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Some Novel Formulas of Lucas Polynomials via Different Approaches

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Abstract: Some new formulas related to the well-known symmetric Lucas polynomials are the primary focus of this article. Different approaches are used for establishing these formulas. A matrix approach to Lucas polynomials is followed in order to obtain some fundamental properties. Particularly, some recurrence relations and determinant forms are determined by suitable Hessenberg matrices. Conjugate Lucas polynomials and generating functions are derived and examined. Several connection problems between the Lucas polynomials and other celebrated symmetric and non-symmetric orthogonal polynomials such as the first and second kinds of Chebyshev polynomials and their shifted counterparts are solved. We prove that several argument-type hypergeometric functions are involved in the connection coefficients. In addition, we construct new formulas for high-order derivatives of Lucas polynomials in terms of their original polynomials, as well as formulas for repeated integrals of Lucas polynomials.

Keywords: Lucas polynomials; matrix analysis; connection coefficients; high-order derivatives

MSC: 11B39; 33C05; 11B83; 11B37

1. Introduction

The investigations regarding different special functions occupy a considerable part of the literature. Numerous papers have been published interested in examining various polynomial sequences theoretically. Examples of these contributions are the studies on Fibonacci polynomials and their generalized polynomials ([1–4]). The interesting books by Koshy [5] and Djordjevic [6] comprehensively studied many sequences of polynomials. Two kinds of orthogonal polynomials have recently been investigated from a theoretical point of view in [7,8]. A note on the Bernoulli and Euler polynomials was given in [9]. The authors in [10] studied three other families of polynomials. Some properties of Horadam polynomials were given in [11]. A class of generalized polynomials associated with Hermite and Euler polynomials was considered and investigated in [12].

Numerous problems in many fields, such as approximation theory and theoretical physics, rely on special functions. Some uses of some special functions are discussed in [13]. In numerical analysis, special functions and polynomials play important roles in developing several approximations of different problems. One can consult the useful books in [14,15] that illustrate the importance of some special functions in numerical analysis. Specifically, obtaining spectral solutions to various differential equations can be aided by using several special functions. For example, Chebyshev polynomials of the fifth kind were used in [16] to handle a type of fractional differential equations. Chebyshev polynomials of the third and fourth kinds have also been used in a variety of papers to treat different types of differential equations. For some papers in this direction, one can consult [17,18].

A wide variety of disciplines, including physics, computer science, biology, and statistics, rely primarily on Lucas polynomials and their generalized sequences of polynomials.

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A wide range of theoretical publications in the literature focus on these sequences. For instance, the authors in [19,20] derived formulas concerning respectively the second and the qth derivative sequences of Fibonacci and Lucas polynomials. For some other articles regarding Lucas polynomials, one may be referred to [21,22]. In addition, some differential equations have been solved numerically using these polynomial sequences. For example, the authors in [23] used Lucas polynomials to approach a certain model that arises in chemical processes. In [24], a numerical solution for a two-dimensional Sobolev equation using mixed Lucas and Fibonacci polynomials was proposed. The authors in [25] obtained numerical solutions for the multi-dimensional sinh-Gordon equation based on Lucas polynomials. A fractional-order model was treated using Lucas polynomials in [26].

The issue of how different polynomials are connected is an essential one. Finding the connection coefficients $A_{i,j}$ such that $F_i(x) = \sum_{j=0}^{i} A_{i,j}\, G_j(x)$ is the solution to the connection problem between the two sets of polynomials $\{F_i(x)\}_{i\geq 0}$ and $\{G_j(x)\}_{j\geq 0}$. Numerous papers have been written on the subject of addressing the connection problems between different orthogonal polynomials. For two examples, see [27,28].

Due to their connections to different classes of special functions, many types of hypergeometric functions play important roles in many areas of mathematics. These functions are frequently used to calculate the connection and linearization coefficients between several polynomials. See, for example ([29,30]).

One further area of curiosity is the development of explicit formulations for high-order derivatives and repeated integrals of different kinds of special functions. More precisely, expressing the derivatives of different celebrated polynomials as combinations of their original ones is of great importance. They are very useful in approximating a variety of differential equations. Specifically, in [31], Abd-Elhameed established new derivatives formulas for Chebyshev polynomials of the sixth kind and used them as basis functions to treat a type of non-linear Burgers' equation. Furthermore, in [32], some other polynomial derivative formulas generalizing Chebyshev polynomials of the third kind were established. Using the spectral Galerkin approach, these polynomials were used to find solutions to linear and non-linear even-order BVPs.

Our main aim in this article is to establish some new formulas of Lucas polynomials and their relationships with some celebrated orthogonal polynomials via two approaches. To this end, we use some fundamental properties of Lucas polynomials as well as some properties of well-known orthogonal polynomials.

Below is an outline of the article's primary aims:

- Follow a matrix approach to Lucas polynomials in order to derive some of the fundamental relations of these polynomials. Important features of a polynomial sequence can be obtained using this approach ([33,34]);
- Solve connection problems between Lucas polynomials and some orthogonal polynomials. We will show that the connection coefficients are expressed in terms of hypergeometric functions of different arguments;
- Establish new high-order derivatives and repeated integral formulas of Lucas polynomials.

The paper is organized as follows: In the next section, we give some elementary properties of Lucas polynomials. Section 3 is devoted to following a matrix approach to obtain some formulas concerned with Lucas polynomials. Section 4 is concerned with establishing new connection formulas between Lucas polynomials and some orthogonal polynomials and their inversion formulas. In Section 5, we derive two new formulas. In the first, we express the high-order derivatives of Lucas polynomials in terms of their original polynomials. In contrast, in the second one, we give a new expression for the repeated integrals of Lucas polynomials in terms of their original ones. Finally, some conclusions are presented in Section 6.

2. An Overview on Lucas Polynomials and Jacobi Polynomials

This section mainly focuses on outlining some important relationships related to Lucas polynomials. Additionally, several characteristics of Jacobi polynomials and their shifted ones are presented.

2.1. Some Properties of Lucas Polynomials

Consider the Lucas polynomials sequence $\{L_n(x)\}_{n\geq 0}$. This recurrence relation can be used to construct the Lucas polynomials:

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x), \quad L_0(x) = 2,\ L_1(x) = x,\ n \geq 2. \tag{1}$$

In the power basis, we can write

$$L_n(x) = \sum_{k=0}^{n} a_{n,k}\, x^k, \quad a_{n,k} \in \mathbb{R},\ n = 0, 1, \cdots,$$

with $a_{0,0} = 2$ and, for $n, k > 0$,

$$a_{n,k} = \begin{cases} \dfrac{2n}{n+k} \dbinom{\frac{n+k}{2}}{\frac{n-k}{2}}, & (n-k) \text{ even}, \\ 0, & (n-k) \text{ odd}, \end{cases} \tag{2}$$

or, alternatively, as

$$L_n(x) = j \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\binom{n-m}{m}}{n-m}\, x^{n-2m}, \quad n \geq 0. \tag{3}$$

From (1), it is clear that the coefficients $a_{n,k}$ satisfy the following recurrence relation:

$$a_{n,k} = a_{n-1,k-1} + a_{n-2,k}, \qquad n \geq k \geq 2, \tag{4}$$

with

$$a_{n,0} = \begin{cases} 2, & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases} \qquad a_{n,1} = \begin{cases} 0, & n \text{ even}, \\ n, & n \text{ odd}, \end{cases} \qquad n \geq 0.$$

The inversion formula to (3) can be written in the form

$$x^n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i\, \xi_{n-2i} \binom{n}{i} L_{n-2i}(x), \quad n \geq 0, \tag{5}$$

where

$$\xi_i = \begin{cases} \frac{1}{2}, & i = 0, \\ 1, & i \geq 1. \end{cases} \tag{6}$$

Furthermore, one of the important relations of Lucas polynomials is its structural formula which can be easily shown from the power form representation (3). This formula is given by

$$L_n(x) = \frac{1}{n+1}\, L'_{n+1}(x) + \frac{1}{n-1}\, L'_{n-1}(x), \quad n \geq 2. \tag{7}$$

2.2. Some Properties of Jacobi Polynomials and Their Shifted Ones

This section focuses on presenting some fundamental properties of the classical Jacobi polynomials and their shifted ones.

