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Gutman Connection Index of Graphs under Operations

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Abstract: In the modern era, mathematical modeling consisting of graph theoretic parameters or invariants applied to solve the problems existing in various disciplines of physical sciences like computer sciences, physics, and chemistry. Topological indices (TIs) are one of the graph invariants which are frequently used to identify the different physicochemical and structural properties of molecular graphs. Wiener index is the first distance-based TI that is used to compute the boiling points of the paraffine. For a graph F, the recently developed Gutman Connection (GC) index is defined on all the unordered pairs of vertices as the sum of the multiplications of the connection numbers and the distance between them. In this note, the GC index of the operation-based symmetric networks called by first derived graph $D_1(F)$ (subdivision graph), second derived graph $D_2(F)$ (vertex-semitotal graph), third derived graph $D_3(F)$ (edge-semitotal graph) and fourth derived graph $D_4(F)$ (total graph) are computed in their general expressions consisting of various TIs of the parent graph F, where these operation-based symmetric graphs are obtained by applying the operations of subdivision, vertex semitotal, edge semitotal and the total on the graph F respectively.

Keywords: connection number; connection distance index; Gutman connection index

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1. Introduction

A Topological index (TI) is a function from the set of graphs on the set of real numbers that associates a numeric number to each graph appearing in the set of graphs. If two graphs are isomorphic to each other, then the numeric values of the obtained TIs remain the same. Moreover, the computed values of the TIs predict the various physical and chemical properties of the understudy graphs, see [1]. In the subject of cheminformatics, TIs are also applied in the studies of the quantitative structures property and activity relationships, see [2–4].

In almost mid of the 20th century, Wiener (1947) [5] discovered a close correlation between the boiling point of paraffine (an alkane) and the sum of the distances between all the unordered pairs of vertices. Later on, this first distance-based mathematical expression is called the name of Wiener index. After the passage of a quarter of the century, Gutman and Trinajsti (1972) [6] discovered the first and second Zagreb indices. These degree-based TIs were utilized to determine the total π-electron energy of the molecules. These developments urged other mathematicians and chemists to develop new TIs for the study of the different chemical properties of molecular graphs (structures). In the class of distance-based TIs, the Gutman index and degree distance index are the most important applicable TIs, see [7,8]. For more details on degree and distance-based indices, we refer to [9–11].

In 2018, first Zagreb connection index (ZC_1), second Zagreb connection index (ZC_2) and the modified first Zagreb connection index (ZC_1^*) were restudied by Ali and Trinajstic [12]. Later on, the connection distance (CD) index and Gutman connection (GC) index are studied in [13]. It is important to mention that the International Academy of Mathematical

Chemistry (IAMC) declared that the Zagreb connection indices are better than the ordinary Zagreb indices for many physicochemical properties of chemical compounds existing in the molecular graphs. Moreover, Javaid et al. [14] presented a comparison of correlation coefficients between different TIs and confirmed that connection number-based indices are very useful TIs for the prediction of entropy, acentric factor, enthalpy of vaporization, and standard enthalpy of vaporization.

Four newly derived graphs are introduced by Yan et al. [15] by applying subdivision-related operations on a graph F and obtained first derived graph $D_1(F)$ (subdivided graph), second derived graph $D_2(F)$ (vertex-total graph), third derived graph $D_3(F)$ (edge-total graph) and the fourth derived graph $D_3(F)$ (total graph). Moreover, for the graphs obtained by different operations of graphs, the various TIs such as omega index [16], sombor index [17,18] and Zagreb indices and coindices [19,20] are computed. In particular, Xu et al. [21] and Bahadur et al. [22] computed the degree distance and Gutman indices of these derived graphs respectively. Recently, the connection distance index of derived graphs are computed in [23]. Motivated by this, in the present note, we computed exact and bounded values of the Gutman connection (GC) index on these derived graphs in the form of the various TIs of the parent graphs.

2. Preliminaries

A connected and simple graph F is taken into consideration throughout this article in which, $V(F) = \{a_k : 1 \leq k \leq r\}$ and $E(F) = \{\eta_l : 1 \leq m \leq s\}$ such that $|V(F)| = r$ and $|E(F)| = s$. The most useful definitions are given below

- The minimum number of consecutive edges that occurred between the two nodes a_k and a_m is called the distance between them and is denoted by $\lambda(a_k, a_m)$ for $1 \leq k$, $m \leq r$.
- The cardinality of is $N_F^1(b) = \{a \in V(F), \lambda(a,b) = 1\}$ is called the degree of node b of graph F and is denoted by $\Delta(b)$.
- The cardinality of $N_F^2(b) = \{a \in V(F), \lambda(a,b) = 2\}$ is called the connection number of node b of graph F and is denoted by $\chi(b)$.
- Degree of an edge $\eta_k = a_m a_n$ is denoted by $\Delta(\eta_k)$ and is equal to $\Delta(a_m) + \Delta(a_n) - 2$, where $1 \leq k \leq s$ for some $1 \leq m, n \leq r$.
- The minimum distance between the corresponding nodes of two edges $\eta_k = a_x a_y$ and $\eta_m = a_z a_w$ is called the distance between the two edges and is denoted by $\lambda_G(\eta_k, \eta_m)$ i.e., $\lambda_G(\eta_k, \eta_m) = min\{\lambda_F(a_x, a_z), \lambda_F(a_x, a_w), \lambda_F(a_y, a_z), \lambda_F(a_y, a_w)\}$, where $1 \leq k, m \leq s$ and $1 \leq x, y, z, w \leq r$.
- The distance between one edge $\eta_m = a_x a_y$ and one node a_k is defined as $\lambda_F(a_k, \eta_m) = min\{\lambda_F(a_k, a_x), \lambda_F(a_k, a_y)\}$, where $1 \leq j \leq s$ and $1 \leq i, x, y \leq r$.

