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Almost Riemann Solitons with Vertical Potential on Conformal Cosymplectic Contact Complex Riemannian Manifolds †

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† Dedicated to the memory of the author's irreplaceable teacher, colleague and friend Prof. Kostadin Gribachev (1938–2022).

Abstract: Almost-Riemann solitons are introduced and studied on an almost contact complex Riemannian manifold, i.e., an almost-contact B-metric manifold, which is obtained from a cosymplectic manifold of the considered type by means of a contact conformal transformation of the Reeb vector field, its dual contact 1-form, the B-metric, and its associated B-metric. The potential of the studied soliton is assumed to be in the vertical distribution, i.e., it is collinear to the Reeb vector field. In this way, manifolds from the four main classes of the studied manifolds are obtained. The curvature properties of the resulting manifolds are derived. An explicit example of dimension five is constructed. The Bochner curvature tensor is used (for a dimension of at least seven) as a conformal invariant to obtain these properties and to construct an explicit example in relation to the obtained results.

Keywords: almost-Riemann soliton; contact conformal transformation; almost-Einstein-like manifold; almost-η-Einstein manifold; almost-contact B-metric manifold; almost-contact complex Riemannian manifold

MSC: Primary 53C25, 53D15, 53C50; Secondary 53C44, 53D35, 70G45

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1. Introduction

The concept of Riemann flow was introduced by C. Udrişte in [1,2]. It refers to the flow associated with the evolution equation

$$\frac{\partial}{\partial t}(g \owedge g)(t) = -4R(g(t)),$$

where R is the Riemann curvature tensor of type $(0,4)$ corresponding to the metric g at time t and $\owedge$ stands for the Kulkarni–Nomizu product of two symmetric tensors of type $(0,2)$; e.g., this product has the following form for order-2 covariant tensors g and h:

$$\begin{aligned}(g \owedge h)(x,y,z,w) = {} & g(y,z)h(x,w) - g(x,z)h(y,w) \\ & + h(y,z)g(x,w) - h(x,z)g(y,w).\end{aligned} \tag{1}$$

Here and further x, y, z, and w stand for arbitrary vector fields on a smooth manifold M.

Riemann solitons were introduced by I. E. Hirică and C. Udrişte in [3]. They are critical metrics for Riemann flow as they are self-similar solutions of its evolution equation, i.e., it evolves over time from a given Riemannian metric on M by means of diffeomorphisms and dilatations.

A Riemannian metric g on a smooth manifold M is said to be a *Riemann soliton* if there exists a differentiable vector field ϑ and a real constant σ such that [3]

$$2R + \sigma g \owedge g + g \owedge \mathcal{L}_\vartheta g = 0,$$

where $\mathcal{L}_\vartheta$ is the Lie derivative along ϑ. Such a vector field ϑ is known as the *potential of the soliton*. In the case in which σ is a differentiable function on M, then g is called an *almost-Riemann soliton*. If ϑ is Killing, i. e., $\mathcal{L}_\vartheta g = 0$, then M is a manifold of constant sectional curvature. In this sense, the Riemann soliton is a generalization of a space of constant curvature.

In early studies [3], the notion of the Riemann soliton was studied in the context of Sasakian geometry and it was known as the Sasaki–Riemann soliton.

In recent years, some interesting results have been obtained for Riemann solitons and almost-Riemann solitons on almost-contact metric manifolds. In [4,5], Venkatesha, Devaraja and Kumara studied the cases of almost-Kenmotsu manifolds and K-contact manifolds. Biswas, Chen and U. C. De characterized almost-co-Kähler manifolds whose metrics are Riemann solitons in [6]. K. De and U. C. De proved in [7] some geometric properties of almost-Riemann solitons on non-cosymplectic normal almost-contact metric manifolds and in particular on quasi-Sasakian 3-dimensional manifolds. In [8], Chidananda and Venkatesha studied Riemann solitons on non-Sasakian (κ, μ)-contact manifolds in relation with the η-Einstein property, where the potential is an infinitesimal contact transformation or collinear to the Reeb vector field.

A.-M. Blaga contributed to the study of Riemann and almost-Riemann solitons in [9] for Riemannian manifolds, together with Laţcu, and in [10] for (α, β)-contact metric manifolds. In the latter case, compact Riemann solitons with constant-length potential were shown to be trivial. This result was extended by Tokura, Barboza, Batista, and Menezes in [11] without additional conditions on the potential.

$\mathcal{D}$-homothetic deformations were introduced by S. Tanno [12] in almost-contact metric geometry, where $\mathcal{D}$ denotes the contact distribution. These transformations preserve the K-contact or Sasakian properties of a structure. In [13], Blaga studied almost-Riemann solitons on a $\mathcal{D}$-homothetically deformed Kenmotsu manifold with different conditions on the potential and explicitly obtain Ricci and scalar curvatures for some cases.

An *almost-contact complex Riemannian* (or accR for short) manifold is an odd-dimensional pseudo-Riemannian manifold M equipped with a B-metric g and an almost-contact structure (φ, ξ, η) and therefore M has a codimension-one distribution $\mathcal{H} = \ker \eta$ equipped with a complex Riemannian structure. These manifolds are also known as *almost-contact B-metric manifolds* [14].

What mainly distinguishes an accR structure from the better-known almost-contact metric structure is the presence of another metric of the same type associated with the given metric. Both B-metrics have a neutral signature on $\mathcal{H}$ and the restriction of φ on $\mathcal{H}$ (actually, an almost-complex structure) acts as an anti-isometry on the metric. Manifolds of this type have been studied and investigated, for example, in [14–27].

The aim of this paper was to investigate the interaction between almost-Riemann solitons and the accR structure. One way to realize this goal is to use conformal transformations of the accR structure. Contact conformal transformations of B-metric were introduced and initially studied in [23,24] by K. Gribachev and the author. The metric deformation depends on both the two B-metrics and their restriction on the vertical distribution determined by ξ. A generalization of these transformations and the $\mathcal{D}$-homothetic deformations of the accR structure (introduced in [28]) that use a triplet of functions on the manifold are the following transformations. Contact conformal transformations of a general type that transform not only the B-metrics but also ξ and η were studied in [18]. According to this work, the class of accR manifolds, which is closed under the action of these transformations, is the direct sum of the four main classes among the eleven basic classes of these manifolds, known from the classification of Ganchev–Mihova–Gribachev presented in [14]. The main classes are designated as those for which the manifolds are characterized by the fact that the covariant derivative of the structure tensors with respect to the Levi–Civita connection of any of the B-metrics is expressed only by a pair of B-metrics and the corresponding traces.

The present paper is organized as follows. Section 2 recalls the basic concepts of accR manifolds and contact conformal transformations of the structure tensors on them. Section 3 introduces the notion of an almost-Riemann soliton with vertical potential on a transformed accR manifold and demonstrates the conditions that imply the flatness of the manifold. Section 4 presents the curvature properties of contact conformal accR manifolds that are transformed from such manifolds of cosymplectic type and admit the studied soliton. Section 5 is devoted to the particular case of the situation discussed in the previous section when the transformed manifold is also of cosymplectic type. The last two sections provide explicit examples for the studied manifolds in relation with the obtained results.