It is well known that the Jacobi polynomials $P_m^{(\alpha,\beta)}(x),\ x \in [-1,1],\ m \geq 0\ \alpha > -1$, $\beta > -1$, (see [35,36]) can be generated by the following Rodrigues formula:

$$P_m^{(\alpha,\beta)}(x) = \frac{(-1)^m}{2^m\, m!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^m}{dx^m}\Big[(1-x)^{\alpha+m}(1+x)^{\beta+m}\Big].$$

These polynomials have the following hypergeometric representation:

$$P_m^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_m}{m!}\ {}_2F_1\left(\begin{matrix} -m, m+\alpha+\beta+1 \\ \alpha+1 \end{matrix}\ \middle|\ \frac{1-x}{2}\right).$$

For convenience, the normalized Jacobi polynomials can be defined as (see, [7])

$$R_m^{(\alpha,\beta)}(x) = {}_2F_1\left(\begin{matrix} -m, m+\alpha+\beta+1 \\ \gamma+1 \end{matrix}\ \middle|\ \frac{1-x}{2}\right). \tag{8}$$

The main characteristic of the polynomials in (8) is that they satisfy the property:

$$R_m^{(\alpha,\beta)}(1) = 1, \quad m = 0,1,2,\ldots.$$

It is an easy matter to transform all identities and relations of the classical Jacobi polynomials $P_m^{(\alpha,\beta)}(x)$ in order to obtain their counterparts for the polynomials $R_m^{(\alpha,\beta)}(x)$. For example, the orthogonality relation for the polynomials $R_m^{(\alpha,\beta)}(x)$ is given by

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\, R_m^{(\alpha,\beta)}\,(x)\, R_n^{(\alpha,\beta)}(x)\, dx = \begin{cases} 0, & m \neq m, \\ h_m^{\alpha,\beta}, & m = n, \end{cases} \tag{9}$$

where

$$h_m^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}\, m!\, \Gamma(m+\beta+1)\,(\Gamma(\alpha+1))^2}{(2m+\alpha+\beta+1)\,\Gamma(m+\alpha+\beta+1)\,\Gamma(m+\alpha+1)}. \tag{10}$$

If the parameters α and β are chosen appropriately, the following six well-known special polynomials can be derived as special polynomials of $R_m^{(\alpha,\beta)}(x)$:

- The first-kind Chebyshev polynomials $T_m(x)$;
- The second-kind Chebyshev polynomials $U_m(x)$;
- The third-kind Chebyshev polynomials $V_m(x)$;
- The fourth-kind Chebyshev polynomials $W_m(x)$;
- Legendre polynomials $L_m(x)$.
- Ultraspherical polynomials $C_m^{(\alpha)}(x)$.

In fact, the above six families of polynomials can be extracted from the polynomials $R_m^{(\alpha,\beta)}(x)$ by the following relations:

$$\begin{aligned} T_m(x) &= R_m^{(-\frac{1}{2},-\frac{1}{2})}(x), & U_m(x) &= (m+1)\, R_m^{(\frac{1}{2},\frac{1}{2})}(x), \\ V_m(x) &= R_m^{(-\frac{1}{2},\frac{1}{2})}(x), & W_m(x) &= (2m+1)\, R_m^{(\frac{1}{2},-\frac{1}{2})}(x), \\ L_m(x) &= R_m^{(0,0)}(x), & C_m^{(\alpha)}(x) &= R_m^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(x). \end{aligned}$$

Now, we define the shifted normalized Jacobi polynomials $\tilde{R}_m^{(\alpha,\beta)}(x)$ on $[0,1]$ as:

$$\tilde{R}_m^{(\alpha,\beta)}(x) = R_m^{(\alpha,\beta)}(2x-1).$$

The orthogonality relation of $\tilde{R}_m^{(\alpha,\beta)}(x)$ can be written as

$$\int_0^1 (1-x)^\alpha\, x^\beta\, \tilde{R}_m^{(\alpha,\beta)}(x)\, \tilde{R}_n^{(\alpha,\beta)}(x)\, dx = \begin{cases} 0, & n \neq m, \\ \tilde{h}_m^{\alpha,\beta}, & n = m, \end{cases} \tag{11}$$

where

$$\tilde{h}_m^{\alpha,\beta} = \frac{m!\,(\Gamma(\alpha+1))^2\,\Gamma(m+\beta+1)}{(2m+\alpha+\beta+1)\,\Gamma(m+\alpha+1)\,\Gamma(m+\alpha+\beta+1)}. \tag{12}$$

The following power form representation of $\tilde{R}_m^{(\alpha,\beta)}(x)$ and its inversion formula are useful in the sequel:

$$\tilde{R}_m^{(\alpha,\beta)}(x) = \sum_{r=0}^{m} \frac{(-1)^r\, m!\, \Gamma(\alpha+1)\, (\beta+1)_m\, (\alpha+\beta+1)_{2m-r}}{r!\,(m-r)!\,\Gamma(m+\alpha+1)\,(\alpha+\beta+1)_m\,(\beta+1)_{m-r}}\, x^{m-r}, \tag{13}$$

and

$$x^m = \sum_{r=0}^{m} \frac{\binom{m}{r}(\alpha+1)_{m-r}(m-r+\beta+1)_r}{(2m-2r+\alpha+\beta+2)_r(m-r+\alpha+\beta+1)_{m-r}}\, \tilde{R}_{m-r}^{(\alpha,\beta)}(x). \tag{14}$$

Jacobi polynomials and the particular families they belong to are covered in detail in [35,37].

3. A Matrix Approach to Lucas Polynomials

In this section, we follow a matrix approach to Lucas polynomials. We denote by A the matrix with entries $a_{i,j}$, $i,j = 0,\ldots,n$, $a_{i,i} \neq 0$, $\forall\, i$ and $a_{i,j} = 0$ for $j > i$. A is an infinite, lower triangular non-singular matrix. For example, for $n = 6$, we obtain

$$A = \begin{pmatrix} 2 & 0 & 0 & \cdots & & \cdots & 0 \\ 0 & 1 & \ddots & & & & \vdots \\ 2 & 0 & 1 & \ddots & & & \vdots \\ 0 & 3 & 0 & 1 & \ddots & & \vdots \\ 2 & 0 & 4 & 0 & 1 & \ddots & \vdots \\ 0 & 5 & 0 & 5 & 0 & 1 & 0 \\ 2 & 0 & 9 & 0 & 6 & 0 & 1 \end{pmatrix}.$$

Note that the classical sequence of Lucas numbers can be computed from the sums of the elements in the rows of A, that is, $\sum_{i=0}^{k} a_{k,i} = L_k$.

Moreover, the elements of the columns of A corresponding to the column sequences of the $(2,1)-$Pascal triangle except for the element $a_{0,0}$ being 2.

