

Article

A Catalog of Enumeration Formulas for Bouquet and Dipole Embeddings under Symmetries

Mark N. Ellingham ¹  and Joanna A. Ellis-Monaghan ^{2,*} ¹ Department of Mathematics, 1326 Stevenson Center, Vanderbilt University, Nashville, TN 37240, USA² Korteweg-de Vries Institute for Mathematics, University of Amsterdam, Science Park 105-107, 1098 XH Amsterdam, The Netherlands

* Correspondence: jellismonaghan@gmail.com

Abstract: Motivated by the problem arising out of DNA origami, we give a general counting framework and enumeration formulas for various cellular embeddings of bouquets and dipoles under different kinds of symmetries. Our algebraic framework can be used constructively to generate desired symmetry classes, and we use Burnside's lemma with various symmetry groups to derive the enumeration formulas. Our results assimilate several existing formulas into this unified framework. Furthermore, we provide new formulas for bouquets with colored edges (and thus for bouquets in nonorientable surfaces) as well as for directed embeddings of directed bouquets. We also enumerate vertex-labeled dipole embeddings. Since dipole embeddings may be represented by permutations, the formulas also apply to certain equivalence classes of permutations and permutation matrices. The resulting bouquet and dipole symmetry formulas enumerate structures relevant to a wide variety of areas in addition to DNA origami, including RNA secondary structures, Feynman diagrams, and topological graph theory. For uncolored objects, we catalog 58 distinct sequences, of which 43 have not, as far as we know, been described previously.



Citation: Ellingham, M.N.; Ellis-Monaghan, J.A. A Catalog of Enumeration Formulas for Bouquet and Dipole Embeddings under Symmetries. *Symmetry* **2022**, *14*, 1793. <https://doi.org/10.3390/sym14091793>

Academic Editors: Erica Flapan and Helen Wong

Received: 22 June 2022

Accepted: 20 August 2022

Published: 29 August 2022

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



Copyright: © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

Keywords: DNA origami; graph embeddings; bouquets; dipoles; chord diagrams; enumeration; upper embedding; edge-outer embedding; permutations

MSC: 05C10; secondary 05A15; 05C45; 05E18

Part I Overview

1. Introduction

We provide a unified framework and new results for enumeration formulas for cellularly embedded bouquets and dipoles under various symmetries. Bouquets and dipoles are graphs that encode information critical in diverse settings. A bouquet is a graph with one vertex and some loops. A dipole is a graph with two vertices and some edges, none of which are loops. Embeddings of these, or other, graphs are determined by cyclic orderings of half-edges around the vertices, possibly also with edge twists. We encode embedded bouquets and dipoles by labeling (and sometimes coloring) their half-edges, and then convert the labeled graph to a simple combinatorial object (a permutation or perfect matching). We use Burnside's lemma to count these objects under equivalence relations defined by various groups of symmetries (such as cyclic shifts and reflections). Computations for multiple groups of symmetries with a common subgroup are handled together using the idea of 'coset averages'.

We catalog here our own as well as existing enumeration formulas for embedded bouquets and dipoles under various symmetry constraints and equivalences, with the goal

of making them readily accessible in one place. For uncolored objects, we list 58 distinct sequences, of which 12 already appear in the Online Encyclopedia of Integer Sequences (OEIS, a very widely used database of integer sequences) [1], and three occur elsewhere in the literature. The remaining 43 sequences have, as far as we know, not been described previously.

We divide this paper into three parts. Part I contains some background, definitions, and descriptions of the various sorts of embeddings and symmetries. For the convenience of the reader, we list all the counting formulas in Part II, followed by a discussion of related work and open problems. We defer the technical details and proofs of the results to Part III. We begin with the formulas and proofs for dipoles because the computational ideas there encompass the simpler analogues for bouquets. Furthermore, we give accessible geometric interpretations and applications for the various symmetries. A reader who simply needs the formulas can go directly to Part II, although the descriptions of the various objects and symmetries in Section 2 may be helpful in identifying the appropriate formulas.