2. Almost-Contact Complex Riemannian Manifolds and Their Contact Conformal Transformations

Here we study *almost-contact complex Riemannian manifolds* or *accR manifolds* for short, also known as *almost-contact B-metric manifolds*. Such a manifold, denoted by (M,φ,ξ,η,g), is a $(2n+1)$-dimensional differentiable manifold, which is equipped with an almost-contact structure (φ,ξ,η) and the B-metric g. This means that φ is an endomorphism of the tangent bundle TM, ξ is a Reeb vector field, and η is its dual contact 1-form. Moreover, g is a pseudo-Riemannian metric of signature $(n+1,n)$ satisfying the following algebraic relations: [14]

$$\begin{gathered}\varphi\xi=0, \qquad \varphi^2=-\iota+\eta\otimes\xi, \qquad \eta\circ\varphi=0, \qquad \eta(\xi)=1,\\ g(\varphi x,\varphi y)=-g(x,y)+\eta(x)\eta(y),\end{gathered} \tag{2}$$

where ι is the identity transformation on the set $\Gamma(TM)$ of vector fields on M.

As consequences of (2), the following equations are known:

$$g(\varphi x,y)=g(x,\varphi y), \qquad g(x,\xi)=\eta(x), \qquad g(\xi,\xi)=1, \qquad \eta(\nabla_x\xi)=0,$$

where ∇ is the Levi-Civita connection of g.

The investigated manifold (M,φ,ξ,η,g) has another B-metric in addition to g. This is the associated metric $\tilde{g}$ of g on M, defined by

$$\tilde{g}(x,y)=g(x,\varphi y)+\eta(x)\eta(y).$$

Obviously, $\tilde{g}$ as well as g satisfies the last condition in (2) as well and has the same signature.

A classification of accR manifolds containing eleven basic classes $\mathcal{F}_1,\mathcal{F}_2,\dots,\mathcal{F}_{11}$ is given in [14]. This classification is made with respect to the tensor F of type $(0,3)$ defined by

$$F(x,y,z)=g\big((\nabla_x\varphi)y,z\big).$$

The following identities are valid:

$$\begin{aligned}&F(x,y,z)=F(x,z,y)=F(x,\varphi y,\varphi z)+\eta(y)F(x,\xi,z)+\eta(z)F(x,y,\xi),\\ &F(x,\varphi y,\xi)=(\nabla_x\eta)y=g(\nabla_x\xi,y).\end{aligned}$$

The special class $\mathcal{F}_0$, determined by the condition $F=0$, is the intersection of the basic classes and it is known as the class of the *cosymplectic accR manifolds*. Sometimes, in the context of classification and for brevity, these manifolds are called $\mathcal{F}_0$-manifolds.

Let $\{e_i;\xi\}$ $(i=1,2,\dots,2n)$ be a basis of T_pM and let (g^{ij}) be the inverse matrix of the matrix (g_{ij}) of g. Then the following 1-forms are associated with F:

$$\theta=g^{ij}F(e_i,e_j,\cdot), \quad \theta^*=g^{ij}F(e_i,\varphi e_j,\cdot), \quad \omega=F(\xi,\xi,\cdot).$$

These 1-forms are known also as the *Lee forms* of the considered manifold. Obviously, the identities $\omega(\xi) = 0$ and $\theta^* \circ \varphi = -\theta \circ \varphi^2$ are always valid.

In [23], the so-called contact conformal transformation of the B-metric g is introduced. It maps g into a new B-metric $\bar{g}$ using both the B-metrics. Later, in [18], this transformation is generalized as a contact conformal transformation that gives an accR structure $(\varphi, \bar{\xi}, \bar{\eta}, \bar{g})$ as follows:

$$\begin{aligned} &\bar{\xi} = e^{-w}\xi, \qquad \bar{\eta} = e^{w}\eta, \\ &\bar{g} = e^{2u}\cos 2v\, g + e^{2u}\sin 2v\, \tilde{g} + \left(e^{2w} - e^{2u}\cos 2v - e^{2u}\sin 2v\right)\eta\otimes\eta, \end{aligned} \tag{3}$$

where u, v, w are differentiable functions on M. The group of these transformations is denoted by G and for brevity we call each of the elements of G a *G-transformation*.

Note that the G-transformations of (φ, ξ, η, g) are a generalization of the *$\mathcal{D}$-homothetic deformations*, where $\mathcal{D}$ denotes the contact distribution $\ker\eta$. Namely, for a positive constant λ, a $\mathcal{D}$-homothetic deformation is defined by [28] (see also [29] (p. 125) for the metric case)

$$\bar{\xi} = \lambda^{-1}\xi, \qquad \bar{\eta} = \lambda\,\eta, \qquad \bar{g} = -\lambda\, g + \lambda(\lambda+1)\eta\otimes\eta.$$

It is clear that $\mathcal{D}$-homothetic deformation is a G-transformation of the accR structure (φ, ξ, η, g) for constants $u = \frac{1}{2}\ln\lambda$, $v = \frac{\pi}{2}$, and $w = \ln\lambda$.

The structure (φ, ξ, η, g) determines two mutually orthogonal distributions with respect to g. They are the horizontal (contact) distribution $\mathcal{H} = \ker\eta$ and the vertical distribution $\mathcal{V} = \operatorname{span}\xi$. They coincide with the respective distributions for the structure $(\varphi, \bar{\xi}, \bar{\eta}, \bar{g})$, i.e., $\mathcal{H} = \bar{\mathcal{H}} = \ker\bar{\eta}$ and $\mathcal{V} = \bar{\mathcal{V}} = \operatorname{span}\bar{\xi}$, due to the equalities in the first line of (3).

The corresponding tensors F and $\bar{F}$ for the accR structures (φ, ξ, η, g) and $(\varphi, \bar{\xi}, \bar{\eta}, \bar{g})$ are related by means of a G-transformation (3) as follows (e.g., [18]; see also [25])

$$\begin{aligned} 2\bar{F}(x,y,z) = {} & 2e^{2u}\cos 2v\, F(x,y,z) + e^{2u}\sin 2v[P(x,y,z) + P(x,z,y)] \\ & + (e^{2w} - e^{2u}\cos 2v)[Q(x,y,z) + Q(x,z,y) + Q(y,z,x) + Q(z,y,x)] \\ & - 2e^{2u}[\gamma(z)g(\varphi x, \varphi y) + \delta(z)g(x,\varphi y) + \gamma(y)g(\varphi x, \varphi z) + \delta(y)g(x, \varphi z)] \\ & + 2e^{2w}\eta(x)[\eta(y)\,\mathrm{d}w(\varphi z) + \eta(z)\,\mathrm{d}w(\varphi y)], \end{aligned} \tag{4}$$

where for brevity we use the following notation:

$$\begin{aligned} &P(x,y,z) = F(\varphi y, z, x) - F(y, \varphi z, x) + F(x, \varphi y, \xi)\eta(z), \\ &Q(x,y,z) = [F(x,y,\xi) + F(\varphi y, \varphi x, \xi)]\eta(z), \\ &\gamma(z) = \cos 2v\,\alpha(z) + \sin 2v\,\beta(z), \qquad \delta(z) = \cos 2v\,\beta(z) - \sin 2v\,\alpha(z), \end{aligned}$$

$$\alpha = \mathrm{d}u\circ\varphi + \mathrm{d}v, \qquad \beta = \mathrm{d}u - \mathrm{d}v\circ\varphi. \tag{5}$$

In the general case, the relations between the Lee forms of the corresponding manifolds $(M, \varphi, \xi, \eta, g)$ and $(M, \varphi, \bar{\xi}, \bar{\eta}, \bar{g})$ are as follows (see [18]):

$$\bar{\theta} = \theta + 2n\,\alpha, \qquad \bar{\theta}^* = \theta^* + 2n\,\beta, \qquad \bar{\omega} = \omega + \mathrm{d}w\circ\varphi. \tag{6}$$

As proven in [20] (Theorem 4.2, p. 62), the class of accR manifolds that is preserved by G-transformations is the direct sum of all main classes $\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_{11}$, denoted here for brevity as $G(\mathcal{F}_0)$. The main classes are the only classes of accR manifolds in the Ganchev–Mihova–Gribachev classification, where F is expressed only by the metric $(0,2)$-tensors g, $\tilde{g}$, $\eta\otimes\eta$, and the Lee forms. The class $G(\mathcal{F}_0)$ obviously contains $\mathcal{F}_0$.