If $L^x = \left[L_0(x), L_1(x), \ldots, L_n(x), \ldots\right]^T$, and $X = \left[1, x, x^2, \ldots, x^n, \ldots\right]^T$, we obtain the matrix form of the Lucas polynomial sequence:

$$L^x = AX.$$

In the following, we will write P.S. to denote a polynomial sequence. Since A is an infinite, lower triangular non-singular matrix, so is also $A^{-1} = B = \left(b_{i,k}\right)_{i,k\geq 0}$ with

$$b_{0,0} = \frac{1}{2}, \qquad b_{n,k} = \begin{cases} b_k(-1)^{\frac{n-k}{2}}\binom{n}{\frac{n-k}{2}}, & (n-k) \text{ even}, \\ 0, & (n-k) \text{ odd}, \end{cases} \tag{15}$$

with $b_0 = \frac{1}{2}$ and $b_k = 1$ for $k \geq 1$. For $n = 6$, we have

$$B = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \cdots & & \cdots & 0 \\ 0 & 1 & \ddots & & & & \vdots \\ -1 & 0 & 1 & \ddots & & & \vdots \\ 0 & -3 & 0 & 1 & \ddots & & \vdots \\ 3 & 0 & -4 & 0 & 1 & \ddots & \vdots \\ 0 & 10 & 0 & -5 & 0 & 1 & 0 \\ -10 & 0 & 15 & 0 & -6 & 0 & 1 \end{pmatrix}.$$

The recurrence relation is satisfied by the elements $b_{n,k}$.

$$\begin{aligned} &b_{0,0} = \tfrac{1}{2}, \quad b_{n,0} = -b_{n-1,1}, \\ &b_{n,k} = c_k b_{n-1,k-1} - b_{n-1,k+1}, \end{aligned} \qquad n,k = 1,2,\ldots, \tag{16}$$

with $c_1 = 2$ and $c_k = 1$ for $k \geq 2$.

3.1. The Conjugate Sequence

The P.S. $\{\widehat{L}_n(x)\}_{n\in\mathbb{N}}$ related to the matrix B, with elements

$$\widehat{L}_n(x) = \sum_{i=0}^{n} b_{n,i} x^i,$$

is called the *conjugate* P.S. of $\{L_n(x)\}_{n\in\mathbb{N}}$. We say that the polynomial sequences $\{L_n(x)\}_{n\in\mathbb{N}}$ and $\{\widehat{L}_n(x)\}_{n\in\mathbb{N}}$ are conjugate to each other and it results in

$$L_n \circ \widehat{L}_n = \widehat{L}_n \circ L_n = i_n,$$

where $\{i_n\}_{n\in\mathbb{N}}$ the P.S. related to the identity matrix I, with $i_n(x) = x^n$.

For the conjugate sequences, $\{L_n(x)\}_{n\in\mathbb{N}}$ and $\{\widehat{L}_n(x)\}_{n\in\mathbb{N}}$, we have

$$L^x = AX = A^2\widehat{L}^x, \qquad \text{and} \qquad \widehat{L}^x = BX = B^2 L^x,$$

where $\widehat{L}^x = \left[\widehat{L}_0(x), \widehat{L}_1(x), \ldots, \widehat{L}_n(x), \ldots\right]^T$.

If $\forall n \in \mathbb{N}$, A_n and B_n are the principal submatrices of order n of A and B, respectively, $X_n = \left[1, x, x^2, \ldots, x^n\right]^T$, $L_n^x = \left[L_0(x), L_1(x), \ldots, L_n(x)\right]^T$, and $\widehat{L}_n^x = \left[\widehat{L}_0(x), \widehat{L}_1(x), \ldots, \widehat{L}_n(x)\right]^T$, then

$$L_n^x = A_n X_n = A_n^2 \widehat{L}_n^x, \quad \text{and} \quad \widehat{L}_n^x = B_n X_n = B_n^2 L_n^x. \tag{17}$$

3.2. Basis for $\mathcal{P}_n$

Let $\mathcal{P}_n$ denote the set of polynomials of degree $\leq n$. From (17), $\forall n \in \mathbb{N}$, we have

$$x^n = \sum_{k=0}^{n} b_{n,k} L_k(x) \qquad \text{and} \qquad x^n = \sum_{k=0}^{n} a_{n,k} \widehat{L}_k(x). \tag{18}$$

Therefore, each of the two sets $\{L_0(x), \ldots, L_n(x)\}$ and $\{\widehat{L}_0(x), \ldots, \widehat{L}_n(x)\}$ forms a basis for $\mathcal{P}_n$.

Hence, if $q_n(x) \in \mathcal{P}_n$, that is $q_n(x) = \sum_{k=0}^{n} q_{n,k} x^{n-k}$, then

$$q_n(x) = \sum_{k=0}^{n} c_{n,k} L_k(x) = \sum_{k=0}^{n} d_{n,k} \widehat{L}_k(x),$$

where $c_{n,k} = \sum_{j=0}^{\lfloor \frac{n-k}{2} \rfloor} q_{n,n-k-2j} b_{k+2j,k}$, $d_{n,k} = \sum_{j=0}^{\lfloor \frac{n-k}{2} \rfloor} q_{n,n-k-2j} a_{k+2j,k}$, $k = 0, 1, \ldots, n$.

From (15), for $k = 0, 1, \ldots, n$, we obtain

$$c_{n,0} = \frac{1}{2} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} q_{n,n-2j} (-1)^j \binom{2j}{j}, \quad c_{n,k} = \sum_{j=0}^{\lfloor \frac{n-k}{2} \rfloor} q_{n,n-k-2j} (-1)^j \binom{k+2j}{j}, \ k > 1,$$

and from (2), we have

$$d_{n,0} = 2 \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} q_{n,n-2j}, \quad d_{n,k} = \sum_{j=0}^{\lfloor \frac{n-k}{2} \rfloor} q_{n,n-k-2j} \frac{k+2j}{k+j} \binom{k+j}{j} \ k > 1.$$

3.3. First Recurrence Relation and Determinant Form

From the first relation in (18), $L_n(x)$ can be written as:

$$L_n(x) = x^n - \sum_{k=1}^{n-1} b_{n,k} L_k(x).$$

Furthermore, the first relation in (18), for $n \geq 0$, can be considered as an infinite linear system in the unknowns $L_j(x)$, $j \geq 0$. Using Cramer's rule, the first $(n+1)$ equations in the unknowns $L_0(x), \ldots, L_n(x)$ can be solved to produce the determinant form shown below.

$$L_0(x) = 2, \quad L_n(x) = 2(-1)^n \begin{vmatrix} 1 & x & x^2 & \cdots & x^{n-1} & x^n \\ b_{0,0} & b_{1,0} & b_{2,0} & \cdots & b_{n-1,0} & b_{n,0} \\ 0 & b_{1,1} & b_{2,1} & \cdots & b_{n-1,1} & b_{n,1} \\ \vdots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & b_{n-1,n-1} & b_{n,n-1} \end{vmatrix}, \quad n \geq 1.$$

Similarly, from the second of (18), we can express the conjugate polynomials as

$$\widehat{L}_n(x) = x^n - \sum_{k=1}^{n-1} a_{n,k} \widehat{L}_k(x),$$

and

$$\widehat{L}_0(x) = \frac{1}{2}, \quad \widehat{L}_n(x) = \frac{(-1)^n}{2} \begin{vmatrix} 1 & x & x^2 & \cdots & x^{n-1} & x^n \\ a_{0,0} & a_{1,0} & a_{2,0} & \cdots & a_{n-1,0} & a_{n,0} \\ 0 & a_{1,1} & a_{2,1} & \cdots & a_{n-1,1} & a_{n,1} \\ \vdots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & a_{n-1,n-1} & a_{n,n-1} \end{vmatrix}, \quad n \geq 1.$$

3.4. Second Recurrence Relation and Determinant Form

The production matrix or Stieltjes matrix Π ([38]) of an infinite lower triangular matrix A is the Hessenberg matrix Π such that

$$A\,\Pi = \mathcal{D}A,$$

with $\mathcal{D} = \left(\beta_{i+1,j}\right)_{i,j \geq 0}$, being $\delta_{i,j}$ the well-known Kroneker's delta function.

Observe that $\mathcal{D}A = \overline{A}$, where $\overline{A}$ is the matrix obtained from A by deleting its first row.

