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A Two-Step Iteration Method for Vertical Linear Complementarity Problems

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Abstract: In this paper, a two-step iteration method was established which can be viewed as a generalisation of the existing modulus-based methods for vertical linear complementarity problems. The convergence analysis of the proposed method is presented, which can enlarge the convergence domain of the parameter matrix compared to the recent results. Numerical examples show that the proposed method is efficient with the two-step technique and confirm the improvement of the theoretical results.

Keywords: vertical linear complementarity problem; two-step splitting; modulus-based method

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1. Introduction

As a generalisation of a linear complementarity problem (LCP) [1], the vertical linear complementarity problem (VLCP) has wide applications in many fields of science and technology, such as control theory, nonlinear networks and economics; see [2–7] for details. Let $A_1, \dots, A_\ell \in \mathbb{R}^{n\times n}$ and $q_1, \dots, q_\ell \in \mathbb{R}^n$, where ℓ is a positive integer. The VLCP$_\ell$ seeks to find $z, w_1, \dots, w_\ell \in \mathbb{R}^n$ to satisfy

$$w_i = A_i z + q_i, i = 1, \dots, \ell, \ \text{ with } \ \min\{z, w_1, \dots, w_\ell\} = 0, \tag{1}$$

where the minimum operation is taken component-wise, which implies that all the involved vectors are non-negative and at least one entry of the ith ($i = 1, 2, \dots, n$) component is zero. Note that the VLCP$_1$ is exactly the LCP.

Recently, for solving the VLCP, a modulus-based formulation was introduced by Mezzadi in [8], which can result in a class of modulus-based matrix splitting (MMS) iteration methods, shown to be more efficient than smooth Newton method [9]. The kind of modulus-based methods can be viewed as the generalisation of the one for solving LCP [10]. For other existing methods of solving the VLCP, one can refer to the recent studies [8,11,12] and the references therein. The MMS methods have been successfully used to solve many kinds of complementarity problems due to the high efficiency in solving linear modulus equations in each iteration. There were also many accelerated techniques, such as double splitting [13], precondition [14], two-step splitting [15–18], and relaxation [19] to improve the convergence rate of MMS methods in recent works.

In this work, we focus on applying the two-step splitting technique to the equivalent modulus equation of the VLCP. The advantage of the two-step splitting is to make full use of the information of the system matrix in each iteration. Such a technique was successfully used in the LCP [15,16], nonlinear complementarity problem [17], and horizontal LCP [18]. Numerical results showed that the computation time could be saved significantly by the two-step technique comparing to the original iteration method. Hence, we aim to construct the two-step modulus-based matrix splitting (TMMS) iteration for the VLCP, which will be done in Section 2. The convergence theorems of the proposed method will be given in Section 3, and shown to improve the existing ones of the MMS method. By numerical tests in the Section 4, the efficiency of the proposed method is presented. Concluding remarks are given in Section 5.

Some necessary notations, definitions and known results are given first. Let $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$. Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ and $A = D_A - C_A = D_A - L_A - U_A$, where $D_A, -C_A, -L_A$ and $-U_A$ denote the diagonal, nondiagonal, strictly lower triangle and strictly upper triangle parts of A, respectively. By $\rho(A)$, we denote the spectral radius of A. For $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, $A > (\geq) 0$ means that $a_{ij} > (\geq) 0$ for all i, j. For two matrices $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{m\times n}$ the order $A \geq (>) B$ means $a_{ij} \geq (>) b_{ij}$ for any i and j. By $|A|$, we denote $|A| = (|a_{ij}|)$ and the comparison matrix of A is $\langle A\rangle = (\langle a_{ij}\rangle)$, where $\langle a_{ij}\rangle = |a_{ij}|$ if $i = j$ and $\langle a_{ij}\rangle = -|a_{ij}|$ if $i \neq j$. A is called a Z-matrix if $a_{ij} \leq 0$ for any $i \neq j$, a nonsingular M-matrix if it is a nonsingular Z-matrix with $A^{-1} \geq 0$, an H-matrix if $\langle A\rangle$ is a nonsingular M-matrix, a strictly diagonal dominant (s.d.d.) matrix if $|a_{ii}| > \sum_{j\neq i} |a_{ij}|$ for all $1 \leq i \leq n$ (e.g., see [20]), and an H_+-matrix if A is an H-matrix with $a_{ii} > 0$ for every i (e.g., see [21]). $A = M - N$ is called an H-splitting if $\langle M\rangle - |N|$ is a nonsingular M-matrix (e.g., see [22]). It is known that the VLCP has a unique solution if the row-representative matrices $\{A_1, A_2, \ldots, A_\ell\}$ satisfy row $\mathcal{W}$-property; see [7]. In the following discussion, we always assume that both the system matrices of the VLCP and their row-representative matrices are H_+-matrices, which is a sufficient condition of the row $\mathcal{W}$-property, including many typical situations where the solution is unique; see [7,8].

2. New Method

First, we introduce the MMS method for solving VLCP$_\ell$.

Let $A_i = F_i - G_i$ $(i = 1, 2, \ldots, \ell)$ be ℓ splittings of A_i, Ω be a diagonal matrix with positive diagonal entries and γ be a positive constant. Then, with

$$\left\{\begin{array}{lcl} z & = & \frac{1}{\gamma}(|x_1| + x_1), \\ w_j & = & \sum\limits_{i=1}^{j} \frac{\Omega}{\gamma}(|x_i| - x_i) + \frac{\Omega}{\gamma}(|x_{j+1}| + x_{j+1}), \ \ j = 1, 2, \ldots, \ell - 1, \\ w_\ell & = & \sum\limits_{i=1}^{\ell} \frac{\Omega}{\gamma}(|x_i| - x_i), \end{array}\right. \tag{2}$$

VLCP$_\ell$ can be equivalently transformed into a system of fixed-point equations

$$\left\{\begin{array}{lcl} (2^{\ell-1}\Omega + \sum\limits_{i=1}^{\ell-1} 2^{\ell-i-1}F_i + F_\ell)x_1 & = & (\sum\limits_{i=1}^{\ell-1} 2^{\ell-i-1}G_i + G_\ell)x_1 + (2^{\ell-1}\Omega - \sum\limits_{i=1}^{\ell-1} 2^{\ell-i-1}A_i \\ & & -A_\ell)|x_1| + \Omega \sum\limits_{i=2}^{\ell} 2^{\ell-i+1}|x_i| - \gamma(\sum\limits_{i=1}^{\ell-1} 2^{\ell-i-1}q_i + q_\ell), \\ x_j & = & \frac{1}{2}\Omega^{-1}\left[(A_{j-1} - A_j)(|x_1| + x_1) + \gamma(|x_{j+1}| + x_{j+1})\right. \\ & & \left.+\gamma q_{j-1} - \gamma q_j\right], j = 2, 3, \ldots, \ell - 1, \\ x_\ell & = & \frac{1}{2}\Omega^{-1}\left[(A_{\ell-1} - A_\ell)(|x_1| + x_1) + \gamma q_{\ell-1} - \gamma q_\ell\right], \end{array}\right. \tag{3}$$

see [8] for more details. Based on (2) and (3), the MMS method is presented as follows:

Method 1 ([8]). **MMS method for VLCP$_\ell$**
Let $A_i = F_i - G_i (i = 1, 2, \ldots, \ell)$ be ℓ splittings of $A_i \in \mathbb{R}^{n\times n}$, $\Omega \in \mathbb{R}^{n\times n}$ be a diagonal matrix with positive diagonal entries and γ be a positive constant. Given $x_1^{(0)} \in \mathbb{R}^n$, for $k = 0, 1, 2, \ldots$, compute $x_2^{(k)}, \ldots, x_\ell^{(k)}$ by

$$\begin{cases} x_\ell^{(k)} &= \frac{1}{2}\Omega^{-1}\left[(A_{\ell-1}-A_\ell)(|x_1^{(k)}|+x_1^{(k)})+\gamma q_{\ell-1}-\gamma q_\ell\right], \\ x_j^{(k)} &= \frac{1}{2}\Omega^{-1}\left[(A_{j-1}-A_j)(|x_1^{(k)}|+x_1^{(k)})+\gamma(|x_{j+1}^{(k)}|+x_{j+1}^{(k)})+\gamma q_{j-1}-\gamma q_j\right], \\ & \quad j=\ell-1,\ell-2,\ldots,2. \end{cases} \tag{4}$$

and compute $x_1^{(k+1)} \in \mathbb{R}^n$ *by*

$$\begin{aligned} & (2^{\ell-1}\Omega+\sum_{i=1}^{\ell-1}2^{\ell-i-1}F_i+F_\ell)x_1^{(k+1)} \\ = & (\sum_{i=1}^{\ell-1}2^{\ell-i-1}G_i+G_\ell)x_1^{(k)}+(2^{\ell-1}\Omega-\sum_{i=1}^{\ell-1}2^{\ell-i-1}A_i-A_\ell)|x_1^{(k)}| \\ & +\Omega\sum_{i=2}^{\ell}2^{\ell-i+1}|x_i^{(k)}|-\gamma(\sum_{i=1}^{\ell-1}2^{\ell-i-1}q_i+q_\ell). \end{aligned}$$

Then, set

$$\begin{cases} z^{(k+1)} &= \frac{1}{\gamma}(|x_1^{(k+1)}|+x_1^{(k+1)}), \\ w_j^{(k+1)} &= \sum\limits_{i=1}^{j}\frac{\Omega}{\gamma}(|x_i^{(k+1)}|-x_i^{(k+1)})+\frac{\Omega}{\gamma}(|x_{j+1}^{(k+1)}|+x_{j+1}^{(k+1)}), \;\; j=1,2,\ldots,\ell-1, \\ w_\ell^{(k+1)} &= \sum\limits_{i=1}^{\ell}\frac{\Omega}{\gamma}(|x_i^{(k+1)}|-x_i^{(k+1)}), \end{cases} \tag{5}$$

until the iteration is convergent.

In order to make full use of the information in the system matrices, by the two-step matrix splitting technique, we construct the two-step modulus-based matrix splitting (TMMS) iteration method as follows:

Method 2. TMMS method for VLCP$_\ell$
Let $\Omega \in \mathbb{R}^{n\times n}$ *be a diagonal matrix with positive diagonal entries,* γ *be a positive constant, and* $A_i = F_i^{(t)} - G_i^{(t)} (t=1,2)$ *be two splittings of* A_i $(i=1,2,\ldots,\ell)$*. Given an initial vector* $x_1^{(0)} \in \mathbb{R}^n$*, for* $k=0,1,2,\ldots$*, compute* $x_1^{(k+1)} \in \mathbb{R}^n$ *by*

$$\begin{cases} (2^{\ell-1}\Omega+\sum\limits_{i=1}^{\ell-1}2^{\ell-i-1}F_i^{(1)}+F_\ell^{(1)})x_1^{(k+\frac{1}{2})} &= (\sum\limits_{i=1}^{\ell-1}2^{\ell-i-1}G_i^{(1)}+G_\ell^{(1)})x_1^{(k)}+(2^{\ell-1}\Omega \\ & \quad -\sum\limits_{i=1}^{\ell-1}2^{\ell-i-1}A_i-A_\ell)|x_1^{(k)}|+\Omega\sum\limits_{i=2}^{\ell}2^{\ell-i+1}|x_i^{(k)}| \\ & \quad -\gamma(\sum\limits_{i=1}^{\ell-1}2^{\ell-i-1}q_i+q_\ell), \\ (2^{\ell-1}\Omega+\sum\limits_{i=1}^{\ell-1}2^{\ell-i-1}F_i^{(2)}+F_\ell^{(2)})x_1^{(k+1)} &= (\sum\limits_{i=1}^{\ell-1}2^{\ell-i-1}G_i^{(2)}+G_\ell^{(2)})x_1^{(k+\frac{1}{2})}+(2^{\ell-1}\Omega \\ & \quad -\sum\limits_{i=1}^{\ell-1}2^{\ell-i-1}A_i-A_\ell)|x_1^{(k+\frac{1}{2})}|+\Omega\sum\limits_{i=2}^{\ell}2^{\ell-i+1}|x_i^{(k+\frac{1}{2})}| \\ & \quad -\gamma(\sum\limits_{i=1}^{\ell-1}2^{\ell-i-1}q_i+q_\ell), \end{cases} \tag{6}$$

where $x_2^{(k)},\ldots,x_\ell^{(k)}$ *are computed by (4). Then, set the same sequences as (5) until the iteration is convergent.*

Clearly, if we take $F_i^{(1)} - G_i^{(1)} = F_i^{(2)} - G_i^{(2)}$, Method 2 reduces to Method 1. For the simplest case, when $\ell = 2$, by (4), the main iteration (6) reduces to

$$\begin{cases} (2\Omega + F_1^{(1)} + F_2^{(1)})x_1^{(k+\frac{1}{2})} & = & (G_1^{(1)} + G_2^{(1)})x_1^{(k)} + (2\Omega - A_1 - A_2)|x_1^{(k)}| \\ & & +|(A_1 - A_2)(|x_1^{(k)}| + x_1^{(k)}) + \gamma q_1 - \gamma q_2| \\ & & -\gamma(q_1 + q_2), \\ (2\Omega + F_1^{(2)} + F_2^{(2)})x_1^{(k+1)} & = & (G_1^{(2)} + G_2^{(2)})x_1^{(k+\frac{1}{2})} + (2\Omega - A_1 - A_2)|x_1^{(k+\frac{1}{2})}| \\ & & +|(A_1 - A_2)(|x_1^{(k+\frac{1}{2})}| + x_1^{(k+\frac{1}{2})}) + \gamma q_1 - \gamma q_2| \\ & & -\gamma(q_1 + q_2), \end{cases} \tag{7}$$

Moreover, by specifically choosing the matrix splittings of the system matrices, one can obtain some TMMS relaxation methods. For $i = 1, 2, \ldots, \ell$, taking

$$\begin{cases} F_i^{(1)} = \frac{1}{\alpha}(D_{A_i}^{(1)} - \beta L_{A_i}^{(1)}), G_i^{(1)} = F_i^{(1)} - A_i, \\ F_i^{(2)} = \frac{1}{\alpha}(D_{A_i}^{(2)} - \beta U_{A_i}^{(2)}), G_i^{(2)} = F_i^{(2)} - A_i, \end{cases} \tag{8}$$

we can obtain the two-step modulus-based accelerated overrelaxation (TMAOR) iteration method. Taking $(\alpha, \beta) = (\alpha, \alpha)$, $(\alpha, \beta) = (1, 1)$ and $(\alpha, \beta) = (1, 0)$, the TMAOR reduces to the two-step modulus-based successive overrelaxation (TMSOR), Gauss–Seidel (TMMGS) and Jacobi (TMJ) iteration methods, respectively.