Previous work on enumerating the embeddings of graphs, including bouquets and dipoles, under various symmetries, was done by Mull, Rieper, and White [2], Rieper [3], Kwak and Lee [4], Mull [5], Kim and Park [6], Feng, Kwak, and Zhou [7,8], Orbanic et al. [9], and Chen, Gao, and Huang [10]. We cite results from some of these sources where appropriate in Part II.

There is also an extensive literature on counting the embeddings of bouquets, dipoles and other graphs where the symmetry considerations central to this work are not taken into account. In this situation, it is possible to examine the *genus distribution* of embeddings of a given graph, and information on the number of embeddings of a given graph can be represented using a *genus polynomial*. We do not study genus distributions, but the interested reader can find early work in this area, for example, in Stahl [11], Gross and Furst [12], Furst, Gross, and Statman [13], and Gross, Robbins, and Tucker [14]. Citation searches on these papers provide access to the recent literature in this area. One paper particularly relevant for bouquets and dipoles is Kwak and Shim [15].

The problem of enumerating bouquets and dipoles arises in surprisingly diverse settings. For example, embedded bouquets are equivalent to chord diagrams, as in Figure 1. The cyclic order of the half-edges about the vertex in the bouquet determines an embedding of the bouquet in an oriented surface and corresponds to the cyclic order of endpoints of the chords in the chord diagram. Chord diagrams are used in genomics [16] and modeling RNA secondary structures [17–19]. Chord diagrams also characterize moduli spaces [20]. Chord diagrams with labeled points and possibly some unpaired points are counted by Feynman integrals. There are also applications to the physics of chord diagrams with a given number of crossings, counted by Touchard [21] and Riordan [22]. Chord diagrams are fundamental to the study of Vassiliev knot invariants; see, for example, [23] and ([24], Section 4). The nLab has an extensive catalog of such applications of chord diagrams in knot theory and physics [25].

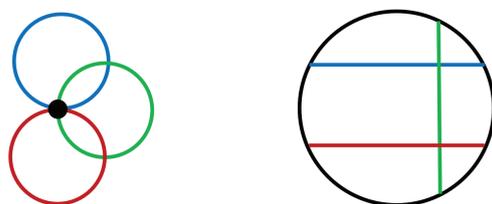


Figure 1. An embedded bouquet (left) and the corresponding chord diagram (right).

To these numerous settings for bouquets and dipoles, we add our own motivating application, that of constructing and analyzing DNA origami molecules. DNA origami uses the Watson–Crick complementarity and self-replicating properties of DNA to use DNA as an engineered building material for nano-scale robotics, electronics, and medical devices. Determining routes for a single strand of DNA through assembly targets (which are often wireframe polyhedra) is integral to both DNA origami [26–28] and experimental

verification of the targeted constructs [29,30]. When the target construct has the shape of a graph, these routes correspond to facial walks in an embedding of the graph.

In the DNA origami method of self assembly, a single-stranded DNA plasmid, called a scaffolding strand, traces a targeted shape. Then some 200–250 short strands of DNA complementary to specific regions of the plasmid are introduced to fold and secure the molecule into the desired shape; see [31–33]. When the target is a wireframe structure modeled as a graph embedded in space, a key design step is to determine a route through the graph that the scaffolding strand will follow and to locate the staple strands, as in Figure 2. A related problem, that of finding a reporter strand (a single strand of DNA that is extracted from a construct and then read to confirm the results of an experiment), corresponds to finding a route through the target graph that traces every edge at least once and at most twice, and when twice in opposite directions.

The resulting route for a DNA strand through a wireframe target molecule corresponds to a facial walk in a special embedding of a graph, called an *edge-outer embedding*, as described below. See [29,30] for further details. We became interested in special edge-outer embeddable graphs with exactly two faces (the *bi-Eulerian* embeddings described below) as a possible design strategy for DNA self-assembly. Recognizing that the surface duals of bi-Eulerian embeddings are embedded dipoles (see Figure 3) prompted this present study.

