big(\Big(\frac{(-2+\gamma)^2\theta^{2\gamma+3}\gamma^2}{\Gamma(2\gamma+4)}-\frac{(-2+\gamma)(2\gamma^2-5\gamma+4)t^{\gamma+2}\gamma}{\Gamma(\gamma+3)}\\
&+\theta\Big((\gamma^2+2)^2-4\gamma^2\Big)\Big)(\eta^6+\xi^6+1440)\Big)-\frac{5\theta(\eta^6+\xi^6+1440)(\gamma^2-2\gamma+2)}{\gamma}\Big),\\
\Theta_3(\eta,\xi,\theta) =& n\Big(\frac{n\hbar\Big(-\gamma\Big(3\theta+\frac{(-2+\gamma)\theta^{\gamma+2}}{\Gamma(\gamma+3)}+\theta(\gamma^2+2)\Big)(\eta^6+\xi^6+1440)\Big)}{(720\gamma)}\\
&+\frac{\hbar^2}{720}\Big(\frac{1}{\gamma^2}\Big(\Big(\frac{(-2+\gamma)^2\theta^{2\gamma+3}\gamma^2}{\Gamma(2\gamma+4)}-\frac{(-2+\gamma)(2\gamma^2-5\gamma+4)t^{\gamma+2}\gamma}{\Gamma(\gamma+3)}\\
&+\theta\Big((\gamma^2+2)^2-4\gamma^2\Big)\Big)(\eta^6+\xi^6+1440)\Big)-\frac{5\theta(\eta^6+\xi^6+1440)(\gamma^2-2\gamma+2)}{\gamma}\Big)\Big)\\
&+h\Big(\frac{1}{720\gamma^3}\Big(\Big(\gamma^2\Big(\frac{(-2+\gamma^2)(3\hbar\gamma^2-7\hbar\gamma+n\gamma+6\hbar)\theta^{2\gamma+3}}{\Gamma(2\gamma+4)}+2\theta(22\hbar-5n)\Big)\\
&+\Big(10n\theta-\frac{\hbar(-2+\gamma)^3\theta^{3\gamma+4}}{\Gamma(3\gamma+5)}\Big)\gamma^3\\
&+\Big(\frac{(-3\gamma^4\hbar+14\gamma^3\hbar-2\gamma^3n-28\gamma^2\hbar+5\gamma^2n+28\gamma\hbar-4\gamma n-12\hbar)(-2+\gamma)\theta^{\gamma+2}}{\Gamma(\alpha+3)}\\
&-\theta n(5\gamma^3-4)\Big)\gamma+(\gamma^6\hbar-7\gamma^5\hbar+\gamma^5n+8\hbar)\theta\Big)(\eta^6+\xi^6+1440)\hbar\Big)\\
&+\frac{(\eta^6+\xi^6+1440)(11\gamma^3-20\gamma^2-14)\hbar^2\theta}{360\gamma^2}\Big),\\
&\vdots
\end{aligned}
\tag{42}
$$

The series solution is given by

$$
\Theta(\eta,\theta)=\Theta_0(\eta,\theta)+\frac{1}{n}\Theta_1(\eta,\theta)+\frac{1}{n^2}\Theta_2(\eta,\theta)+\frac{1}{n^3}\Theta(\eta,\theta)+\cdots. \tag{43}
$$

Putting the values of $\Theta_m(\eta,\theta)$ $(m=0,1,2,\cdots)$ in Equation (43), we have