3. Convergence Analysis

Lemma 3 ([20])**.** *Assume that A is a Z-matrix. Then, the following three statements are equivalent:*

(1) *A is a nonsingular M-matrix;*

(2) *There exists a diagonal matrix D with positive diagonal entries, such that AD is an s.d.d. matrix with positive diagonal entries.*

(3) *If $A = F - G$ satisfy $F^{-1} \geq 0$ and $G \geq 0$, then $\rho(F^{-1}G) < 1$.*

Lemma 4 ([23])**.** *Let A be an H-matrix. Then $|A^{-1}| \leq \langle A \rangle^{-1}$.*

Lemma 5 ([24])**.** *Let $B \in \mathbb{R}^{n\times n}$ be an s.d.d. matrix. Then, $\forall C \in \mathbb{R}^{n\times n}$,*

$$||B^{-1}C||_\infty \leq \max_{1\leq i\leq n} \frac{(|C|e)_i}{(\langle B \rangle e)_i}.$$

We first give the convergence result for $\ell = 2$.

Theorem 6. *Let A_1, A_2 and all their row-representative matrices be H_+-matrices. Let $A_i = F_i^{(t)} - G_i^{(t)} (t = 1, 2)$ be two splittings of A_i $(i = 1, 2)$. Assume that:*

(I) *For $t = 1, 2$, $A_1 = F_1^{(t)} - G_1^{(t)}$ is a splitting of A_1 satisfying $D_{F_1^{(t)}} > 0$, and $A_2 = F_2^{(t)} - G_2^{(t)}$ is an H-splitting of A_2;*

(II) *For $t = 1, 2$, $\langle F_1^{(t)} \rangle \geq \langle F_2^{(t)} \rangle$ and $|G_2^{(t)}| \geq |G_1^{(t)}|$;*

(III) *There exists a positive diagonal matrix T with positive diagonal entries such that both $(\langle F_2^{(1)} \rangle - |G_2^{(1)}|)T$ and $(\langle F_2^{(2)} \rangle - |G_2^{(2)}|)T$ are s.d.d. matrices;*

(IV)

$$\Omega Te > \max_{t=1,2} \left\{ [\frac{1}{2}(D_{F_1^{(t)}} + D_{F_2^{(t)}}) - (\langle F_2^{(t)} \rangle - |G_2^{(t)}|)] Te \right\}. \tag{9}$$

Then, Method 2 converges to the solution of the $VLCP_2$.

Proof. For $t = 1, 2$, by Assumptions (II) and (III), we have

$$\begin{array}{rl} & \langle 2\Omega + F_1^{(t)} + F_2^{(t)} \rangle Te \\ > & \langle F_1^{(t)} + F_2^{(t)} \rangle Te \\ \geq & (\langle F_1^{(t)} \rangle + \langle F_2^{(t)} \rangle) Te \\ \geq & 2\langle F_2^{(t)} \rangle Te \\ \geq & (2\langle F_2^{(t)} \rangle - |G_2^{(t)}|) Te \\ > & 0. \end{array}$$

Therefore, $\langle 2\Omega + F_1^{(t)} + F_2^{(t)} \rangle T$ is an s.d.d matrix, which implies that $2\Omega + F_1^{(t)} + F_2^{(t)}$ is an H-matrix. Then, by Lemma 4, we have the error at iteration $k+1$:

Let x_1^* be the solution of (3). By the first equation of (7), we can obtain the error at the iteration $(k+1)$:

$$\left\{ \begin{array}{rcl} (2\Omega + F_1^{(1)} + F_2^{(1)})(x_1^{(k+\frac{1}{2})} - x_1^*) & = & (G_1^{(1)} + G_2^{(1)})(x_1^{(k)} - x_1^*) + (2\Omega - A_1 - A_2) \\ & & \times(|x_1^{(k)}| - |x_1^*|) + |(A_1 - A_2)(|x_1^{(k)}| + x_1^{(k)}) \\ & & +\gamma q_1 - \gamma q_2| - |(A_1 - A_2)(|x_1^*| + x_1^*) \\ & & +\gamma q_1 - \gamma q_2|, \\ (2\Omega + F_1^{(2)} + F_2^{(2)})(x_1^{(k+1)} - x_1^*) & = & (G_1^{(2)} + G_2^{(2)})(x_1^{(k+\frac{1}{2})} - x_1^*) + (2\Omega - A_1 - A_2) \\ & & \times(|x_1^{(k+\frac{1}{2})}| - |x_1^*|) + |(A_1 - A_2)(|x_1^{(k+\frac{1}{2})}| + x_1^{(k+\frac{1}{2})}) \\ & & +\gamma q_1 - \gamma q_2| - |(A_1 - A_2)(|x_1^*| + x_1^*) \\ & & +\gamma q_1 - \gamma q_2|, \end{array} \right.$$

Then, by Lemma 4, we obtain

$$|x_1^{(k+1)} - x_1^*| \leq \mathcal{P}^{(2)} \mathcal{P}^{(1)} |x_1^{(k)} - x_1^*|, \tag{10}$$

where

$$\left\{ \begin{array}{ll} \mathcal{P}^{(t)} & = \mathcal{F}^{(t)^{-1}} \mathcal{G}^{(t)}, \\ \mathcal{F}^{(t)} & = \langle 2\Omega + F_1^{(t)} + F_2^{(t)} \rangle, \\ \mathcal{G}^{(t)} & = |G_1^{(t)} + G_2^{(t)}| + |2\Omega - A_1 - A_2| + 2|A_1 - A_2|. \end{array} \right.$$

By Lemma 5, we have

$$||T^{-1}\mathcal{P}^{(t)}T||_\infty = ||T^{-1}\mathcal{F}^{(t)^{-1}}\mathcal{G}^{(t)}T||_\infty = ||(\mathcal{F}^{(t)}T)^{-1}(\mathcal{G}^{(t)}T)||_\infty \leq \max_{1 \leq i \leq n} \frac{(\mathcal{G}^{(t)}Te)_i}{(\mathcal{F}^{(t)}Te)_i}. \tag{11}$$

Still considering assumption (II), we can obtain

$$\left\{ \begin{array}{l} |G_1^{(t)} + G_2^{(t)}| + |G_1^{(t)} - G_2^{(t)}| = 2|G_2^{(t)}|, \\ |C_{F_1^{(t)}} + C_{F_2^{(t)}}| + |C_{F_1^{(t)}} - C_{F_2^{(t)}}| = 2|C_{F_2^{(t)}}|, \\ |D_{F_1^{(t)}} - D_{F_2^{(t)}}| = D_{F_1^{(t)}} - D_{F_2^{(t)}}. \end{array} \right. \tag{12}$$

