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Optimal Control Problems for Hilfer Fractional Neutral Stochastic Evolution Hemivariational Inequalities

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Abstract: In this paper, we concentrate on a control system with a non-local condition that is governed by a Hilfer fractional neutral stochastic evolution hemivariational inequality (HFNSEHVI). By using concepts of the generalized Clarke sub-differential and a fixed point theorem for multivalued maps, we first demonstrate adequate requirements for the existence of mild solutions to the concerned control system. Then, using limited Lagrange optimal systems, we demonstrate the existence of optimal state-control pairs that are regulated by an HFNSEHVI with a non-local condition. In order to demonstrate the existence of fixed points, the symmetric structure of the spaces and operators that we create is essential. Without considering the uniqueness of the control system's solutions, the best control results are established. Lastly, an illustration is used to demonstrate the major result.

Keywords: optimal control; Hilfer fractional derivative; stochastic evolution equation; neutral system; hemivariational inequalities; non-local condition

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1. Introduction

The hemivariational inequality (HVI) was originally established by Panagiotopoulos in 1981 as a weak formulation for a number of kinds of mechanical problems using non-smooth and non-convex energy functionals [1,2]. Recently, many researchers have paid a lot of attention to the control problems of hemivariational inequalities (HVIs). Migórski and Ochal [3] explored the direct approach of the calculus of variations along with the Galerkin method in 2000 when it came to the optimal control issues of the parabolic HVIs. Using the Faedo-Galerkin technique and the direct method of the arithmetic of modification, the authors [4] showed in 2007 that there are optimum control pairs for a hyperbolic quasi-linear HVI. Researchers [5] recently employed a convergence point approach for multivalued maps to look at the approximative controllability of HVIs. The optimum control issue of second-order stochastic evolution hemivariational inequalities (SEHVIs) with Poisson jumps was recently addressed by Muthukumar et al. [6] using the fixed point technique of multivalued maps and Balder's theorem. In order to investigate the optimal controls and solvability of impulsive HF delay evolution inclusions with Clarke sub-differential, Harrat et al. [7] used fractional calculus, semigroup theory, fixed point strategy, and multivalued analysis.

Understanding partial differential equations is greatly aided by symmetry analysis, especially when dealing with equations derived from mathematical concepts relating to accounting. Even though symmetry is absent from the majority of natural observations, it is the secret of nature. The occurrence of unexpected symmetry-breaking is a better way to conceal symmetry. The two categories are finite and infinitesimal symmetry. There are two types of discrete and continuous finite symmetries. Natural symmetries like parity and temporal inversion are discrete whereas space is a continuous change. On the other hand, fractional calculus has grown in importance in mathematics during the recent several decades. Fractional-order differential equations are better suitable for various physical issues than integer-order differential equations. Due to their applications in viscoelasticity materials, electrical circuits, neural networks, control theory, chemistry, engineering, biology, mechanics, and physics, fractional differential equations (FDEs) have become a popular research topic ([8–14]). We point out that during the past three decades, FDEs have undergone a substantial evolution (see, for instance, [15–20]) and are a useful instrument for the description of specific materials and processes [21–30].

Stochastic differential equations (SDEs) are useful tools in several branches of science and engineering for describing some systems and processes with stochastic disturbances. See the monograph [31] for information on the general concept of stochastic differential equations. Furthermore, it should be noted that both natural and artificial systems exhibit stochastic discomfort or noise. Stochastic differential systems have garnered a lot of interest because of their numerous uses in the biological, physical, and pharmaceutical sciences (see [32–34]); they are crucial for simulating real-world processes when an element of randomness is required. Stochastic evolution equations (SEEs) in infinite-dimensional spaces are inspired by the random events studied in the biological sciences, such as thermodynamics, molecular biology, and operations research. Many authors have extensively studied the existence of mild solutions for various types of SEEs and their optimum management in Hilbert spaces (see [35–37]).

The Reimann–Liouville (R-L) and Caputo fractional derivatives are also included in the Hilfer fractional derivative (HFD), which was first developed by Hilfer [9]. In theoretical electromagnetic simulations of glass-forming components, it is used. The presence of mild solutions to an evolution equation with HFD was looked at by Gu and Trujillo in [38]. The solvability and best controls of impulsive Hilfer fractional (HF) delay evolution systems with Clarke sub-differential were investigated by Harrat et al. in [7]. Non-local conditions for HF evolution equations provide some fascinating findings. Yang and Wang, for instance, looked at the approximability of controllability of an HF differential system with non-local conditions in [39]. The existence of mild solutions to an HF differential equation with non-local conditions was investigated by the researchers in [40,41].

Due to their vast applicability in numerous fields of pragmatic mathematics, neutral systems have attracted increased interest in recent years. With or without delay, various neutral systems, such as thermal expansion in substances, stretchability, surface waves, and several organic improvements, profit from neutral systems. Readers can consult [19,33,42,43] for more information on the neutral system and its use.

The idea of controllability is crucial to the study and design of control systems, as is well known. Furthermore, it is important to research hemivariational inequalities with fractional derivatives since they are related to applicable disciplines including evaporation in heat exchangers, thermoviscoelasticity, and selective memory thermodynamics. The authors [44] investigate the HF evolution hemivariational inequalities with non-local initial conditions and optimal controls for condensing multivalued maps. The fixed point theory for multivalued maps has been used to generate the optimum control issues for Hilfer fractional neutral stochastic evolution hemivariational inequalities, which were inspired by the aforementioned work.Both the evaluation of hemivariational inequality and the consideration of the controllability of the control systems described by a class of stochastic HVI with fractional derivatives appear to have been neglected.

We believe that the literature has not yet addressed the presence of and optimal controls for the HF neutral stochastic evolution hemivariational inequality (HFNSEHVI).

This work's main focus will be on the existence and optimal control of the subsequent control system, which is governed by HFNSEHVI.

$$\begin{cases} \langle D_{0^+}^{\delta,\eta}[\varkappa(\varrho)-\sigma(\varrho,\varkappa(\varrho))]+\widetilde{A}[\varkappa(\varrho)-\sigma(\varrho,\varkappa(\varrho))]+\mathcal{B}(\varrho)\mathfrak{u}(\varrho)+\mathsf{F}(\varrho,\varkappa(\varrho))\frac{dW(\varrho)}{d\varrho},\omega\rangle_{\mathbb{Z}} \\ \qquad\qquad +\mathcal{G}^0(\varrho,\varkappa(\varrho);\omega)\geq 0, \quad \varrho\in\mathcal{W}=[0,\grave{a}],\ \forall\,\omega\in\mathbb{Z}, \\ \mathcal{I}_{0^+}^{(1-\delta)(1-\eta)}[\varkappa(\varrho)]_{\varrho=0}+\hbar(\varkappa)=\varkappa_0, \end{cases} \tag{1}$$

where $D_{0^+}^{\delta,\eta}$ denotes the HFD, $\varrho\in\mathcal{W}'=[0,\grave{a}]$, $\delta\in[0,1]$, $\eta\in(0,1)$. The state variables $\varkappa(\cdot)$ takes values in the Hilbert space $\mathbb{Z}$ with the norm $\|\cdot\|_{\mathbb{Z}}$ and the inner product $\langle\cdot,\cdot\rangle_{\mathbb{Z}}$. The infinitesimal generator of the strongly continuous cosine family $\mathcal{N}(\varrho)$, $(\varrho>0)$ in $\mathbb{Z}$ is $\widetilde{A}:D(\widetilde{A})\subset\mathbb{Z}\to\mathbb{Z}$. Let $\mathfrak{u}$ be a control function, and J be the set of all admissible controls that is also a Hilbert space. $\mathcal{B}:J\to\mathbb{Z}$ is a bounded linear operator. The function $\sigma:\mathcal{W}\times\mathbb{Z}\to\mathbb{Z}$ and $\mathsf{F}(\varrho,\varkappa(\varrho)):\mathcal{W}\times\mathbb{Z}\to L_2^0$ are the appropriate function. Let $(\Lambda,\mathscr{E},P)$ be a complete probability spaces, and let K be the other separable Hilbert space. Assume the Wiener process $\{W(\varrho):\varrho>0\}$ is a K-Wiener process with nuclear covariance operator $Q\geq 0$ having a finite trace. The norm of $L(\mathrm{K},\mathbb{Z})$ is denoted by the same notations, $\|\cdot\|$, where $L(\mathrm{K},\mathbb{Z})$ represents the space of all bounded operators from K into $\mathbb{Z}$. Simply as $L(\mathrm{K},\mathbb{Z})=L(\mathbb{Z})$ if $\mathrm{K}=\mathbb{Z}$. The Clarke sub-differential of a globally Lipschitz function $\mathcal{G}(\varrho,\cdot):\mathbb{Z}\to\mathbb{R}$ is denoted by the notation $\mathcal{G}^0(\varrho,\cdot;\cdot)$. Let $\mathscr{A}_{ad}$ be the set of all admissible state control pairs $(\varkappa,\mathfrak{u})$ and E denote the expectation of a random variable or the Lebesgue integral with regard to the probability measure P. The cost functional on the set $\mathscr{A}_{ad}$ is provided by

$$\mathscr{K}(\varkappa,\mathfrak{u})=E\int_0^{\grave{a}}\mathscr{L}(\varrho,\varkappa^{\mathfrak{u}}(\varrho),\mathfrak{u}(\varrho))d\varrho. \tag{2}$$

The plan of this paper is organized into five sections. We list some important preliminaries in Section 2. In Section 3 we demonstrate that the system (1) has a mild solution given a few reasonable assumptions. We answer the optimal control problem governed by (1) under acceptable criteria in Section 4. In Section 5, a specific illustration is given to demonstrate our primary findings.

2. Preliminaries

This section provides the basic material as well as the essential fractional calculus ideas, notations, and lemmas that are necessary to establish the main findings.

Suppose that $\mathbb{Z}$ is a separable Hilbert space and its norm represented by $\|\cdot\|_{\mathbb{Z}}$. Consider $(\Lambda,\mathscr{E},P)$ denotes the complete probability space with the usual classification $\{\mathscr{E}_\varrho,\ \varrho>0\}$. $L^2(\mathscr{E},\mathbb{Z})=L^2(\Lambda,\mathscr{E},P,\mathbb{Z})$ denotes the Hilbert space of all strongly $\mathscr{E}$-measurable square integrable $\mathbb{Z}$-valued random variable satisfying $E\|\varkappa\|_{\mathbb{Z}}^2<\infty$. Suppose that $C(\mathcal{W},L^2(\mathscr{E},\mathbb{Z}))$ is the Banach space of all continuous maps from $\mathcal{W}\to L^2(\mathscr{E},\mathbb{Z})$ with $\|\varkappa\|_{L^2}=\left[\sup_{\varrho\in[0,\grave{a}]}E\|\varkappa(\varrho)\|_{\mathbb{Z}}^2\right]^{\frac{1}{2}}<\infty$. $L_{\mathscr{E}}^2(\mathcal{W},\mathbb{Z})$ represent the Hilbert space of all stochastic processes $\mathscr{E}_\varrho$-adapted measurable determined on $\mathcal{W}$ using values in $\mathbb{Z}$ and a norm $\|\varkappa\|_{L_{\mathscr{E}}^2(\mathcal{W},\mathbb{Z})}=\left[\int_0^{\grave{a}}E\|\varkappa(\varrho)\|_{\mathbb{Z}}^2d\varrho\right]^{\frac{1}{2}}<\infty$. The space $L_{\mathscr{E}}^2(\mathcal{W},J)$ represents the Hilbert space of all stochastic processes $\mathscr{E}_\varrho$-adapted measurable determined on $\mathcal{W}$ assuming values in J and a norm $\|\mathfrak{u}\|_{L_{\mathscr{E}}^2(\mathcal{W},J)}=\left[\int_0^{\grave{a}}E\|\mathfrak{u}(\varrho)\|_J^2d\varrho\right]^{\frac{1}{2}}<\infty$.