Theorem 1 ([39,40]). *Let $\{p_n(x)\}_{n\in\mathbb{N}}$ be a P.S. with matrix $A = (a_{i,k})_{i,k\geq 0}$ and let $\Pi = (\pi_{ij})_{i,j\geq 0}$ be the production matrix of A^{-1}. Then,*

$$p_0(x) = a_{0,0}, \qquad \pi_{n,n+1}p_{n+1}(x) = (x-\pi_{n,n})p_n(x) - \sum_{i=0}^{n-1}\pi_{n,i}p_i(x), \tag{19}$$

where $\sum_{i=0}^{-1}\cdot = 0$.

We require the production matrices of B and A in order to obtain further recurrence relations and determinant forms for the elements of $\{L_n(x)\}_{n\in\mathbb{N}}$ and $\{\widehat{L}_n(x)\}_{n\in\mathbb{N}}$.

Proposition 1. *The production matrix Π of B is tridiagonal with elements*

$$\pi_{0,1} = 2, \quad \pi_{i,i-1} = -1, \quad \pi_{i,i} = 0, \quad \pi_{i,i+1} = 1, \;\; i \geq 1. \tag{20}$$

Proof. The proof follows by induction, taking into account relations (4) and (16). □

From (20) and Theorem 1, we obtain the recurrence relation (1).
The production matrix of A can be obtained by the same technique used in Proposition 1:

Proposition 2. *The production matrix $\widehat{\Pi}$ of A is a lower Hessenberg matrix with elements*

$$\widehat{\pi}_{0,1} = \frac{1}{2}, \qquad \widehat{\pi}_{0,k} = 0, \quad k \neq 1, \qquad \widehat{\pi}_{1,2} = 1,$$

$$\widehat{\pi}_{i,k} = \begin{cases} 4b_{i-1,0}, & i \geq 1, \quad k = 0, \\ \widehat{\pi}_{i-1,0}, & i \geq 1, \quad k = 1, \\ \dfrac{\widehat{\pi}_{i-1,1}}{i-1}, & i \geq 2, \quad k = 2, \\ \widehat{\pi}_{i-k+2,2}, & i \geq 2, \quad 2 < k \leq i+1, \\ 0, & i \geq 2, \quad k > i+1. \end{cases}$$

The following recurrence relation can be obtained from Proposition 2 and Theorem 1:

$$\widehat{L}_0(x) = \frac{1}{2}, \;\; \widehat{L}_1(x) = x, \;\; \widehat{L}_{n+1}(x) = x\widehat{L}_n(x) - \frac{1}{n}\sum_{i=0}^{n-1} b_{n-i-1,0}\widehat{L}_i(x).$$

From Theorem 8 in [39], the recurrence relation (19) is equivalent to the determinant of a suitable Hessenberg matrix. That is, every P.S. $p_n \in \{p_n(x)\}_{n\in\mathbb{N}}$ with matrix A can be written as $p_0(x) = a_{0,0}$,

$$p_n(x) = \frac{(-1)^n p_0(x)}{\prod_{i=0}^{n-1}\pi_{i,i+1}} \begin{vmatrix} \pi_{0,0}-x & \pi_{0,1} & 0 & 0 & \cdots & 0 \\ \pi_{1,0} & \pi_{1,1}-x & \pi_{1,2} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \pi_{n-2,n-1} \\ \pi_{n-1,0} & \pi_{n-1,1} & \cdots & \cdots & \pi_{n-1,n-2} & \pi_{n-1,n-1}-x \end{vmatrix}, \quad n > 0. \tag{21}$$

Hence, from (21), Propositions 1 and 2, we obtain the following determinant forms for $L_n(x)$ and $\hat{L}_n(x)$:

$$L_0(x) = 2, \qquad L_n(x) = (-1)^n \begin{vmatrix} -x & 2 & 0 & 0 & \cdots & 0 \\ -1 & -x & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & -1 & -x \end{vmatrix}, \quad n \geq 1;$$

$$\hat{L}_0(x) = 2, \qquad \hat{L}_n(x) = (-1)^n \begin{vmatrix} -x & \frac{1}{2} & 0 & 0 & \cdots & 0 \\ 1 & -x & 1 & 0 & \cdots & 0 \\ b_{1,0} & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 \\ b_{n-2,0} & b_{n-3,0} & \cdots & b_{1,0} & 1 & -x \end{vmatrix}, \quad n \geq 1.$$

For the zeros z of $L_n(x)$, from (20) and the Gershgorin theorem, we have

$$|z - \pi_{s,s}| \leq \sum_{\substack{j=0 \\ j \neq s}}^{s} |\pi_{s,j}| = 2, \quad s = 1, \dots, n.$$

It is known ([41]) that, except for $n = 0$, all of the roots of $L_n(x)$ are $z_k = 2i \cos \frac{(2k+1)\pi}{2n}$, $k = 0, \dots, n-1$. Hence, their absolute values are less than 2.

3.5. Generating Function

The following theorem allows for determining the generating function of a P.S. under appropriate hypotheses.

Theorem 2 ([42])**.** *Let $\{p_n\}_{n \in \mathbb{N}}$ be a P.S. with a three-term recurrence relation of the type*

$$p_{n+1}(x) = axp_n(x) + bp_{n-1}(x), \qquad a, b \in \mathbb{R}, \quad n > 1,$$

with $p_0(x) = \alpha$, $p_1(x) = \beta_1 x + \beta_2$, $\alpha, \beta_1, \beta_2 \in \mathbb{R}$. Then, the generating function is

$$G(x,t) = \frac{\alpha(1 - axt) + tp_1(x)}{1 - axt - bt^2}.$$

The generating function of Lucas P.S. $\{L_n\}_{n \in \mathbb{N}}$ can be obtained from Theorem 2 with $a = b = 1$ and $\alpha = L_0(x) = 2$:

$$\sum_{n=0}^{\infty} L_n(x) t^n = \frac{2 - xt}{1 - xt - t^2}.$$

This is in accordance with the literature ([11,43]).

4. Connection Formulas between Lucas Polynomials and Other Classes of Polynomials

Let $\{p_n(x)\}_{n \in \mathbb{N}}$ and $\{q_n(x)\}_{n \in \mathbb{N}}$ be two polynomial sequences with matrix, respectively, $T = \left(t_{i,j}\right)_{i,j \geq 0}$ and S. With the usual notations, we obtain $P^x = TX$, $Q^x = SX$, from which

$$P^x = \left(T\,S^{-1}\right) Q^x = CQ^x, \tag{22}$$

with $C = TS^{-1} = \left(c_{i,j}\right)_{i,j\geq 0}$.

Thus, $\forall n \in \mathbb{N}$, we can write

$$p_n(x) = \sum_{j=0}^{n} c_{n,j} q_j(x), \qquad \text{with} \qquad c_{i,j} = \sum_{k=j}^{i} t_{i,k} \bar{s}_{k,j}, \tag{23}$$

being $S^{-1} = \left(\bar{s}_{i,j}\right)_{i,j\geq 0}$. C is the *connection matrix* of $\{p_n\}_{n\in\mathbb{N}}$ and $\{q_n\}_{n\in\mathbb{N}}$.

From (23), we can easily obtain

$$q_n(x) = \frac{1}{c_{n,n}} \left\{ p_n(x) - \sum_{j=0}^{n-1} c_{n,j} q_j(x) \right\},$$

and the determinant form

$$q_0(x) = \frac{1}{c_{0,0}} p_0(x), \qquad q_n(x) = \frac{(-1)^n}{\prod_{i=0}^{n} c_{i,i}} \begin{vmatrix} p_0(x) & p_1(x) & \cdots & p_n(x) \\ \bar{c}_{0,0} & \bar{c}_{1,0} & \cdots & \bar{c}_{n,0} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \bar{c}_{n-1,n-1} & \bar{c}_{n,n-1} \end{vmatrix}, \quad n > 0,$$

where $\overline{C} = \left(\bar{c}_{i,k}\right)_{i,k\geq 0}$ is the inverse matrix of C.