More detailed knowledge can be obtained from [24–26]. Some related TIs are the followings:

Definition 1 ([5])**.** *Wiener index of a connected and simple graph F is*

$$W(F) = \frac{1}{2} \sum_{a_k, a_m \in V(F)} \lambda_F(a_k, a_m).$$

Definition 2 ([6])**.** *First and second Zagreb index of a connected and simple graph F are defined as*

$$M_1(F) = \sum_{a_k a_m \in E(F)} [\Delta_F(a_k) + \Delta_F(a_m)] = \sum_{a_k \in V(F)} [\Delta_F(a_k)]^2.$$

and

$$M_2(F) = \sum_{a_k a_m \in E(F)} [\Delta_F(a_k) \Delta_F(a_m)].$$

Definition 3 ([27])**.** *Edge version of Wiener index of a connected and simple graph F is defined as*

$$W_e(F) = \sum_{\{\eta_k,\eta_m\}\subseteq E(F)} [\lambda_F(\eta_k,\eta_m) + 1].$$

Definition 4 ([7])**.** *The degree distance index of a connected and simple graph F is*

$$DD(F) = \frac{1}{2} \sum_{a_k,a_m\in V(F)} \{\lambda_F(a_k,a_m)(\Delta_F(a_k) + \Delta_F(a_m))\}.$$

The degree distance index of P_n is $DD(P_n) = \dfrac{n(n-1)(2n-1)}{3}$.

Definition 5 ([22])**.** *Edge version of degree distance index of a connected and simple graph F is*

$$DD_e(F) = \sum_{\{\eta_k,\eta_m\}\subseteq E(F)} [\lambda_e(\eta_k,\eta_m) + 1][\Delta(\eta_k) + \Delta(\eta_m))].$$

Definition 6 ([8])**.** *Gutman index of a connected and simple graph F is*

$$Gut(F) = \frac{1}{2} \sum_{a_k,a_m\in V(F)} \{\lambda_F(a_k,a_m)(\Delta_F(a_k)\Delta_F(a_m))\}.$$

Definition 7 ([22])**.** *Edge version of Gutman index of a connected and simple graph F is*

$$Gut_e(F) = \sum_{\{\eta_k,\eta_m\}\subseteq E(F)} [\lambda_e(\eta_k,\eta_m) + 1][\Delta(\eta_k)\Delta(\eta_m))].$$

Definition 8 ([13])**.** *Connection Distance (CD) of a connected and simple graph F is*

$$CD(F) = \sum_{\{a_k,a_m\}\subseteq V(F)} \lambda_F(a_k,a_m)[\chi_F(a_k) + \chi_F(a_m)]$$

or

$$CD(F) = \frac{1}{2} \sum_{a_k,a_m\in V(F)} \{\lambda_F(a_k,a_m)(\chi_F(a_k) + \chi_F(a_m))\}.$$

Definition 9 ([13])**.** *Gutman Connection (GC) of a connected and simple graph F is defined as*

$$GC(F) = \sum_{\{a_k,a_m\}\subseteq V(F)} \lambda_F(a_k,a_m)[\chi(a_k)\chi(a_m)]$$

or

$$GC(F) = \frac{1}{2} \sum_{a_k,a_m\in V(F)} \{\lambda_F(a_k,a_m)(\chi_F(a_k)\chi_F(a_m))\}.$$

Gutman index of P_n is $GM(P_n) = \dfrac{(n-1)(2n^2-4n+3)}{3}$.
Edge version of P_n is $Gut_e(P_n) = Gut(P_{n-1}) = \dfrac{(n-2)(2n^2-8n+9)}{3}$
$CD(P_n) = \dfrac{2n^3-6n^2+10n-12}{3}$
$GC(P_n) = \dfrac{2n^3-12n^2+34n-42}{3}$

$$DD(C_n) = GM(C_n) = Gut_e(C_n) = CD(C_n) = GC(C_n) = \begin{cases} \dfrac{n^3}{2} & \text{if } n \text{ is even} \\ \dfrac{n(n^2-1)}{2} & \text{if } n \text{ is odd} \end{cases}$$

Four new graphs were obtained from the four operations D_1, D_2, D_3 and D_4 on the graph F by Yan et al. [15] which are defined as follows:

- First derived graph $D_1(F)$ is established from F when every edge $\eta_k = a_m a_n$ of F is upgraded by a path of length 2 by including a new node c_k in it. The newly included nodes c_k are also called white or new vertices while a_m and a_n are called old/black nodes.
- Second derived graph $D_2(F)$ is established from $D_1(F)$ when a new node c_k is again joined with the end nodes a_m and a_n of the respective edge η_k.
- Third derived graph $D_3(F)$ is established from $D_1(F)$ when two white nodes c_k and c_m are further joined together if their respective edges η_k and η_m have one common end node in graph F.
- Fourth derived graph $D_4(F)$ is established from $D_2(F)$ when two white nodes c_k and c_m are further joined together if their respective edges η_k and η_m have one common end node in graph F.

Faiz Farid et al. [23] derived the relation between the connection numbers of derived graphs and the connection numbers or degrees of graphs in the following lemmas.