We suppose that $\exists$ a complete orthonormal system $\{e_n\}$ in K, a bounded sequence of non-negative real integers $\{\beta_n\}$, such that $Qe_n=\beta_ne_n$, $n=1,2,\cdots$, and a sequence $\{\mu_n\}$ of independent Wiener process, such that

$$\langle W(\varrho),\varepsilon\rangle=\sum_{n=1}^{\infty}\sqrt{\beta_n}\langle e_n,\varepsilon\rangle\mu_n(\varrho), \quad \varepsilon\in\mathrm{K},\ \varrho>0.$$

Let $\phi \in L(\mathrm{K}, \mathbb{Z})$ and defined by

$$\|\phi\|_Q^2 = Tr(\phi Q \phi^*) = \sum_{n=1}^{\infty} \left\| \sqrt{\beta_n} \phi e_n \right\|^2.$$

Suppose that $\|\phi\|_Q < \infty$, then ϕ is called a Q-Hilbert Schmidt operator. Let the space of all Q-Hilbert Schmidt operators $\phi : \mathrm{K} \to \mathbb{Z}$ be defined as $L_Q(\mathrm{K}, \mathbb{Z})$. The fulfilment $L_Q(\mathrm{K}, \mathbb{Z})$ of $L(\mathrm{K}, \mathbb{Z})$ with regard to the geometry caused by $\| \cdot \|_Q$, with $\|\phi\|_Q^2 = \langle \phi, \phi \rangle$ is a Hilbert space with the above norm geometry.

The concepts from fractional calculus are introduced below. For more information [10,13].

Definition 1 (see [10,13])**.** *For a function g, the fractional integral of order $\delta > 0$ with the lower bound zero is defined as*

$$I_\varrho^\delta g(\varrho) = \frac{1}{\Gamma(\delta)} \int_0^\varrho (\varrho - \varpi)^{\delta - 1} g(\varpi) d\varpi, \; \varrho > 0,$$

if the right side is point-wise defined on $[0, +\infty)$, where Γ is the gamma function. We make the unassumed assumption that the gamma functions utilised in this work are real without loss of generality.

Definition 2 (see [10,13])**.** *For a function $g : [0, +\infty) \to \mathbb{R}$, the R-L derivative of order $\delta > 0$ with lower limit zero can be denoted as*

$${}^L D_\varrho^\delta g(\varrho) = \frac{1}{\Gamma(m - \delta)} \frac{d^m}{d\varrho^m} \int_0^\varrho \frac{g(\varpi)}{(\varrho - \varpi)^{\delta - m + 1}} d\varpi, \; m - 1 < \delta < m, \; m \in \mathbb{N}.$$

Definition 3 (see [10,13])**.** *The Caputo fractional derivative of order $\delta > 0$ can be defined as*

$$D_{0^+}^\delta g(\varrho) = ^L D_\varrho^\delta \left(g(\varrho) - \sum_{n=1}^{m-1} \frac{\varrho^n}{n!} g^n(0) \right), \; m = 1 < \delta < m, \; m \in \mathbb{N},$$

where the derivative of the function g is completely continuous up to order $m - 1$.

Definition 4 (see [9])**.** *The HFD of order $0 \le \delta \le 1$, $0 < \eta < 1$ for the function $\varkappa$ is defined by*

$$D_{0^+}^{\delta, \eta} g(\varrho) = [\mathcal{I}_{0^+}^{\delta(1-\eta)} D(\mathcal{I}_{0^+}^{(1-\delta)(1-\eta)} g)](\varrho).$$

We now go over numerous fundamental properties of a multivalued map; for further information, please see the works by [45,46].

In the case of a Banach space Y with $\| \cdot \|$, Y^* designates the dual of Y, and $\langle \cdot, \cdot \rangle$ the pairing of Y and Y^*. For our satisfaction, we will be using the following conditions:

$$\mathcal{P}_{g(c)}(Y) = \{\Lambda \subseteq Y : \Lambda \text{ is non-empty, closed (convex)}\},$$
$$\mathcal{P}_{(w)k(c)}(Y) = \{\Lambda \subseteq Y : \Lambda \text{ is non-empty, (weakly) compact (convex)}\}.$$

We will now define the generalised Clarke gradient for a globally Lipschitzian functional $\mathcal{G} : Y \to \mathbb{R}$. $\mathcal{G}_0(\varkappa; \omega)$ represents the Clarke geometric derivative of $\mathcal{G}$ at $\varkappa$ in the direction ω, i.e.,

$$\mathcal{G}^0(\varkappa, \omega) = \lim_{\varkappa' \to \varkappa} \sup_{\beta \to 0^+} \frac{\mathcal{G}(\varkappa' + \beta\omega) - \mathcal{G}(\varkappa')}{\beta}.$$

As you may recall, the Clarke sub-differential of $\mathcal{G}$ at $\varkappa$, denoted by $\partial\mathcal{G}$, is a subset of Y^* generated by

$$\partial\mathcal{G}(\varkappa) = \{\varkappa^* \in Y^* : \mathcal{G}^0(\varkappa, \omega) \ge \langle \varkappa^*, \omega \rangle, \; \forall \, \omega \in Y\}.$$

The upcoming fundamental characteristics of the generalised geometric derivative and the generalised gradient are crucial to our main conclusions.

Proposition 1 (see [47]). *If $g : \Lambda \to \mathbb{R}$ is a globally Lipschitz function on an open set Λ of $\mathbb{Z}$, then*

(i) $\forall\, z \in \mathbb{Z}$, *one has* $g^0(\varkappa, \omega) = \max\{\langle \varkappa^*, \omega\rangle : \forall\, \varkappa \in \partial g(\varkappa)\}$;

(ii) $\forall\, \varkappa \in \Lambda$, *the derivative $\partial g(x)$ is a convex, non-empty, weak*-compact subset of $\mathbb{Z}^*$ and $\|\varkappa^*\|_{\mathbb{Z}^*} \leq \mathbb{K}\ \forall\, \varkappa \in g(\varkappa)$ (where $\mathbb{K}$ is the Lipschitz constant of g near $\varkappa$);*

(iii) The graph of the generalized derivative ∂g is closed in $\Lambda \times \mathbb{Z}^*_{v^*}$ topology, i.e., suppose that $\{y_n\} \subset \Lambda$ and $\{y^*_n\} \subset \mathbb{Z}$ are sequences, such that $\varkappa_n \in \partial g(\varkappa_n)$ and $\varkappa_n \to \varkappa$ in $\mathbb{Z}$, $\varkappa_n \to$ weakly* in $\mathbb{Z}$, then $\varkappa \in \partial g(\varkappa)$ (where $\mathbb{Z}^*_{v^*}$ represent the Banach space $\mathbb{Z}$ related with the v^*-topology);

(iv) The multi-valued function Λ such that $\varkappa \to \partial g(\varkappa) \subseteq \mathbb{Z} : \Lambda \to \mathbb{Z}^*_{v^*}$ is upper semi-continuous.

Lemma 1 (see [48]). *Let $G : [0, \grave{a}] \times \Lambda \to L^2_0$ be a strongly measurable mapping such that $\int_0^{\grave{a}} E\|G(\varrho)\|^p_{L^2_0} d\varrho < \infty$. Then*

$$E\left\| \int_0^{\grave{a}} G(\varpi) dW(\varpi) \right\|^p \leq L_G \int_0^{\grave{a}} E\|G(\varpi)\|^p_{L^2_0} d\varpi,$$

$\forall\, 0 \leq \varrho \leq \grave{a}$ and $p \geq 2$, where L_G is the constant employing p and $\grave{a}$.

Theorem 1 (see [49]). *Let Y be a globally convex Banach space and $\mathcal{G} : Y \to 2^Y$ be a compact convex valued, upper semi-continuous multivalued map such that $\exists$ a closed neighborhood V of 0 for which $\mathcal{G}(V)$ is a relatively compact set. Assume that*

$$\Lambda = \{\varkappa \in Y : \lambda \varkappa \in \mathcal{G}(\varkappa)\ \forall\, \lambda > 1\}$$

is bounded, then $\mathcal{G}$ has a fixed point.

3. Existence

The following fractional evolution inclusion can be taken into account while analyzing system (1):

$$\begin{cases} D^{\delta,\eta}_{0^+}[\varkappa(\varrho) - \sigma(\varrho, \varkappa(\varrho))] \in \widetilde{A}[\varkappa(\varrho) - \sigma(\varrho, \varkappa(\varrho))] + \mathcal{B}(\varrho)\mathfrak{u}(\varrho) + \mathsf{F}(\varrho, \varkappa(\varrho))\frac{dW(\varrho)}{d\varrho} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \partial\mathcal{G}(\varrho, \varkappa(\varrho)), \quad \varrho \in \mathcal{W}', \\ \mathcal{I}^{(1-\delta)(1-\eta)}_{0^+}[\varkappa(\varrho)]_{\varrho=0} + \hbar(\varkappa) = \varkappa_0, \end{cases} \tag{3}$$

where $\partial\mathcal{G}$ denotes the generalized Clarke sub-differential of a globally Lipschitz functional $\mathcal{G}(\varrho, \cdot) : \mathbb{Z} \to \mathbb{R}$. The control function $\mathfrak{u}(\cdot)$ is a stochastic process provided in $L^2_{\mathscr{E}}(\mathcal{W}, J)$ of admissible control functions, and the set J is a Hilbert space, $\mathcal{B} : J \to \mathbb{Z}$ is a bounded linear operator. $\mathsf{F} : \mathcal{W} \times \mathscr{E} \to L^2_0$ is a appropriate functions and $\varkappa_0$ is measurable $\mathbb{Z}$-valued random variables independent of W.

It is clear that each solution to system (3) also solves system (1). In reality, suppose $\varkappa(\varrho) \in C(\mathcal{W}, L^2(\mathscr{E}, \mathbb{Z}))$ is a solution of the system (1), then $\exists$ a function $g(\varrho) \in \partial\mathcal{G}(\varrho, \varkappa(\varrho))$, a.e., $\varrho \in \mathcal{W}$, and satisfies the following equation:

$$\begin{cases} D^{\delta,\eta}_{0^+}[\varkappa(\varrho) - \sigma(\varrho, \varkappa(\varrho))] \in \widetilde{A}[\varkappa(\varrho) - \sigma(\varrho, \varkappa(\varrho))] + \mathcal{B}(\varrho)\mathfrak{u}(\varrho) + \mathsf{F}(\varrho, \varkappa(\varrho))\frac{dW(\varrho)}{d\varrho} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \partial\mathcal{G}(\varrho, \varkappa(\varrho)), \quad \varrho \in \mathcal{W}', \\ \mathcal{I}^{(1-\delta)(1-\eta)}_{0^+}[\varkappa(\varrho)]_{\varrho=0} + \hbar(\varkappa) = \varkappa_0. \end{cases}$$

In view of above equation, we obtain

$$\begin{cases} \langle D_{0^+}^{\delta,\eta}[\varkappa(\varrho)-\sigma(\varrho,\varkappa(\varrho))]+\widetilde{A}[\varkappa(\varrho)-\sigma(\varrho,\varkappa(\varrho))]+\mathcal{B}(\varrho)\mathrm{u}(\varrho)+\mathsf{F}(\varrho,\varkappa(\varrho))\frac{dW(\varrho)}{d\varrho},\omega\rangle_{\mathbb{Z}} \\ \qquad\qquad +\langle\mathcal{G}(\varrho),\omega\rangle_{\mathbb{Z}}=0, \quad \text{a.e. } \varrho\in\mathcal{W}',\ \forall\,\omega\in\mathbb{Z}, \\ \mathcal{I}_{0^+}^{(1-\delta)(1-\eta)}[\varkappa(\varrho)]_{\varrho=0}+\hbar(\varkappa)=\varkappa_0. \end{cases}$$

Since $g(\varrho)\in\partial\mathcal{G}(\varrho,\varkappa(\varrho))$ and $\langle\mathcal{G}(\varrho),\omega\rangle_{\mathbb{Z}}\le\mathcal{G}^0(\varrho,\varkappa(\varrho);\omega)$, we obtain

$$\begin{cases} \langle D_{0^+}^{\delta,\eta}[\varkappa(\varrho)-\sigma(\varrho,\varkappa(\varrho))]+\widetilde{A}[\varkappa(\varrho)-\sigma(\varrho,\varkappa(\varrho))]+\mathcal{B}(\varrho)\mathrm{u}(\varrho)+\mathsf{F}(\varrho,\varkappa(\varrho))\frac{dW(\varrho)}{d\varrho},\omega\rangle_{\mathbb{Z}} \\ \qquad\qquad +\mathcal{G}^0(\varrho,\varkappa(\varrho);\omega)\ge 0, \quad \varrho\in\mathcal{W}',\ \forall\,\omega\in\mathbb{Z}, \\ \mathcal{I}_{0^+}^{(1-\delta)(1-\eta)}[\varkappa(\varrho)]_{\varrho=0}+\hbar(\varkappa)=\varkappa_0. \end{cases}$$

It is proved that by using the equivalent evolution inclusion system (3), we may refer the system (1).