Similarly, we have

$$p_n(x) = \frac{1}{\bar{c}_{n,n}} \left\{ q_n(x) - \sum_{j=0}^{n-1} \bar{c}_{n,j} p_j(x) \right\},$$

and

$$p_0(x) = \frac{1}{\bar{c}_{0,0}} q_0(x), \qquad p_n(x) = \frac{(-1)^n}{\prod_{i=0}^{n} \bar{c}_{i,i}} \begin{vmatrix} q_0(x) & q_1(x) & \cdots & q_n(x) \\ c_{0,0} & c_{1,0} & \cdots & c_{n,0} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & c_{n-1,n-1} & c_{n,n-1} \end{vmatrix}, \quad n > 0.$$

The coefficients $c_{i,j}$ in (23) can be determined by different techniques. In the following, we are going to develop some new connection formulas between Lucas polynomials and some other orthogonal polynomials. In addition, we present the inversion of these formulas.

4.1. Connection Formulas between Lucas Polynomials and Ultraspherical Polynomials

We now establish novel connection formulas between various orthogonal polynomials and Lucas polynomials. More specifically, we address the following two connection problems:

$$L_n(x) = \sum_{j=0}^{n} a_j\, C_j^{(\alpha)}(x), \qquad C_n^{(\alpha)}(x) = \sum_{j=0}^{n} a_j\, L_j(x), \tag{24}$$

where $C_n^{(\alpha)}(x)$ is the ultraspherical polynomial of degree n.

With the use of the connection formulae in (24), one may obtain the connection formulas for the Lucas-first-type Chebyshev, Lucas-second-kind Chebyshev, Lucas–Legendre, and their inversions.

Theorem 3. *Let n be a non-negative integer. The following connection formula holds:*

$$L_n(x) = \frac{\sqrt{\pi}\, n!\, 2^{-n-2\lambda+1}}{\Gamma\left(\lambda+\frac{1}{2}\right)} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n+\lambda-2j)\Gamma(n-2j+2\lambda)}{j!(n-2j)!\Gamma(n-j+\lambda+1)} \times {}_2F_1\left(\begin{matrix} -j, -n+j-\lambda \\ 1-n \end{matrix} \middle| -4 \right) C_{n-2j}^{(\lambda)}(x). \tag{25}$$

Proof. We should find the coefficients $c_{n,j}$ such that

$$L_n(x) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} c_{n,j}\, C_{n-2j}^{(\lambda)}(x). \tag{26}$$

Based on the orthogonality relation of $C_n^{(\lambda)}(x)$, the coefficients $c_{n,j}$ in (26) are

$$c_{n,j} = \frac{1}{h_{n-2j}^{\lambda}} \int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}}\, L_n(x)\, C_{n-2j}^{(\lambda)}(x)\, dx,$$

where h_n^{λ} is given by $h_n^{\lambda} = \dfrac{\sqrt{\pi}\, n!\, \Gamma\left(\lambda+\frac{1}{2}\right) \Gamma(2\lambda+1)}{2(n+\lambda)\,\Gamma(\lambda+1)\,\Gamma(n+2\lambda)}$. Following the Rodrigues formula, $c_{n,j}$ can be given as

$$c_{n,j} = \frac{(-1)^n\, 2^{-n-2\lambda+2j+1}\, (n+\lambda-2j)\,\Gamma(n-2j+2\lambda)}{\Gamma\left(\lambda+\frac{1}{2}\right)(n-2j)!\,\Gamma\left(n-2j+\lambda+\frac{1}{2}\right)} \int_{-1}^{1} L_n(x)\, D^{n-2j}\left(1-x^2\right)^{n-2j+\lambda-\frac{1}{2}} dx. \tag{27}$$

Integrating the right side of (27) $(n-2j)$ times yields

$$c_{n,j} = \frac{2^{-n-2\lambda+2j+1}\, (n+\lambda-2j)\,\Gamma(n-2j+2\lambda)}{\Gamma\left(\lambda+\frac{1}{2}\right)(n-2j)!\,\Gamma\left(n-2j+\lambda+\frac{1}{2}\right)} \int_{-1}^{1} \left(1-x^2\right)^{n-2j+\lambda-\frac{1}{2}} D^{n-2j} L_n(x)\, dx.$$

With the aid of the analytic form of Lucas polynomials (3), the last equation turns into

$$c_{n,j} = \frac{n\, 2^{-n-2\lambda+2j+1}\, (n+\lambda-2j)\,\Gamma(n-2j+2\lambda)}{\Gamma\left(\lambda+\frac{1}{2}\right)(n-2j)!\,\Gamma\left(n-2j+\lambda+\frac{1}{2}\right)} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-i-1)!}{i!(2j-2i)!} \int_{-1}^{1} x^{2j-2i} \left(1-x^2\right)^{n+\lambda-2j-\frac{1}{2}} dx.$$

It is easy to show that

$$\int_{-1}^{1} x^{2j-2i} \left(1-x^2\right)^{n+\lambda-2j-\frac{1}{2}} dx = \frac{\Gamma\left(-i+j+\frac{1}{2}\right)\Gamma\left(n-2j+\lambda+\frac{1}{2}\right)}{\Gamma(n-i-j+\lambda+1)},$$

and consequently, the coefficients $c_{n,j}$ are given by

$$c_{n,j} = \frac{n\, 2^{-n-2\lambda+2j+1}\, (n+\lambda-2j)\,\Gamma(n-2j+2\lambda)}{\Gamma\left(\lambda+\frac{1}{2}\right)(n-2j)!} \sum_{i=0}^{j} \frac{(n-i-1)!\,\Gamma\left(j-i+\frac{1}{2}\right)}{i!\,(2j-2i)!\,\Gamma(n-i-j+\lambda+1)}. \tag{28}$$

Now, it is not difficult to show that the last sum in (28) takes the following form:

$$\sum_{i=0}^{j} \frac{(n-i-1)!\,\Gamma\left(-i+j+\frac{1}{2}\right)}{i!\,(2j-2i)!\,\Gamma(n-i-j+\lambda+1)} = \frac{\sqrt{\pi}\, 2^{-2j}\, (n-1)!}{j!\,\Gamma(n-j+\lambda+1)}\, {}_2F_1\left(\begin{matrix} -j, -n+j-\lambda \\ 1-n \end{matrix} \middle| -4 \right),$$

and therefore, the coefficients $c_{n,j}$ are equivalent to

$$c_{n,j} = \frac{\sqrt{\pi}\, n!\, 2^{-n-2\lambda+1}\,(n+\lambda-2j)\Gamma(n-2j+2\lambda)}{\Gamma\left(\lambda+\frac{1}{2}\right) j!(n-2j)!\Gamma(n-j+\lambda+1)}\; {}_2F_1\left(\begin{matrix} -j, -n+j-\lambda \\ 1-n \end{matrix}\,\middle|\, -4\right).$$

With this, we have proved Theorem 3. □

The following three corollaries are direct consequences of Theorem 3.