Then, we have

$$
\begin{aligned}
& \mathcal{F}^{(t)}Te-\mathcal{G}^{(t)}Te \\
= & \big(\langle 2\Omega+F_1^{(t)}+F_2^{(t)}\rangle-|G_1^{(t)}+G_2^{(t)}|-|2\Omega-A_1-A_2|-2|A_1-A_2|\big)Te \\
= & \big[2\Omega+D_{F_1^{(t)}}+D_{F_2^{(t)}}-|C_{F_1^{(t)}}+C_{F_2^{(t)}}|-|G_1^{(t)}+G_2^{(t)}| \\
& -|2\Omega-D_{F_1^{(t)}}-D_{F_2^{(t)}}-C_{F_1^{(t)}}-C_{F_2^{(t)}}+G_1^{(t)}+G_2^{(t)}| \\
& -2|D_{F_1^{(t)}}+C_{F_1^{(t)}}-G_1^{(t)}-(D_{F_2^{(t)}}+C_{F_2^{(t)}}-G_2^{(t)})|\big]Te \\
\geq & \big(2\Omega+D_{F_1^{(t)}}+D_{F_2^{(t)}}-|C_{F_1^{(t)}}+C_{F_2^{(t)}}|-|G_1^{(t)}+G_2^{(t)}| \\
& -|2\Omega-D_{F_1^{(t)}}-D_{F_2^{(t)}}|-|C_{F_1^{(t)}}+C_{F_2^{(t)}}|+2|G_1^{(t)}+G_2^{(t)}| \\
& -2|D_{F_1^{(t)}}-D_{F_2^{(t)}}|-2|C_{F_1^{(t)}}-C_{F_2^{(t)}}|-2|G_1^{(t)}-G_2^{(t)}|\big)Te \\
= & \big(2\Omega+3D_{F_2^{(t)}}-D_{F_1^{(t)}}-|2\Omega-D_{F_1^{(t)}}-D_{F_2^{(t)}}| \\
& -2|G_1^{(t)}+G_2^{(t)}|-2|G_1^{(t)}-G_2^{(t)}|-2|C_{F_1^{(t)}}+C_{F_2^{(t)}}|-2|C_{F_1^{(t)}}-C_{F_2^{(t)}}|\big)Te \\
= & \big(2\Omega+3D_{F_2^{(t)}}-D_{F_1^{(t)}}-|2\Omega-D_{F_1^{(t)}}-D_{F_2^{(t)}}|-4|G_2^{(t)}|-4|C_{F_2^{(t)}}|\big)Te, \qquad (13)
\end{aligned}
$$

where the last two equalities hold by (12).

When

$$\Omega \geq \frac{1}{2}(D_{F_1^{(t)}}+D_{F_2^{(t)}}),$$

by (13), we have

$$
\begin{aligned}
& \mathcal{F}^{(t)}Te-\mathcal{G}^{(t)}Te \\
\geq & \big[2\Omega+3D_{F_2^{(t)}}-D_{F_1^{(t)}}-(2\Omega-D_{F_1^{(t)}}-D_{F_2^{(t)}})-4|G_2^{(t)}|-4|C_{F_2^{(t)}}|\big]Te \\
= & \big(4D_{F_2^{(t)}}-4|G_2^{(t)}|-4|C_{F_2^{(t)}}|\big)Te \\
= & 4(\langle F_2^{(t)}\rangle-|G_2^{(t)}|)Te>0. \qquad (14)
\end{aligned}
$$

When

$$\big[\frac{1}{2}(D_{F_1^{(t)}}+D_{F_2^{(t)}})-(\langle F_2^{(t)}\rangle-|G_2^{(t)}|)\big]Te<\Omega Te<\frac{1}{2}(D_{F_1^{(t)}}+D_{F_2^{(t)}})Te,$$

by (13), we have

$$
\begin{aligned}
& \mathcal{F}^{(t)}Te-\mathcal{G}^{(t)}Te \\
\geq & \big[2\Omega+3D_{F_2^{(t)}}-D_{F_1^{(t)}}-(D_{F_1^{(t)}}+D_{F_2^{(t)}}-2\Omega)-4|G_2^{(t)}|-4|C_{F_2^{(t)}}|\big]Te \\
= & \big(4\Omega+2D_{F_2^{(t)}}-2D_{F_1^{(t)}}-4|G_2^{(t)}|-4|C_{F_2^{(t)}}|\big)Te \\
= & 4\big[\Omega-\frac{1}{2}(D_{F_1^{(t)}}+D_{F_2^{(t)}})+\langle F_2^{(t)}\rangle-|G_2^{(t)}|\big]Te>0. \qquad (15)
\end{aligned}
$$

Combining (14) and (15), we have $\mathcal{F}^{(t)}Te-\mathcal{G}^{(t)}Te>0$ provided that the assumption (IV) holds. Then, by (11), we have $||T^{-1}{\mathcal{F}^{(t)}}^{-1}\mathcal{G}^{(t)}T||_\infty<1$. Then, we have the next inequality:

$$||\rho(\mathcal{P}^{(2)}\mathcal{P}^{(1)})||\leq||T^{-1}\mathcal{P}^{(2)}\mathcal{P}^{(1)}T||_\infty\leq||T^{-1}\mathcal{P}^{(2)}T||_\infty||T^{-1}\mathcal{P}^{(1)}T||_\infty<1,$$

which implies that $\lim\limits_{k\to+\infty} x_1^{(k)} = x_1^*$ by (10), ending the proof. $\square$

Remark 7. *By the proof of Theorem 6, (II) is relevant in the derivation of the first chain of inequalities. A simple example can be given to make readers understand it easily. Let*

$$A_1 = \begin{pmatrix} 4 & -1 & & \\ -1 & 4 & & \\ & & 4 & -1 \\ & & -1 & 4 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 4 & -1 & -1 & \\ -1 & 4 & & -1 \\ -1 & & 4 & -1 \\ & -1 & -1 & 4 \end{pmatrix}.$$

Consider the two-step SOR splitting, where

$$F_1^{(1)} = \begin{pmatrix} \frac{4}{\alpha} & & & \\ -1 & \frac{4}{\alpha} & & \\ & & \frac{4}{\alpha} & \\ & & -1 & \frac{4}{\alpha} \end{pmatrix}, G_1^{(1)} = \begin{pmatrix} \frac{4}{\alpha}-4 & 1 & & \\ & \frac{4}{\alpha}-4 & & \\ & & \frac{4}{\alpha}-4 & 1 \\ & & & \frac{4}{\alpha}-4 \end{pmatrix},$$

$$F_1^{(2)} = \begin{pmatrix} \frac{4}{\alpha} & -1 & & \\ & \frac{4}{\alpha} & & \\ & & \frac{4}{\alpha} & -1 \\ & & & \frac{4}{\alpha} \end{pmatrix}, G_1^{(2)} = \begin{pmatrix} \frac{4}{\alpha}-4 & & & \\ 1 & \frac{4}{\alpha}-4 & & \\ & & \frac{4}{\alpha}-4 & \\ & & 1 & \frac{4}{\alpha}-4 \end{pmatrix},$$