We now define the mild solution to system (3) using the Wright function:

$$\mathfrak{M}_\eta(\theta)=\sum_{n=1}^{\infty}\frac{(-\theta)^{n-1}}{(n-1)\Gamma(1-\mu n)}, \quad 0<\mu<1,\ \theta\in\mathbb{C},$$

that fulfills the equality

$$\int_0^\infty \theta^\iota\mathfrak{M}_\eta(\theta)d\theta=\frac{\Gamma(1+\iota)}{\Gamma(1+\eta\iota)}, \quad \theta\ge 0.$$

Lemma 2 (see [19]). *The operators $\mathcal{M}_{\delta,\eta}$, $\mathcal{N}_\eta$ and $\mathbb{Q}_\eta$ admit the following conditions:*

(*a*) *For any fixed $\varrho>0$, $\mathcal{M}_{\delta,\eta}(\varrho)$, $\mathcal{N}_\eta(\varrho)$ and $\mathbb{Q}_\eta(\varrho)$ are bounded linear operators such that, $\forall$ $\varkappa\in\mathbb{Z}$,*

$$\|\mathcal{M}_{\delta,\eta}(\varrho)\varkappa\|\le\frac{M\varrho^{\eta-1}}{\Gamma(\delta(1-\eta)+\eta)}\|\varkappa\|, \quad \|\mathcal{N}_\eta(\varrho)\varkappa\|\le\frac{M\varrho^{\eta-1}}{\Gamma(\eta)}\|\varkappa\| \text{ and}$$
$$\|\mathbb{Q}_\eta(\varrho)\varkappa\|\le\frac{M\varrho^{\eta-1}}{\Gamma(\eta)}\|\varkappa\|.$$

(*b*) *$\{\mathcal{M}_{\delta,\eta}(\varrho),\ \varrho>0\}$, $\{\mathcal{N}_\eta(\varrho),\ \varrho>0\}$ and $\{\mathbb{Q}_\eta(\varrho),\ \varrho>0\}$ are strongly continuous.*

(*c*) *If $T(\varrho)$ is compact, then $\forall\ \varrho>0$, $\mathcal{M}_{\delta,\eta}(\varrho)$, $\mathcal{N}_\eta(\varrho)$ and $\mathbb{Q}_\eta(\varrho)$ are also compact operators.*

Lemma 3 (see [48]). *Suppose $\{T(\varrho)\}_{\varrho>0}$ is a compact C_0-semigroup $\forall\ \varrho>0$, then it is uniformly continuous $\forall\ \varrho>0$.*

Proposition 2. *Consider $\eta\in(0,1)$, $q\in(0,1]$ and $\forall\ \varkappa\in D(\widetilde{A})$, then $\exists$ a $\kappa_q>0$ such that*

$$\|\widetilde{A}^q\mathbb{Q}_\eta(\varrho)\varkappa\|\le\frac{\eta\kappa_q\Gamma(2-q)}{\varrho^{\eta q}\Gamma(1+\eta(1-q))}\|\varkappa\|,\ 0<\varrho<\grave{a}.$$

Definition 5. *For each $\mathrm{u}\in L^2_{\mathscr{E}}(\mathcal{W},J)$, an $\mathscr{E}_\varrho$-adapted stochastic process $\varkappa\in C(\mathcal{W},L^2(\mathscr{E},\mathbb{Z}))$ is called a mild solution of the control system (3) suppose $\varkappa(0)=\varkappa_0\in\mathbb{Z}$ and $\exists$ a $g\in L^2_{\mathscr{E}}(\mathcal{W},\mathbb{Z})$, such that $g(\varrho)\in\partial\mathcal{G}(\varrho,\varkappa(\varrho))$, a.e., $\varrho\in\mathcal{W}$ and*

$$\begin{aligned}\varkappa(\varrho)=&\ \mathcal{M}_{\delta,\eta}(\varrho)[\varkappa(0)-\hbar(\varkappa)-\sigma(0,\varkappa(0))]+\sigma(\varrho,\varkappa(\varrho))\\&+\int_0^\varrho(\varrho-\varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho-\varpi)[g(\varpi)+\mathcal{B}\mathrm{u}(\varpi)]d\varpi\\&+\int_0^\varrho(\varrho-\varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho-\varpi)\mathsf{F}(\varpi,\varkappa(\varpi))dW(\varpi),\end{aligned}$$

$\forall\ \varrho \in \mathcal{W}$, *where*

$$\mathcal{M}_{\delta,\eta}(\varrho) = \mathcal{I}^{\eta(1-\delta)}\theta\mathcal{N}_{\eta}(\varrho),\ \mathcal{N}_{\eta}(\varrho) = \varrho^{\eta-1}\mathbb{Q}_{\eta}(\varrho) \text{ and } \mathbb{Q}_{\eta}(\varrho) = \int_0^{\infty} \eta\xi\mathfrak{M}_{\eta}(\xi)\mathcal{N}(\varrho^{\eta}\xi)d\xi.$$

The following hypotheses are used throughout this paper:

(H_0) *The operator* $\mathcal{M}(\varrho)$ *is compact* $\forall\ \varrho > 0$.

(H_1) *The function* $\varrho \to \mathcal{M}(\varrho)$ *is continuous in* $\mathcal{B}(\mathbb{Z})\ \forall\ \varrho > 0$, *and* $\exists$ *a constant* $M > 1$, *such that* $\|\mathcal{M}(\varrho)\| \leq M$.

(H_2) $\mathcal{G} : \mathcal{W} \times \mathbb{R}$ *fulfils the following requirements:*

- (a) $\forall\ \varkappa \in \mathbb{Z}$, $\mathcal{G}(\cdot, \varkappa)$ *is measurable;*
- (b) *For a.e.* $\varrho \in \mathcal{W}$, $\mathcal{G}(\varrho; \cdot)$ *is globally Lipschitz continuous;*
- (c) $\exists$ *a* $b \in L^{\frac{1}{\gamma}}(\mathcal{W}, \mathbb{R}^+)$, $\gamma \in (0, 2\eta - 1)$ *and a constant* $c \geq 0$, *such that*

$$E\|\partial\mathcal{G}(\varrho, \varkappa)\|^2 = \sup\{E\|g(\varrho)\|^2 : g(\varrho) \in \partial\mathcal{G}(\varrho, \varkappa)\} \leq b(\varrho) + c\|\varkappa\|^2,$$

for a.e. $\varrho \in \mathcal{W}$ *and* $\forall\ \varkappa \in \mathbb{Z}$.

(H_3) $\mathsf{F} : \mathcal{W} \times \mathbb{Z} \to L_0^2$ *is continuous in the second variable for a.e.* $\varrho \in \mathcal{W}$ *and* $\exists$ *a function* $d \in L^{\frac{1}{\gamma}}(\mathcal{W}, \mathbb{R}^+)$, $\gamma \in (0, 2\eta - 1)$ *and a constant* $e \geq 0$, *such that*

$$E\|\mathsf{F}(\varrho, \varkappa)\|^2 \leq d(\varrho) + e\|\varkappa\|^2.$$

(H_4) $\exists\ L_{\hbar}$, *a constant such that* $\forall\ \varkappa_1; \varkappa_2 \in C$,

$$E\|\hbar(\varkappa_1) - \hbar(\varkappa_2)\|^2 \leq L_{\hbar}\|\varkappa_1 - \varkappa_2\|^2.$$

(H_5) $\sigma : \mathcal{W} \times \mathbb{Z} \to \mathbb{Z}$ *is a continuous function and* $\exists$ *constants* $q \in (0, 1)$ *and* $M_\sigma > 0$, *such that* σ *is* $\mathbb{Z}_q$*-valued and fulfils the following requirements:*

$$E\|\sigma(\varrho, \varkappa)\|^2 \leq M_\sigma(1 + \|\varkappa\|^2_{\mathbb{Z}}),\ \varkappa \in \mathbb{Z},\ \varrho \in \mathcal{W}.$$

Define the admissible set as follows:

$$\mathscr{U}_{ad} = \left\{\mathrm{u}(\cdot) \in L^p_{\mathscr{E}}(\mathcal{W}, J); \mathrm{u}(\varrho) \in J \text{ a.e. } \varrho \in \mathcal{W}\right\}.$$

Then, by Proposition 2.1.7 and Lemma 2.3.2 of [46], we know that $\mathscr{U}_{ad} \neq \emptyset$; *and* $\mathscr{U}_{ad}$ *is bounded, convex, and closed subset of* $L^p(\mathcal{W}, J)$ *with* $1 < p < \infty$. *Clearly,* $\mathcal{B}\mathrm{u} \in L^p(\mathcal{W}, \mathbb{Z})\ \forall\ \mathrm{u} \in \mathscr{U}_{ad}$. *Next, define an operator* $\Upsilon : L^2_{\mathscr{E}}(\mathcal{W}, \mathbb{Z})$ *by*

$$\Upsilon(\varkappa) = \{W \in L^2_{\mathscr{E}}(\mathcal{W}, \mathbb{Z}) : W(\varrho) \in \partial\mathcal{G}(\varrho, \varkappa(\varrho)) \text{ a.e. } \varrho \in \mathcal{W}\ \forall\ \varkappa \in L^2_{\mathscr{E}}(\mathcal{W}, \mathbb{Z})\}.$$

We also require the following lemmas in order to reach our main results:

Lemma 4 (see [35])**.** *Provided that* (H_2) *holds, then* $\forall\ \varkappa \in L^2(\mathcal{W}, \mathbb{Z})$, *the set* $\Upsilon(\varkappa)$ *has non-empty, convex and weakly compact values.*

Lemma 5 (see [35])**.** *Suppose that* (H_2) *holds,* Υ *satisfies: if* $\varkappa_n \to \varkappa \in L^2(\mathcal{W}, \mathbb{Z})$, $z_k \to z$ *weakly in* $L^2(\mathcal{W}, \mathbb{Z})$ *and* $z_k \in \Upsilon(\varkappa_k)$, *therefore* $z \in \Upsilon(\varkappa)$.

Lemma 6 (see [6])**.** *Suppose that* (H_2) *holds and the operator* Υ *fulfills: if* $\varkappa_n \to \varkappa$ *in* $L^2_{\mathscr{E}}(\mathcal{W}, \mathbb{Z})$, $W_n \to W$ *weakly in* $L^2_{\mathscr{E}}(\mathcal{W}, \mathbb{Z})$ *and* $W_n \in \Upsilon(\varkappa_n)$, *then* $W \in \Upsilon(\varkappa)$.

Theorem 2. *Suppose that $(H_0) - (H_5)$ holds, then the HF stochastic system (1) has a mild solution on $\mathcal{W}$ given by*

$$S = \left(\frac{2M\grave{a}^{\eta-1}}{\Gamma(\eta)}\right)^2 \frac{\grave{a}^{2\eta-1}}{2\eta-1} + (\grave{a}c + L_G e) < 1.$$

Proof. $\forall\ \varkappa \in C \subset L^2(\mathcal{W}, L^2(\mathscr{E}, \mathbb{Z}))$, by corresponding Lemma 4, consider the multi-operator $\Psi : C \to 2^C$ as follows:

$$\Psi(\varkappa) = \begin{cases} z \in C, \\ z(\varrho) = \mathcal{M}_{\delta,\eta}(\varrho)[\varkappa(0) - \hbar(\varkappa) - \sigma(0, \varkappa(0))] + \sigma(\varrho, \varkappa(\varrho)) \\ \qquad + \int_0^\varrho (\varrho - \varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho - \varpi)[g(\varpi) + \mathcal{B}\mathfrak{u}(\varpi)]d\varpi \\ \qquad + \int_0^\varrho (\varrho - \varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho - \varpi)\mathsf{F}(\varpi, \varkappa(\varpi))dW(\varpi). \end{cases}$$

We have now come to the conclusion that the goal of our concentrated effort was to identify a fixed point of Ψ. We now prove that Ψ satisfies each and every necessary premise of Theorem 1. We organised our evidence into six phases, as shown below, to make it easier to use.

Step 1: Now, we'll demonstrate that $\Psi(\varkappa)$ has convex, non-empty, and weakly compact values $\forall\ \varkappa \in C$. By using Lemma 4, we may simply demonstrate that $\Psi(\varkappa)$ has non-empty and weakly compact values. Additionally, the values of $\Upsilon(\varkappa)$ are convex; by giving $\chi_1, \chi_2 \in \Upsilon(\varkappa)$ and then $\alpha\chi_1 + (1-\alpha)\chi_2 \in \Upsilon(\varkappa)$, we can now draw a result $\forall\ \alpha \in [0,1]$. The function $\Psi(\varkappa)$ is convex.

Step 2: Ψ is bounded in C, where $\mathfrak{D}_q = \{\varkappa \in C : \|\varkappa\|^2 \le q\}$, $\forall\ q > 0$. Certainly, $\mathfrak{D}_q$ is the closed, convex and bounded set of C.