3. Almost Riemann Solitons with Vertical Potential on Contact Conformal accR Manifolds

Definition 1. *It can be said that the B-metric $\bar{g}$ generates a* Riemann soliton *with potential $\bar{\vartheta}$ and constant $\bar{\sigma}$, denoted $(\bar{g}, \bar{\vartheta}, \bar{\sigma})$, on an accR manifold $(M, \varphi, \bar{\xi}, \bar{\eta}, \bar{g})$, if the following condition is satisfied:*

$$2\bar{R} + \bar{\sigma}\,\bar{g} \owedge \bar{g} + \bar{g} \owedge \mathcal{L}_{\bar{\vartheta}}\bar{g} = 0, \tag{7}$$

where $\bar{R}$ is the Riemannian curvature tensor of $(M, \varphi, \bar{\xi}, \bar{\eta}, \bar{g})$ for $\bar{g}$. If $\bar{\sigma}$ is a differentiable function on M, then the generated soliton is called an almost-Riemann soliton *$(\bar{g}, \bar{\vartheta}, \bar{\sigma})$ on $(M, \varphi, \bar{\xi}, \bar{\eta}, \bar{g})$.*

In this work, we consider the case in which the potential $\bar{\vartheta}$ is a vertical vector field, i.e., $\bar{\vartheta}$ is collinear to $\bar{\xi}$. Then we have the expression $\bar{\vartheta} = \bar{k}\,\bar{\xi}$ for a differentiable function $\bar{k}$ on the manifold. Obviously, the equality $\bar{k} = \bar{\eta}(\bar{\vartheta})$ holds. We require that the potential $\bar{\vartheta}$ does not degenerate at any point on the manifold $(M, \varphi, \bar{\xi}, \bar{\eta}, \bar{g})$. This means that $\bar{k}$ does not vanish anywhere, i.e., $\bar{k} \neq 0$.

The following expression of the Lie derivative in terms of the covariant derivative with respect to the Levi-Civita connection $\bar{\nabla}$ of $\bar{g}$ is well-known:

$$\left(\mathcal{L}_{\bar{\xi}}\bar{g}\right)(x, y) = \bar{g}\big(\bar{\nabla}_x\bar{\xi}, y\big) + \bar{g}\big(x, \bar{\nabla}_y\bar{\xi}\big).$$

Similarly, the following formula can be obtained:

$$(\mathcal{L}_{\bar{\vartheta}}\bar{g})(x, y) = \bar{g}\big(\bar{\nabla}_x\bar{\vartheta}, y\big) + \bar{g}\big(x, \bar{\nabla}_y\bar{\vartheta}\big). \tag{8}$$

For a vertical potential we have $\bar{\nabla}_x\bar{\vartheta} = \bar{\nabla}\big(\bar{k}\,\bar{\xi}\big) = \bar{k}\bar{\nabla}\bar{\xi} + \mathrm{d}\bar{k}(x)\bar{\xi}$. Then, the latter two equalities imply the formula

$$(\mathcal{L}_{\bar{\vartheta}}\bar{g})(x, y) = \bar{k}\left(\mathcal{L}_{\bar{\xi}}\bar{g}\right)(x, y) + \bar{h}_1(x, y), \tag{9}$$

where we use the following notation

$$\bar{h}_1(x, y) = \mathrm{d}\bar{k}(x)\bar{\eta}(y) + \mathrm{d}\bar{k}(y)\bar{\eta}(x). \tag{10}$$

Obviously, the $(0, 2)$-tensor $\bar{h}_1$ is symmetric and has the properties

$$\bar{h}_1(\varphi x, \varphi y) = 0, \qquad \bar{h}_1(\bar{\xi}, \bar{\xi}) = \operatorname{tr}\bar{h}_1 = 2\,\mathrm{d}\bar{k}(\bar{\xi}).$$

Therefore, it vanishes on $\mathcal{H}$. Furthermore, $\bar{h}_1$ vanishes if and only if $\bar{k}$ is a constant.

The following theorem holds for an arbitrary $\mathcal{F}_0$-manifold M. It is not necessary to assume that the structure of M is obtained by means of some G-transformation.

Theorem 1. *Every $\mathcal{F}_0$-manifold admitting an almost-Riemann soliton with vertical potential is flat.*

Proof. Let us consider an $\mathcal{F}_0$-manifold $(M, \varphi, \xi, \eta, g)$ admitting an almost-Riemann soliton (g, ϑ, σ) with vertical potential $\vartheta = k\,\xi$. Then its curvature tensor for g has the following form, similar to (7):

$$R = \frac{1}{2}\sigma\,g \owedge g - \frac{1}{2}g \owedge \mathcal{L}_{\vartheta}g.$$

Since $\nabla\xi$ vanishes on an $\mathcal{F}_0$-manifold, then we have $\nabla_x\vartheta = \nabla_x(k\xi) = \mathrm{d}k(x)\xi$. Due to the equality $(\mathcal{L}_{\vartheta}g)(x, y) = g(\nabla_x\vartheta, y) + g\big(x, \nabla_y\vartheta\big)$, which is the analogue of (8) on $(M, \varphi, \xi, \eta, g)$, we can observe in this case that $\mathcal{L}_{\vartheta}g = h_1$, where we use the following notation, similarly to (10):

$$h_1(x, y) = \mathrm{d}k(x)\eta(y) + \mathrm{d}k(y)\eta(x). \tag{11}$$

Thus, the curvature tensor of such a manifold $(M, \varphi, \xi, \eta, g)$ takes the form

$$R = -\frac{1}{2}\sigma\, g \owedge g - \frac{1}{2} g \owedge h_1. \tag{12}$$

Using (11) and (12), we obtain the Ricci tensor and scalar curvatures for g and $\tilde{g}$, respectively, given in the following expressions:

$$\rho = -\{2n\,\sigma + \mathrm{d}k(\xi)\}g - \frac{1}{2}(2n-1)h_1, \tag{13}$$

$$\tau = -2n\{(2n+1)\sigma + 2\,\mathrm{d}k(\xi)\}, \qquad \tilde{\tau} = 0. \tag{14}$$

The Riemannian curvature tensor R of an $\mathcal{F}_0$-manifold has the Kähler property

$$R(x, y, \varphi z, \varphi w) = -R(x, y, z, w) \tag{15}$$

since φ, ξ, and η are covariant constant on M with respect to ∇ [24]. As consequences of (15) and (11) we have $\rho(\xi, \xi) = 0$ and $h_1(\xi, \xi) = 2\,\mathrm{d}k(\xi)$, respectively, which, together with (13), imply that

$$\mathrm{d}k(\xi) = -\sigma. \tag{16}$$

On the other hand, by virtue of (12) and (1), we obtain

$$R(x, y, \varphi z, \varphi w) = -\frac{1}{2}\sigma(g^* \owedge g^*)(x, y, z, w) - \frac{1}{2}(g^* \owedge h_1^*)(x, y, z, w), \tag{17}$$

where we use the notations $g^* = g(\cdot, \varphi\cdot)$, $h_1^* = h_1(\cdot, \varphi\cdot)$. Then, taking into account the fact that g^* and h_1^* are traceless due to the properties of φ, the equalities (15) and (17) consequently yield

$$R = -\frac{1}{2}\sigma\, g^* \owedge g^* + \frac{1}{2} g^* \owedge h_1^*, \qquad \rho = \sigma[g - \eta \otimes \eta], \qquad \tau = 2n\sigma, \qquad \tilde{\tau} = 0. \tag{18}$$