$$F_2^{(1)} = \begin{pmatrix} \frac{4}{\alpha} & & & \\ -1 & \frac{4}{\alpha} & & \\ -1 & & \frac{4}{\alpha} & \\ & -1 & -1 & \frac{4}{\alpha} \end{pmatrix}, G_2^{(1)} = \begin{pmatrix} \frac{4}{\alpha}-4 & 1 & 1 & \\ & \frac{4}{\alpha}-4 & & 1 \\ & & \frac{4}{\alpha}-4 & 1 \\ & & & \frac{4}{\alpha}-4 \end{pmatrix},$$

$$F_2^{(2)} = \begin{pmatrix} \frac{4}{\alpha} & -1 & -1 & \\ & \frac{4}{\alpha} & & -1 \\ & & \frac{4}{\alpha} & -1 \\ & & & \frac{4}{\alpha} \end{pmatrix}, G_2^{(2)} = \begin{pmatrix} \frac{4}{\alpha}-4 & & & \\ 1 & \frac{4}{\alpha}-4 & & \\ 1 & & \frac{4}{\alpha}-4 & \\ & 1 & 1 & \frac{4}{\alpha}-4 \end{pmatrix}.$$

Clearly, for the matrices presented above, (II) is satisfied. Moreover, by simple computation, one can easily determine that $\langle A_1\rangle \geq \langle A_2\rangle$ with two-step AOR splittings is a sufficient condition of (II).

Remark 8. *If we take $F_i^{(1)} - G_i^{(1)} = F_i^{(2)} - G_i^{(2)}$, all the assumptions in Theorem 6 can reduce to those in Theorem 4.1 of [8]. Clearly, (IV) is weaker than the corresponding one in Theorem 4.1 of [8], where Ω was assumed to satisfy $\Omega \geq \frac{1}{2}(D_{F_1} + D_{F_2})$. On the other hand, $2\Omega + F_1^{(t)} + F_2^{(t)}$ is proved to be an H-matrix in Theorem 6, not set to be an assumption as that in [8].*

Remark 9. *In view of the assumptions in Theorem 6, Assumption (III) seems to be a special one. In fact, for some special cases, the matrix T given in the Assumption (III) can be computed. Taking the TMAOR method where the matrix splittings are given by (8), for example, we have*

$$\langle F_2^{(1)}\rangle - |G_2^{(1)}| = \langle F_2^{(2)}\rangle - |G_2^{(2)}| = \frac{1-|1-\alpha|}{\alpha} D_{A_2} - |C_{A_2}|.$$

Since A_2 is an H_+-matrix, by Lemma 3, we have $\rho(D_{A_2}^{-1}|C_{A_2}|) < 1$. By simple computation, if $0 < \beta \leq \alpha < \frac{2}{1+\rho(D_{A_2}^{-1}|C_{A_2}|)}$, we can obtain $\frac{1-|1-\alpha|}{\alpha} D_{A_2} - |C_{A_2}|$ is an M-matrix. Then, letting

$$T = \operatorname{diag}\Big[\big(\frac{1-|1-\alpha|}{\alpha} D_{A_2} - |C_{A_2}|\big)^{-1} e\Big], \tag{16}$$

Assumption (III) of Theorem 6 can be satisfied.

By the similar proof technique, we can obtain the convergence theorem for $\mathrm{VLCP}_\ell(\ell \geq 3)$. We then first show the idea of the proof when $\ell = 3$.

First, by (6) and the first equation of (3), we can determine the error at the iteration $(k+1)$:

$$\begin{cases} |x_1^{(k+\frac{1}{2})} - x_1^*| & \leq & |4\Omega + 2F_1^{(1)} + F_2^{(1)} + F_3^{(1)}|^{-1}\big(|2G_1^{(1)} + G_2^{(1)} + G_3^{(1)}| \\ & & +|4\Omega - 2A_1 - A_2 - A_3| + 2|2A_1 - A_2 - A_3| + 4|A_2 - A_3|\big) \\ & & \times|x_1^{(k)} - x_1^*|, \\ |x_1^{(k+1)} - x_1^*| & \leq & |4\Omega + 2F_1^{(2)} + F_2^{(2)} + F_3^{(2)}|^{-1}\big(|2G_1^{(2)} + G_2^{(2)} + G_3^{(2)}| \\ & & +|4\Omega - 2A_1 - A_2 - A_3| + 2|2A_1 - A_2 - A_3| + 4|A_2 - A_3|\big) \\ & & \times|x_1^{(k+\frac{1}{2})} - x_1^*|, \end{cases}$$

If there exists a diagonal matrix T with positive diagonal entries such that $(\langle F_3^{(t)}\rangle - |G_3^{(t)}|)T, t = 1, 2,$ are s.d.d. matrices, we obtain

$$|x_1^{(k+1)} - x_1^*| \leq \mathcal{P}^{(2)}\mathcal{P}^{(1)}|x_1^{(k)} - x_1^*|,$$

where

$$\begin{cases} \mathcal{P}^{(t)} & = \mathcal{F}^{(t)^{-1}}\mathcal{G}^{(t)}, \\ \mathcal{F}^{(t)} & = \langle 4\Omega + 2F_1^{(t)} + F_2^{(t)} + F_3^{(t)}\rangle, \\ \mathcal{G}^{(t)} & = |2G_1^{(t)} + G_2^{(t)} + G_3^{(t)}| + |4\Omega - 2A_1 - A_2 - A_3| + 4|A_2 - A_3|. \end{cases}$$

If $2\langle F_1^{(t)}\rangle \geq \langle F_2^{(t)} + F_3^{(t)}\rangle, \langle F_2^{(t)}\rangle \geq \langle F_3^{(t)}\rangle, 2|G_1^{(t)}| \leq |G_2^{(t)} + G_3^{(t)}|$, and $|G_2^{(t)}| \leq |G_3^{(t)}|$ hold, we can also have that $\langle 4\Omega + 2F_1^{(t)} + F_2^{(t)} + F_3^{(t)}\rangle T$ is an s.d.d. matrix and

$$||T^{-1}\mathcal{F}^{(t)^{-1}}\mathcal{G}^{(t)}T||_\infty \leq \max_{1\leq i\leq n} \frac{(\mathcal{G}^{(t)}Te)_i}{(\mathcal{F}^{(t)}Te)_i}.$$

Similarly to (13), we can obtain

$$\begin{aligned} & \mathcal{F}^{(t)}Te - \mathcal{G}^{(t)}Te \\ \geq\ & \big(4\Omega - 2D_{F_1^{(t)}} - D_{F_2^{(t)}} + 7D_{F_3^{(t)}} - |4\Omega - 2D_{F_1^{(t)}} - D_{F_2^{(t)}} - D_{F_3^{(t)}}| - 8|G_3^{(t)}| - 8|C_{F_3^{(t)}}|\big)Te. \end{aligned}$$

Then, we can also distinguish two cases with respect to Ω, where

$$\Omega \geq \frac{1}{4}(2D_{F_1^{(t)}} + D_{F_2^{(t)}} + D_{F_3^{(t)}})$$

and

$$\big[\frac{1}{4}(2D_{F_1^{(t)}} + D_{F_2^{(t)}} + D_{F_3^{(t)}}) - (\langle F_3^{(t)}\rangle - |G_3^{(t)}|)\big]Te < \Omega Te < \big[\frac{1}{4}(2D_{F_1^{(t)}} + D_{F_2^{(t)}} + D_{F_3^{(t)}})\big]Te,$$

and obtain

$$\mathcal{F}^{(t)}Te - \mathcal{G}^{(t)}Te \geq 8(\langle F_3^{(t)}\rangle - |G_3^{(t)}|)Te > 0$$

and

$$\mathcal{F}^{(t)}Te - \mathcal{G}^{(t)}Te = 8\big[\Omega - \frac{1}{4}(2D_{F_1^{(t)}} + D_{F_2^{(t)}} + D_{F_3^{(t)}}) + \langle F_3^{(t)}\rangle - |G_3^{(t)}|\big]Te > 0,$$

respectively.