In practise, it is sufficient to demonstrate the existence of a positive constant r^*, such that $\|\vartheta\| \le r^*$, $\forall\ \Upsilon \in \Psi(\varkappa)$, and $\varkappa \in \mathfrak{D}_q$. Suppose that $\vartheta \in \Psi(\varkappa)$, then $\exists$ a function $g \in \Upsilon(\varkappa)$ such that

$$\begin{aligned} \vartheta(\varrho) =& \mathcal{M}_{\delta,\eta}(\varrho)[\varkappa_0 - \hbar(\varkappa) - \sigma(0, \varkappa(0))] + \sigma(\varrho, \varkappa(\varrho)) \\ &+ \int_0^\varrho (\varrho - \varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho - \varpi)g(\varpi)d\varpi + \int_0^\varrho (\varrho - \varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho - \varpi)\mathcal{B}\mathfrak{u}(\varpi)d\varpi \\ &+ \int_0^\varrho (\varrho - \varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho - \varpi)\mathsf{F}(\varpi, \varkappa(\varpi))dW(\varpi), \quad \varrho \in \mathcal{W}'. \end{aligned}$$

From $(H_0) - (H_5)$, Lemma 1 and the Hölder inequality, we obtain

$$\begin{aligned} E\|\vartheta\|^2 \le& 5\Big[E\|\mathcal{M}_{\delta,\eta}(\varrho)[\varkappa_0 - \hbar(\varkappa) - \sigma(0, \varkappa(0))]\|^2 + E\|\sigma(\varrho, \varkappa(\varrho))\|^2 \\ &+ \int_0^\varrho E\|(\varrho - \varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho - \varpi)g(\varpi)d\varpi\|^2 \\ &+ \int_0^\varrho (\varrho - \varpi)^{2(\eta-1)}d\varpi \int_0^\varrho E\|\mathbb{Q}_\eta(\varrho - \varpi)\mathcal{B}\mathfrak{u}(\varpi)\|^2 d\varpi \\ &+ L_G \int_0^\varrho E\|(\varrho - \varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho - \varpi)\mathsf{F}(\varpi, \varkappa(\varpi))d\varpi\|^2\Big] \\ \le& 5\Big[\left(\frac{M\grave{a}^{\eta-1}}{\Gamma(\delta(1-\eta)+\eta)}\right)^2 (2E[\|\varkappa_0\|^2 + \|\hbar(0)\|^2] + L_\hbar r + 2M_\sigma(1 + \|\varkappa_0\|^2)) \\ &+ M_\sigma(1 + \|\varkappa\|^2) + \left(\frac{M\grave{a}^{\eta-1}}{\Gamma(\eta)}\right)^2 \Big\{\left(\frac{1-\gamma}{2\eta-1-\gamma}\right)^{\gamma-1} \grave{a}^{2\eta-1-\gamma}\big[\|b\|_{L^{\frac{1}{\gamma}}(\mathcal{W},\mathbb{R}^+)} \\ &+ L_G\|d\|_{L^{\frac{1}{\gamma}}(\mathcal{W},\mathbb{R}^+)}\big] + \frac{\grave{a}^{2\eta-1}}{2\eta-1}\big[(c + L_G e)q + \|\mathcal{B}\|^2\|\mathfrak{u}\|^2_{L^2_{\mathscr{E}}(\mathcal{W},J)}\big]\Big\} := r^*. \end{aligned}$$

Thus, $\Psi(\mathfrak{D}_q)$ is bounded in $C(\mathcal{W}, L^2(\mathscr{E}, \mathbb{Z}))$.

Step 3: $\{\Psi(\varkappa) : \varkappa \in \mathfrak{D}_q\}$ is equicontinuous.
Firstly, $\forall\ \varkappa \in \mathfrak{D}_q$,

$$\begin{aligned}\vartheta(\varrho) =& \mathcal{M}_{\delta,\eta}(\varrho)[\varkappa_0 - \hbar(\varkappa) - \sigma(0,\varkappa(0))] + \sigma(\varrho,\varkappa(\varrho)) + \int_0^{\varrho}(\varrho-\varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho-\varpi)g(\varpi)d\varpi \\ &+ \int_0^{\varrho}(\varrho-\varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho-\varpi)\mathcal{B}\mathrm{u}(\varpi)d\varpi + \int_0^{\varrho}(\varrho-\varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho-\varpi)\mathsf{F}(\varpi,\varkappa(\varpi))dW(\varpi).\end{aligned}$$

Next, for $\epsilon > 0$ small enough and $0 < \varrho_1 < \varrho_2 \leq \grave{a}$, we get

$$\begin{aligned}
&E\|\vartheta(\varrho_2) - \vartheta(\varrho_1)\|^2 \\
&\leq E\bigg\|\bigg[\mathcal{M}_{\delta,\eta}(\varrho_2)[\varkappa_0 - \hbar(\varkappa) - \sigma(0,\varkappa(0))] + \sigma(\varrho_2,\varkappa(\varrho_2)) + \int_0^{\varrho_2}(\varrho_2-\varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho_2-\varpi)g(\varpi)d\varpi \\
&\quad + \int_0^{\varrho_2}(\varrho_2-\varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho_2-\varpi)\mathcal{B}\mathrm{u}(\varpi)d\varpi + \int_0^{\varrho_2}(\varrho_2-\varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho_2-\varpi)\mathsf{F}(\varpi,\varkappa(\varpi))dW(\varpi)\bigg] \\
&\quad - \bigg[\mathcal{M}_{\delta,\eta}(\varrho_1)[\varkappa_0 - \hbar(\varkappa) - \sigma(0,\varkappa(0))] + \sigma(\varrho_1,\varkappa(\varrho_1)) + \int_0^{\varrho_1}(\varrho_1-\varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho_1-\varpi)g(\varpi)d\varpi \\
&\quad + \int_0^{\varrho_1}(\varrho_1-\varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho_1-\varpi)\mathcal{B}\mathrm{u}(\varpi)d\varpi + \int_0^{\varrho_1}(\varrho_1-\varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho_1-\varpi)\mathsf{F}(\varpi,\varkappa(\varpi))dW(\varpi)\bigg]\bigg\|^2 \\
&\leq 5E\Big\|[\mathcal{M}_{\delta,\eta}(\varrho_2) - \mathcal{M}_{\delta,\eta}(\varrho_1)](\varkappa_0 - \hbar(\varkappa) - \sigma(0,\varkappa(0))\Big\|^2 \\
&\quad + 5E\|\sigma(\varrho_2,\varkappa(\varrho_2)) - \sigma(\varrho_1,\varkappa(\varrho_1))\|^2 \\
&\quad + 5E\bigg\|\int_0^{\varrho_1}[(\varrho_2-\varpi)^{\eta-1} - (\varrho_1-\varpi)^{\eta-1}]\mathbb{Q}_\eta(\varrho_2-\varpi)g(\varpi)d\varpi \\
&\quad + \int_0^{\varrho_1}(\varrho_1-\varpi)^{\eta-1}[\mathbb{Q}_\eta(\varrho_2-\varpi) - \mathbb{Q}_\eta(\varrho_1-\varpi)]g(\varpi)d\varpi \\
&\quad + \int_{\varrho_1}^{\varrho_2}(\varrho_2-\varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho_2-\varpi)g(\varpi)d\varpi\bigg\|^2 \\
&\quad + 5E\bigg\|\int_0^{\varrho_1}[(\varrho_2-\varpi)^{\eta-1} - (\varrho_1-\varpi)^{\eta-1}]\mathbb{Q}_\eta(\varrho_2-\varpi)\mathcal{B}\mathrm{u}(\varpi)d\varpi \\
&\quad + \int_0^{\varrho_1}(\varrho_1-\varpi)^{\eta-1}[\mathbb{Q}_\eta(\varrho_2-\varpi) - \mathbb{Q}_\eta(\varrho_1-\varpi)]\mathcal{B}\mathrm{u}(\varpi)d\varpi \\
&\quad + \int_{\varrho_1}^{\varrho_2}(\varrho_2-\varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho_2-\varpi)\mathcal{B}\mathrm{u}(\varpi)d\varpi\bigg\|^2 \\
&\quad + 5E\bigg\|\int_0^{\varrho_1}[(\varrho_2-\varpi)^{\eta-1} - (\varrho_1-\varpi)^{\eta-1}]\mathbb{Q}_\eta(\varrho_2-\varpi)\mathsf{F}(\varpi,\varkappa(\varpi))dW(\varpi) \\
&\quad + \int_0^{\varrho_1}(\varrho_1-\varpi)^{\eta-1}[\mathbb{Q}_\eta(\varrho_2-\varpi) - \mathbb{Q}_\eta(\varrho_1-\varpi)]\mathsf{F}(\varpi,\varkappa(\varpi))dW(\varpi) \\
&\quad + \int_{\varrho_1}^{\varrho_2}(\varrho_2-\varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho_2-\varpi)\mathsf{F}(\varpi,\varkappa(\varpi))dW(\varpi)\bigg\|^2 \\
&= \sum_{i=1}^{5} J_i.
\end{aligned}$$

By the strong continuity of $\mathcal{M}_{\delta,\eta}(\varrho)$, we get $J_1 \to 0$ as $\varrho_2 \to \varrho_1$. Similarly,

$$J_2 = 5E\|\sigma(\varrho_2,\varkappa(\varrho_2)) - \sigma(\varrho_1,\varkappa(\varrho_1))\|^2 \to 0 \text{ as } \varrho_2 \to \varrho_1.$$

By assumptions $(H_0)-(H_4)$ and the same method used in Lemma 3.1 of [32], we get

$$\begin{aligned}
J_3 \leq 12\Big[&\int_0^{\varrho_1}[(\varrho_2-\varpi)^{\eta-1}-(\varrho_1-\varpi)^{\eta-1}]^2 d\varpi \int_0^{\varrho_1}\|\mathbb{Q}_\eta(\varrho_2-\varpi)\|^2 E\|g(\varpi)\|^2 d\varpi \\
&+\int_0^{\varrho_1}(\varrho_1-\varpi)^{2(\eta-1)}d\varpi \int_0^{\varrho_1}\|\mathbb{Q}_\eta(\varrho_2-\varpi)-\mathbb{Q}_\eta(\varrho_1-\varpi)\|^2 E\|g(\varpi)\|^2 d\varpi \\
&+\int_{\varrho_1}^{\varrho_2}(\varrho_2-\varpi)^{2(\eta-1)}d\varpi \int_{\varrho_1}^{\varrho_2}\|\mathbb{Q}_\eta(\varrho_2-\varpi)\|^2 E\|g(\varpi)\|^2 d\varpi\Big] \\
\leq 12\Big[&\left(\frac{M\varrho_2^{\eta-1}}{\Gamma(\eta)}\right)^2\int_0^{\varrho_1}[(\varrho_2-\varpi)^{\eta-1}-(\varrho_1-\varpi)^{\eta-1}]^2 d\varpi \int_0^{\varrho_1} E\|g(\varpi)\|^2 d\varpi \\
&+\frac{\varrho_1^{2\eta-1}}{2\eta-1}\sup_{\varpi\in[0,\varrho_1]}\|\mathbb{Q}_\eta(\varrho_2-\varpi)-\mathbb{Q}_\eta(\varrho_1-\varpi)\|^2 E\|g(\varpi)\|^2 d\varpi \\
&+\left(\frac{M\varrho_2^{\eta-1}}{\Gamma(\eta)}\right)^2\frac{(\varrho_2-\varrho_1)^{2\eta-1}}{2\eta-1}\int_{\varrho_1}^{\varrho_2}E\|g(\varpi)\|^2 d\varpi\Big].
\end{aligned}$$