Lemma 1 ([23])**.** *Let $D_1(F)$ be first derived graph of connected and simple graph F. Then*
(i) $\chi_{D_1(F)}(a_k) = \Delta_{(F)}(a_k)$ and
(ii) $\chi_{D_1(F)}(c_k) = \Delta_{(F)}(a_m) + \Delta_{(F)}(a_n) - 2 = \Delta(\eta_k)$ where c_k is a white node with respective edge $\eta_k = a_m a_n$.

Lemma 2 ([23])**.** *Let $D_2(F)$ be second derived graph of connected and simple graph F and*
(a) If F is a $\{C_3, C_4\}-$ free graph, then
(i) $\chi_{D_2(F)}(a_k) = 2\chi_F(a_k)$ and
(ii) $\chi_{D_2(F)}(c_k) = 2[\Delta_F(a_m) + \Delta_F(a_n)] - 4 = 2[\Delta_F(\eta_k)]$
(b) If F is a $\{C_3, C_4\}-$ graph, then
(i) $\chi_{D_2(F)}(a_k) \leq 2\chi_F(a_k) + p$, where $p = max\{p_k\}$ and p_k are number of C_3 and C_4 cycles joined with a_k in F
(ii) $\chi_{D_2(F)}(c_k) \leq 2[\Delta_{(F)}(a_m) + \Delta_{(F)}(a_n)] - 4 - q = 2[\Delta_{(F)}(\eta_k)] - q$, where $q = max\{q_k\}$ and q_k are number of C_3 cycles joined with c_k in F

Lemma 3 ([23])**.** *Let $D_3(F)$ be third derived graph of connected and simple graph F and*
(a) If F is a $\{C_3, C_4\}-$ free graph, then
(i) $\chi_{D_3(F)}(a_i) = \Delta_{(F)}(a_k) + \chi_{(F)}(a_k)$ and
(ii) $\chi_{D_3(F)}(c_k) = \chi_{(F)}(d_k) + \chi_{(F)}(e_k)$.
(b) If F is a $\{C_3, C_4\}-$ graph, then
(i) $\chi_{D_3(F)}(a_k) \leq \Delta_{(F)}(a_k) + \chi_{(F)}(a_k) + p$ where $p = max\{p_k\}$ and p_k is the number of C_3 and C_4 cycles joined with vertex a_i .
(ii) $\chi_{D_3(F)}(c_k) \leq \chi_{(F)}(d_k) + \chi_{(F)}(e_k) + q$ where $q = max\{q_k\}$ and q_k is the number of C_3 cycles in graph F joined with edge η_k .

Lemma 4 ([23])**.** *Let $D_4(F)$ be fourth derived graph of connected and simple graph F and*
(a) If F is a $\{C_3, C_4\}-$ free graph, then
(i) $\chi_{D_4(F)}(a_k) = 2\chi_{(F)}(a_k)$ and
(ii) $\chi_{D_4(F)}(c_k) = \chi_{(F)}(a_m) + \chi_{(F)}(a_n)$
(b) If F is a $\{C_3, C_4\}-$ graph, then
(i) $\chi_{D_4(F)}(a_k) \leq 2\chi_{(F)}(a_k) + p$ where $p = max\{p_k\}$ and vertex a_k is connected with p_k number of C_3 and C_4 cycles and
(ii) $\chi_{D_4(F)}(c_k) \leq \chi_{(F)}(a_m) + \chi_{(F)}(a_n) + q$ where $q = max\{q_k\}$ and edge η_k is connected with the q_k number of C_3 cycles in graph F

3. Mian Results

This section covers the main results of the Gutman connection index on the four types of derived graphs.

Theorem 1. *Let $D_1(F)$ be first derived graph of connected and simple graph F, then*

$$GC(D_1(F)) = 2Gut(F) + 2Gut_e(F) + m(M_1 - 2m) + \sum_{k=1}^{r}\sum_{m=1}^{s}[\Delta_(a_k)\Delta_F(\eta_m)]\lambda_G(a_k, \eta_m)$$

Proof. $\chi_{D_1(F)}(a_k) = \Delta_F(a_k)$ and $\chi_{D_1(F)}(c_k) = \Delta_F(a_m) + \Delta_F(a_n) - 2 = \Delta_{D_1(F)}(\eta_i)$

$$\lambda_{D_1(F)}(a_k, a_m) = 2\lambda_F(a_k, a_m)$$

$$\lambda_{D_1(F)}(c_k, c_m) = 2[\lambda_F(\eta_k, \eta_m) + 1]$$

$$\lambda_{D_1(F)}(a_k, c_m) = 2\lambda_F(a_k, \eta_m) + 1$$

$$GC(D_1(F)) = \sum_{\{a_k,a_m\}\subseteq V(F)}[\chi_{D_1(F)}(a_k)\chi_{D_1(F)}(a_m)]\lambda_{D_1(F)}(a_k, a_m).$$

$$= \frac{1}{2}\sum_{k,m=1}^{r}[\chi_{D_1(F)}(a_k)\chi_{D_1(F)}(a_m)]\lambda_{D_1(F)}(a_k, a_m) + \frac{1}{2}\sum_{k,m=1}^{s}[\chi_{D_1(F)}(c_k)\chi_{D_1(F)}(c_m)]\lambda_{D_1(F)}(c_k, c_m)$$

$$+ \frac{1}{2}\sum_{k=1}^{r}\sum_{m=1}^{s}[\chi_{D_1(F)}(a_k)\chi_{D_1(F)}(c_m)]\lambda_{D_1(F)}(a_k, c_m)$$

$$= \frac{1}{2}\sum_{k,m=1}^{r}[\Delta_F(a_k)\Delta_F(a_m)]2\lambda_F(a_k, a_m) + \frac{1}{2}\sum_{k,m=1}^{s}[\Delta_F(\eta_k)\Delta_F(\eta_m)]2[\lambda_F(\eta_k, \eta_m) + 1]$$