Comparing the values of τ in (18) and (14), we obtain $\mathrm{d}k(\xi) = -(n+1)\sigma$, which due to (16) gives $\mathrm{d}k(\xi) = \sigma = 0$. Therefore, from (18) it follows that $\rho = 0$, which, together with (13), implies that $h_1 = 0$ and then $R = 0$, bearing in mind (12). $\square$

4. $G(\mathcal{F}_0)$-Manifolds Admitting the Studied Solitons

In this section, we consider $(M, \varphi, \xi, \eta, g)$ as an $\mathcal{F}_0$-manifold, i.e., $F = 0$. Let the resulting accR manifold $(M, \varphi, \bar{\xi}, \bar{\eta}, \bar{g})$ via a G-transformation be called a *$G(\mathcal{F}_0)$-manifold*. Then, the following expression follows from (4) and gives the form of the fundamental tensor of $(M, \varphi, \bar{\xi}, \bar{\eta}, \bar{g})$:

$$\begin{aligned}2\bar{F}(x, y, z) = &-2e^{2u}\{\gamma(z)g(\varphi x, \varphi y) + \delta(z)g(x, \varphi y) + \gamma(y)g(\varphi x, \varphi z) + \delta(y)g(x, \varphi z)\}\\ &+ 2e^{2w}\eta(x)\{\eta(y)\,\mathrm{d}w(\varphi z) + \eta(z)\,\mathrm{d}w(\varphi y)\}.\end{aligned}$$

Then, using (5) and (6), the corresponding Lee forms are specialized as follows:

$$\bar{\theta} = 2n\{\mathrm{d}u \circ \varphi + \mathrm{d}v\}, \qquad \bar{\theta}^* = 2n\{\mathrm{d}u - \mathrm{d}v \circ \varphi\}, \qquad \bar{\omega} = \mathrm{d}w \circ \varphi.$$

Theorem 2. *A $G(\mathcal{F}_0)$-manifold $(M, \varphi, \bar{\xi}, \bar{\eta}, \bar{g})$ admitting an almost-Riemann soliton $(\bar{g}, \bar{\vartheta}, \bar{\sigma})$ with vertical potential has a curvature tensor of the following form:*

$$\begin{aligned}\bar{R} = &-\left[\tfrac{\bar{\sigma}}{2} + \bar{k}\,\mathrm{d}u(\bar{\xi})\right]\bar{g} \owedge \bar{g} - \bar{k}\{\mathrm{d}v(\bar{\xi})\,\bar{g} \owedge \tilde{\bar{g}} - [\mathrm{d}u(\bar{\xi}) + \mathrm{d}v(\bar{\xi})]\bar{g} \owedge (\bar{\eta} \otimes \bar{\eta})\}\\ &- \tfrac{1}{2}\,\bar{g} \owedge \bar{h}_1 - \tfrac{1}{2}\,\bar{k}\,\bar{g} \owedge \bar{h}_2,\end{aligned} \tag{19}$$

where

$$\bar{h}_2(x,y) = \bar{\eta}(x)\,\mathrm{d}w(\varphi^2 y) + \bar{\eta}(y)\,\mathrm{d}w(\varphi^2 x). \tag{20}$$

Proof. Bearing in mind (7), we have to determine $\mathcal{L}_{\bar{\vartheta}}\bar{g}$. The expression of the Lie derivative of $\bar{g}$ along $\bar{\xi}$ for a $G(\mathcal{F}_0)$-manifold is given in [22] in the form:

$$\begin{aligned}\left(\mathcal{L}_{\bar{\xi}}\bar{g}\right)(x,y) =& -2e^{2u-w}[\cos 2v\ \mathrm{d}u(\xi) - \sin 2v\ \mathrm{d}v(\xi)]g(\varphi x,\varphi y)\\ &+ 2e^{2u-w}[\cos 2v\ \mathrm{d}v(\xi) + \sin 2v\ \mathrm{d}u(\xi)]g(x,\varphi y)\\ &+ e^{w}\Big[\eta(x)\,\mathrm{d}w(\varphi^2 y) + \eta(y)\,\mathrm{d}w(\varphi^2 x)\Big].\end{aligned} \tag{21}$$

Using the second line in (3), we derive the following formulas:

$$\begin{aligned}g(\varphi x,\varphi y) &= e^{-2u}[\cos 2v\,\bar{g}(\varphi x,\varphi y) + \sin 2v\,\bar{g}(x,\varphi y)],\\ g(x,\varphi y) &= e^{-2u}[\cos 2v\,\bar{g}(x,\varphi y) - \sin 2v\,\bar{g}(\varphi x,\varphi y)],\end{aligned}$$

which we apply in (21), together with the first line in (3). In this way, we obtian

$$\left(\mathcal{L}_{\bar{\xi}}\bar{g}\right)(x,y) = -2\big[\mathrm{d}u(\bar{\xi})\bar{g}(\varphi x,\varphi y) - \mathrm{d}v(\bar{\xi})\bar{g}(x,\varphi y)\big] + \bar{h}_2(x,y), \tag{22}$$

where we introduce the notation (20). Obviously, $\bar{h}_2$ is a symmetric $(0,2)$-tensor having the following properties:

$$\bar{h}_2(\varphi x,\varphi y) = \bar{h}_2(\bar{\xi},\bar{\xi}) = \operatorname{tr}\bar{h}_2 = 0, \qquad \bar{h}_2(x,y) = \bar{h}_2(x,\bar{\xi})\bar{\eta}(y) + \bar{\eta}(x)\bar{h}_2(\bar{\xi},y).$$

Moreover, the formula $\bar{h}_2(x,\bar{\xi}) = \mathrm{d}w(\varphi^2 x)$ is valid. It is easy to conclude that $\bar{h}_2$ vanishes if and only if the function w is constant on $\mathcal{H}$, i.e., $\mathrm{d}w \circ \varphi^2 = 0$.

The formula in (22) can be rewritten in the following form:

$$\mathcal{L}_{\bar{\xi}}\bar{g} = 2\big\{\mathrm{d}u(\bar{\xi})\bar{g} + \mathrm{d}v(\bar{\xi})\tilde{\bar{g}} - \big[\mathrm{d}u(\bar{\xi}) + \mathrm{d}v(\bar{\xi})\big]\bar{\eta}\otimes\bar{\eta}\big\} + \bar{h}_2.$$

Then we substitute the last equality in (9) and get the following

$$\mathcal{L}_{\bar{\vartheta}}\bar{g} = 2\bar{k}\big\{\mathrm{d}u(\bar{\xi})\bar{g} + \mathrm{d}v(\bar{\xi})\tilde{\bar{g}} - \big[\mathrm{d}u(\bar{\xi}) + \mathrm{d}v(\bar{\xi})\big]\bar{\eta}\otimes\bar{\eta}\big\} + \bar{h}_1 + \bar{k}\bar{h}_2.$$

Using the Kulkarni–Nomizu product for $\bar{g}$ and the last obtained Lie derivative, we obtain

$$\begin{aligned}\bar{g}\owedge\mathcal{L}_{\bar{\vartheta}}\bar{g} =& 2\bar{k}\big\{\mathrm{d}u(\bar{\xi})\,\bar{g}\owedge\bar{g} + \mathrm{d}v(\bar{\xi})\,\bar{g}\owedge\tilde{\bar{g}} - \big[\mathrm{d}u(\bar{\xi}) + \mathrm{d}v(\bar{\xi})\big]\bar{g}\owedge(\bar{\eta}\otimes\bar{\eta})\big\}\\ &+ \bar{g}\owedge\bar{h}_1 + \bar{k}\,\bar{g}\owedge\bar{h}_2.\end{aligned} \tag{23}$$