Further, using a similar way, we can get

$$\begin{aligned}
J_4 \leq 12\Big[&\int_0^{\varrho_1}[(\varrho_2-\varpi)^{\eta-1}-(\varrho_1-\varpi)^{\eta-1}]^2 d\varpi \int_0^{\varrho_1}\|\mathbb{Q}_\eta(\varrho_2-\varpi)\|^2\|\mathcal{B}\|^2\|\mathfrak{u}\|^2_{L^2_{\mathscr{E}}(\mathcal{W},J)} d\varpi \\
&+\int_0^{\varrho_1}(\varrho_1-\varpi)^{2(\eta-1)}d\varpi \int_0^{\varrho_1}\|\mathbb{Q}_\eta(\varrho_2-\varpi)-\mathbb{Q}_\eta(\varrho_1-\varpi)\|^2\|\mathcal{B}\|^2\|\mathfrak{u}\|^2_{L^2_{\mathscr{E}}(\mathcal{W},J)} d\varpi \\
&+\int_{\varrho_1}^{\varrho_2}(\varrho_2-\varpi)^{2(\eta-1)}d\varpi \int_{\varrho_1}^{\varrho_2}\|\mathbb{Q}_\eta(\varrho_2-\varpi)\|^2\|\mathcal{B}\|^2\|\mathfrak{u}\|^2_{L^2_{\mathscr{E}}(\mathcal{W},J)} d\varpi\Big] \\
\leq 12\Big[&\left(\frac{M\varrho_2^{\eta-1}}{\Gamma(\eta)}\right)^2\|\mathcal{B}\|^2\|\mathfrak{u}\|^2_{L^2_{\mathscr{E}}(\mathcal{W},J)}\varrho_1\int_0^{\varrho_1}[(\varrho_2-\varpi)^{\eta-1}-(\varrho_1-\varpi)^{\eta-1}]^2 \\
&+\frac{\varrho_1^{2\eta-1}}{2\eta-1}\|\mathcal{B}\|^2\|\mathfrak{u}\|^2_{L^2_{\mathscr{E}}(\mathcal{W},J)}\varrho_1\sup_{\varpi\in[0,\varrho_1]}\|\mathbb{Q}_\eta(\varrho_2-\varpi)-\mathbb{Q}_\eta(\varrho_1-\varpi)\|^2 \\
&+\left(\frac{M\varrho_2^{\eta-1}}{\Gamma(\eta)}\right)^2\frac{(\varrho_2-\varrho_1)^{2\eta-1}}{2\eta-1}\|\mathcal{B}\|^2\|\mathfrak{u}\|^2_{L^2_{\mathscr{E}}(\mathcal{W},J)}(\varrho_2-\varrho_1)\Big],
\end{aligned}$$

$$\begin{aligned}
J_5 \leq 12L_G\Big[&\int_0^{\varrho_1}[(\varrho_2-\varpi)^{\eta-1}-(\varrho_1-\varpi)^{\eta-1}]^2 d\varpi \int_0^{\varrho_1}\|\mathbb{Q}_\eta(\varrho_2-\varpi)\|^2 E\|\mathsf{F}(\varpi,\varkappa(\varpi))\|^2_{L_0^2}d\varpi \\
&+\int_0^{\varrho_1}(\varrho_1-\varpi)^{2(\eta-1)}d\varpi \int_0^{\varrho_1}\|\mathbb{Q}_\eta(\varrho_2-\varpi)-\mathbb{Q}_\eta(\varrho_1-\varpi)\|^2 E\|\mathsf{F}(\varpi,\varkappa(\varpi))\|^2_{L_0^2}d\varpi \\
&+\int_{\varrho_1}^{\varrho_2}(\varrho_2-\varpi)^{2(\eta-1)}d\varpi \int_{\varrho_1}^{\varrho_2}\|\mathbb{Q}_\eta(\varrho_2-\varpi)\|^2 E\|\mathsf{F}(\varpi,\varkappa(\varpi))\|^2_{L_0^2}d\varpi\Big] \\
\leq 12L_G\Big[&\left(\frac{M\varrho_2^{\eta-1}}{\Gamma(\eta)}\right)^2\int_0^{\varrho_1}[(\varrho_2-\varpi)^{\eta-1}-(\varrho_1-\varpi)^{\eta-1}]^2 d\varpi \int_0^{\varrho_1} E\|\mathsf{F}(\varpi,\varkappa(\varpi))\|^2_{L_0^2}d\varpi \\
&+\frac{\varrho_1^{2\eta-1}}{2\eta-1}\sup_{\varpi\in[0,\varrho_1]}\|\mathbb{Q}_\eta(\varrho_2-\varpi)-\mathbb{Q}_\eta(\varrho_1-\varpi)\|^2 E\|\mathsf{F}(\varpi,\varkappa(\varpi))\|^2_{L_0^2}d\varpi \\
&+\left(\frac{M\varrho_2^{\eta-1}}{\Gamma(\eta)}\right)^2\frac{(\varrho_2-\varrho_1)^{2\eta-1}}{2\eta-1}\int_{\varrho_1}^{\varrho_2}E\|\mathsf{F}(\varpi,\varkappa(\varpi))\|^2_{L_0^2}d\varpi\Big].
\end{aligned}$$

Hence, using Lebesgue's dominated convergence theorem, we deduce that the right side of the above inequalities tends to zero as $\varrho_2-\varrho_1\to 0$. Therefore, we deduce that $\Psi(\varkappa)(\varrho)$ is continuous from the right in $(0,\grave{a}]$. Which is likewise continuous from the left in $(0,\grave{a}]$, as shown by a similar argument.

In a similar manner, for $\varrho_1 = 0$ and $0 < \varrho_2 \leq \grave{a}$, we may demonstrate $E\|\vartheta(\varrho_2) - \varkappa_0\|^2 \to 0$ independently of $\varkappa \in \mathfrak{D}_q$ as $\varrho_2 \to 0$.

Therefore, it appears from the above explanations that $\{\Psi(\varkappa) : \varkappa \in \mathfrak{D}_q\}$ is an equicontinuous set of functions in $C(\mathcal{W}, L^2(\mathscr{E}, \mathbb{Z}))$.

Step 4: Ψ is completely continuous.

Suppose that $\varrho \in \mathcal{W}$ be fixed. We prove the set $\Sigma(\varrho) = \{\vartheta(\varrho) : \vartheta \in \Psi(\mathfrak{D}_q)\}$ is relatively compact in $\mathbb{Z}$. It is clear that $\Sigma(0) = \{\varkappa_0\}$ is compact.

Therefore, only $\varrho > 0$ must be taken into account. Let $0 < \varrho \leq \grave{a}$ be fixed and $\forall$ $\varkappa \in \mathfrak{D}_q$, $\vartheta \in \Psi(\varkappa)$, $\exists$ a $g \in \Upsilon(\varkappa)$, such that

$$\begin{aligned}\vartheta(\varrho) =& \mathcal{M}_{\delta,\eta}(\varrho)[\varkappa_0 - \hbar(\varkappa) - \sigma(0, \varkappa(0))] + \sigma(\varrho, \varkappa(\varrho)) \\ &+ \int_0^{\varrho} (\varrho - \varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho - \varpi)[g(\varpi) + \mathcal{B}\mathrm{u}(\varpi)]d\varpi \\ &+ \int_0^{\varrho} (\varrho - \varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho - \varpi)\mathsf{F}(\varpi, \varkappa(\varpi))dW(\varpi), \quad \varrho \in \mathcal{W}.\end{aligned}$$

$\forall\, \epsilon \in (0, \varrho)$, $\varrho \in (0, \grave{a}]$ and any $\varkappa \in \mathfrak{D}_q$, we define

$$\begin{aligned}\vartheta^{\epsilon}(\varrho) =& \mathcal{M}_{\delta,\eta}(\varrho)[\varkappa_0 - \hbar(\varkappa) - \sigma(0, \varkappa(0))] + \sigma(\varrho, \varkappa(\varrho)) \\ &+ \int_0^{\varrho-\epsilon} (\varrho - \varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho - \varpi)[g(\varpi) + \mathcal{B}\mathrm{u}(\varpi)]d\varpi \\ &+ \int_0^{\varrho-\epsilon} (\varrho - \varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho - \varpi)\mathsf{F}(\varpi, \varkappa(\varpi))dW(\varpi).\end{aligned}$$

From the boundedness of $\int_0^{\varrho-\epsilon}(\varrho - \varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho - \varpi)[g(\varpi) + \mathcal{B}\mathrm{u}(\varpi)]d\varpi$, $\int_0^{\varrho-\epsilon}(\varrho - \varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho - \varpi)\mathsf{F}(\varpi, \varkappa(\varpi))dW(\varpi)$, and the compactness of $\mathcal{M}_{\delta,\eta}(\varrho)$, $\mathbb{Q}_\eta$, we obtain the set $\Sigma_\epsilon(\varrho)\{\vartheta^\epsilon(\varrho) : \vartheta \in \Psi(\mathfrak{D}_q)\}$ is relatively compact in $\mathbb{Z}$ $\forall\, \epsilon \in (0, \varrho)$. Furthermore, $\forall\, \vartheta \in \Psi(\mathfrak{D}_q)$, we get

$$\begin{aligned}E\|\vartheta(\varrho) - \vartheta^\epsilon(\varrho)\|^2 \leq& E\Bigg\|\Bigg(\mathcal{M}_{\delta,\eta}(\varrho)[\varkappa_0 - \hbar(\varkappa) - \sigma(0, \varkappa(0))] + \sigma(\varrho, \varkappa(\varrho)) \\ &+ \int_0^{\varrho} (\varrho - \varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho - \varpi)[g(\varpi) + \mathcal{B}\mathrm{u}(\varpi)]d\varpi \\ &+ \int_0^{\varrho} (\varrho - \varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho - \varpi)\mathsf{F}(\varpi, \varkappa(\varpi))dW(\varpi)\Bigg) \\ &- \Bigg(\mathcal{M}_{\delta,\eta}(\varrho)[\varkappa_0 - \hbar(\varkappa) - \sigma(0, \varkappa(0))] + \sigma(\varrho, \varkappa(\varrho)) \\ &+ \int_0^{\varrho-\epsilon} (\varrho - \varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho - \varpi)[g(\varpi) + \mathcal{B}\mathrm{u}(\varpi)]d\varpi \\ &+ \int_0^{\varrho-\epsilon} (\varrho - \varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho - \varpi)\mathsf{F}(\varpi, \varkappa(\varpi))dW(\varpi)\Bigg)\Bigg\|^2 \\ \leq& 2\left(\frac{M\epsilon^{\eta-1}}{\Gamma(\eta)}\right)^2\Bigg[\frac{\epsilon^{2\eta-1}}{2\eta-1}\int_{\varrho-\epsilon}^{\varrho} E\|g(\varpi) + \mathcal{B}\mathrm{u}(\varpi)\|^2 d\varpi \\ &+ \int_{\varrho-\epsilon}^{\varrho} (\varrho - \varpi)^{2(\eta-1)}E\|\mathsf{F}(\varpi, \varkappa(\varpi))\|^2_{L_0^2}d\varpi\Bigg] \\ \leq& 2\left(\frac{M\epsilon^{\eta-1}}{\Gamma(\eta)}\right)^2\Bigg[\frac{2\epsilon^{2\eta-1}}{2\eta-1}(\epsilon^{1-\gamma}\|b\|_{L^{\frac{1}{\gamma}}(\mathcal{W},\mathbb{R}^+)} + cq\epsilon + \|\mathcal{B}\|^2\|\mathrm{u}\|^2_{L^2_{\mathscr{E}(\mathcal{W},J)}}\epsilon) \\ &+ L_G\left(\frac{1-\gamma}{2\eta-1-\gamma}\right)^{\gamma-1}\epsilon^{2\eta-1-\gamma}\|d\|_{L^{\frac{1}{\gamma}}(\mathcal{W},\mathbb{R}^+)} + L_G\frac{\epsilon^{2\eta-1}}{2\eta-1}eq\Bigg],\end{aligned}$$

$\Longrightarrow$ the set $\Sigma_\epsilon(\varrho)\{\vartheta^\epsilon(\varrho) : \vartheta \in \Psi(\mathfrak{D}_q)\}$ is totally bounded. The Ascoli–Arzela theorem allows us to prove Ψ is completely continuous.