$$+ \frac{1}{2}\sum_{k=1}^{r}\sum_{m=1}^{s}[\Delta_F(a_k)\Delta_F(\eta_m)][2\lambda_F(a_k, \eta_m) + 1]$$

$$= 2Gut(F) + 2Gut_e(F) + \frac{1}{2}\sum_{k=1}^{r}\sum_{m=1}^{s}\Delta_F(a_k)\Delta_F(\eta_m)$$

$$+ \sum_{k=1}^{r}\sum_{m=1}^{s}[\Delta_F(a_k)\Delta_F(\eta_m)]\lambda_F(a_k, \eta_m)$$

$$= 2Gut(F) + 2Gut_e(F) + \frac{1}{2}(\sum_{k=1}^{r}[\Delta_F(a_k)])(\sum_{m=1}^{s}[\Delta_F(\eta_m)])$$

$$+ \sum_{k=1}^{r}\sum_{m=1}^{s}[\Delta_F(a_k)\Delta_F(\eta_m)]\lambda_F(a_k, \eta_m)$$

$$= 2Gut(F) + 2Gut_e(F) + s(M_1 - 2s) + \sum_{k=1}^{r}\sum_{m=1}^{s}[\Delta_F(a_k)\Delta_F(\eta_m)]\lambda_F(a_k, \eta_m)$$

□

Theorem 2. *Let $D_2(F)$ be second derived graph of connected and simple graph F and*

$$(a) GC(D_2(F)) \leq 4CD(F) + 2pCD(F) + p^2W(F) + 4Gut_e(F) + 2(M_1 - 2s)^2$$

$$+ 2\sum_{k=1}^{r}\sum_{m=1}^{s}\chi_F(a_k)\Delta_F(\eta_m)\lambda_F(a_k, \eta_m) + r\sum_{k=1}^{r}\sum_{m=1}^{s}\Delta_F(\eta_m)\lambda_F(a_k, \eta_m)$$

$$+2(M_1-2s)(\sum_{k=1}^{r}\chi_F(a_k))+rp(M_1-2s)$$

$$(b)GC(D_2(F))\geq 4GC(F)+4Gut_e(F)+2(M_1-2s)^2-2qDD_e(F)-2sq(M_1-2s)$$
$$+q^2W_e(F)+\frac{q^2s^2}{2}+2\sum_{k=1}^{r}\sum_{m=1}^{s}[\chi_F(a_k)\Delta_F(\eta_m)]\lambda_F(a_k,\eta_m)$$
$$+2(M_1-2s)\sum_{k=1}^{r}\chi_F(a_k)-s\sum_{k=1}^{r}\sum_{m=1}^{s}[\chi_F(a_k)(\lambda_F(a_k,\eta_m)+1)$$

Proof. (a) For upper bounds, $\chi_{D_2(F)}(a_k)\geq 2\chi_{(F)}(a_k)+p$ and $\chi_{D_2(F)}(c_k)=\Delta_{(F)}(a_m)+\Delta_{(F)}(a_n)-2=\Delta_F(\eta_i)$
Also $\lambda_{D_2(F)}(a_k,a_m)=\lambda_F(a_k,a_m)$, for $a_k,a_m\in V(F)$
$\lambda_{D_2(F)}(c_k,c_m)=\lambda_F(\eta_k,\eta_m)+2$, for $\eta_k,\eta_m\in E(F)$
$\lambda_{D_2(F)}(a_k,c_m)=\lambda_F(a_k,\eta_m)+1$, for $a_k\in V(F)$ and $\eta_m\in E(F)$

$$GC(D_2(F))=\sum_{\{a_k,a_m\}\subseteq V(G)}\lambda_{D_2(F)}(a_k,a_m)[\chi_{D_2(F)}(a_k)\chi_{D_2(F)}(a_m)].$$