In summary, we have the next result.

Theorem 10. *Let A_1, A_2, A_3 and all their row-representative matrices be H_+-matrices. Let $A_i = F_i^{(t)} - G_i^{(t)} (t = 1, 2)$ be two splittings of A_i $(i = 1, 2, 3)$. Assume that:*

(I) $D_{F_1^{(t)}} > 0, D_{F_2^{(t)}} > 0$, *and $A_3 = F_3^{(t)} - G_3^{(t)}$ are an H-splitting of A_3;*

(II) $2\langle F_1^{(t)}\rangle \geq \langle F_2^{(t)} + F_3^{(t)}\rangle, \langle F_2^{(t)}\rangle \geq \langle F_3^{(t)}\rangle, 2|G_1^{(t)}| \leq |G_2^{(t)} + G_3^{(t)}|$, *and* $|G_2^{(t)}| \leq |G_3^{(t)}|$;

(III) There exists a diagonal matrix T with positive diagonal entries such that $(\langle F_3^{(t)}\rangle - |G_3^{(t)}|)T, t = 1, 2,$ are s.d.d. matrices;

(IV) $\Omega Te \geq \left[\frac{1}{4}(2D_{F_1^{(t)}} + D_{F_2^{(t)}} + D_{F_3^{(t)}}) - (\langle F_3^{(t)}\rangle - |G_3^{(t)}|)\right] Te.$

Then, Method 2 converges to the solution of the VLCP$_3$.

Furthermore, by deduction, for a general ℓ, we can also show the main steps of the proof.

In fact, the errors at the iteration $(k+1)$ are

$$\begin{cases} |x_1^{(k+\frac{1}{2})} - x_1^*| & \leq & |2^{\ell-1}\Omega + \sum\limits_{i=1}^{\ell-1} 2^{\ell-i-1}F_i^{(1)} + F_\ell^{(1)}|^{-1}\Big(|\sum\limits_{i=1}^{\ell-1} 2^{\ell-i-1}G_i^{(1)} + G_\ell^{(1)}| \\ & & +|2^{\ell-1}\Omega - \sum\limits_{i=1}^{\ell-1} 2^{\ell-i-1}A_i - A_\ell| \\ & & +2\big[|A_{\ell-1} - A_\ell| + \sum\limits_{j=2}^{\ell-2} 2^{\ell-j-1}|2^{j-1}A_{\ell-j} - \sum\limits_{s=\ell-j+1}^{\ell-1} 2^{\ell-s-1}A_s - A_\ell| \\ & & +|2^{\ell-2}A_1 - \sum\limits_{s=2}^{\ell-1} 2^{\ell-s-1}A_s - A_\ell|\big]\Big)|x_1^{(k)} - x_1^*|, \\ |x_1^{(k+1)} - x_1^*| & \leq & |2^{\ell-1}\Omega + \sum\limits_{i=1}^{\ell-1} 2^{\ell-i-1}F_i^{(2)} + F_\ell^{(2)}|^{-1}\Big(|\sum\limits_{i=1}^{\ell-1} 2^{\ell-i-1}G_i^{(2)} + G_\ell^{(2)}| \\ & & +|2^{\ell-1}\Omega - \sum\limits_{i=1}^{\ell-1} 2^{\ell-i-1}A_i - A_\ell| \\ & & +2\big[|A_{\ell-1} - A_\ell| + \sum\limits_{j=2}^{\ell-2} 2^{\ell-j-1}|2^{j-1}A_{\ell-j} - \sum\limits_{s=\ell-j+1}^{\ell-1} 2^{\ell-s-1}A_s - A_\ell| \\ & & +|2^{\ell-2}A_1 - \sum\limits_{s=2}^{\ell-1} 2^{\ell-s-1}A_s - A_\ell|\big]\Big)|x_1^{(k+\frac{1}{2})} - x_1^*|. \end{cases}$$

If there exists a diagonal matrix T with positive diagonal entries such that $(\langle F_\ell^{(t)}\rangle - |G_\ell^{(t)}|)T, t = 1, 2,$ are s.d.d. matrices, we obtain

$$|x_1^{(k+1)} - x_1^*| \leq \mathcal{P}^{(2)}\mathcal{P}^{(1)}|x_1^{(k)} - x_1^*|,$$

where

$$\begin{cases} \mathcal{P}^{(t)} & = & {\mathcal{F}^{(t)}}^{-1}\mathcal{G}^{(t)}, \\ \mathcal{F}^{(t)} & = & \langle 2^{\ell-1}\Omega + \sum\limits_{i=1}^{\ell-1} 2^{\ell-i-1}F_i^{(t)} + F_\ell^{(t)}\rangle, \\ \mathcal{G}^{(t)} & = & |\sum\limits_{i=1}^{\ell-1} 2^{\ell-i-1}G_i^{(1)} + G_\ell^{(1)}| + |2^{\ell-1}\Omega - \sum\limits_{i=1}^{\ell-1} 2^{\ell-i-1}A_i - A_\ell| \\ & & +2\big[|A_{\ell-1} - A_\ell| + \sum\limits_{j=2}^{\ell-2} 2^{\ell-j-1}|2^{j-1}A_{\ell-j} - \sum\limits_{s=\ell-j+1}^{\ell-1} 2^{\ell-s-1}A_s - A_\ell| \\ & & +|2^{\ell-2}A_1 - \sum\limits_{s=2}^{\ell-1} 2^{\ell-s-1}A_s - A_\ell|\big]. \end{cases}$$

If

$$\begin{cases} 2^{\ell-j}\langle F_{j-1}^{(t)}\rangle \geq \langle \sum\limits_{i=j}^{\ell-1} 2^{\ell-i-1}F_i^{(t)} + F_\ell^{(t)}\rangle, \quad (j = 2, 3, \ldots, \ell-1) \\ \langle F_{\ell-1}^{(t)}\rangle \geq \langle F_\ell^{(t)}\rangle, \end{cases} \tag{17}$$

and

$$\begin{cases} 2^{\ell-j}|G_j^{(t)}| \leq |\sum_{i=j}^{\ell-1} 2^{\ell-i-1} G_i^{(t)} + G_\ell^{(t)}|, \quad (j = 2, 3, \dots, \ell-1) \\ |G_{\ell-1}^{(t)}| \leq |G_\ell^{(t)}|; \end{cases} \tag{18}$$

hold, we can also have $\mathcal{F}^{(t)}T$ as an s.d.d. matrix and

$$||T^{-1}\mathcal{F}^{(t)^{-1}}\mathcal{G}^{(t)}T||_\infty \leq \max_{1\leq i\leq n} \frac{(\mathcal{G}^{(t)}Te)_i}{(\mathcal{F}^{(t)}Te)_i}.$$