$$
\begin{aligned}
\Theta(\eta,\xi,\theta) =& 2+\frac{\eta^6}{6!}+\frac{\xi^6}{6!}+\frac{1}{n}\Big(\frac{\hbar\Big(-\gamma\Big(3\theta+\frac{(-2+\gamma)\theta^{\gamma+2}}{\Gamma(\gamma+3)}+\theta(\gamma^2+2)\Big)(\eta^6+\xi^6+1440)\Big)}{(720\gamma)}\Big)\\
&+\frac{1}{n^2}\Big(\frac{n\hbar\Big(-\gamma\Big(3\theta+\frac{(-2+\gamma)\theta^{\gamma+2}}{\Gamma(\gamma+3)}+\theta(\gamma^2+2)\Big)(\eta^6+\xi^6+1440)\Big)}{(720\gamma)}\\
&+\frac{\hbar^2}{720}\Big(\frac{1}{\gamma^2}\Big(\Big(\frac{(-2+\gamma)^2\theta^{2\gamma+3}\gamma^2}{\Gamma(2\gamma+4)}-\frac{(-2+\gamma)(2\gamma^2-5\gamma+4)t^{\gamma+2}\gamma}{\Gamma(\gamma+3)}\\
&+\theta\Big((\gamma^2+2)^2-4\gamma^2\Big)\Big)(\eta^6+\xi^6+1440)\Big)-\frac{5\theta(\eta^6+\xi^6+1440)(\gamma^2-2\gamma+2)}{\gamma}\Big)\Big)\\
&+\frac{1}{n^3}\Big(n\Big(\frac{n\hbar\Big(-\gamma\Big(3\theta+\frac{(-2+\gamma)\theta^{\gamma+2}}{\Gamma(\gamma+3)}+\theta(\gamma^2+2)\Big)(\eta^6+\xi^6+1440)\Big)}{(720\gamma)}\\
&+\frac{\hbar^2}{720}\Big(\frac{1}{\gamma^2}\Big(\Big(\frac{(-2+\gamma)^2\theta^{2\gamma+3}\gamma^2}{\Gamma(2\gamma+4)}-\frac{(-2+\gamma)(2\gamma^2-5\gamma+4)t^{\gamma+2}\gamma}{\Gamma(\gamma+3)}\\
&+\theta\Big((\gamma^2+2)^2-4\gamma^2\Big)\Big)(\eta^6+\xi^6+1440)\Big)-\frac{5\theta(\eta^6+\xi^6+1440)(\gamma^2-2\gamma+2)}{\gamma}\Big)\Big)\\
&+h\Big(\frac{1}{720\gamma^3}\Big(\Big(\gamma^2\Big(\frac{(-2+\gamma^2)(3\hbar\gamma^2-7\hbar\gamma+n\gamma+6\hbar)\theta^{2\gamma+3}}{\Gamma(2\gamma+4)}+2\theta(22\hbar-5n)\Big)\\
&+\Big(10n\theta-\frac{\hbar(-2+\gamma)^3\theta^{3\gamma+4}}{\Gamma(3\gamma+5)}\Big)\gamma^3\\
&+\Big(\frac{(-3\gamma^4\hbar+14\gamma^3\hbar-2\gamma^3n-28\gamma^2\hbar+5\gamma^2n+28\gamma\hbar-4\gamma n-12\hbar)(-2+\gamma)\theta^{\gamma+2}}{\Gamma(\alpha+3)}\\
&-\theta n(5\gamma^3-4)\Big)\gamma+(\gamma^6\hbar-7\gamma^5\hbar+\gamma^5n+8\hbar)\theta\Big)(\eta^6+\xi^6+1440)\hbar\Big)\\
&+\frac{(\eta^6+\xi^6+1440)(11\gamma^3-20\gamma^2-14)\hbar^2\theta}{360\gamma^2}\Big)\Big)+\cdots.
\end{aligned}
\tag{44}
$$

Substituting $n=1$, $\hbar=-1$ and $\gamma=1$, we obtain

$$
\begin{aligned}
\Theta(\eta,\xi,\theta) =& (2+\frac{\eta^6}{720}+\frac{\xi^6}{720})\theta-\frac{\theta^3(\eta^6+\xi^6+1440)}{1440}+\frac{(\frac{1}{120}\theta^5+\frac{1}{6}\theta^3+5\theta)(\eta^6+\xi^6+1440)}{360}\\
&+\frac{\theta(\eta^6+\xi^6+1440)}{20}+\frac{(\frac{-1}{120}\theta^5-46\theta-\frac{1}{5040}\theta^7)(\eta^6+\xi^6+1440)}{720}+\cdots,\\
\Theta(\eta,\xi,\theta) =& (2+\frac{\eta^6}{720}+\frac{\xi^6}{720})\theta+(\frac{-\eta^6}{4320}-\frac{\xi^6}{4320}-\frac{1}{3})\theta^3+(\frac{\eta^6}{86400}+\frac{\xi^6}{86400}+\frac{1}{60})\theta^5\\
&+(\frac{-\eta^6}{3628800}-\frac{\xi^6}{3628800}-\frac{1}{2520})\theta^7+\cdots,\\
\Theta(\eta,\xi,\theta) =& (2+\frac{\eta^6}{720}+\frac{\xi^6}{720})\Big(\theta-\frac{\theta^3}{6}+\frac{\theta^5}{120}-\frac{\theta^7}{5040}+\cdots\Big),\\
\Theta(\eta,\xi,\theta) =& \Big(2+\frac{\eta^6}{720}+\frac{\xi^6}{720}\Big)\sin(\theta).
\end{aligned}
\tag{45}
$$

which is the exact solution of Equation (34).