Corollary 1. *The Lucas-first-kind Chebyshev connection formula is:*

$$L_n(x) = \frac{n!}{2^{n-1}} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{j!\,(n-j)!}\; {}_2F_1\left(\begin{matrix} -j, -n+j \\ 1-n \end{matrix}\,\middle|\, -4\right) T_{n-2j}(x), \quad n \geq 0. \tag{29}$$

Corollary 2. *The Lucas-second-kind Chebyshev connection formula is:*

$$L_n(x) = \frac{n!}{2^n} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-2j+1)!}{j!\,(n-2j)!\,(n-j+1)!}\; {}_2F_1\left(\begin{matrix} -j, -n+j-1 \\ 1-n \end{matrix}\,\middle|\, -4\right) U_{n-2j}(x), \quad n \geq 0. \tag{30}$$

Corollary 3. *The Lucas–Legendre connection formula is:*

$$L_n(x) = \frac{\sqrt{\pi}\, n!}{2^n} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n-2j+\frac{1}{2}}{j!\,\Gamma\left(n-j+\frac{3}{2}\right)}\; {}_2F_1\left(\begin{matrix} -j, -n+j-\frac{1}{2} \\ 1-n \end{matrix}\,\middle|\, -4\right) P_{n-2j}(x), \quad n \geq 0. \tag{31}$$

In the following, we give the inversion formula for (25). Due to the non-orthogonality of Lucas polynomials, we will use another method to obtain the desired relation. In fact, the suggested method is based on the use of the power form representation of $C_j^{(\lambda)}(x)$ given by

$$C_j^{(\lambda)}(x) = \frac{j!\,\Gamma(2\lambda+1)}{2\,\Gamma(\lambda+1)\,\Gamma(j+2\lambda)} \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} \frac{(-1)^r\, 2^{j-2r}\,\Gamma(j-r+\lambda)}{r!\,(j-2r)!}\, x^{j-2r}, \tag{32}$$

and the inversion formula of Lucas polynomials (5).

Theorem 4. *Let j be a non-negative integer. The following connection formula holds:*

$$C_j^{(\lambda)}(x) = \frac{j!\, 2^{j+2\lambda-1}\,\Gamma\left(\lambda+\frac{1}{2}\right)\,\Gamma(j+\lambda)}{\sqrt{\pi}\,\Gamma(j+2\lambda)} \sum_{p=0}^{\lfloor \frac{j}{2} \rfloor} \frac{(-1)^p\,\xi_{j-2p}}{p!\,(j-p)!}\; {}_2F_1\left(\begin{matrix} -p, p-j \\ -j-\lambda+1 \end{matrix}\,\middle|\, -\frac{1}{4}\right) L_{j-2p}(x), \tag{33}$$

and ξ_i is as defined in (6).

Proof. From (32) and the inversion Formula (5), the following relation can be obtained

$$\begin{aligned} C_j^{(\lambda)}(x) = & \frac{j!\,\Gamma(2\lambda+1)}{2\Gamma(\lambda+1)\,\Gamma(j+2\lambda)} \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} \frac{(-1)^r\, 2^{j-2r}\,\Gamma(j-r+\lambda)}{r!\,(j-2r)!} \times \\ & \sum_{t=0}^{\lfloor \frac{j}{2} \rfloor - r} \frac{(-1)^t\,\xi_{j-2r-2t}(j-2r-t+1)_t}{t!}\, L_{j-2r-2t}(x). \end{aligned}$$

Rearranging the terms in the last formula and performing some calculations leads to the following relation

$$C_j^{(\lambda)}(x) = \sum_{p=0}^{\lfloor \frac{j}{2} \rfloor} \frac{\xi_{j-2p}\,(-1)^p\, j!\,\Gamma\left(\lambda+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(j+2\lambda)} \sum_{r=0}^{p} \frac{2^{j+2\lambda-2r-1}\,\Gamma(j-r+\lambda)}{r!\,(p-r)!\,(j-p-r)!}\, L_{j-2p}(x). \tag{34}$$

It is not difficult to show the identity

$$\sum_{r=0}^{p} \frac{2^{j+2\lambda-2r-1}\Gamma(j-r+\lambda)}{r!\,(p-r)!\,(j-p-r)!} = \frac{2^{j+2\lambda-1}\,\Gamma(j+\lambda)}{p!\,(j-p)!}\; {}_2F_1\left(\begin{matrix}-p,\, p-j\\ -j-\lambda+1\end{matrix}\,\middle|\, -\frac{1}{4}\right),$$

and hence, the connection Formula (33) can be obtained. □

The following three corollaries are direct consequences of Theorem 4.

Corollary 4. *The first-kind Chebyshev–Lucas connection formula is:*

$$T_j(x) = 2^{j-1} j! \sum_{p=0}^{\lfloor \frac{j}{2} \rfloor} \frac{(-1)^p\, \xi_{j-2p}}{p!(j-p)!}\; {}_2F_1\left(\begin{matrix}-p,\, p-j\\ 1-j\end{matrix}\,\middle|\, -\frac{1}{4}\right) L_{j-2p}(x), \quad j \geq 0. \tag{35}$$

Corollary 5. *Let j be a non-negative integer. The second-kind Chebyshev–Lucas connection formula is:*

$$U_j(x) = 2^j j! \sum_{p=0}^{\lfloor \frac{j}{2} \rfloor} \frac{(-1)^p\, \xi_{j-2p}}{p!(j-p)!}\; {}_2F_1\left(\begin{matrix}-p,\, p-j\\ -j\end{matrix}\,\middle|\, -\frac{1}{4}\right) L_{j-2p}(x), \quad j \geq 0. \tag{36}$$

Corollary 6. *Let j be a non-negative integer. The Legendre–Lucas connection formula is:*

$$P_j(x) = \frac{2^j \Gamma\left(j+\frac{1}{2}\right)}{\sqrt{\pi}} \sum_{p=0}^{\lfloor \frac{j}{2} \rfloor} \frac{(-1)^p\, \xi_{j-2p}}{p!(j-p)!}\; {}_2F_1\left(\begin{matrix}-p,\, p-j\\ \frac{1}{2}-j\end{matrix}\,\middle|\, -\frac{1}{4}\right) L_{j-2p}(x), \quad j \geq 0. \tag{37}$$

4.2. Some Other Connection Formulas

In this section, we present some other connection formulas between Lucas polynomials and some other orthogonal polynomials. Due to the similarities of proofs with the proofs given in the previous section, the details are omitted.

Theorem 5. *The Lucas–Hermite connection formula is:*

$$L_j(x) = \frac{j!}{2^j} \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} \frac{1}{r!\,(j-2r)!}\; {}_1F_1(-r; 1-j; 4)\, H_{j-2r}(x), \quad j \geq 0. \tag{38}$$

Theorem 6. *The Hermite–Lucas connection formula is:*

$$H_j(x) = j! \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} \xi_{j-2r}\, \frac{2^{j-2r}}{r!(j-r)!}\, U(-r, j-2r+1, -4)\, L_{j-2r}(x), \quad j \geq 0, \tag{39}$$

where $U(a,b,z)$ is the confluent hypergeometric function (see [35]).

Theorem 7. *The Lucas-shifted Jacobi connection formula is:*

$$
\begin{aligned}
L_j(x) =& \frac{j!\,\Gamma(j+\beta+1)}{\Gamma(\alpha+1)} \sum_{r=0}^{\left\lfloor \frac{j}{2} \right\rfloor} \frac{(\alpha+\beta+2j-4r+1)\Gamma(j-2r+\alpha+1)\Gamma(j-2r+\alpha+\beta+1)}{(2r)!\,(j-2r)!\,\Gamma(j-2r+\beta+1)\,\Gamma(2j-2r+\alpha+\beta+2)} \times \\
& {}_4F_3\left(\begin{matrix} -r, \frac{1}{2}-r, -\frac{\alpha}{2}-\frac{\beta}{2}-j+r-\frac{1}{2}, -\frac{\alpha}{2}-\frac{\beta}{2}-j+r \\ 1-j, -\frac{\beta}{2}-\frac{j}{2}, -\frac{\beta}{2}-\frac{j}{2}+\frac{1}{2} \end{matrix} \middle| -\frac{1}{4} \right) \tilde{R}^{(\alpha,\beta)}_{j-2r}(x) \\
& + \frac{j!\,\Gamma(j+\beta+1)}{\Gamma(\alpha+1)} \sum_{r=0}^{\left\lfloor \frac{j-1}{2} \right\rfloor} \frac{(\alpha+\beta+2j-4r-1)\,\Gamma(j-2r+\alpha)\Gamma(j-2r+\alpha+\beta)}{(2r+1)!\,(j-2r-1)!\,\Gamma(j-2r+\beta)\,\Gamma(2j-2r+\alpha+\beta+1)} \times \\
& {}_4F_3\left(\begin{matrix} -r, -r-\frac{1}{2}, -\frac{\alpha}{2}-\frac{\beta}{2}-j+r, -\frac{\alpha}{2}-\frac{\beta}{2}-j+r+\frac{1}{2} \\ 1-j, -\frac{\beta}{2}-\frac{j}{2}, -\frac{\beta}{2}-\frac{j}{2}+\frac{1}{2} \end{matrix} \middle| -4 \right) \tilde{R}^{(\alpha,\beta)}_{j-2r-1}(x), \quad j \geq 0.
\end{aligned} \tag{40}
$$