Similarly to (13), we can obtain

$$\begin{aligned} & \mathcal{F}^{(t)}Te - \mathcal{G}^{(t)}Te \\ \geq \ & \Big[2^{\ell-1}\Omega + (2^\ell - 1)D_{F_\ell^{(t)}} - \sum_{i=1}^{\ell-1} 2^{\ell-i-1} D_{F_i^{(t)}} - 2^\ell |C_{F_\ell^{(t)}}| - 2^\ell |G_\ell^{(t)}| \\ & -|2^{\ell-1}\Omega - \sum_{i=1}^{\ell-1} 2^{\ell-i-1} D_{F_i^{(t)}} - D_{F_\ell^{(t)}}|\Big] Te. \end{aligned}$$

Then, we can also distinguish two cases with respect to Ω, where

$$\Omega \geq 2^{1-\ell}(\sum_{i=1}^{\ell-1} 2^{\ell-i-1} D_{F_i^{(t)}} + D_{F_\ell^{(t)}})$$

and

$$\begin{aligned} & [2^{1-\ell}(\sum_{i=1}^{\ell-1} 2^{\ell-i-1} D_{F_i^{(t)}} + D_{F_\ell^{(t)}}) - (\langle F_\ell^{(t)} \rangle - |G_\ell^{(t)}|)] Te \\ < \ & \Omega Te < 2^{1-\ell}(\sum_{i=1}^{\ell-1} 2^{\ell-i-1} D_{F_i^{(t)}} + D_{F_\ell^{(t)}}) Te, \end{aligned}$$

and obtain

$$\mathcal{F}^{(t)}Te - \mathcal{G}^{(t)}Te \geq 2^\ell(\langle F_\ell^{(t)} \rangle - |G_\ell^{(t)}|)Te > 0$$

and

$$\mathcal{F}^{(t)}Te - \mathcal{G}^{(t)}Te \geq \Big[2^\ell\Omega - 2(\sum_{i=1}^{\ell-1} 2^{\ell-i-1} D_{F_i^{(t)}} + D_{F_\ell^{(t)}}) + 2^\ell(\langle F_\ell^{(t)} \rangle - |G_\ell^{(t)}|)\Big] Te > 0,$$

respectively. Finally, we have the next theorem.

Theorem 11. *Let $A_1, A_2, \dots, A_\ell$ and all their row-representative matrices be H_+-matrices. Let $A_i = F_i^{(t)} - G_i^{(t)} (t = 1, 2)$ be two splittings of A_i $(i = 1, 2, \dots, \ell)$. Assume that:*

(I) $D_{F_i^{(t)}} > 0, i = 1, 2, \dots, \ell-1$, *and* $A_\ell = F_\ell^{(t)} - G_\ell^{(t)}$ *are an H-splitting of* A_ℓ*;*

(II) *(17) and (18) are satisfied;*

(III) *There exists a diagonal matrix T with positive diagonal entries such that* $(\langle F_\ell^{(t)} \rangle - |G_\ell^{(t)}|)T, t = 1, 2,$ *are s.d.d. matrices;*

(IV) $\Omega Te \geq [2^{1-\ell}(\sum_{i=1}^{\ell-1} 2^{\ell-i-1} D_{F_i^{(t)}} + D_{F_\ell^{(t)}}) - (\langle F_\ell^{(t)} \rangle - |G_\ell^{(t)}|)] Te.$

Then, Method 2 converges to the solution of the VLCP$_\ell$.

Same comments as in Remarks 8 and 9 can be given for Theorems 10 and 11.

4. Numerical Examples

In this section, numerical examples are given to show the efficiency of the proposed method.

Consider the two following examples similar to [8], where Examples 12 and 13 are of the symmetry and asymmetry cases, respectively.

Example 12. *Let $n = m^2$. Consider the VLCP whose system matrices are given by*

$$A_1 = \begin{pmatrix} S & & & \\ & S & & \\ & & \ddots & \\ & & & S \end{pmatrix} + I_n \in \mathbb{R}^{n\times n}, A_2 = \begin{pmatrix} S & -I_m & & \\ -I_m & S & \ddots & \\ & \ddots & \ddots & -I_m \\ & & -I_m & S \end{pmatrix} \in \mathbb{R}^{n\times n},$$

where

$$S = \begin{pmatrix} 4 & -1 & & \\ -1 & 4 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 4 \end{pmatrix} \in \mathbb{R}^{m\times m}.$$

Example 13. *Let $n = m^2$. Consider the VLCP whose system matrices are given by*

$$A_1 = \begin{pmatrix} S & & & \\ & S & & \\ & & \ddots & \\ & & & S \end{pmatrix} + I_n \in \mathbb{R}^{n\times n}, A_2 = \begin{pmatrix} S & -0.5I_m & & \\ -1.5I_m & S & \ddots & \\ & \ddots & \ddots & -0.5I_m \\ & & -1.5I_m & S \end{pmatrix} \in \mathbb{R}^{n\times n},$$

where

$$S = \begin{pmatrix} 4 & -0.5 & & \\ -1.5 & 4 & \ddots & \\ & \ddots & \ddots & -0.5 \\ & & -1.5 & 4 \end{pmatrix} \in \mathbb{R}^{m\times m}.$$

The numerical tests are performed on a computer, which has Intel(R) Core(TM) i7-9700 CPU 3.00 GHz with 8 GB memory. Denote the total computation time (in seconds) and the iteration steps by T and IT, respectively. Let $\gamma = 1$, $x_1^{(0)} = e$ and the tolerance be 10^{-6}. By "$SAVE$", we denote the per centum of total computation time saved by the TMSOR method from the MSOR method, where

$$SAVE = \frac{T_{MSOR} - T_{TMSOR}}{T_{MSOR}} \times 100\%.$$

The numerical results are presented in Tables 1–3, where the notations "MSOR$_\alpha$" and "TMSOR$_\alpha$" denote the MSOR and TMSOR methods with relaxation parameter α, respectively, and the parameter matrix Ω is chosen as

$$\Omega = \frac{\tau}{2}(D_{F_1} + D_{F_2}),$$

$\tau = 0.8, 0.9, 1.0$.

One can see that all methods are convergent for different dimensions. Since there are two linear systems solved in each iteration of the TMMS method, most of the number of iteration steps of the MMS method is nearly twice or a little less than twice as long as that of the TMMS method in each comparison. Meanwhile, the TMMS method converges faster than the MMS method except for a few cases. Specially, we can see that the CPU time is saved larger than 20% for most cases. Therefore, the two-step technique works for the improvement of the MMS method. On the other hand, one can see that the relaxation parameter may affect the computation efficiencies of the MMS and TMMS.