$$=\frac{1}{2}\sum_{k,m=1}^{r}[\chi_{D_2(F)}(a_k)\chi_{D_2(F)}(a_m)]\lambda_{D_2}(a_k,a_m)+\frac{1}{2}\sum_{k,m=1}^{s}[\chi_{D_2(F)}(c_k)\chi_{D_2(F)}(c_m)]\lambda_{D_2(F)}(c_k,c_m)$$
$$+\frac{1}{2}\sum_{k=1}^{r}\sum_{m=1}^{s}[\chi_{D_2(F)}(a_k)][\chi_{D_2(F)}(c_m)]\lambda_{D_2(F)}(a_k,c_m)$$
$$\leq\sum\frac{1}{2}\sum_{k,m=1}^{r}[2\chi_F(a_k)+p][2\chi_F(a_m)+p]\lambda_F(a_k,a_m)+\frac{1}{2}\sum_{k,m=1}^{s}[2\Delta_F(\eta_k)][2\Delta_F(\eta_m)][\lambda_F(\eta_k,\eta_m)+2]$$
$$+\frac{1}{2}\sum_{k=1}^{r}\sum_{m=1}^{s}[2\chi_F(a_k)+p][2\Delta_F(\eta_m)][\lambda_F(a_k,\eta_m)+1]$$
$$=2\sum_{k,m=1}^{r}[\chi_F(a_k)\chi_F(a_m)]\lambda_F(a_k,a_m)+p\sum_{k,m=1}^{r}[\chi_F(a_k)+\chi_F(a_m)]\lambda_F(a_k,a_m)$$
$$+\frac{p^2}{2}\sum_{k,m=1}^{r}\lambda_F(a_k,a_m)+2\sum_{k,m=1}^{s}[\Delta_F(\eta_k)\Delta_F(\eta_m)](\lambda_F(\eta_k,\eta_m)+1)$$
$$+2\sum_{k,m=1}^{s}[\Delta_F(\eta_k)\Delta_F(\eta_m)]+2\sum_{k=1}^{r}\sum_{m=1}^{s}\chi_F(a_k)\Delta_F(\eta_m)\lambda_F(a_k,\eta_m)+p\sum_{k=1}^{r}\sum_{m=1}^{s}\Delta_F(\eta_m)\lambda_F(a_k,\eta_m)$$
$$+2\sum_{k=1}^{r}\sum_{m=1}^{s}\chi_F(a_k)\Delta_F(\eta_m)+p\sum_{k=1}^{r}\sum_{m=1}^{s}\Delta_F(\eta_m)$$
$$=4GC(F)+2pCD(F)+p^2W(F)+4Gut_e(F)+2(M_1-2s)^2$$
$$+2\sum_{k=1}^{r}\sum_{m=1}^{s}\chi_F(a_k)\Delta_F(\eta_m)\lambda_F(a_k,\eta_m)+p\sum_{k=1}^{r}\sum_{m=1}^{s}\Delta_F(\eta_k)\lambda_F(a_k,\eta_m)$$
$$+2(\sum_{k=1}^{r}\chi_F(a_k)(\sum_{m=1}^{s}\Delta_F\eta_m))+pr(M_1-2s)$$
$$=4GC(F)+2pCD(F)+p^2W(F)+4Gut_e(F)+2(M_1-2s)^2+$$
$$+2\sum_{k=1}^{r}\sum_{m=1}^{s}\chi_F(a_k)\Delta_F(\eta_m)\lambda_F(a_k,\eta_m)+p\sum_{k=1}^{r}\sum_{m=1}^{s}\Delta_F(\eta_m)\lambda_F(a_k,\eta_m)$$

$$+2(M_1-2s)(\sum_{k=1}^{r}\chi_F(a_k))+pr(M_1-2s)$$

(b) For lower bounds, $\chi_{D_2(F)}(a_k)=2\chi_{(F)}(a_k)$ and $\chi_{D_2(F)}(c_k)=[\Delta_{(F)}(d_k)+\Delta_{(F)}(e_k)-2]-q=\Delta_F(\eta_k)-q$

$$GC(D_2(F))=\sum_{\{a,b\}\subseteq V(F)}[\chi_{D_2(F)}(a)\chi_{D_2(F)}(b)]\lambda_{D_2(F)}(a,b).$$

$$=\frac{1}{2}\sum_{k,m=1}^{r}[\chi_{D_2(F)}(a_k)\chi_{D_2(F)}(a_m)]\lambda_{D_2(F)}(a_k,a_m)+\frac{1}{2}\sum_{k,m=1}^{s}[\chi_{D_2(F)}(c_k)\chi_{D_2(F)}(c_m)][\lambda_{D_2(F)}(c_k,c_m)+1]$$

$$+\frac{1}{2}\sum_{k=1}^{r}\sum_{m=1}^{s}[\chi_{D_2(F)}(a_k)\chi_{D_2(F)}(c_m)]\lambda_{D_2(F)}(a_k,c_m)$$

$$\geq\frac{1}{2}\sum_{k,m=1}^{r}[2\chi_F(a_k)][2\chi_F(a_m)]\lambda_F(a_k,a_m)+\frac{1}{2}\sum_{k,m=1}^{s}[2\Delta_F(\eta_k)-q][2\Delta_F(\eta_m)-q][\lambda_F(\eta_k,\eta_m)+2]$$

$$+\frac{1}{2}\sum_{k=1}^{r}\sum_{m=1}^{s}[2\chi_F(a_k)][2\Delta_F(\eta_m)-q][\lambda_F(a_k,\eta_m)+1]$$

$$=2\sum_{k,m=1}^{r}[\chi_F(a_k)\chi_F(a_m)]\lambda_F(a_k,a_m)+2\sum_{k,m=1}^{s}[\Delta_F(\eta_k)\Delta_F(\eta_m)][\lambda_F(\eta_k,\eta_m)+1]$$

$$+2\sum_{k,m=1}^{s}[\Delta_F(\eta_k)\Delta_F(\eta_m)]-q\sum_{k,m=1}^{s}[\Delta_F(\eta_k)+\Delta_F(\eta_m)](\lambda_F(\eta_k,\eta_m)+1)-q\sum_{k,m=1}^{s}[\Delta_F(\eta_k)+\Delta_F(\eta_m)]$$

$$+\frac{q^2}{2}\sum_{k,m=1}^{s}(\lambda_F(\eta_k,\eta_m)+1)+\frac{q^2}{2}\sum_{k,m=1}^{s}+2\sum_{k=1}^{r}\sum_{m=1}^{s}[\chi_F(a_k)\Delta_G(\eta_m)]\lambda_F(a_k,\eta_m)$$

$$+2\sum_{k=1}^{r}\sum_{m=1}^{s}\chi_F(a_k)\Delta_F(\eta_m)-q\sum_{k=1}^{r}\sum_{m=1}^{s}[\chi_F(a_k)(\lambda_F(a_k,\eta_m)+1)$$

$$=4GC(F)+4Gut_e(F)+2(M_1-2s)^2-2qDD_e(F)-2sq(M_1-2s)$$

$$+q^2W_e(F)+\frac{q^2s^2}{2}+2\sum_{k=1}^{r}\sum_{m=1}^{s}[\chi_F(a_k)\Delta_F(\eta_m)]\lambda_F(a_k,\eta_m)$$

$$+2(M_1-2s)\sum_{k=1}^{r}\chi_F(a_k)-q\sum_{k=1}^{r}\sum_{m=1}^{s}[\chi_F(a_k)(\lambda_F(a_k,\eta_m)+1)$$