Then, according to (7) and (23), we can establish the truthfulness of the statement. □

Taking the trace of (19), we obtain the expression of the Ricci tensor of the almost-Riemann soliton satisfying the conditions of Theorem 2 as follows:

$$\begin{aligned}\bar{\rho} =& -\big[2n\,\bar{\sigma} + \mathrm{d}\bar{k}(\bar{\xi}) + (4n-1)\bar{k}\,\mathrm{d}u(\bar{\xi})\big]\bar{g}\\ &- (2n-1)\bar{k}\big\{\mathrm{d}v(\bar{\xi})\,\tilde{\bar{g}} - \big[\mathrm{d}u(\bar{\xi}) + \mathrm{d}v(\bar{\xi})\big]\,\bar{\eta}\otimes\bar{\eta}\big\} - \tfrac{1}{2}(2n-1)\big[\bar{h}_1 + \bar{k}\,\bar{h}_2\big].\end{aligned} \tag{24}$$

Now, we take the trace of the Ricci tensor in (24) to obtain the scalar curvature of $(M,\varphi,\bar{\xi},\bar{\eta},\bar{g})$ as follows

$$\bar{\tau} = -2n\big[(2n+1)\bar{\sigma} + 2\,\mathrm{d}\bar{k}(\bar{\xi}) + 4n\bar{k}\,\mathrm{d}u(\bar{\xi})\big]. \tag{25}$$

Then, we compute the associated quantity $\bar{\tau}^*$ of $\bar{\tau}$ defined by $\bar{\tau}^* = \bar{g}^{ij}\varphi_j^k\bar{\rho}_{ik}$ and using (24), we obtain

$$\bar{\tau}^* = 2n(2n-1)\bar{k}\,\mathrm{d}v(\bar{\xi}). \tag{26}$$

For every $\mathcal{F}_0$-manifold, the relation $\tilde{\tau} = -\tau^*$ is known from [17], where $\tilde{\tau}$ is the scalar curvature for $\tilde{g}$. However, for $G(\mathcal{F}_0)$-manifolds, which are outside of $\mathcal{F}_0$, this is not true, so there we use the so-called $*$*-scalar curvature*.

Corollary 1. *A $G(\mathcal{F}_0)$-manifold $(M,\varphi,\bar{\xi},\bar{\eta},\bar{g})$ with an almost-Riemann soliton $(\bar{g},\bar{\vartheta},\bar{\sigma})$ and a vertical potential has vanishing $*$-scalar curvature if and only if the function v is a vertical constant, i.e., $\mathrm{d}v(\bar{\xi}) = 0$.*

Proof. This statement follows from (26) and the condition that $\bar{k}$ is not identically zero; otherwise, it would lead to a degeneration of the potential $\bar{\vartheta}$. □

In [21], the notion of an *Einstein-like* accR manifold (M,φ,ξ,η,g) was introduced by means of the following condition for its Ricci tensor:

$$\rho = a\,g + b\,\tilde{g} + c\,\eta\otimes\eta, \tag{27}$$

where (a,b,c) is some triplet of constants. In particular, when $b=0$ and $b=c=0$, the manifold is called an *η-Einstein manifold* and an *Einstein manifold*, respectively. If a, b, c are functions on M, then the manifold satisfying condition (27) is called *almost-Einstein-like*, and in particular for $b=0$ and $b=c=0$ it is called an *almost-η-Einstein* and *almost-Einstein* manifold, respectively.

Theorem 3. *A $G(\mathcal{F}_0)$-manifold $(M,\varphi,\bar{\xi},\bar{\eta},\bar{g})$ with an almost Riemann soliton $(\bar{g},\bar{\vartheta},\bar{\sigma})$ and a vertical potential is an almost Einstein-like manifold if and only if the condition $\bar{k} = \lambda e^w$ ($\lambda = const$) is satisfied on $\mathcal{H}$.*

The almost-Einstein-like manifold $(M,\varphi,\bar{\xi},\bar{\eta},\bar{g})$ has the following Ricci tensor:

$$\begin{aligned}\bar{\rho} = &-\left[2n\,\bar{\sigma} + \mathrm{d}\bar{k}(\bar{\xi}) + (4n-1)\bar{k}\,\mathrm{d}u(\bar{\xi})\right]\bar{g} - (2n-1)\bar{k}\,\mathrm{d}v(\bar{\xi})\,\tilde{\bar{g}}\\ &-(2n-1)\left\{\mathrm{d}\bar{k}(\bar{\xi}) - \bar{k}\left[\mathrm{d}u(\bar{\xi}) + \mathrm{d}v(\bar{\xi})\right]\right\}\bar{\eta}\otimes\bar{\eta},\end{aligned} \tag{28}$$

and the expressions of the scalar curvatures are the same as those in (25) and (26).

Proof. Bearing in mind (24) and the definition of an almost-Einstein-like accR manifold, we can conclude that the considered manifold is almost Einstein-like if and only if $\bar{h}_1 + \bar{k}\,\bar{h}_2$ is a function multiple of $\bar{\eta}\otimes\bar{\eta}$. Therefore, we have the following condition due to (10) and (20)

$$\mathrm{d}\bar{k} + \bar{k}\,\mathrm{d}w\circ\varphi^2 = f\,\bar{\eta}, \tag{29}$$

where f is an arbitrary function on the manifold. An immediate consequence of (29) for $\bar{\xi}$ is $f = \mathrm{d}\bar{k}(\bar{\xi})$. Then, we obtain

$$\bar{h}_1 + \bar{k}\,\bar{h}_2 = 2\,\mathrm{d}\bar{k}(\bar{\xi})\bar{\eta}\otimes\bar{\eta}, \tag{30}$$

and in particular for $f=0$ the function $\bar{k}$ is a vertical constant.

Applying φ to the argument of (29), we can observe the following consequence:

$$\mathrm{d}\bar{k}\circ\varphi - \bar{k}\,\mathrm{d}w\circ\varphi = 0.$$

Since $\bar{k}$ is not zero, the last equation has the following solution: $\bar{k} = \lambda e^w$ restricted on $\mathcal{H}$, where λ is an arbitrary constant.

The formula in (28) follows from (24) and (30). It implies the same expressions of $\bar{\tau}$ and $\bar{\tau}^*$ as in (25) and (26). □

Theorem 4. *Let $(M, \varphi, \bar{\xi}, \bar{\eta}, \bar{g})$ be a $G(\mathcal{F}_0)$-manifold with an almost-Riemann soliton $(\bar{g}, \bar{\vartheta}, \bar{\sigma})$ and a vertical potential. Then $(M, \varphi, \bar{\xi}, \bar{\eta}, \bar{g})$ is:*

(i) an almost-η-Einstein manifold if and only if v is a vertical constant, i.e., $\mathrm{d}v(\bar{\xi}) = 0$;

(ii) an almost-Einstein manifold if and only if v is a vertical constant and the condition $\bar{k} = \mu e^u$ ($\mu = const$) is satisfied on $\mathcal{V}$.