Step 5: Ψ has a closed graph.
Suppose that $\varkappa_n \to \varkappa_*$ in $C(\mathcal{W}, L^2(\mathscr{E}, \mathbb{Z}))$, $\vartheta_n \in \Psi(\varkappa_n)$ and $\vartheta_n \to \vartheta_*$ in $C(\mathcal{W}, L^2(\mathscr{E}, \mathbb{Z}))$. We will prove that $\vartheta_* \in \Psi(\varkappa_*)$. Indeed, $\vartheta_n \in \Psi(\varkappa_n)$ means that $\exists$ a $g_n \in \Upsilon(\varkappa_n)$, such that

$$\begin{aligned} \vartheta_n(\varrho) =& \mathcal{M}_{\delta,\eta}(\varrho)[\varkappa_0 - \hbar(\varkappa) - \sigma(0, \varkappa(0))] + \sigma(\varrho, \varkappa_n(\varrho)) \\ &+ \int_0^{\varrho} (\varrho - \varpi)^{\eta-1} \mathbb{Q}_\eta(\varrho - \varpi)[g_n(\varpi) + \mathcal{B}\mathrm{u}(\varpi)] d\varpi \\ &+ \int_0^{\varrho} (\varrho - \varpi)^{\eta-1} \mathbb{Q}_\eta(\varrho - \varpi) \mathsf{F}(\varpi, \varkappa_n(\varpi)) dW(\varpi). \end{aligned} \tag{4}$$

From $(H_3)(iii)$ and (H_4), we may prove that $\{(g_n, f(\cdot, \varkappa_n))\}_{n \geq 1} \subseteq L^2_{\mathscr{E}}(\mathcal{W}, \mathbb{Z}) \times L^2_0$ is bounded. Therefore, we may suppose, proceeding on if required to a subsequent thought, that

$$(g_n, f(\cdot, \varkappa_n)) \to (g_*, f(\cdot, \varkappa_*)) \text{ weakly in } L^2_{\mathscr{E}}(\mathcal{W}, \mathbb{Z}) \times L^2_0. \tag{5}$$

By the compactness of $\mathbb{Q}_\eta$, (H_4), (4) and (5), we obtain

$$\begin{aligned} \vartheta_n(\varrho) \to& \mathcal{M}_{\delta,\eta}(\varrho)[\varkappa_0 - \hbar(\varkappa) - \sigma(0, \varkappa(0))] + \sigma(\varrho, \varkappa_*(\varrho)) \\ &+ \int_0^{\varrho} (\varrho - \varpi)^{\eta-1} \mathbb{Q}_\eta(\varrho - \varpi)[g_*(\varpi) + \mathcal{B}\mathrm{u}(\varpi)] d\varpi \\ &+ \int_0^{\varrho} (\varrho - \varpi)^{\eta-1} \mathbb{Q}_\eta(\varrho - \varpi) \mathsf{F}(\varpi, \varkappa_*(\varpi)) dW(\varpi). \end{aligned} \tag{6}$$

It should be noted that $\vartheta_n \to \vartheta_*$ in $C(\mathcal{W}, L^2(\mathscr{E}, \mathbb{Z}))$ and $g_n \in \Upsilon(\varkappa_n)$. According to Lemma 6 and Equation (6), we get $g_* \in \Upsilon(\varkappa_*)$. Consequently, we have shown that $\vartheta_* \in \Psi(\varkappa_*)$, $\Longrightarrow \Psi$ has a closed graph. From [47], it may be concluded Ψ is upper semi-continuous.
Step 6: A priori estimate.
It is evident from Steps 1–5 that Ψ is compact convex value and ϑ, $\Psi(\mathfrak{D}_q)$ is a relatively compact set. We continue from Theorem 1 to demonstrate the collection

$$\Pi = \{\varkappa \in C(\mathcal{W}, L^2(\mathscr{E}, \mathbb{Z})) : \lambda\varkappa \in \Psi(\varkappa),\ \lambda > 1\},$$

is bounded to get a fixed point of Ψ.

Consider $\varkappa \in \Pi$ and suppose $\exists$ a $f \in \Upsilon(\varkappa)$, such that

$$\begin{aligned} \vartheta(\varrho) =& \lambda^{-1} \mathcal{M}_{\delta,\eta}(\varrho)[\varkappa_0 - \hbar(\varkappa) - \sigma(0, \varkappa(0))] + \lambda^{-1} \sigma(\varrho, \varkappa(\varrho)) \\ &+ \lambda^{-1} \int_0^{\varrho} (\varrho - \varpi)^{\eta-1} \mathbb{Q}_\eta(\varrho - \varpi)[g(\varpi) + \mathcal{B}\mathrm{u}(\varpi)] d\varpi \\ &+ \lambda^{-1} \int_0^{\varrho} (\varrho - \varpi)^{\eta-1} \mathbb{Q}_\eta(\varrho - \varpi) \mathsf{F}(\varpi, \varkappa(\varpi)) dW(\varpi). \end{aligned}$$

From $(H_0) - (H_5)$, Lemma 1 and the Hölder inequality, we obtain

$$
\begin{aligned}
E\|\varkappa(\varrho)\|^2 \leq\ & 5\Big[E\|\mathcal{M}_{\delta,\eta}(\varrho)[\varkappa_0 - \hbar(\varkappa) - \sigma(0,\varkappa(0))]\|^2 + E\|\sigma(\varrho,\varkappa(\varrho))\|^2 \\
&+ E\|\int_0^{\varrho}(\varrho-\varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho-\varpi)g(\varpi)d\varpi\|^2 \\
&+ E\|\int_0^{\varrho}(\varrho-\varpi)^{2(\eta-1)}d\varpi\int_0^{\varrho}\mathbb{Q}_\eta(\varrho-\varpi)\mathcal{B}\mathfrak{u}(\varpi)\|^2 d\varpi \\
&+ E\|\int_0^{\varrho}(\varrho-\varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho-\varpi)\mathsf{F}(\varpi,\varkappa(\varpi))dW(\varpi)\|^2\Big] \\
\leq\ & 5\left(\frac{M\grave{a}^{\eta-1}}{\Gamma(\delta(1-\eta)+\eta)}\right)^2[E\|\varkappa_0\|^2 + E\|\hbar(0)\|^2 + L_{\hbar}r + M_\sigma(1+\|\varkappa_0\|^2)] + 5M_\sigma(1+\|\varkappa\|^2) \\
&+ 5\left(\frac{M\grave{a}^{\eta-1}}{\Gamma(\eta)}\right)^2\Bigg[\left(\frac{\grave{a}^{2\eta-1}}{2\eta-1}\int_0^{\varrho}[b(\varpi)+cE\|\varkappa(\varpi)\|^2]\right. \\
&+ \left(\frac{\grave{a}^{2\eta-1}}{2\eta-1}\int_0^{\varrho}E\|\mathcal{B}\mathfrak{u}(\varkappa)\|^2 d\varpi + L_G\int_0^{\varrho}(\varrho-\varpi)^{2(\eta-1)}[d(\varpi)+eE\|\varkappa(\varpi)\|^2]\right]\\
\leq\ & 5\left(\frac{M\grave{a}^{\eta-1}}{\Gamma(\delta(1-\eta)+\eta)}\right)^2[E\|\varkappa_0\|^2 + E\|\hbar(0)\|^2 + L_{\hbar}r + M_\sigma(1+\|\varkappa_0\|^2)] + 5M_\sigma(1+\|\varkappa\|^2) \\
&+ 5\left(\frac{M\grave{a}^{\eta-1}}{\Gamma(\eta)}\right)^2\Bigg[\frac{\grave{a}^{2\eta-1}}{2\eta-1}\left(\grave{a}^{\eta-1}\|b\|_{L^{\frac{1}{\gamma}}(\mathcal{W},\mathbb{R}^+)} + (\grave{a}c + L_G e)\|\varkappa\|^2 + \|\mathcal{B}\|^2\|\mathfrak{u}\|^2_{L^2_{\mathscr{E}}(\mathcal{W},J)}\right) \\
&+ L_G\left(\frac{1-\gamma}{2\eta-1-\gamma}\right)^{\gamma-1}\grave{a}^{2\eta-1-\gamma}\|d\|_{L^{\frac{1}{\gamma}}(\mathcal{W},\mathbb{R}^+)}\Bigg] \\
\leq\ & \ell + S\|\varkappa\|^2,
\end{aligned}
\tag{7}
$$

where

$$
\begin{aligned}
\ell =\ & 5\left(\frac{M\grave{a}^{\eta-1}}{\Gamma(\delta(1-\eta)+\eta)}\right)^2[E\|\varkappa_0\|^2 + E\|\hbar(0)\|^2 + L_{\hbar}r + M_\sigma(1+\|\varkappa_0\|^2)] + 5M_\sigma(1+\|\varkappa\|^2) \\
&+ 5\left(\frac{M\grave{a}^{\eta-1}}{\Gamma(\eta)}\right)^2\Bigg[\frac{\grave{a}^{2\eta-1}}{2\eta-1}\left(\grave{a}^{\eta-1}\|b\|_{L^{\frac{1}{\gamma}}(\mathcal{W},\mathbb{R}^+)} + \frac{\grave{a}^{2\eta-1}}{2\eta-1}\|\mathcal{B}\|^2\|\mathfrak{u}\|^2_{L^2_{\mathscr{E}}(\mathcal{W},J)}\right) \\
&+ L_G\left(\frac{1-\gamma}{2\eta-1-\gamma}\right)^{\gamma-1}\grave{a}^{2\eta-1-\gamma}\|d\|_{L^{\frac{1}{\gamma}}(\mathcal{W},\mathbb{R}^+)}\Bigg].
\end{aligned}
$$

Hence, from $S < 1$, the inequality (7),

$$
\Longrightarrow \|\varkappa\|^2 = \sup_{\varrho\in\mathcal{W}} E\|\varkappa(\varrho)\|^2 \leq \ell + S\|\varkappa\|^2 \leq \frac{\ell}{S-1} := p_0.
$$

Therefore, the set Π is bounded. We determined that Ψ has a fixed point from Theorem 1. Hence, completed the proof. □

4. Optimal Controls

In this segment, we look at the preceding Lagrange problem (LP):
(P) Find a pair $(\varkappa^0, \mathfrak{u}^0) \in C(\mathcal{W}, L^2(\mathscr{E},\mathbb{Z})) \times \mathscr{U}_{ad}$ such that

$$
\mathscr{K}(\varkappa^0,\mathfrak{u}^0) \leq \mathscr{K}(\varkappa,\mathfrak{u}),\ \forall\ (\varkappa,\mathfrak{u}) \in C(\mathcal{W}, L^2(\mathscr{E},\mathbb{Z})) \times \mathscr{U}_{ad},
$$

where

$$
\mathscr{K}(\varkappa,\mathfrak{u}) = E\int_0^{\grave{a}} \mathscr{L}(\varrho,\varkappa^{\mathfrak{u}}(\varrho),\mathfrak{u}(\varrho))d\varrho.
$$

Here, $\varkappa^{\mathfrak{u}}$ represents the mild solution of system (1) relating to the control $\mathfrak{u} \in \mathscr{U}_{ad}$. We base our analysis on the following assumption (H_6) to denote the Lagrange problem (P):

(a) The functional $\mathscr{L} : \mathcal{W} \times \mathbb{Z} \times J \to \mathbb{R} \bigcup \{\infty\}$ is Borel measurable;
(b) For almost all $\varrho \in \mathcal{W}$, $\mathscr{L}$ is sequentially l.s.c. on $\mathbb{Z} \times J$;
(c) $\forall\ \varkappa \in \mathbb{Z}$ and almost all $\varrho \in \mathcal{W}$, $\mathscr{L}(\varrho, \varkappa, \cdot)$ is convex on J;
(d) $\exists$ constants $r_1 \geq 0$, $r_2 > 0$, ξ is positive and $\xi \in L^1(\mathcal{W}, \mathbb{R})$, such that

$$\mathscr{L}(\varrho, \varkappa, \mathfrak{u}) \geq \xi(\varrho) + r_1 E\|\varkappa\|^2_{\mathbb{Z}} + r_2 E\|\mathfrak{u}\|^p_J.$$

Theorem 3. *Assume that $(H_0) - (H_6)$ are fulfilled. The optimal control problem (P) permits at least one optimal pair if $\mathcal{B}$ is a strongly continuous operator.*

Proof. If $\inf\{\mathscr{K}(\varkappa, \mathfrak{u}) | \mathfrak{u} \in \mathscr{U}_{ad}\} = +\infty$, then we can simply determine Lagrange problem (P) has a single optimal pair. Without loss of consensus, we might assume that $\inf\{\mathscr{K}(\varkappa, \mathfrak{u}) : \mathfrak{u} \in \mathscr{U}_{ad}\} = v < +\infty$. Then condition $(H_6)(d)$ implies that $v > -\infty$. By definition of infimum, $\exists$ a minimizing sequence of possible pair $\{(\varkappa^n, \mathfrak{u}^n)\} \subset \mathscr{A}_{ad}$, such that $\mathscr{L}(\varkappa^n, \mathfrak{u}^n) \to v$ as $n \to +\infty$. Since $\{\mathfrak{u}^n\} \subseteq \mathscr{U}_{ad}$, $n = 1, 2, \cdots$, $\{\mathfrak{u}^n\}$ is bounded on $L^p_{\mathscr{E}}(\mathcal{W}, J)$, due to the reflexivity of $L^p_{\mathscr{E}}(\mathcal{W}, J)$, $\exists$ a subsequence of $\{\mathfrak{u}^n\}$, represented again by $\{\mathfrak{u}^n\}$, and $\mathfrak{u}^* \in L^p_{\mathscr{E}}(\mathcal{W}, J)$ satisfying

$$\mathfrak{u}^n \xrightarrow{\text{weakly}} \mathfrak{u}^* \in L^p_{\mathscr{E}}(\mathcal{W}, J).$$

Since $\mathscr{U}_{ad}$ is convex and closed, it follows from Mazur's lemma that $\mathfrak{u}^* \in \mathscr{U}_{ad}$. Consider the related sequence of solutions to the following integral equation be denoted by the symbol $\{\varkappa^n\}$:

$$\begin{aligned}\varkappa^n(\varrho) =& \mathcal{M}_{\delta,\eta}(\varrho)[\varkappa_0 - \hbar(\varkappa) - \sigma(0, \varkappa(0))] + \sigma(\varrho, \varkappa^n(\varrho)) \\ &+ \int_0^{\varrho} (\varrho - \varpi)^{\eta-1} \mathbb{Q}_\eta(\varrho - \varpi)[g^n(\varpi) + \mathcal{B}\mathfrak{u}^n(\varpi)]d\varpi \\ &+ \int_0^{\varrho} (\varrho - \varpi)^{\eta-1} \mathbb{Q}_\eta(\varrho - \varpi) \mathsf{F}(\varpi, \varkappa^n(\varpi))dW(\varpi),\end{aligned} \tag{8}$$

where $f(\varpi, \varkappa^n(\varpi)) \in S_{\mathsf{F},\varkappa^n}$ and $g^n \in \Upsilon(\varkappa^n)$.