□

Corollary 1. *If F be a* $\{C_3,C_4\}-$ *free graph,then*

$$GC(D_2(F))=4GC(F)+4Gut_e(F)+2(M_1-2s)^2+2\sum_{k=1}^{r}\sum_{m=1}^{s}[\chi_F(a_k)\Delta_F(\eta_m)]\lambda_F(a_k,\eta_m)$$

$$+2(M_1-2s)\sum_{k=1}^{r}\chi_F(a_k)$$

Proof. By taking $r=0$ and $s=0$, we can get the required result. □

Theorem 3. *Let* $D_3(F)$ *be third derived graph of connected and simple graph F and*

$$GC(D_3(F))\leq Gut(F)+GC(F)+p^2W(F)+pDD(F)+pCD(F)+2r^2s^2$$

$$+\frac{1}{2}\sum_{k,m=1}^{r}[\Delta_F(a_k)\chi_F(a_m)+\chi_F(a_k)\Delta_F(a_m)][\lambda_F(a_k,a_m)]+2prs$$

$$+\frac{1}{2}\sum_{k,m=1}^{r}[\chi_F(a_k)\chi_F(a_m)]+\frac{p}{2}\sum_{k,m=1}^{r}[\chi_F(a_k)+\chi_F(a_m)]+\frac{p^2r^2}{2}$$

$$+\frac{1}{2}\sum_{k,m=1}^{r}[\Delta_F(a_k)\chi_F(a_m)+\chi_F(a_k)\Delta_F(a_m)]+q^2W_e(F)$$

$$+\frac{q}{2}\sum_{k,m=1}^{s}[\chi_F(d_k)+\chi_F(e_k)+\chi_F(d_m)+\chi_F(e_m)][\lambda_F(\eta_k,\eta_m)+1]$$

$$+\frac{1}{2}\sum_{k,m=1}^{s}[\chi_F(d_k)+\chi_F(e_m)][\chi_F(d_k)+\chi_F(e_m)][\lambda_F(\eta_k,\eta_m)+1]$$

$$+\frac{1}{2}\sum_{k=1}^{r}\sum_{m=1}^{s}[\Delta_F(a_k)+\chi_F(a_k)+p][\chi_F(d_m)+\chi_F(e_m)+q][\lambda_F(a_k,\eta_m)+1]$$

Proof. $\chi_{D_3(F)}(a_k)\leq\Delta_F(a_k)+\chi_F(a_k)+p$ and $\chi_{D_3(F)}(c_k)\leq\chi_{(F)}(d_k)+\chi_{(F)}(e_k)+q$

$\lambda_{D_3(F)}(a,b)=\lambda_F(a,b)+1$

$\lambda_{D_3(F)}(c_k,c_m)=\lambda_F(\eta_k,\eta_m)+1$

$\lambda_{D_3(F)}(a_k,c_m)=\lambda_F(a_k,\eta_m)+1$

$$GC(D_3(F))=\sum_{\{a,b\}\subseteq V(F)}[\chi_{D_3(F)}(a)\chi_{D_3(F)}(b)]\lambda_{D_3}(a,b).$$

$$=\frac{1}{2}\sum_{k,m=1}^{r}[\chi_{D_3(F)}(a_k)\chi_{D_3(F)}(a_m)]\lambda_{D_3(F)}(a_k,a_m)+\frac{1}{2}\sum_{k,m=1}^{s}[\chi_(c_k)\chi_(c_m)]\lambda_{D_3(F)}(c_k,c_m)$$

$$+\frac{1}{2}\sum_{k=1}^{r}\sum_{m=1}^{s}[\chi_{D_3}(a_k)\chi_{D_3}(c_m)]\lambda_{D_3}(a_k,c_m)$$

$$\leq\frac{1}{2}\sum_{k,m=1}^{r}[\Delta_F(a_k)+\chi_F(a_k)+p][\Delta_F(a_m)+\chi_F(a_m)+p][\lambda_F(a_k,a_)+1]$$

$$+\frac{1}{2}\sum_{k,m=1}^{s}[\chi_F(d_k)+\chi_F(e_k)+q][\chi_F(d_m)+\chi_F(e_m)+q][\lambda_F(\eta_k,\eta_m)+1]$$

$$+\frac{1}{2}\sum_{k=1}^{r}\sum_{m=1}^{s}[\Delta_F(a_k)+\chi_F(a_k)+p][\chi_F(d_m)+\chi_F(e_m)+q][\lambda_F(a_k,\eta_m)+1]$$

$$=\frac{1}{2}\sum_{k,m=1}^{r}[\Delta_F(a_k)\Delta_F(a_m)][\lambda_F(a_k,a_m)]+\frac{1}{2}\sum_{k,m=1}^{r}[\chi_F(a_k)\chi_F(a_m)][\lambda_F(a_k,a_m)]$$

$$+\frac{1}{2}\sum_{k,m=1}^{r}[\Delta_F(a_k)\chi_F(a_m)+\chi_F(a_k)\Delta_F(a_m)][\lambda_F(a_k,a_m)]+\frac{p^2}{2}\sum_{k,m=1}^{r}[\lambda_F(a_k,a_m)]$$