Proof. The statements in (i) and (ii) are easily derived by considering the particular cases of (27) that are reflected in (28). The equality in (ii) is a solution of $\mathrm{d}\bar{k}(\bar{\xi}) = \bar{k}\,\mathrm{d}u(\bar{\xi})$. □

Corollary 2. *Let $(M, \varphi, \bar{\xi}, \bar{\eta}, \bar{g})$ be a $G(\mathcal{F}_0)$-manifold with an almost-Riemann soliton $(\bar{g}, \bar{\vartheta}, \bar{\sigma})$ and a vertical potential. Then the Ricci tensor and the scalar curvatures are the following when $(M, \varphi, \bar{\xi}, \bar{\eta}, \bar{g})$ is:*

(i) an almost-η-Einstein manifold:

$$\bar{\rho} = -\left[2n\,\bar{\sigma} + \mathrm{d}\bar{k}(\bar{\xi}) + (4n-1)\bar{k}\,\mathrm{d}u(\bar{\xi})\right]\bar{g} - (2n-1)\left[\mathrm{d}\bar{k}(\bar{\xi}) - \bar{k}\,\mathrm{d}u(\bar{\xi})\right]\bar{\eta}\otimes\bar{\eta},$$

$\bar{\tau}$ as in (25) and $\bar{\tau}^* = 0$.

(ii) an almost-Einstein manifold:

$$\bar{\rho} = -2n\left[\bar{\sigma} + 2\bar{k}\,\mathrm{d}u(\bar{\xi})\right]\bar{g}, \qquad \bar{\tau} = -2n(2n+1)\left[\bar{\sigma} + 2\bar{k}\,\mathrm{d}u(\bar{\xi})\right], \qquad \bar{\tau}^* = 0.$$

Corollary 3. *A $G(\mathcal{F}_0)$-manifold $(M, \varphi, \bar{\xi}, \bar{\eta}, \bar{g})$ with an almost-Riemann soliton $(\bar{g}, \bar{\vartheta}, \bar{\sigma})$ and a vertical potential is an Einstein-like manifold if and only if the functions $\bar{\sigma}$, $\bar{k}$, u and v satisfy the following conditions:*

$$\bar{\sigma} + 2\bar{k}\,\mathrm{d}u(\bar{\xi}) = const, \qquad \mathrm{d}\bar{k}(\bar{\xi}) - \bar{k}\,\mathrm{d}u(\bar{\xi}) = const, \qquad \bar{k}\,\mathrm{d}v(\bar{\xi}) = const. \tag{31}$$

Moreover, $(\bar{g}, \bar{\vartheta}, \bar{\sigma})$ is a Riemann soliton with a vertical potential on the Einstein-like manifold if and only if $\mathrm{d}\bar{k}(\bar{\xi})$, $\bar{k}\,\mathrm{d}u(\bar{\xi})$, and $\bar{k}\,\mathrm{d}v(\bar{\xi})$ are constants.

Proof. The considered manifold is Einstein-like if and only if the three coefficients of $(0,2)$-tensors in (28) are constants. This system of equations is equivalent to the equations in (31). The case for the Riemann soliton follows from $\bar{\sigma} = const$ and (31). □

4.1. Example of an $\mathcal{F}_5$-Manifold of Dimension 5

A trivial example of an $\mathcal{F}_0$-manifold $\left(\mathbb{R}^{2n+1}, \varphi, \xi, \eta, g\right)$ of an arbitrary dimension is given in [14]. An accR structure is defined in the space $\mathbb{R}^{2n+1} = \left\{\left(x^1, \ldots, x^n; y^1, \ldots, y^n; t\right)\right\}$ in the following way:

$$\varphi\frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^i}, \qquad \varphi\frac{\partial}{\partial y^i} = -\frac{\partial}{\partial x^i}, \qquad \varphi\frac{\partial}{\partial t} = 0, \qquad \xi = \frac{\partial}{\partial t}, \qquad \eta = \mathrm{d}t,$$
$$g(z,z) = -\delta_{ij}\lambda^i\lambda^j + \delta_{ij}\mu^i\mu^j + \nu^2,$$

where $z = \lambda^i\dfrac{\partial}{\partial x^i} + \mu^i\dfrac{\partial}{\partial y^i} + \nu\dfrac{\partial}{\partial t}$ and δ_{ij} is the Kronecker delta.

In [19] (see also [20] (Example 5, p. 105)), we give an example of a pair of functions (u, v) on $\mathbb{R}^{2n+1}$ for dimension 5, i.e., $n = 2$; it can be written as follows

$$u = \ln\sqrt{\frac{|t|}{(x^1+y^2)^2 + (x^2-y^1)^2}}, \qquad v = \arctan\frac{x^1+y^2}{x^2-y^1}, \tag{32}$$

where $x^1 + y^2 \neq 0$, $x^2 - y^1 \neq 0$ and $t \neq 0$.

It is shown that (u, v) satisfy the condition $\mathrm{d}v = -\mathrm{d}u \circ \varphi$; therefore, the G-transformation determined by $(u, v, w = 0)$ deforms the given $\mathcal{F}_0$-manifold into an $\mathcal{F}_5$-manifold $\left(\mathbb{R}^{2n+1}, \varphi, \xi, \eta, \bar{g}\right)$ defined by $\bar{F}(x,y,z) = -\frac{1}{4}\bar{\theta}^*(\xi)\{\bar{g}(x,\varphi y)\eta(z) + \bar{g}(x,\varphi z)\eta(y)\}$, where

$\bar{\theta}^*(\xi) = 4\,\mathrm{d}u(\xi) = \frac{2}{t}$. Its curvature tensor, the scalar curvature, and the $*$-scalar curvature are given in the form

$$\bar{R} = \frac{\bar{\tau}}{32}\{\bar{g} \owedge \bar{g} - \bar{g} \owedge (\eta \otimes \eta)\}, \qquad \bar{\tau} = -\frac{8}{t^2}, \qquad \bar{\tau}^* = 0. \tag{33}$$

Now, let us introduce an almost-Riemann soliton $(\bar{g}, \bar{\vartheta}, \bar{\sigma})$ with vertical potential $\bar{\vartheta} = \bar{k}\eta$ on $\left(\mathbb{R}^{2n+1}, \varphi, \xi, \eta, \bar{g}\right)$, assuming that we have the following functions:

$$\bar{\sigma} = \frac{1}{3t^2}, \qquad \bar{k} = -\frac{1}{6t}, \tag{34}$$

which determine the soliton.

Bearing in mind Theorem 2, we can check the expression of $\bar{R}$ in (19). Using (32) and (34), we compute successively $\bar{h}_2 = 0$ due to $w = 0$, $\bar{h}_1 = -\frac{1}{3t^2}\eta \otimes \eta$, $\mathrm{d}u(\xi) = \frac{1}{2t}$, $\mathrm{d}v(\xi) = 0$, and obtain for the coefficients in (19) the following:

$$-\left[\tfrac{\bar{\sigma}}{2} + \bar{k}\,\mathrm{d}u(\bar{\xi})\right] = -\tfrac{1}{4t^2}, \qquad -\bar{k}\,\mathrm{d}v(\bar{\xi}) = 0, \qquad \bar{k}\left[\mathrm{d}u(\bar{\xi}) + \mathrm{d}v(\bar{\xi})\right] = \tfrac{1}{12t^2}.$$

Then (19) takes the form

$$\bar{R} = -\frac{1}{4t^2}\{\bar{g} \owedge \bar{g} - \bar{g} \owedge (\eta \otimes \eta)\}, \tag{35}$$

which is in agreement with (33). Thus, we verify Theorem 2 and Corollary 1.

Using (35), we can observe the following consequences:

$$\bar{\rho} = -\frac{1}{4t^2}\{7\bar{g} - 3\eta \otimes \eta\}, \qquad \bar{\tau} = -\frac{8}{t^2}, \qquad \bar{\tau}^* = 0.$$

We can thus conclude that the constructed manifold has negative scalar curvature and zero $*$-scalar curvature, and that it is almost η-Einstein-like (a particular case of almost Einstein-like manifolds), which is not almost Einstein. These results support Corollary 1, Theorem 3, Theorem 4(i), and Corollary 2(i).