We then demonstrate that $\{\varkappa^n\}$ is a relatively compact subset of $C(\mathcal{W}, L^2(\mathscr{E}, \mathbb{Z}))$. Firstly, in a similar manner that the proof of Equation (7), we get

$$\begin{aligned}E\|\varkappa(\varrho)\|^2 \leq& 5\left(\frac{M\grave{a}^{\eta-1}}{\Gamma(\delta(1-\eta)+\eta)}\right)^2 [E\|\varkappa_0\|^2 + E\|\hbar(0)\|^2 + L_\hbar r + M_\sigma(1 + \|\varkappa_0\|^2)] + 5M_\sigma(1 + \|\varkappa\|^2) \\ &+ 5\left(\frac{M\grave{a}^{\eta-1}}{\Gamma(\eta)}\right)^2 \left[\frac{\grave{a}^{2\eta-1}}{2\eta-1}\left(\grave{a}^{\eta-1}\|b\|_{L^{\frac{1}{\gamma}}(\mathcal{W},\mathbb{R}^+)} + \grave{a}c\|\varkappa\|^2 + \|\mathcal{B}\|^2\|\mathfrak{u}\|^2_{L^2_{\mathscr{E}}(\mathcal{W},J)}\right)\right. \\ &\left.+ L_G\left(\frac{1-\gamma}{2\eta-1-\gamma}\right)^{\gamma-1} \grave{a}^{2\eta-1-\gamma}\|d\|_{L^{\frac{1}{\gamma}}(\mathcal{W},\mathbb{R}^+)} + L_G e\frac{\grave{a}^{2\eta-1}}{2\eta-1}\|\varkappa\|^2\right].\end{aligned} \tag{9}$$

Because of the boundedness of $\{\mathfrak{u}_n\}$, (9) and Gronwall's inequality, we infer that $\exists$ a constant $\mu > 0$, such that $\|\varkappa^n\| \leq \mu$, $\Longrightarrow \{\varkappa^n\}$ is uniformly bounded.

Then, according to the argument of Steps 3 and 4 in Theorem 2, we may prove $\{\varkappa^n(\varrho)\}$ is equicontinuous on $\mathcal{W}$ and $\{\varkappa^n(\varrho)\}$ is relatively compact $\forall\ \varrho \in \mathcal{W}$. Thus, the Ascoli–Arzelà theorem $\Longrightarrow \{\varkappa^n\}$ is a relatively compact subset of $C(\mathcal{W}, L^2(\mathscr{E}, \mathbb{Z}))$ and so $\exists$ a function $\varkappa^* \in C(\mathcal{W}, L^2(\mathscr{E}, \mathbb{Z}))$, such that

$$\varkappa^n \to \varkappa^* \text{ in } C(\mathcal{W}, L^2(\mathscr{E}, \mathbb{Z})) \subset L^2(\mathcal{W}, L^2(\mathscr{E}, \mathbb{Z})). \tag{10}$$

The boundedness of $\{\mathfrak{u}_n\}$ and compactness of $\mathcal{N}_\eta(\varrho-\varpi)$ together with the dominated convergence theorem

$$\Longrightarrow \int_0^\varrho (\varrho-\varpi)^{\eta-1}\mathcal{N}_\eta(\varrho-\varpi)\mathcal{B}\mathfrak{u}^n(\varpi)d\varpi \to \int_0^\varrho (\varrho-\varpi)^{\eta-1}\mathcal{N}_\eta(\varrho-\varpi)\mathcal{B}\mathfrak{u}^*(\varpi)d\varpi. \quad (11)$$

Equivalent to the proof of Step 5 in Theorem 2, according to the compactness of $\mathbb{Q}_\eta$, $(H_2)(c)$, (H_3), (10) and Lemmas 6, one has

$$\begin{aligned}
\mathcal{M}_{\delta,\eta}(\varrho)[\varkappa_0 - \hbar(\varkappa) - \sigma(0,\varkappa(0))] + \sigma(\varrho,\varkappa^n(\varrho)) + \int_0^\varrho (\varrho-\varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho-\varpi)g^n(\varpi)d\varpi \\
+ \int_0^\varrho (\varrho-\varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho-\varpi)\mathsf{F}(\varpi,\varkappa^n(\varpi))dW(\varpi) \\
\to \mathcal{M}_{\delta,\eta}(\varrho)[\varkappa_0 - \hbar(\varkappa) - \sigma(0,\varkappa(0))] + \sigma(\varrho,\varkappa^*(\varrho)) + \int_0^\varrho (\varrho-\varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho-\varpi)g^*(\varpi)d\varpi \\
+ \int_0^\varrho (\varrho-\varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho-\varpi)\mathsf{F}(\varpi,\varkappa^*(\varpi))dW(\varpi),
\end{aligned} \quad (12)$$

where $\mathsf{F}(\varpi,\varkappa^*(\varpi)) \in S_{\mathsf{F},\varkappa^*}$ and $g^* \in \Upsilon(\varkappa^*)$. Hence, it concludes from (11) and (12) that

$$\begin{aligned}
\varkappa^*(\varrho) =& \mathcal{M}_{\delta,\eta}(\varrho)[\varkappa_0 - \hbar(\varkappa) - \sigma(0,\varkappa(0))] + \sigma(\varrho,\varkappa^*(\varrho)) \\
&+ \int_0^\varrho (\varrho-\varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho-\varpi)[g^*(\varpi) + \mathcal{B}\mathfrak{u}^*(\varpi)]d\varpi \\
&+ \int_0^\varrho (\varrho-\varpi)^{\eta-1}\mathbb{Q}_\eta(\varrho-\varpi)\mathsf{F}(\varpi,\varkappa^*(\varpi))dW(\varpi).
\end{aligned} \quad (13)$$

This proves that $\varkappa^*$ is a mild solution of (1) following to the control $\mathfrak{u} \in \mathscr{U}_{ad}$.

We discern that $(a)-(d)$ satisfy all the hypotheses of Balder's theorem (see Theorem 2.1 of [50]). Therefore, Balder's theorem shows that the functional

$$(\varkappa,\mathfrak{u}) \mapsto E\int_0^{\grave{a}} \mathscr{L}(\varrho,\varkappa^{\mathfrak{u}}(\varrho),\mathfrak{u}(\varrho))d\varrho,$$

is sequentially lower semi-continuous in the strong topology of $L^1_{\mathscr{E}}(\mathcal{W},\mathbb{Z})$ and weak topology of $L^p_{\mathscr{E}}(\mathcal{W},\mathbb{Z}) \subset L^1_{\mathscr{E}}(\mathcal{W},J)$. Since $L^p_{\mathscr{E}}(\mathcal{W},J) \subset L^1_{\mathscr{E}}(\mathcal{W},J)$, we deduce that $\mathscr{K}$ is weakly lower semi-continuous on $L^p_{\mathscr{E}}(\mathcal{W},J)$. From the hypotheses $(H_5)(d)$, we know that $\mathscr{K} > -\infty$. Thus, we show that $\mathscr{K}$ reaches its infimum at $\mathfrak{u}^* \in \mathscr{U}_{ad}$ and so

$$v = \lim_{n\to\infty} E\int_0^{\grave{a}} \mathscr{L}(\varrho,\varkappa^n(\varrho),\mathfrak{u}^n(\varrho))d\varrho \geq E\int_0^{\grave{a}} \mathscr{L}(\varrho,\varkappa^*(\varrho),\mathfrak{u}^*(\varrho))d\varrho \geq v.$$

This completes the proof. □

5. Example

We wrap up this discussion with a straightforward illustration. We can provide references [7,51] for mathematical induction of HFD and approximate solutions of various fractional differential systems. Take into account the preceding inclusion problem:

$$\begin{cases}
D_{0^+}^{\delta,\frac{4}{7}}[\varkappa(\varrho,\xi) - \Bbbk(\varrho,\varkappa(\varrho,\xi))] \in \frac{\partial^2\varkappa}{\partial\xi^2}[(\varrho,\xi) - \Bbbk(\varrho,\varkappa(\varrho,\xi))] + \partial\mathcal{G}(\varrho,\varkappa(\varrho,\xi)) + \mathfrak{u}(\varrho,\xi) \\
\qquad + \mathsf{F}(\varrho,\varkappa(\varrho,\xi))\frac{dW(\varrho)}{d\varrho},\ \varrho \in (0,1],\ \xi \in [0,\pi], \\
\varkappa(\varrho,0) = \varkappa(\varrho,\pi) = 0,\ \varrho \in [0,1], \\
\mathcal{I}_{0^+}^{(\frac{3}{7})(1-\delta)}(\varkappa(0,\xi)) + \sum_{i=1}^m \int_0^\pi k(\varrho,\varpi)\varkappa(\varrho_i,\varpi)d\varpi = \varkappa_0(\xi),\ \xi \in [0,\pi],
\end{cases} \quad (14)$$

where $D_{0^+}^{\delta,\frac{4}{7}}$ is the HFD of order $\frac{4}{7}$ and type $\delta \in [0,1]$, $\mathcal{I}_{0^+}^{(\frac{3}{7})(1-\delta)}$ is the R-L integral of order $(\frac{3}{7})(1-\delta)$. $k(\varrho,\varpi) \in L^2([0,\pi]\times[0,\pi])$, m is a non-negative integer and $0 < \varrho_1 < \varrho_2 <$

$\cdots < \varrho_m \leq 1$. Take $\mathbb{Z} = Y = L^2[0, \pi]$. Consider $\varkappa(\cdot)(\xi) = \varkappa(\cdot, \xi)$, $\mathcal{B}(\cdot)\mathfrak{u}(\cdot)(\xi) = \mathfrak{u}(\cdot, \xi)$, and

$$\mathfrak{F}(\varkappa, \mathfrak{u}) = \int_0^\pi \int_0^1 |\varkappa(\varrho, \xi)|^2 d\varrho d\xi + \int_0^\pi \int_0^1 |\mathfrak{u}(\varrho, \xi)|^2 d\varrho d\xi.$$

Here, $W(\varrho)$ is a two-sided, one-dimensional Brownian motion in $\mathbb{Z}$ defined on filtered probability space $(\Lambda, \mathscr{E}, P)$, and $\partial\mathcal{G}$ denotes the generalized gradient of a globally Lipschitz function $\mathcal{G}$. A straightforward illustration of $\mathcal{G}$ satisfying the condition (H_2) is $\mathcal{G}(\varrho, \mu) = \mathcal{G}(\mu) = \min\{g_1(\mu), g_2(\mu)\}$ where $g_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, are convex quadratic functions (see [18,47]). The function $\sigma(\varrho, \varkappa(\varrho)) = \Bbbk(\varrho, \varkappa(\varrho, \xi))$, satisfies the condition (H_5).