$$+\frac{p}{2}\sum_{k,m=1}^{r}[\Delta_F(a_k)+\Delta_F(a_m)][\lambda_F(a_k,a_m)]+\frac{p}{2}\sum_{k,m=1}^{r}[\chi_F(a_k)+\chi_F(a_m)][\lambda_F(a_k,a_m)]$$

$$+\frac{1}{2}\sum_{k,m=1}^{r}[\Delta_F(a_k)\Delta_F(a_m)]+\frac{p}{2}\sum_{k,m=1}^{r}[\Delta_F(a_k)+\Delta_F(a_m)]+\frac{1}{2}\sum_{k,m=1}^{r}[\chi_F(a_k)\chi_F(a_m)]$$

$$+\frac{1}{2}\sum_{k,m=1}^{r}[\Delta_F(a_k)\chi_F(a_m)+\chi_F(a_k)\Delta_F(a_m)]+\frac{p}{2}\sum_{k,m=1}^{r}[\chi_F(a_k)+\chi_F(a_m)]+\frac{p^2}{2}\sum_{k,m=1}^{r}$$

$$+\frac{q^2}{2}\sum_{k,m=1}^{s}[\lambda_F(\eta_k,\eta_m)+1]+\frac{q}{2}\sum_{k,m=1}^{s}[\chi_F(d_k)+\chi_F(e_k)+\chi_F(d_m)+\chi_F(e_m)][\lambda_F(\eta_k,\eta_m)+1]$$

$$+\frac{1}{2}\sum_{k,m=1}^{s}[\chi_F(d_k)+\chi_F(e_k)][\chi_F(d_m)+\chi_F(e_m)][\lambda_F(\eta_k,\eta_m)+1]$$

$$+\frac{1}{2}\sum_{k=1}^{r}\sum_{m=1}^{s}[\Delta_F(a_k)+\chi_F(a_k)+p][\chi_F(d_m)+\chi_F(e_m)+q][\lambda_F(a_k,\eta_m)+1]$$

$$= Gut(F)+GC(F)+p^2W(F)+pDD(F)+pCD(F)+2r^2s^2$$

$$+\frac{1}{2}\sum_{k,m=1}^{r}[\Delta_F(a_k)\chi_F(a_m)+\chi_F(a_k)\Delta_F(a_m)][\lambda_F(a_k,a_m)]+2prs$$

$$+\frac{1}{2}\sum_{k,m=1}^{r}[\chi_F(a_k)\chi_F(a_m)]+\frac{p}{2}\sum_{k,m=1}^{r}[\chi_F(a_k)+\chi_F(a_m)]+\frac{p^2r^2}{2}$$

$$+\frac{1}{2}\sum_{k,m=1}^{r}[\Delta_F(a_k)\chi_F(a_m)+\chi_F(a_k)\Delta_F(a_m)]+q^2W_e(F)$$

$$+\frac{q}{2}\sum_{k,m=1}^{s}[\chi_F(d_k)+\chi_F(e_k)+\chi_F(d_m)+\chi_F(e_m)][\lambda_F(\eta_k,\eta_m)+1]$$

$$+\frac{1}{2}\sum_{k,m=1}^{s}[\chi_F(d_k)+\chi_F(e_k)][\chi_F(d_m)+\chi_F(e_m)][\lambda_F(\eta_k,\eta_m)+1]$$

$$+\frac{1}{2}\sum_{k=1}^{r}\sum_{m=1}^{s}[\Delta_F(a_k)+\chi_F(a_k)+p][\chi_F(d_m)+\chi_F(e_m)+q][\lambda_F(a_k,\eta_m)+1]$$

□

Corollary 2. *If F is a $\{C_3, C_4\}-$ free graph, then*

$$GC(D_3(F)=Gut(F)+GC(F)+\frac{1}{2}\sum_{k,m=1}^{r}[\Delta_F(a_k)\chi_F(a_m)+\chi_F(a_k)\Delta_F(a_m)][\lambda_F(a_k,a_m)]$$

$$+\frac{1}{2}\sum_{k,m=1}^{r}[\chi_F(a_k)\chi_F(a_m)]+\frac{1}{2}\sum_{k,m=1}^{r}[\Delta_F(a_k)\chi_F(a_m)+\chi_F(a_k)\Delta_F(a_m)]$$

$$+2r^2s^2+\frac{1}{2}\sum_{k,m=1}^{s}[\chi_F(d_k)+\chi_F(e_k)][\chi_F(d_m)+\chi_F(e_m)][\lambda_F(\eta_k,\eta_m)+1]$$

$$+\frac{1}{2}\sum_{k=1}^{r}\sum_{m=1}^{s}[\Delta_F(a_k)+\chi_F(a_k)][\chi_F(d_m)+\chi_F(e_m)][\lambda_F(a_k,\eta_m)+1]$$

Proof. By taking $p=0$ and $q=0$, we can get the required result.
□

Theorem 4. *Let $D_4(F)$ be fourth derived graph of connected and simple graph F and*

$$GC(D_4(F))\leq 4GC(F)+rCD(F)+p^2W(F)+q^2W_e(F)$$

$$+\frac{1}{2}\sum_{k,m=1}^{s}[\chi_F(d_m)+\chi_F(e_m)][\chi_F(d_m)+\chi_F(e_m)][\lambda_F(\eta_k,\eta_m)+1]$$

$$+\frac{q}{2}\sum_{k,m=1}^{s}[\chi_F(d_m)+\chi_F(e_m)+\chi_F(d_m)+\chi_F(e_m)][\lambda_F(\eta_k,\eta_m)+1]$$

$$+\frac{1}{2}\sum_{k=1}^{r}\sum_{m=1}^{s}[2\chi_F(a_k)+p][\chi_F(d_m)+\chi_F(e_m)+q][\lambda_F(a_k,\eta_m)+1]$$