Figure 3, three dimensional graphs of q-HATM solution at (a) $\gamma=1$ and (b) exact Section 4.2. Figure 4, three dimensional graphs of Section 4.2 for different values of γ. Figure 5, two dimensional plots of the Section 4.2 for various values of γ and Exact solution. Table 2, absolute error (AE) at $n=1$, $\hbar=-1$ and various fractional order of Section 4.2.

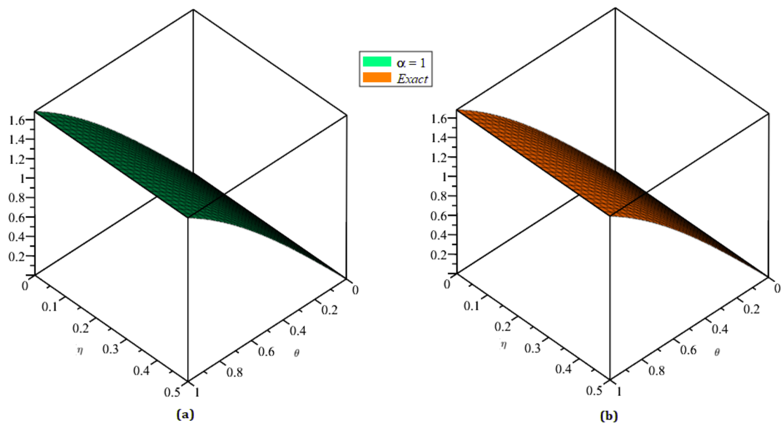

Figure 3. 3D graphs of q-HATM solution at (**a**) $\gamma = 1$ and (**b**) exact Section 4.2.

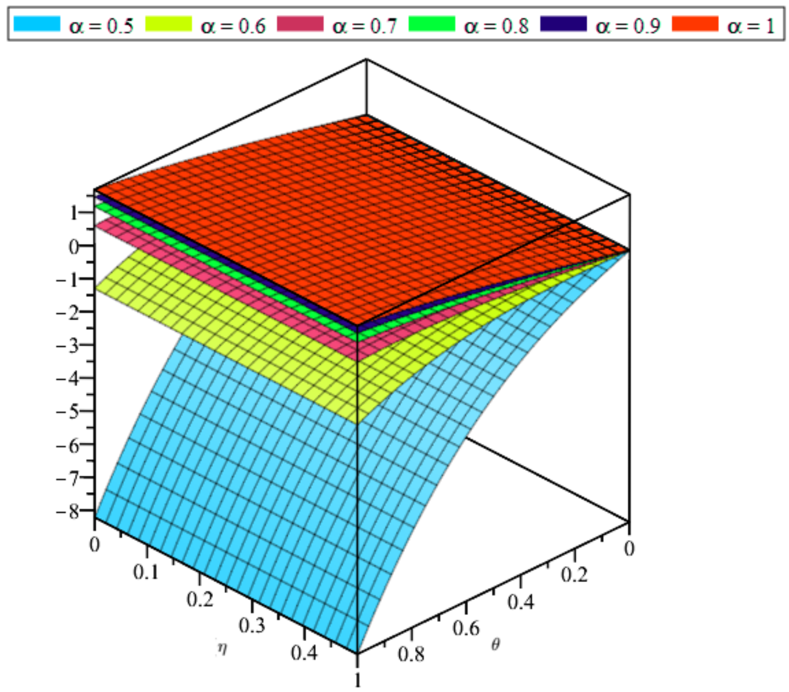

Figure 4. 3D graphs of Section 4.2 for different values of γ.

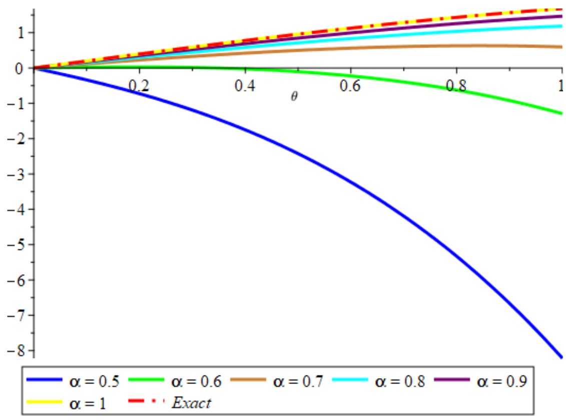

Figure 5. 2D plots of the Section 4.2 for various values of γ and Exact solution.