Taking into consideration the special classes of shifted Jacobi polynomials, the following connection formulas can be deduced.

Corollary 7. *The Lucas-shifted ultraspherical connection formula is:*

$$
\begin{aligned}
L_j(x) =& \frac{2j!\,\Gamma\left(j+\alpha+\frac{1}{2}\right)}{\Gamma\left(\alpha+\frac{1}{2}\right)} \sum_{r=0}^{\left\lfloor \frac{j}{2} \right\rfloor} \frac{(\alpha+j-2r)\Gamma(j-2r+2\alpha)}{(2r)!(j-2r)!\Gamma(2j-2r+2\alpha+1)} \times \\
& {}_4F_3\left(\begin{matrix} -r, \frac{1}{2}-r, -\alpha-j+r, -\alpha-j+r+\frac{1}{2} \\ 1-j, -\frac{\alpha}{2}-\frac{j}{2}+\frac{1}{4}, -\frac{\alpha}{2}-\frac{j}{2}+\frac{3}{4} \end{matrix} \middle| -4 \right) \tilde{C}^{(\alpha)}_{j-2r}(x) \\
& + \frac{2\,j!\,\Gamma\left(j+\alpha+\frac{1}{2}\right)}{\Gamma\left(\alpha+\frac{1}{2}\right)} \sum_{r=0}^{\left\lfloor \frac{j-1}{2} \right\rfloor} \frac{(\alpha+j-2r-1)\,\Gamma(j-2r+2\alpha-1)}{(2r+1)!\,(j-2r-1)!\,\Gamma(2(j-r+\alpha))} \times \\
& {}_4F_3\left(\begin{matrix} -r, -r-\frac{1}{2}, -\alpha-j+r+\frac{1}{2}, -\alpha-j+r+1 \\ 1-j, -\frac{\alpha}{2}-\frac{j}{2}+\frac{1}{4}, -\frac{\alpha}{2}-\frac{j}{2}+\frac{3}{4} \end{matrix} \middle| -4 \right) \tilde{C}^{(\alpha)}_{j-2r-1}(x), \quad j \geq 0.
\end{aligned} \tag{41}
$$

Corollary 8. *The Lucas-shifted first-kind Chebyshev connection formula is:*

$$
\begin{aligned}
L_j(x) =& \frac{(2j)!}{2^{2j-1}} \sum_{r=0}^{\left\lfloor \frac{j}{2} \right\rfloor} \frac{\xi_{j-2r}}{(2r)!\,(2j-2r)!}\ {}_4F_3\left(\begin{matrix} -r, \frac{1}{2}-r, r-j, -j+r+\frac{1}{2} \\ 1-j, \frac{1}{4}-\frac{j}{2}, \frac{3}{4}-\frac{j}{2} \end{matrix} \middle| -4 \right) T^*_{j-2r}(x) \\
& + \frac{(2j)!}{2^{2j-1}} \sum_{r=0}^{\left\lfloor \frac{j-1}{2} \right\rfloor} \frac{\xi_{j-2r-1}}{(2r+1)!(2j-2r-1)!} \times \\
& {}_4F_3\left(\begin{matrix} -r, -r-\frac{1}{2}, -j+r+\frac{1}{2}, -j+r+1 \\ 1-j, \frac{1}{4}-\frac{j}{2}, \frac{3}{4}-\frac{j}{2} \end{matrix} \middle| -4 \right) T^*_{j-2r-1}(x), \quad j \geq 0.
\end{aligned} \tag{42}
$$

Corollary 9. *The Lucas-shifted second-kind Chebyshev connection formula is:*

$$
\begin{aligned}
L_j(x) =& \frac{(2j+1)!}{2^{2j-1}} \sum_{r=0}^{\left\lfloor \frac{j}{2} \right\rfloor} \frac{j-2r+1}{(2r)!\,(2j-2r+2)!}\ {}_4F_3\left(\begin{matrix} -r, \frac{1}{2}-r, -j+r-1, -j+r-\frac{1}{2} \\ 1-j, -\frac{j}{2}-\frac{1}{4}, \frac{1}{4}-\frac{j}{2} \end{matrix} \middle| -4 \right) U^*_{j-2r}(x) \\
& + \frac{(2j+1)!}{2^{2j-1}} \sum_{r=0}^{\left\lfloor \frac{j-1}{2} \right\rfloor} \frac{(j-2r)}{(2r+1)!(2j-2r+1)!}\ {}_4F_3\left(\begin{matrix} -r, -r-\frac{1}{2}, -j+r-\frac{1}{2}, r-j \\ 1-j, -\frac{j}{2}-\frac{1}{4}, \frac{1}{4}-\frac{j}{2} \end{matrix} \middle| -4 \right) U^*_{j-2r-1}(x), \quad j \geq 0.
\end{aligned} \tag{43}
$$

Corollary 10. *The Lucas-shifted Legendre connection formula is:*

$$L_j(x) =(j!)^2 \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} \frac{1+2j-4r}{(2r)!\,(2j-2r+1)!}\, {}_4F_3\left(\begin{matrix} -r, \frac{1}{2}-r, -j+r-\frac{1}{2}, r-j \\ 1-j, \frac{1}{2}-\frac{j}{2}, -\frac{j}{2} \end{matrix} \,\middle|\, -4 \right) P^*_{j-2r}(x)$$
$$+ (j!)^2 \sum_{r=0}^{\lfloor \frac{j-1}{2} \rfloor} \frac{-1+2j-4r}{(2r+1)!\,(2j-2r)!}\, {}_4F_3\left(\begin{matrix} -r-\frac{1}{2}, -r, r-j, -j+r+\frac{1}{2} \\ 1-j, \frac{1}{2}-\frac{j}{2}, -\frac{j}{2} \end{matrix} \,\middle|\, -4 \right) P^*_{j-2r-1}(x), \quad j \geq 0. \tag{44}$$

5. High-Order Derivatives and Repeated Integrals of Lucas Polynomials

This section is concerned with deriving the high-order derivatives of Lucas polynomials in terms of their original polynomials. Moreover, a new formula for repeated integrals of Lucas polynomials is also given.

5.1. Derivation Matrix

Derivation matrices are used in several areas of numerical analysis, such as in the solution of differential equations ([44,45]) and in many engineering and physical problems.

Let $\{p_n(x)\}_{n\in\mathbb{N}}$ be a P.S. with matrix $A = (a_{ij})_{i,j\geq 0}$ and let $B = A^{-1} = (b_{ij})_{i,j\geq 0}$. Moreover, let $P_n^x = [p_0(x), p_1(x), \ldots, p_n(x)]^T$. If $P_n^{x\prime} = [p_0'(x), p_1'(x), \ldots, p_n'(x)]^T$, it is known that the derivation matrix of the P.S. $\{p_n(x)\}_{n\in\mathbb{N}}$ is the matrix D such that

$$P_n^{x\prime} = D\, P_n^x. \tag{45}$$

Note that D is a lower triangular matrix whose diagonal elements are zeros.