Although there are some cases of Example 13 where the values of "SAVE" are small or negative, the values of "SAVE" can be larger than 15% for the "optimal" relaxation parameters of both two examples, set to bold. Nevertheless, by Tables 2 and 3, both the MMS and TMMS methods are convergent for all cases when $\tau < 1$, which confirms the improvement of the proposed convergence theorem as Remark 8 commented. However, the theoretical analysis of the relaxation parameter is still difficult even for the LCP. It may be an interesting work in the future.

Table 1. Numerical results when $\tau = 1$ (the "optimal" computation times of the MSOR and TMSOR are set to bold for each dimension and each example).

Example	Method	$m = 128$			$m = 256$			$m = 512$		
		IT	T	$SAVE$	IT	T	$SAVE$	IT	T	$SAVE$
Example 12	$MSOR_{0.9}$	48	0.1999		49	0.9145		51	5.1585	
	$TMSOR_{0.9}$	24	0.1335	33%	25	0.6761	26%	26	4.0739	21%
	$MSOR_{1.0}$	41	0.1818		42	0.7773		44	4.3167	
	$TMSOR_{1.0}$	21	0.1149	36%	21	**0.5360**	31%	22	3.0526	29%
	$MSOR_{1.1}$	35	**0.1569**		36	**0.6566**		38	**3.9291**	
	$TMSOR_{1.1}$	18	**0.1107**	29%	20	0.5843	11%	19	**2.7612**	30%
	$MSOR_{1.2}$	52	0.2079		52	0.9520		54	5.5109	
	$TMSOR_{1.2}$	26	0.1374	34%	29	0.7632	20%	28	4.0793	26%
Example 13	$MSOR_{0.9}$	42	0.1792		36	0.6613		45	4.6083	
	$TMSOR_{0.9}$	24	0.1321	26%	20	0.5607	15%	25	3.6112	22%
	$MSOR_{1.0}$	35	0.1539		30	0.5444		38	3.8549	
	$TMSOR_{1.0}$	20	0.1317	14%	17	0.4273	22%	22	3.0377	21%
	$MSOR_{1.1}$	30	**0.1417**		25	**0.4919**		32	**3.5243**	
	$TMSOR_{1.1}$	18	**0.1070**	24%	15	**0.4131**	16%	19	**2.7911**	21%
	$MSOR_{1.2}$	42	0.1899		29	0.5140		45	4.5890	
	$TMSOR_{1.2}$	28	0.1828	4%	16	0.4884	5%	30	4.3746	5%

Table 2. Numerical results when $\tau = 0.9$ (the "optimal" computation times of the MSOR and TMSOR are set to bold for each dimension and each example).

Example	Method	$m = 128$			$m = 256$			$m = 512$		
		IT	T	$SAVE$	IT	T	$SAVE$	IT	T	$SAVE$
Example 12	$MSOR_{0.9}$	44	0.1426		46	0.8142		47	4.6364	
	$TMSOR_{0.9}$	22	0.1030	27%	23	0.6159	24%	24	3.2768	29%
	$MSOR_{1.0}$	38	0.1378		39	**0.6936**		40	**3.9927**	
	$TMSOR_{1.0}$	19	**0.0881**	36%	20	**0.5315**	23%	20	**2.8169**	29%
	$MSOR_{1.1}$	42	**0.1375**		41	0.7498		44	4.4324	
	$TMSOR_{1.1}$	22	0.0944	31%	24	0.6519	13%	23	3.3152	25%
	$MSOR_{1.2}$	78	0.2284		81	1.4617		79	8.0223	
	$TMSOR_{1.2}$	37	0.1443	36%	41	1.1061	24%	40	5.7297	28%
Example 13	$MSOR_{0.9}$	39	0.1391		33	0.5767		41	4.1800	
	$TMSOR_{0.9}$	22	0.0981	29%	19	0.4896	15%	24	3.4328	17%
	$MSOR_{1.0}$	32	**0.1116**		27	0.4928		34	**3.4535**	
	$TMSOR_{1.0}$	19	**0.0897**	19%	16	0.4133	16%	20	**2.8645**	17%
	$MSOR_{1.1}$	36	0.1236		25	**0.4295**		39	4.0004	
	$TMSOR_{1.1}$	23	0.1098	11%	14	**0.3647**	15%	24	3.5276	11%
	$MSOR_{1.2}$	55	0.1740		35	0.6576		59	5.8532	
	$TMSOR_{1.2}$	41	0.1664	4%	20	0.5234	20%	44	6.2346	-6%

Table 3. Numerical results when $\tau = 0.8$ (the "optimal" computation times of the MSOR and TMSOR are set to bold for each dimension and each example).

Example	Method	$m = 128$			$m = 256$			$m = 512$		
		IT	T	$SAVE$	IT	T	$SAVE$	IT	T	$SAVE$
Example 12	$\mathrm{MSOR}_{0.9}$	41	0.1400		42	0.7212		44	4.645	
	$\mathrm{TMSOR}_{0.9}$	21	0.0977	30%	21	0.5731	21%	22	3.3496	28%
	$\mathrm{MSOR}_{1.0}$	35	**0.1218**		36	**0.6060**		38	**3.9238**	
	$\mathrm{TMSOR}_{1.0}$	18	**0.0853**	29%	20	**0.5141**	15%	20	**2.9174**	26%
	$\mathrm{MSOR}_{1.1}$	59	0.1803		59	0.9876		61	6.3516	
	$\mathrm{TMSOR}_{1.1}$	29	0.1177	35%	32	0.8357	15%	31	4.6185	27%
	$\mathrm{MSOR}_{1.2}$	184	0.5213		192	3.3582		184	18.8629	
	$\mathrm{TMSOR}_{1.2}$	80	0.3141	40%	83	2.3453	30%	80	11.6375	38%
Example 13	$\mathrm{MSOR}_{0.9}$	35	0.1238		30	0.5136		38	3.9445	
	$\mathrm{TMSOR}_{0.9}$	20	0.0963	22%	17	0.4464	13%	22	3.2718	17%
	$\mathrm{MSOR}_{1.0}$	31	**0.1100**		24	**0.4548**		33	**3.3400**	
	$\mathrm{TMSOR}_{1.0}$	19	**0.0911**	17%	15	**0.3597**	21%	20	**2.9164**	13%
	$\mathrm{MSOR}_{1.1}$	46	0.1566		31	0.5378		49	4.9155	
	$\mathrm{TMSOR}_{1.1}$	32	0.1368	13%	17	0.4551	15%	34	5.0096	-2%
	$\mathrm{MSOR}_{1.2}$	79	0.2616		46	0.8248		85	8.5703	
	$\mathrm{TMSOR}_{1.2}$	93	0.3573	-37%	28	0.7918	4%	93	13.7466	-60%

5. Concluding Remarks

The two-step splittings are successfully applied to the MMS iteration method for solving the VLCP. The convergence analysis is given where the convergence domain of the parameter matrix is larger than the existing one. Numerical results show that the proposed method can improve the convergence rate of the MMS iteration method. In two recent works [25,26], the modulus-based transformation was also used for tensor complementarity problems (TCP). One can thus expect that some accelerated technique such as two-step splittings can be also used for the TCP.

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