Table 2. Absolute error (AE) at $n = 1, \hbar = -1$ and various fractional order of Section 4.2.

θ	η	AE ($\gamma = 0.6$)	AE ($\gamma = 0.7$)	AE ($\gamma = 0.9$)	AE ($\gamma = 1$)
0.1	0.1	0.1781757909	0.0847577008	0.0220435308	2.0×10^{-10}
	0.2	0.1781757987	0.0847577046	0.0220435318	2.0×10^{-10}
	0.3	0.1781758810	0.0847577437	0.0220435419	2.0×10^{-10}
	0.4	0.1781762972	0.0847579415	0.0220435931	2.0×10^{-10}
	0.5	0.1781777226	0.0847586190	0.0220437685	2.0×10^{-10}

4.3. Problem

Consider the fractional-order two-dimensional parabolic equation [34]:

$$ {}^{ABC}D_{\theta}^{\gamma+1}\Theta + \left(\frac{\xi+z}{2cos\eta} - 1\right)\frac{\partial^4\Theta}{\partial\eta^4} + \left(\frac{\eta+z}{2cos\xi} - 1\right)\frac{\partial^4\Theta}{\partial\xi^4} + \left(\frac{\xi+\xi}{2cosz} - 1\right)\frac{\partial^4\Theta}{\partial z^4} = 0,\ 0 < \gamma \le 1,\ \theta > 0, \tag{46}$$

with initial conditions

$$\begin{aligned} \Theta(\eta,\xi,z,0) &= \eta+\xi+z-(cos\eta+cos\xi+cosz), \\ \Theta_\theta(\eta,\xi,z,0) &= cos\eta+cos\xi+cosz-(\eta+\xi+z), \end{aligned} \tag{47}$$

with boundary conditions

$$\begin{aligned} &\Theta(0,\xi,z,\theta) = (-1+\xi+z-cos\xi-cosz)e^{-\theta},\ \Theta(\frac{\pi}{3},\xi,z,\theta) = \left(\frac{2\pi-3}{6}+\xi+z-cos\xi-cosz\right)e^{-\theta}, \\ &\Theta(\eta,0,z,\theta) = (\eta-1+z-cos\eta-cosz)e^{-\theta},\ \Theta(\eta,\frac{\pi}{3},z,\theta) = \left(\frac{2\pi-3}{6}+\eta+z-cos\eta-cosz\right)e^{-\theta}, \\ &\Theta(\eta,\xi,0,\theta) = (-1+\eta+\xi-cos\eta-cos\xi)e^{-\theta},\ \Theta(\eta,\xi,\frac{\pi}{3},\theta) = \left(\frac{2\pi-3}{6}+\eta+\xi-cos\eta-cos\xi\right)e^{-\theta}, \\ &\Theta_\eta(0,\xi,z,\theta) = \Theta_\xi(\eta,0,z,\theta) = \Theta_z(\eta,\xi,0,\theta) = e^{-\theta},\ \Theta_\eta(\frac{\pi}{3},\xi,z,\theta) = \Theta_\xi(\eta,\frac{\pi}{3},z,\theta) = \left(\frac{\sqrt{3}+2}{2}\right)e^{-\theta} \\ &\Theta_z(\eta,\xi,\frac{\pi}{3},\theta) = \left(\frac{\sqrt{3}+2}{2}\right)e^{-\theta}. \end{aligned} \tag{48}$$

Using Aboodh tranform on Equation (46) and then applying Equation (47), we obtain

$$\begin{aligned} &\mathcal{A}_\theta[\Theta(\eta,\xi,z,\theta)] - \frac{(\eta+\xi+z-(cos\eta+cos\xi+cosz))}{s} - \frac{(cos\eta+cos\xi+cosz-(\eta+\xi+z))}{s^2} \\ &+ \left(\frac{1-\gamma+\gamma s^{-\gamma}}{N(\gamma)}\right)\mathcal{A}_\theta\left[\left(\frac{\xi+z}{2cos\eta}-1\right)\frac{\partial^4\Theta}{\partial\eta^4} + \left(\frac{\eta+z}{2cos\xi}-1\right)\frac{\partial^4\Theta}{\partial\xi^4} + \left(\frac{\xi+\xi}{2cosz}-1\right)\frac{\partial^4\Theta}{\partial z^4}\right] = 0, \end{aligned} \tag{49}$$