In addition, we can calculate the scalar curvature $\tilde{\bar{\tau}}$ with respect to $\tilde{\bar{g}}$. In [17] (see also [20] (Corollary 2.4, p. 38)), the relation of this quantity to the $*$-scalar curvature is expressed. For the case under consideration, the given formula can be read in the following way: $\tilde{\bar{\tau}} = -\bar{\tau}^* - \frac{5}{4}(\theta^*(\xi))^2 - 2\xi(\theta^*(\xi))$. Using the fact that $\theta^*(\xi) = \frac{2}{t}$ in the present example, we get $\tilde{\bar{\tau}} = -\frac{1}{t^2}$, i.e., it is also negative as $\bar{\tau}$.

5. $\mathcal{F}_0$-Manifolds That Are $G(\mathcal{F}_0)$-Manifolds and Admit the Studied Solitons

In this section, we consider an $\mathcal{F}_0$-manifold $(M, \varphi, \xi, \eta, g)$, i.e., $F = 0$. Moreover, the resulting manifold $(M, \varphi, \bar{\xi}, \bar{\eta}, \bar{g})$, via a G-transformation, is again in $\mathcal{F}_0$, i.e., $\bar{F} = 0$.

To ensure that both considered manifolds are in $\mathcal{F}_0$, the transformation between them must be of a subgroup G_0 of the group G and defined by the following conditions [18]:

$$\mathrm{d}u \circ \varphi = \mathrm{d}v \circ \varphi^2, \qquad \mathrm{d}u(\xi) = \mathrm{d}v(\xi) = \mathrm{d}w \circ \varphi = 0. \tag{36}$$

In this case, the relationship between the curvature tensors R and $\bar{R}$ for g and $\bar{g}$, respectively, is known from [20] (p. 83) (see also [18,24]) and can be written as follows:

$$\bar{R} = R - g \owedge S + g^* \owedge S^* + (\eta \otimes \eta) \owedge S, \tag{37}$$

where $S^* = S(\cdot, \varphi\cdot)$ and

$$\begin{aligned} S = \nabla\,\mathrm{d}u + \mathrm{d}u \otimes \mathrm{d}u + (\mathrm{d}u \circ \varphi) \otimes (\mathrm{d}u \circ \varphi) + \tfrac{1}{2}\,\mathrm{d}u(\operatorname{grad} u)[g - \eta \otimes \eta] \\ - \tfrac{1}{2}\,\mathrm{d}u(\varphi \operatorname{grad} u)[\tilde{g} - \eta \otimes \eta]. \end{aligned} \tag{38}$$

Theorem 5. *Let (M,φ,ξ,η,g) and its image via a G_0-transformation $(M,\varphi,\bar{\xi},\bar{\eta},\bar{g})$ be $\mathcal{F}_0$-manifolds. Then the corresponding scalar curvatures for the pair of B-metrics satisfy the relations*

$$\begin{aligned}\bar{\tau} &= e^{-4u}\cos 4v\{\tau-4(n-1)\operatorname{tr}S\}+e^{-4u}\sin 4v\{\tilde{\tau}+4(n-1)\operatorname{tr}S^*\},\\ \bar{\tilde{\tau}} &= e^{-4u}\cos 4v\{\tilde{\tau}+4(n-1)\operatorname{tr}S^*\}-e^{-4u}\sin 4v\{\tau-4(n-1)\operatorname{tr}S\},\end{aligned}\tag{39}$$

where

$$\operatorname{tr}S=\delta(\mathrm{d}u)+2n\,\mathrm{d}u(\operatorname{grad}u),\qquad \operatorname{tr}S^*=\tilde{\delta}(\mathrm{d}u)+2n\,\mathrm{d}u(\varphi\operatorname{grad}u)\tag{40}$$

for $\delta(\mathrm{d}u)=g^{ij}(\nabla\,\mathrm{d}u)_{ij}$ and $\tilde{\delta}(\mathrm{d}u)=\tilde{g}^{ij}(\nabla\,\mathrm{d}u)_{ij}$.

Proof. Using (37) with (38) for the corresponding curvature tensors R and $\bar{R}$, through lengthy but standard calculations, we obtain expressions for the corresponding scalar curvatures given in (39). □

Obviously, the trace $\delta(\mathrm{d}u)$ involved in (40) is actually the Laplacian of u for g, usually denoted by Δu or $\nabla^2 u$, whereas $\tilde{\delta}(\mathrm{d}u)$ is some kind of associated quantity of Δu using $\tilde{g}$.

Corollary 4. *Let $(\bar{g},\bar{\vartheta},\bar{\sigma})$ be an almost-Riemann soliton with vertical potential on $(M,\varphi,\bar{\xi},\bar{\eta},\bar{g})$ and let the requirements of Theorem 5 be fulfilled. Then (M,φ,ξ,η,g) has constant scalar curvatures for both B-metrics g and $\tilde{g}$.*

Proof. According to Theorem 1, $(M,\varphi,\bar{\xi},\bar{\eta},\bar{g})$ is flat, i.e., $\bar{R}=0$ and therefore we have $\bar{\tau}=\bar{\tilde{\tau}}=\bar{\sigma}=\mathrm{d}\bar{k}=0$. Substituting the last equalities into (39) and considering (40), we get

$$\begin{aligned}\tau &= 4(n-1)\Big\{\delta(\mathrm{d}u)+2n\,\mathrm{d}u(\operatorname{grad}u)\Big\},\\ \tilde{\tau} &= -4(n-1)\big\{\tilde{\delta}(\mathrm{d}u)+2n\,\mathrm{d}u(\varphi\operatorname{grad}u)\big\}.\end{aligned}\tag{41}$$

In this way, we can obtain the conditions that the scalar curvatures of an $\mathcal{F}_0$-manifold must satisfy in order to be mapped by a G_0-transformation to an $\mathcal{F}_0$-manifold, admitting an almost-Riemann soliton under study.

As a consequence of Theorem 5.2 in [20] (p. 81), we can deduce for an $\mathcal{F}_0$-manifold that the functions $\arctan\frac{\tilde{\tau}}{\tau}$ and $\ln\sqrt{\tau^2+\tilde{\tau}^2}$ are constants, which implies that τ and $\tilde{\tau}$ are constants. □

As is well known, the Bochner curvature tensor B on a Kähler manifold can be considered in some sense as an analogue of the Weyl curvature tensor, and the vanishing of B has remarkable geometric interpretations.

In [24], the *Bochner curvature tensor of φ-holomorphic type* for a curvature tensor with the Kähler property is introduced on an arbitrary accR manifold of dimension at least 7, i. e., $n\geq 3$. The Riemannian curvature tensor R of an $\mathcal{F}_0$-manifold has the Kähler property (15) and the definition of the Bochner curvature tensor $B(R)$ as a tensor of type $(0,4)$ corresponding to R can be written in the following form:

$$\begin{aligned}B(R) = R &- \tfrac{1}{2(n-2)}\{g\owedge\rho-g^*\owedge\rho^*-(\eta\otimes\eta)\owedge\rho\}\\ &+\tfrac{1}{8(n-1)(n-2)}\{\tau[g\owedge g-g^*\owedge g^*-2(\eta\otimes\eta)\owedge g]\\ &\qquad+2\tilde{\tau}[g\owedge g^*-(\eta\otimes\eta)\owedge g^*]\},\end{aligned}\tag{42}$$

where $\rho^*=\rho(\cdot,\varphi\cdot)$.