Let us consider the operator $\widetilde{A} : D(\widetilde{A}) \subset \mathbb{Z} \to \mathbb{Z}$ which is defined in $\widetilde{A}v = v''$ with $D(\widetilde{A}) := \{v \in \mathbb{Z} : v \in L^2([0, \pi]),\ v(0) = v(\pi) = 0\}$. The strongly continuous semigroup $\{\mathcal{M}(\varrho)\}_{\varrho \geq 0}$, which is compact $\forall\ \varrho > 0$, algebraic, and identity, is then produced by $\widetilde{A}$. $\widetilde{A}$ is known to have a discrete spectrum with eigenvalues of the kind $-n^2$, $n \in \mathbb{N}$, and the corresponding normalized eigenvectors are given by $e_n(\varpi) := \sqrt{\frac{2}{\pi}} \sin(n\varpi)$. Similarly, since $\{e_n : n \in \mathbb{N}\}$ is an orthonormal basis for $\mathbb{Z}$, and also $\widetilde{A}$ may be denoted by $\widetilde{A}z = \sum_{n=1}^\infty n^2 \langle z, e_n \rangle e_n$, $z \in D(\widetilde{A})$. In particular, $\|\mathcal{M}(\varrho)\| \leq e^{-\varrho}$ (see [48] for more information). If we assume that $\hbar(\varkappa)(y) = \sum_{i=0}^m \int_0^\pi k(y, z)\varkappa(\varrho_i, z)dz$, then $\hbar$ fulfills condition (C50) (see [39]). Emphasize that the problem (14) may be denoted as (3), an abstract form. From Theorems 2–3, Equation (14) has a mild solution for ρ, $L_\hbar$ appropriately small, and its corresponding limited Lagrange problem admits at least one optimal possible pair.

6. Conclusions

For a class of HFNSEHVI with non-local circumstances, this work investigates whether mild solutions and ideal controls exist. We first established sufficient conditions for the existence of mild solutions to the relevant control system using notions from the extended Clarke sub-differential and a fixed point theorem for multivalued maps. The existence of optimum state-control pairings that are governed by an HFNSEHVI with a non-local condition was then shown using restricted Lagrange optimal systems. The optimum control outcomes are attained without taking into account how distinctive the solutions of the control system are. Finally, an example is used to demonstrate the major conclusion. In the next paper, it will be explored if there are any mild solutions and what the best controls are for HF stochastic integro-differential evolution HVIs with non-local conditions.

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Abbreviations

The following abbreviations are used in this manuscript:

HF	Hilfer fractional
HFD	Hilfer fractional derivative
HVI	Hemivational inequality
HVIs	Hemivational inequalities
FDEs	Fractional differential equations
SDEs	Stochastic differential equations
SEEs	Stochastic evolution equations
SEHVIs	Stochastic evolution hemivational inequalities
HFNSEHVI	Hilfer fractional neutral tochastic evolution hemivational inequality
R-L	Riemann–Liouville

References

1. Panagiotopoulos, P.D. Nonconvex super potentials in sense of F. H. Clarke and applications. *Mech. Res. Commun.* **1981**, *8*, 335–340. [CrossRef]
2. Panagiotopoulos, P.D. *Hemivariational Inequalities, Applications in Mechanics and Engineering*; Springer: Berlin/Heidelberg, Germany, 1993.
3. Migórski, S.; Ochal, A. Quasi-static hemivariational inequality via vanishing acceleration approach. *J. Math. Anal.* **2009**, *41*, 1415–1435.
4. Park, J.Y.; Park, S.H. Optimal control problems for anti-periodic quasilinear hemivariational inequalities. *Optim. Control Appl. Methods* **2007**, *28*, 275–287. [CrossRef]
5. Liu, Z.H.; Li, X.W. Approximate controllability for a class of hemivariational inequalities. *Nonlinear Anal. Real World Appl.* **2015**, *22*, 581–591. [CrossRef]
6. Muthukumar, P.; Durga, N.; Rihan, F.A.; Rajivganthi, C. Optimal control of second order stochastic evolution hemivariational inequalities with Poisson jumps. *Taiwan. J. Math.* **2017**, *21*, 1455–1475. [CrossRef]
7. Harrat, A.; Nieto, J.J.; Debbouche, A. Solvability and optimal controls of impulsive Hilfer fractional delay evolution inclusions with Clarke subdifferential. *J. Comput. Appl. Math.* **2018**, *344*, 725–737. [CrossRef]
8. Diethelm, K. *The Analysis of Fractional Differential Equations*; An application-oriented exposition using differential operators of Caputo type: Lecture Notes in Mathematics; Springer: Berlin/Heidelberg, Germany, 2010.
9. Hilfer, R. *Application of Fractional Calculus in Physics*; World Scientific: Singapore, 2000.
10. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006; p. 204.
11. Lakshmikantham, V.; Vatsala, A.S. Basic Theory of Fractional Differential Equations. *Nonlinear Anal. Theory Methods Appl.* **2008**, *69*, 2677–2682. [CrossRef]
12. Miller, K.S.; Ross, B. *An Introduction to the Fractional Calculus and Differential Equations*; John Wiley: New York, NY, USA, 1993.
13. Podlubny, I. *Fractional Differential Equations*; Academic Press: San Diego, CA, USA, 1999.
14. Zhou, Y. *Basic Theory of Fractional Differential Equations*; World Scientific: Singapore, 2014.
15. Deimling, K. *Multivalued Differential Equations*; Walter de Gruyter: Berlin, Germany, 1992.
16. Jiang, Y.R. Optimal feedback control problems driven by fractional evolution hemivariational inequalities. *Math. Methods Appl. Sci.* **2018**, *41*, 4305–4326. [CrossRef]
17. Kumar, S. Mild solution and fractional optimal control of semilinear system with fixed delay. *J. Optim. Theory Appl.* **2017**, *174*, 108–121. [CrossRef]
18. Lu, L.; Liu, Z.; Jiang, W.; Luo, J. Solvability and optimal controls for semilinear fractional evolution hemivariational inequalities. *Math. Methods Appl. Sci.* **2016**, *39*, 5452–5464. [CrossRef]
19. Zhou, Y.; Jiao, F. Existence of mild solutions for fractional neutral evolution equations. *Comput. Math. Appl.* **2010**, *59*, 1063–1077. [CrossRef]
20. Varun Bose, C.S.; Udhayakumar, R.; Elshenhab, A.M.; Sathish Kumar, M.; Ro, J.-S. Discussion on the Approximate Controllability of Hilfer Fractional Neutral Integro-Differential Inclusions via Almost Sectorial Operators. *Fractal Fract.* **2022**, *6*, 607. [CrossRef]
21. Dokuyucu, M.A. Analysis of a novel finance chaotic model via ABC fractional derivative. *Numer. Methods Partial Differ. Equ.* **2020**, *37*, 1583–1590. [CrossRef]
22. Koca, I.; Akcetin, E.; Yaprakdal, P. Numerical approximation for the spread of SIQR model with Caputo fractional order derivative. *Turk. J. Sci.* **2020**, *5*, 124–139. [CrossRef]
23. Nikan, O.; Golbabai, A.; Machado, J.A.T.; Nikazad, T. Numerical approximation of the time fractional cable model arising in neuronal dynamics. *Eng. Comput.* **2022**, *38*, 155–173. [CrossRef]
24. Phong, T.T.; Long, L.D. Well-posed results for nonlocal fractional parabolic equation involving Caputo-Fabrizio operator. *J. Math. Comput. Sci.* **2022**, *26*, 357–367. [CrossRef]

25. Hattaf, K.; Mohsen, A.A.; Al-Husseiny, H.F. Gronwall inequality and existence of solutions for differential equations with generalized Hattaf fractional derivative. *J. Math. Comput. Sci.* **2022**, *27*, 18–27. [CrossRef]
26. El-sayed, E.; Öğrekci, S. On Hyers-Ulam-Rassias stability of fractional differential equations with Caputo derivative. *J. Math. Comput. Sci.* **2021**, *22*, 325–332. [CrossRef]
27. Khater, M.M.A.; Elagan, S.K.; El-Shorbagy, M.A.; Alfalqi, S.H.; Alzaidi, J.F.; Alshehri, N.A. Folded novel accurate analytical and semi-analytical solutions of a generalized Calogero–Bogoyavlenskii–Schiff equation. *Commun. Theor. Phys.* **2021**, *73*, 095003. [CrossRef]
28. Khater, M.M.A.; Lu, D. Analytical versus numerical solutions of the nonlinear fractional time–space telegraph equation. *Mod. Phys. Lett. B* **2021**, *35*, 2150324. [CrossRef]
29. Khater, M.M.A. Diverse solitary and Jacobian solutions in a continually laminated fluid with respect to shear flows through the Ostrovsky equation. *Mod. Phys. Lett. B* **2021**, *35*, 2150220. [CrossRef]
30. Khater, M.M.A. Nonlinear biological population model; computational and numerical investigations. *Chaos Solitons Fractals* **2022**, *162*, 112388. [CrossRef]
31. Mao, X. *Stochastic Differential Equations and Applications*; Horwood: Chichester, UK, 1997. [CrossRef]
32. Sakthivel, R.; Suganya, S.; Anthoni, S.M. Approximate controllability of fractional stochastic evolution equations. *Comput. Math. Appl.* **2012**, *63*, 660–668.
33. Sivasankar, S.; Udhayakumar, R. Hilfer Fractional Neutral Stochastic Volterra Integro-Differential Inclusions via Almost Sectorial Operators. *Mathematics* **2022**, *10*, 2074. [CrossRef]
34. Sivasankar S.; Udhayakumar, R. New Outcomes Regarding the Existence of Hilfer Fractional Stochastic Differential Systems via Almost Sectorial Operators. *Fractal Fract.* **2022**, *6*, 522. [CrossRef]
35. Lu, L.; Liu, Z. Existence and controllability results for stochastic fractional evolution hemivariational inequalities. *Appl. Math. Comput.* **2015**, *268*, 1164–1176. [CrossRef]
36. Zhou, J. Infinite horizon optimal control problem for stochastic evolution equations in Hilbert spaces. *J. Dyn. Control Syst.* **2016**, *22*, 531–554.
37. Zhou, J.; Liu, B. Optimal control problem for stochastic evolution equations in Hilbert spaces. *Int. J. Control* **2010**, *83*, 1771–1784. [CrossRef]
38. Gu, H.; Trujillo, J.J. Existence of mild solution for evolution equation with Hilfer fractional derivative. *Appl. Math. Comput.* **2015**, *257*, 344–354. [CrossRef]
39. Yang, M.; Wang, Q.R. Approximate controllability of Hilfer fractional differential inclusions with nonlocal conditions. *Math. Methods Appl. Sci.* **2017**, *40*, 1126–1138. [CrossRef]
40. Yang, M.; Wang, Q.R. Existence of mild soltions for a class of Hilfer fractional evolution equations with nonlocal conditions. *Fract. Calc. Appl. Anal.* **2017**, *20*, 679–705. [CrossRef]
41. Sivasankar, S.; Udhayakumar, R.; Subramanian, V.; AlNemer, G.; Elshenhab, A.M. Existence of Hilfer Fractional Stochastic Differential Equations with Nonlocal Conditions and Delay via Almost Sectorial Operators. *Mathematics* **2022**, *10*, 4392.
42. Sivasankar, S.; Udhayakumar, R. A note on approximate controllability of second-order neutral stochastic delay integro-differential evolution inclusions with impulses. *Math. Methods Appl. Sci.* **2022**, *45*, 6650–6676. [CrossRef]
43. Varun Bose, C.S.; Udhayakumar, R. Existence of Mild Solutions for Hilfer Fractional Neutral Integro-Differential Inclusions via Almost Sectorial Operators. *Fractal Fract.* **2022**, *6*, 532. [CrossRef]
44. Pei, Y.; Chang, Y. Hilfer fractional evolution hemivariational inequalities with nonlocal initial conditions and optimal controls. *Nonlinear Anal. Model. Control.* **2019**, *24*, 189–209. [CrossRef]
45. Denkowski, Z.; Migórski, S.; Papageorgiou, N.S. *An Introduction to Nonlinear Analysis (Theory)*; Kluwer Academic: Boston, MA, USA, 2003. [CrossRef]
46. Hu, S.; Papageorgiou, N.S. *Handbook of Multivalued Analysis (Theory)*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1997. [CrossRef]
47. Migórski, S.; Ochal, A.; Sofonea, M. *Nonlinear Inclusions and Hemivariational Inequalities: Models and Analysis of Contact Problems*. Springer: New York, NY, USA, 2013. [CrossRef]
48. Pazy, A. *Semigroups of Linear Operators and Applications to Partial Differential Equations*; Applied Mathematical Sciences; Springer: New York, NY, USA, 1983; Volume 44. [CrossRef]
49. Ma, T.W. Topological degrees for set-valued compact vector fields in locally convex spaces. *Diss. Math.* **1972**, *92*, 1–43. [CrossRef]
50. Balder, E.J. Necessary and sufficient conditions for L^1-strong weak lower semicontinuity of integral functionals. *Nonlinear Anal.* **1987**, *11*, 1399–1404. [CrossRef]
51. Agarwal, R.P.; Baleanu, D.; Nieto, J.J.; Torres, D.F.M.; Zhou, Y. A survey on fuzzy fractional differential and optimal control nonlocal evolution equations. *J. Comput. Appl. Math.* **2018**, *399*, 3–29. [CrossRef]