The nonlinear function $\mathcal{N}$ is represented with the aid of future method as below

$$\begin{aligned} \mathcal{N}[\phi(\eta,\xi,z,\theta;q)] =& \mathcal{A}_\theta[\phi(\eta,\xi,z,\theta;q)] - \frac{(\eta+\xi+z-(cos\eta+cos\xi+cosz))}{s} \\ &- \frac{(cos\eta+cos\xi+cosz-(\eta+\xi+z))}{s^2} + \frac{1}{\mathcal{N}[\gamma]}\Big(1-\gamma+\frac{\gamma}{s^{\gamma+1}}\Big)\mathcal{A}_\theta\Big[\left(\frac{\xi+z}{2cos\eta}-1\right) \\ &\times \frac{\partial^4\phi(\eta,\xi,z,\theta;q)}{\partial\eta^4} + \left(\frac{\eta+z}{2cos\xi}-1\right)\frac{\partial^4\phi(\eta,\xi,z,\theta;q)}{\partial\xi^4} + \left(\frac{\xi+\xi}{2cosz}-1\right)\frac{\partial^4\phi(\eta,\xi,z;q)}{\partial z^4}\Big]. \end{aligned} \tag{50}$$

The deformation equation of m-th order with the help of $q-HATM$ at $\mathbb{H}(\eta,\theta) = 1$, is given by

$$\mathcal{A}_\theta[\Theta_m(\eta,\xi,z,\theta) - k_m\Theta_{m-1}(\eta,\xi,z,\theta)] = \hbar\Re_m[\overrightarrow{\Theta}_{m-1}], \tag{51}$$

where

$$\Re_m[\overrightarrow{\Theta}_{m-1}] = \mathcal{A}_\theta[\Theta(\eta,\xi,z,\theta)] - \frac{(\eta+\xi+z-(cos\eta+cos\xi+cosz))}{s} - \frac{(cos\eta+cos\xi+cosz-(\eta+\xi+z))}{s^2} + \left(\frac{1-\gamma+\gamma s^{-\gamma}}{N(\gamma)}\right)\mathcal{A}_\theta\left[\left(\frac{\xi+z}{2cos\eta}-1\right)\frac{\partial^4\Theta}{\partial\eta^4} + \left(\frac{\eta+z}{2cos\xi}-1\right)\frac{\partial^4\Theta}{\partial\xi^4} + \left(\frac{\xi+\xi}{2cosz}-1\right)\frac{\partial^4\Theta}{\partial z^4}\right]. \tag{52}$$

Applying inverse AT on Equation (51), it reduces to

$$\Theta_m(\eta,\xi,z,\theta) = k_m\Theta_{m-1}(\eta,\xi,z,\theta) + \mathcal{A}_\theta^{-1}\left[\hbar\Re_m[\overrightarrow{\Theta}_{m-1}]\right]. \tag{53}$$

Using initial conditions to simplify the above equation, we can find the terms of the series solution as