The following proposition gives the entries of D.

Proposition 3 ([39]). *The derivation matrix of the P.S. $\{p_n\}_{n\in\mathbb{N}}$ with matrix $A = (a_{i,j})_{i,j\geq 0}$ is $D = (d_{i,j})_{i,j\geq 0}$ with*

$$d_{i,j} = \begin{cases} \sum_{k=j}^{i} (k+1) a_{i,k+1} b_{k,j}, & i > j, \\ 0, & \text{otherwise}. \end{cases} \tag{46}$$

From (45), (46), (2), and (15), we have

$$L_n^{x\prime} = D\, L_n^x \tag{47}$$

where $L_n^{x\prime}(x) = [L_0'(x), L_1'(x), \ldots, L_n'(x)]^T$, and $D = (d_{i,j})_{i,j=0}^{n}$ is the lower triangular matrix with

$$d_{i,0} = \begin{cases} (-1)^{\frac{i-1}{2}} \dfrac{i}{2}, & i \text{ odd}, \\ 0 & i \text{ even}, \end{cases}, \qquad d_{i,j} = \begin{cases} (-1)^{\frac{i-j-1}{2}}\, i, & i+j \text{ odd}, \\ 0 & i+j \text{ even}, \end{cases} \quad i > j. \tag{48}$$

From (47), for higher order derivatives we obtain

$$L_n^{x\,(s)} = D^s\, L_n^x, \quad s \geq 1.$$

Remark 1. *From (48), the matrix D is an upper triangular matrix whose main diagonal elements are zeros. Therefore, it is a nilpotent matrix, that is, $D^s = 0$ for $s > n$.*

5.2. New High-Order Derivatives and Repeated Integral Formulas

In this section, we will derive an explicit formula for the high-order derivatives of Lucas polynomials in terms of their original polynomials.

Lemma 1. *For every $r \geq 1$, we obtain*

$$L'_r(x) = r \sum_{m=0}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^m \, \xi_{r-2m-1} L_{r-2m-1}(x),$$

where ξ_j, $j \geq 0$, are defined in (6).

Proof. The proof follows from (45) and (48). □

This allows us to prove a theorem about how to derive high-order derivatives of Lucas polynomials in terms of their original polynomials. First, the following preliminary lemma is useful.

Lemma 2. *For every non-negative q, one has*

$$\sum_{p=0}^{\ell} \binom{p+q-1}{p} (-2p-q+r)\,(-p-q+r+1)_{q-1} = \frac{(\ell+1)\,(-\ell+r-1)\,(\ell+q)!\,(-\ell+r-2)!}{(\ell+1)!\,q!\,(-\ell-q+r-1)!}.$$

Proof. The proof can be easily followed by induction. □

Theorem 8. *The q-th derivative of Lucas polynomials $L_r(x)$ can be expressed as:*

$$L_r^{(q)}(x) = r \sum_{\ell=0}^{\lfloor \frac{r-q}{2} \rfloor} (-1)^\ell \, \xi_{r-2\ell-q} \binom{\ell+q-1}{\ell} (-\ell-q+r+1)_{q-1} \, L_{r-2\ell-q}(x). \tag{49}$$

Proof. We will prove (49) by induction. From Lemma 1, the theorem is true for $q = 1$. Now, assume the validity of (49). We will show the validity of the following relation:

$$L_r^{(q+1)}(x) = r \sum_{\ell=0}^{\lfloor \frac{r-q-1}{2} \rfloor} (-1)^\ell \, \xi_{r-2\ell-q-1} \binom{\ell+q}{\ell} (r-\ell-q)_q \, L_{r-2\ell-q-1}(x). \tag{50}$$

Differentiating both sides of relation (49) with respect to x and making use of Lemma 1 yields

$$L_r^{(q+1)}(x) = r \sum_{\ell=0}^{\lfloor \frac{r-q-1}{2} \rfloor} (-1)^\ell \, \xi_{r-2\ell-q-1} \sum_{p=0}^{\ell} \binom{p+q-1}{p} \times \\ (-2p-q+r)\,(-p-q+r+1)_{q-1} \, L_{r-2\ell-q-1}(x).$$

The application of Lemma 2 allows one to obtain relation (49). □

Now, we are going to state and prove a theorem in which repeated integrals of $L_j(x)$ are expressed in terms of their original polynomials.

Theorem 9. *Let $I_j^q(x)$ be defined as $I_j^q(x) = \underbrace{\int \int \ldots \int}_{q \text{ times}} L_j(x) \underbrace{dx\, dx \ldots dx}_{q \text{ times}}$. Then,*

$$I_j^q(x) = j \sum_{\ell=0}^{q} \frac{\binom{q}{\ell}}{(j-\ell)_{1+q}} \, L_{j+q-2\ell}(x) + \rho_{q-1}(x), \tag{51}$$

where $\rho_{q-1}(x)$ is a polynomial of degree not exceeding $(q-1)$.

Proof. We will prove (51) by induction on q. The theorem holds for $q = 1$ following the structure Formula (7). Assume that (51) holds. We will show that the following relation holds:

$$I_j^{q+1}(x) = j \sum_{\ell=0}^{q+1} \frac{\binom{q+1}{\ell}}{(j-\ell)_{2+q}} L_{j+q-2\ell+1}(x) + \bar{\rho}_q(x),$$

with $\bar{\rho}_q(x)$—a polynomial of degree not exceeding q.

Now, making use of relation (51) together with the structure Formula (7) yields

$$I_j^{(q+1)}(x) = j \sum_{\ell=0}^{q} \frac{\binom{q}{\ell}}{(j-\ell)_{1+q}} \left(\frac{L_{j+q+1-2\ell}(x)}{j-2\ell+q+1} + \frac{L_{j+q-1-2\ell}(x)}{j-2\ell+q-1} \right).$$

The last formula can be written alternatively as

$$\begin{aligned} I_j^{q+1}(x) =& j \sum_{\ell=1}^{q} \frac{1}{1+j-2\ell+q} \left\{ \frac{\binom{q}{\ell}}{(j-\ell)_{1+q}} + \frac{\binom{q}{-1+\ell}}{(1+j-\ell)_{1+q}} \right\} L_{j-2\ell+q+1}(x) \\ &+ \frac{(j-q-2)!}{(j-1)!} L_{j-q-1}(x) + \frac{j!}{(j+q+1)!} L_{j+q+1}(x). \end{aligned} \tag{52}$$

It is not difficult to show the identity

$$\frac{1}{1+j-2\ell+q} \left\{ \frac{\binom{q}{\ell}}{(j-\ell)_{1+q}} + \frac{\binom{q}{-1+\ell}}{(1+j-\ell)_{1+q}} \right\} = \frac{\binom{q+1}{\ell}}{(j-\ell)_{2+q}},$$

hence, relation (52) turns into

$$I_j^{q+1}(x) = j \sum_{\ell=0}^{q+1} \frac{\binom{q+1}{\ell}}{(j-\ell)_{2+q}} L_{j+q-2\ell+1}(x) + \bar{\rho}_q(x).$$

This concludes the proof. □

6. Conclusions

In this paper, we establish some new formulas concerned with Lucas polynomials. Different approaches are followed. A matrix approach is followed for some fundamental properties. Several connection formulas between Lucas polynomials and different orthogonal polynomials are derived. Furthermore, high-order derivatives formulas and repeated integrals formula of Lucas polynomials are established. As far as we know, most of the formulas derived in this paper are new and they may be useful in various applications. As future work, we hope to investigate some types of generalized and modified Lucas polynomials.

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