Corollary 5. *Let $(\bar{g},\bar{\vartheta},\bar{\sigma})$ be an almost-Riemann soliton with vertical potential on $(M,\varphi,\bar{\xi},\bar{\eta},\bar{g})$ of dimension at least 7 and let the requirements of Theorem 5 be fulfilled. Then the Ricci tensor of (M,φ,ξ,η,g) has the following form:*

$$\rho=-2(n-2)S+\tfrac{1}{4(n-1)}\{\tau[g-\eta\otimes\eta]+\tilde{\tau}[\tilde{g}-\eta\otimes\eta]\},\tag{43}$$

where S is determined by (38). Moreover, $\operatorname{tr} S$, $\operatorname{tr} S^*$, *τ and $\tilde{\tau}$ are constants.*

Proof. It is known from [18] that $B(R)$ on an $\mathcal{F}_0$-manifold is a contact conformal invariant of the group G_0, i.e., $B(\bar{R}) = B(R)$.

For the considered manifold $(M, \varphi, \bar{\xi}, \bar{\eta}, \bar{g})$ we obtained $\bar{R} = 0$. Hence, $B(\bar{R})$ also vanishes and this means $B(R) = 0$ for $(M, \varphi, \xi, \eta, g)$. Then, due to (42), we can obtain an expression of R as follows:

$$\begin{aligned} R = &\frac{1}{2(n-2)}\{g \owedge \rho - g^* \owedge \rho^* - (\eta \otimes \eta) \owedge \rho\} \\ &- \frac{1}{8(n-1)(n-2)}\{\tau[g \owedge g - g^* \owedge g^* - 2(\eta \otimes \eta) \owedge g] + 2\tilde{\tau}[g \owedge g^* - (\eta \otimes \eta) \owedge g^*]\} \end{aligned} \tag{44}$$

where τ and $\tilde{\tau}$ are constants, according to Corollary 4 and have values given in (41).

Using the fact that the Ricci tensor is hybrid with respect to φ, i.e., $\rho = -\rho(\varphi\cdot, \varphi\cdot)$, on an $\mathcal{F}_0$-manifold, we can rewrite (44) in the following more compact form:

$$R = -g \owedge L + g^* \owedge L^* + (\eta \otimes \eta) \owedge L, \tag{45}$$

where L is defined by

$$L = \frac{1}{2(n-2)}\rho - \frac{1}{8(n-1)(n-2)}\{\tau[g - \eta \otimes \eta] + \tilde{\tau}[\tilde{g} - \eta \otimes \eta]\}. \tag{46}$$

The vanishing of $\bar{R}$ and (37) imply the following

$$R = g \owedge S - g^* \owedge S^* - (\eta \otimes \eta) \owedge S. \tag{47}$$

Comparing (47) with (45), we deduce that $S = -L$ and consequently (43) holds. Equalities (40) and (41) imply that $\operatorname{tr} S$ and $\operatorname{tr} S^*$ are also constants like τ and $\tilde{\tau}$. $\square$

5.1. Example of an $\mathcal{F}_0$-Manifold of Arbitrary Dimension

Let us consider again the $\mathcal{F}_0$-manifold $\left(\mathbb{R}^{2n+1}, \varphi, \xi, \eta, g\right)$ that was described at the beginning of Section 4.1.

In [23], the following example of a pair of functions (u, v) on an accR manifold is given as follows:

$$u = \sum_{i=1}^{n} \ln \sqrt{\left(x^i\right)^2 + \left(y^i\right)^2}, \qquad v = \sum_{i=1}^{n} \arctan \frac{y^i}{x^i}. \tag{48}$$

It is shown that (u, v) is a φ-holomorphic pair of functions, i.e., the conditions for them in (36) are satisfied.

Let w be the function e^t. Then, we have $\mathrm{d}w = w\eta$, which implies $\mathrm{d}w \circ \varphi = 0$. As a result, (36) holds and (u, v, w) determine a contact conformal transformation from G_0. This transformation deforms $\left(\mathbb{R}^{2n+1}, \varphi, \xi, \eta, g\right)$ into $\left(\mathbb{R}^{2n+1}, \varphi, \bar{\xi}, \bar{\eta}, \bar{g}\right)$, which is again an $\mathcal{F}_0$-manifold.

Bearing in mind (37) and the fact that $\left(\mathbb{R}^{2n+1}, \varphi, \xi, \eta, g\right)$ is flat, we obtain the curvature tensor of the resulting manifold in the form

$$\bar{R} = -g \owedge S + g^* \owedge S^* + (\eta \otimes \eta) \owedge S,$$

where S is denoted in (38) and here u is given in (48). Then, we compute the scalar curvatures and they have the following values:

$$\begin{aligned} \bar{\tau} &= 4(n-1)e^{-4u}\{\sin 4v \operatorname{tr} S^* - \cos 4v \operatorname{tr} S\}, \\ \bar{\tilde{\tau}} &= 4(n-1)e^{-4u}\{\cos 4v \operatorname{tr} S^* + \sin 4v \operatorname{tr} S\}, \end{aligned}$$

where u and v are given in (48), and

$$\operatorname{tr} S = -2(n-1)\sum_{i=1}^{n}\frac{\left(x^i\right)^2-\left(y^i\right)^2}{\left[\left(x^i\right)^2-\left(y^i\right)^2\right]^2}, \qquad \operatorname{tr} S^* = -4(n-1)\sum_{i=1}^{n}\frac{x^i y^i}{\left[\left(x^i\right)^2-\left(y^i\right)^2\right]^2}. \tag{49}$$

This result supports Theorem 5.

Bearing in mind (49), we obtain vanishing scalar curvatures $\bar{\tau}$ and $\tilde{\bar{\tau}}$ for $n=1$, i.e., $(\mathbb{R}^3, \varphi, \bar{\xi}, \bar{\eta}, \bar{g})$ is scalar-flat.

Since the two considered $\mathcal{F}_0$-manifolds are related by a transformation from G_0, $(\mathbb{R}^{2n+1}, \varphi, \xi, \eta, g)$ is flat and the Bochner curvature tensor is an invariant of G_0 for dimension at least 7, we deduce that $B(\bar{R}) = 0$.

Then, bearing in mind (42), the curvature tensor $\bar{R}$ has an expression corresponding to (45) with (46), namely,

$$\bar{R} = -\bar{g} \owedge \bar{L} + \bar{g}^* \owedge \bar{L}^* + (\bar{\eta}\otimes\bar{\eta}) \owedge \bar{L},$$

$$\bar{L} = \frac{1}{2(n-2)}\bar{\rho} - \frac{1}{8(n-1)(n-2)}\{\bar{\tau}[\bar{g}-\bar{\eta}\otimes\bar{\eta}] + \tilde{\bar{\tau}}[\tilde{\bar{g}}-\bar{\eta}\otimes\bar{\eta}]\}.$$

Let us recall from [18] that if ℓ is a G-transformation determined by (3) for functions (u, v, w), then its inverse transformation ℓ^{-1} is the G-transformation determined for the functions $(-u, -v, -w)$. Then, the present example is in unison with Corollary 4 and Corollary 5.

6. Conclusions

Steady-state solutions of geometric flows, including almost-Riemann solitons, are still the subject of intense research and interest in differential geometry. In the present paper, we introduced almost-Riemann solitons on almost-contact complex Riemannian manifolds and achieved the first results in the study the coexistence of these structures on an odd-dimensional manifold. More precisely, the most important curvature properties of the manifolds obtained from manifolds of the cosymplectic type by means of conformal transformations of the considered structures have been described. Since the study of Riemann solitons is still in its early stages, any contribution in this direction may introduce new perspectives on the geometry of the manifold.

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