$$\begin{aligned}
\Theta_0(\eta,\xi,z,\theta) =& (\eta+\xi+z-(cos\eta+cos\xi+cosz)) + (cos\eta+cos\xi+cosz-(\eta+\xi+z))\theta,\\
\Theta_1(\eta,\xi,z,\theta) =& \frac{1}{\mathcal{N}[\gamma]}\Big(\hbar\Big(-\eta-\xi-z+\gamma\Big(\eta+\xi+z+\Big(-\frac{t^{\gamma+2}}{\Gamma(\alpha+3)}+\frac{t^{\gamma+1}}{\Gamma(\alpha+2)}\Big)\\
&\times\big(\cos(\eta)+\cos(\xi)+\cos(z)-\eta-\xi-z\big)+(\theta-1)\big(\cos(\eta)+\cos(\xi)+\cos(z)\big)\\
&-\theta(\eta+\xi+z)\Big)+\theta(\eta+\xi+z)-(\theta-1)\big(\cos(\eta)+\cos(\xi)+\cos(z)\big)\Big)\Big),\\
\Theta_2(\eta,\xi,z,\theta) =& \frac{1}{\mathcal{N}[\gamma]}\Big(\hbar\Big(-\eta-\xi-z+\gamma\Big(\eta+\xi+z+\Big(-\frac{t^{\gamma+2}}{\Gamma(\alpha+3)}+\frac{t^{\gamma+1}}{\Gamma(\alpha+2)}\Big)\\
&\times\big(\cos(\eta)+\cos(\xi)+\cos(z)-\eta-\xi-z\big)+(\theta-1)\big(\cos(\eta)+\cos(\xi)+\cos(z)\big)\\
&-\theta(\eta+\xi+z)\Big)+\theta(\eta+\xi+z)-(\theta-1)\big(\cos(\eta)+\cos(\xi)+\cos(z)\big)\Big)\Big)\\
&+\frac{1}{\mathcal{N}[\gamma]^2}\Big(\hbar^2\Big(\Big(1-\theta+\Big(-1-\frac{t^{\gamma+2}}{\Gamma(\gamma+3)}+\theta+\frac{\theta^{\gamma+1}}{\Gamma(\gamma+2)}\Big)\gamma\Big)\mathcal{N}[\gamma]-1+\theta\\
&+\gamma^2\Big(-1+\frac{\theta^{2\alpha+3}}{\Gamma(2\gamma+4)}+\theta+\frac{2\theta^{\alpha+1}}{\Gamma(\gamma+2)}-\frac{2\theta^{\alpha+2}}{\Gamma(\gamma+3)}-\frac{\theta^{2\alpha+2}}{\Gamma(2\gamma+3)}\Big)\\
&+2\gamma\Big(1+\frac{\theta^{\alpha+2}}{\Gamma(\gamma+3)}-\theta-\frac{\theta^{\alpha+1}}{\Gamma(\gamma+2)}\Big)\Big)\big(\cos(\eta)+\cos(\xi)+\cos(z)-\eta-\xi-z\big)\Big),\\
&\vdots
\end{aligned} \tag{54}$$

The series solution is given by

$$\Theta(\eta,\theta) = \Theta_0(\eta,\theta) + \frac{1}{n}\Theta_1(\eta,\theta) + \frac{1}{n^2}\Theta_2(\eta,\theta) + \frac{1}{n^3}\Theta(\eta,\theta) + \cdots. \tag{55}$$

Putting the values of $\Theta_m(\eta,\theta)$ $(m=0,1,2,\cdots)$ in Equation (55), we have

$$\begin{aligned}\Theta(\eta,\xi,z,\theta) =&(\eta+\xi+z-(cos\eta+cos\xi+cosz))+(cos\eta+cos\xi+cosz-(\eta+\xi+z))\theta,\\&+\frac{1}{n}\Big(\frac{1}{\mathcal{N}[\gamma]}\Big(\hbar\Big(-\eta-\xi-z+\gamma\Big(\eta+\xi+z+\Big(-\frac{t^{\gamma+2}}{\Gamma(\alpha+3)}+\frac{t^{\gamma+1}}{\Gamma(\alpha+2)}\Big)\\&\times\big(\cos(\eta)+\cos(\xi)+\cos(z)-\eta-\xi-z\big)+(\theta-1)\big(\cos(\eta)+\cos(\xi)+\cos(z)\big)\\&-\theta(\eta+\xi+z)\Big)+\theta(\eta+\xi+z)-(\theta-1)\big(\cos(\eta)+\cos(\xi)+\cos(z)\big)\Big)\Big)\Big)\\&+\frac{1}{n^2}\Big(\frac{1}{\mathcal{N}[\gamma]}\Big(\hbar\Big(-\eta-\xi-z+\gamma\Big(\eta+\xi+z+\Big(-\frac{t^{\gamma+2}}{\Gamma(\alpha+3)}+\frac{t^{\gamma+1}}{\Gamma(\alpha+2)}\Big)\\&\times\big(\cos(\eta)+\cos(\xi)+\cos(z)-\eta-\xi-z\big)+(\theta-1)\big(\cos(\eta)+\cos(\xi)+\cos(z)\big)\\&-\theta(\eta+\xi+z)\Big)+\theta(\eta+\xi+z)-(\theta-1)\big(\cos(\eta)+\cos(\xi)+\cos(z)\big)\Big)\Big)\\&+\frac{1}{\mathcal{N}[\gamma]^2}\Big(\hbar^2\Big(\Big(1-\theta+\Big(-1-\frac{t^{\gamma+2}}{\Gamma(\gamma+3)}+\theta+\frac{\theta^{\gamma+1}}{\Gamma(\gamma+2)}\Big)\gamma\Big)\mathcal{N}[\gamma]-1+\theta\\&+\gamma^2\Big(-1+\frac{\theta^{2\alpha+3}}{\Gamma(2\gamma+4)}+\theta+\frac{2\theta^{\alpha+1}}{\Gamma(\gamma+2)}-\frac{2\theta^{\alpha+2}}{\Gamma(\gamma+3)}-\frac{\theta^{2\alpha+2}}{\Gamma(2\gamma+3)}\Big)\\&+2\gamma\Big(1+\frac{\theta^{\alpha+2}}{\Gamma(\gamma+3)}-\theta-\frac{\theta^{\alpha+1}}{\Gamma(\gamma+2)}\Big)\Big)\big(\cos(\eta)+\cos(\xi)+\cos(z)-\eta-\xi-z\big)\Big)\Big)+\cdots.\end{aligned}\tag{56}$$