Proof. $\chi_{D_4(F)}(a_k)=2\chi_F(a_k)+p$ and $\chi_{D_4(F)}(c_k)=\chi_{(F)}(d_k)+\chi_{(F)}(e_k)+q=\chi_{D_4(F)}(\eta_k)$ Also

$$\lambda_{D_4(F)}(a_k,a_m)=\lambda_F(a_k,a_m)$$

$$\lambda_{D_4(F)}(c_k,c_m)=\lambda_F(\eta_k,\eta_m)+1$$

$$\lambda_{D_4(F)}(a_k,c_m)=\lambda_F(a_k,\eta_m)+1$$

$$GC(D_4(F))=\sum_{\{a,b\}\subseteq V(F)}\lambda_{D_4}(a,b)[\chi_{D_4}(a)\chi_{D_3}(b)].$$

$$=\frac{1}{2}\sum_{k,m=1}^{r}[\chi_{D_4(F)}(a_k)\chi_{D_4(F)}(a_m)]\lambda_{D_4(F)}(a_k,a_m)+\frac{1}{2}\sum_{k,m=1}^{s}[\chi_{D_4(F)}(c_k)\chi_{D_4(F)}(c_m)]\lambda_{D_4(F)}(c_k,c_m)$$

$$+\frac{1}{2}\sum_{i=1}^{r}\sum_{m=1}^{s}[\chi_{D_4(F)}(a_k)+\chi_{D_4(F)}(c_m)]\lambda_{D_4(F)}(a_k,c_m)$$

$$\leq\frac{1}{2}\sum_{k,m=1}^{r}[2\chi_F(a_k)+p][2\chi_F(a_m)+p][\lambda_F(a_k,a_m)]$$

$$+\frac{1}{2}\sum_{k,m=1}^{s}[\chi_F(d_k)+\chi_F(e_k)+q][\chi_F(d_m)+\chi_F(e_m)+q][\lambda_F(\eta_k,\eta_m)+1]$$

$$+\frac{1}{2}\sum_{k=1}^{r}\sum_{m=1}^{s}[2\chi_F(a_k)+p][\chi_F(d_m)+\chi_F(e_m)+q][\lambda_F(a_k,\eta_m)+1]$$

$$=2\sum_{k,m=1}^{r}[\chi_F(a_k)\chi_F(a_m)][\lambda_F(a_k,a_m)]+p\sum_{k,m=1}^{r}[\chi_F(a_k)+\chi_F(a_m)][\lambda_F(a_k,a_m)]+\frac{p^2}{2}\sum_{k,m=1}^{r}[\lambda_F(a_k,a_m)]$$

$$+\frac{q^2}{2}\sum_{k,m=1}^{s}[\lambda_F(\eta_k,\eta_m)+1]+\frac{q}{2}\sum_{k,m=1}^{s}[\chi_F(d_k)+\chi_F(e_k)+\chi_F(d_m)+\chi_F(e_m)][\lambda_F(\eta_k,\eta_m)+1]$$

$$+\frac{1}{2}\sum_{k,m=1}^{s}[\chi_F(d_k)+\chi_F(e_k)][\chi_F(d_m)+\chi_F(e_m)][\lambda_F(\eta_k,\eta_m)+1]$$

$$+\frac{1}{2}\sum_{k=1}^{r}\sum_{m=1}^{s}[2\chi_F(a_k)+r][\chi_F(d_m)+\chi_F(e_m)+s][\lambda_F(a_k,\eta_m)+1]$$

$$=4GC(F)+2pCD(F)+r^2W(F)+p^2W_e(F)$$

$$+\frac{1}{2}\sum_{k,m=1}^{s}[\chi_F(d_k)+\chi_F(e_k)][\chi_F(d_m)+\chi_F(e_m)][\lambda_F(\eta_k,\eta_m)+1]$$

$$+\frac{q}{2}\sum_{k,m=1}^{s}[\chi_F(d_k)+\chi_F(e_k)+\chi_F(d_m)+\chi_F(e_m)][\lambda_F(\eta_k,\eta_m)+1]$$

$$+\frac{1}{2}\sum_{k=1}^{r}\sum_{m=1}^{s}[2\chi_F(a_k)+p][\chi_F(d_m)+\chi_F(e_m)+q][\lambda_F(a_k,\eta_m)+1]$$

□

Corollary 3. *If F is a* $\{C_3, C_4\}-$ *free graph, then*

$$GC(D_4(F)) = 4GC(F) + \frac{1}{2}\sum_{k,m=1}^{s}[\chi_F(d_k)+\chi_F(e_k)][\chi_F(d_m)+\chi_F(e_m)][\lambda_F(\eta_k,\eta_m)+1]$$

$$+\frac{1}{2}\sum_{k=1}^{r}\sum_{m=1}^{s}[2\chi_F(a_k)][\chi_F(d_m)+\chi_F(e_m)][\lambda_F(a_k,\eta_m)+1]$$

Proof. By taking $r = 0$ and $s = 0$, we can get the required result. □

4. Conclusions

In this paper, we studied the four types of derived graphs subdivision graph, vertex-semitotal graph, edge-semitotal graph, and total graph with the help of the Gutman connection index, where the derived are obtained under the four different operations of subdivision. All the obtained results are expressed in the terms of the different TIs of the parent graph. Moreover, the results are also deduced for the derived graphs being free from the cycles of the order of three and four. However, the problem is still open to computing the Gutman connection index for the derived graphs obtained by the various operations of the product of graphs.

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