Substituting $n=1$, $\hbar=-1$ and $\gamma=1$, we obtain

$$\begin{aligned}\Theta(\eta,\xi,z,\theta) =&\eta+\xi+z-\cos(\eta)-\cos(\xi)-\cos(z)+(\cos(\eta)+\cos(\xi)+\cos(z)-\eta-\xi-z)\theta\\&-3(\frac{-1}{6}\theta^3+\frac{1}{2}\theta^2)(\cos(\eta)+\cos(\xi)+\cos(z)-\eta-\xi-z)+2\Big(\frac{-1}{6}\theta^3+\frac{1}{2}\theta^2+\frac{1}{120}\theta^5\\&-\frac{1}{24}\theta^4\Big)\big(\cos(\eta)+\cos(\xi)+\cos(z)-\eta-\xi-z\big)+\Big(\frac{-1}{120}\theta^5+\frac{1}{24}\theta^4-\frac{1}{720}\theta^6+\frac{1}{5040}\theta^7\Big)\\&\times(\cos(\eta)+\cos(\xi)+\cos(z)-\eta-\xi-z)+\cdots,\\\Theta(\eta,\xi,z,\theta) =&\eta+\xi+z-\cos(\eta)-\cos(\xi)-\cos(z)+(\cos(\eta)+\cos(\xi)+\cos(z)-(\eta+\xi+z))\theta\\&+\Big(\frac{-\cos(\eta)}{2}-\frac{\cos(\xi)}{2}-\frac{\cos(z)}{2}+\frac{\eta}{2}+\frac{\xi}{2}+\frac{z}{2}\Big)\theta^2+\Big(\frac{-\cos(\eta)}{6}-\frac{\cos(\xi)}{6}-\frac{\cos(z)}{6}\\&+\frac{\eta}{6}+\frac{\xi}{6}+\frac{z}{6}\Big)\theta^3+\Big(\frac{-\cos(\eta)}{24}-\frac{\cos(\xi)}{24}-\frac{\cos(z)}{24}+\frac{\eta}{24}+\frac{\xi}{24}+\frac{z}{24}\Big)\theta^4\\&+\Big(\frac{-\cos(\eta)}{120}-\frac{\cos(\xi)}{120}-\frac{\cos(z)}{120}+\frac{\eta}{120}+\frac{\xi}{120}+\frac{z}{120}\Big)\theta^5++\Big(\frac{-\cos(\eta)}{720}-\frac{\cos(\xi)}{720}-\frac{\cos(z)}{720}\\&+\frac{\eta}{720}+\frac{\xi}{720}+\frac{z}{720}\Big)\theta^6+\Big(\frac{-\cos(\eta)}{5040}-\frac{\cos(\xi)}{5040}-\frac{\cos(z)}{5040}+\frac{\eta}{5040}+\frac{\xi}{5040}+\frac{z}{5040}\Big)\theta^7+\cdots.\end{aligned}\tag{57}$$

$$\begin{aligned}\Theta(\eta,\xi,z,\theta) =&(\eta+\xi+z-\cos(\eta)-\cos(\xi)-\cos(z))\Big(1-\theta+\frac{\theta^2}{2}-\frac{\theta^3}{6}+\frac{\theta^4}{24}-\frac{\theta^5}{120}+\frac{\theta^6}{720}-\frac{\theta^7}{5040}+\cdots\Big),\\\Theta(\eta,\xi,z,\theta) =&(\eta+\xi+z-\cos(\eta)-\cos(\xi)-\cos(z))exp(-\theta).\end{aligned}\tag{58}$$

which is the exact solution of Section 4.3.

Figure 6, three dimensional graphs of q-HATM solution at (**a**) $\gamma=1$ and (**b**) exact Section 4.3. Figure 7, three dimensional graphs of Section 4.3 for different values of γ. Figure 8, two dimensional plots of the Section 4.3 for various values of γ and Exact solution.

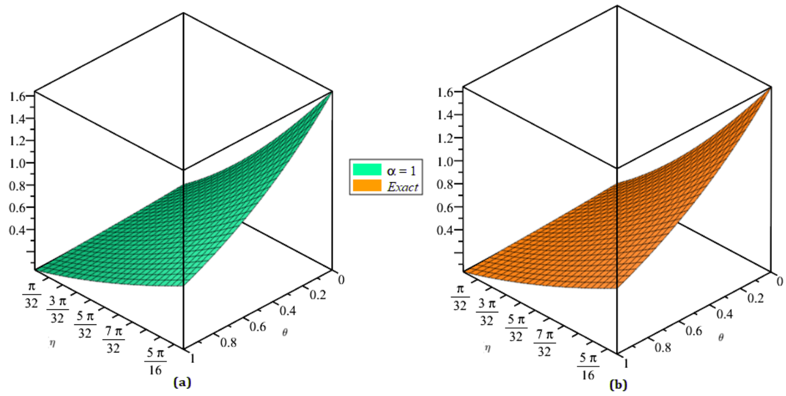

Figure 6. 3D graphs of q-HATM solution at (**a**) $\gamma = 1$ and (**b**) exact Section 4.3.

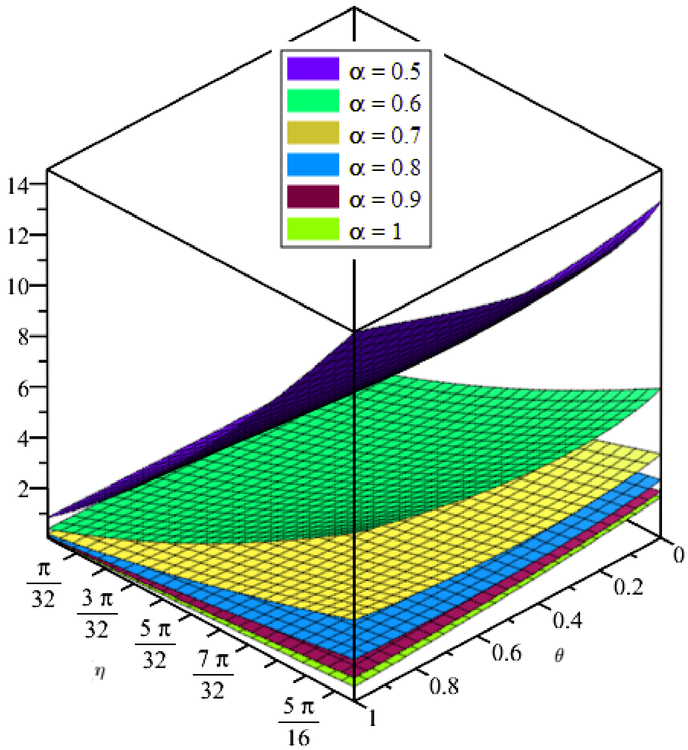

Figure 7. 3D graphs of Section 4.3 for different values of γ.

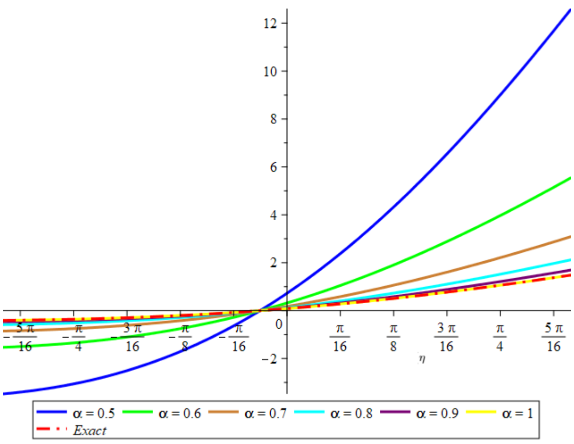

Figure 8. 2D plots of the Section 4.3 for various values of γ and Exact solution.

5. Conclusions

In this study, the q-homotopy analysis transform method is successfully employed to solve fractional parabolic equations numerically. The collected findings indicate the method's dependability and simplicity. The parameter h provided by the proposed approach allows us to regulate the convergence zone of the series solution. As the q-homotopy analysis transform approach does not require linearization, tiny perturbations, or discretization, computations are drastically reduced. Compared to other methods, the q-homotopy analysis transform method is a competent instrument for obtaining numerical solutions to linked nonlinear fractional partial differential equations.

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