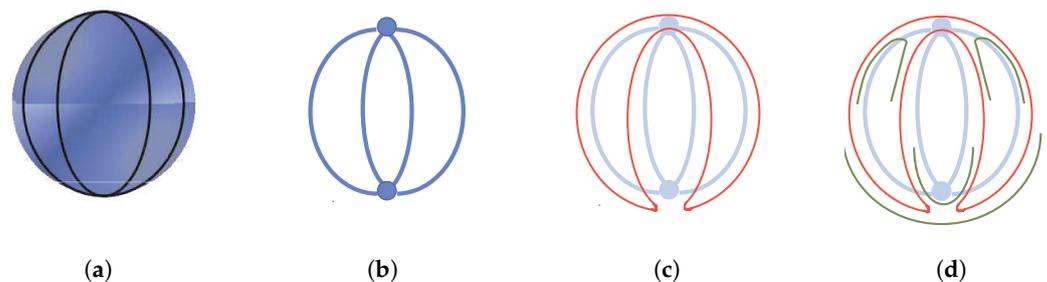


Figure 2. Basic design principle for DNA origami. (a) Target structure. (b) Target modeled as a graph. (c) Route for scaffolding strand (Euler circuit). (d) Placing the staple strands.

In the theoretical setting of graph embeddings, the bouquets and dipoles characterize some important classes of graphs. An *upper-embeddable* graph is a graph that can be cellularly embedded in an orientable surface with only one or two faces (see [34,35]). An *edge-outer embeddable* graph is a graph that can be cellularly embedded so that every edge lies on a single distinguished face, although there may be other faces as well (see [29]). These graphs, and their special sub-classes, can be characterized by their surface duals, as shown in Figure 3 (see [36–38] for explanations of duality for graph embeddings). For simplicity, we show planar examples, but in general, these embeddings are not necessarily planar. For example, the half-edges of a loop need not occur consecutively around a vertex.

Upper-embedded graphs with exactly one face are characterized by having bouquets as their surface duals, and edge-outer embeddings with exactly two faces that are Euler circuits (*bi-Eulerian* embeddings) are characterized by having dipoles as their surface duals. To the best of our knowledge there has not yet been any effort to enumerate the looped dipoles in Figure 3c or graphs of the form in Figure 3d. However, those counting problems are likely to build on the formulas given here, and thus this work lays the necessary foundations for this enumeration.

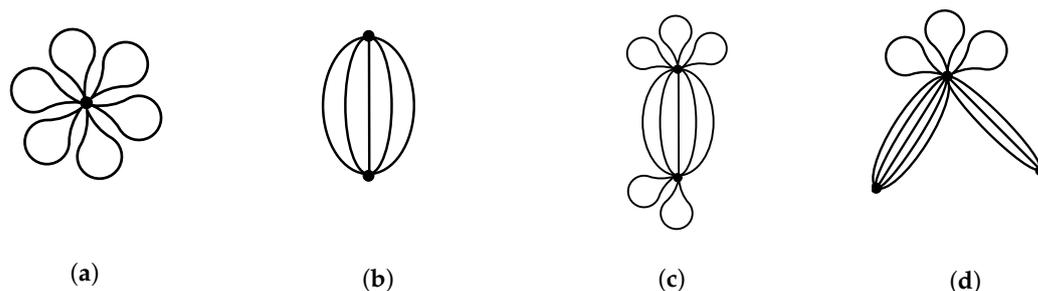


Figure 3. Surface duals of classes of embedded graphs. (a) Dual of an upper embeddable graph with one face. (b) Dual of a bi-Eulerian edge-outer embeddable graph. (c) Dual of an upper embeddable graph with two faces. (d) Dual of an edge-outer embeddable graph.

In all these settings where bouquets and dipoles play a central role, good enumeration formulas are essential. In addition to the obvious theoretic interest, these inform the experimental designs, algorithmic solutions, and estimations of solution space size.

However, each setting requires careful consideration of what symmetries are relevant to the application. For example, linear RNA secondary structures are often modeled by chord diagrams with a designated point on the circle boundary that indicates where the chord diagram should be ‘cut open’ to form a linear structure. See Figure 4. The presence or absence of such a symmetry-breaking point significantly alters the counting problem. Similarly, chirality often plays a role, and this too changes the enumeration problem. The possibility of directions on the edges, or of edges with various attributes (here colors), lead to further enumeration problems.

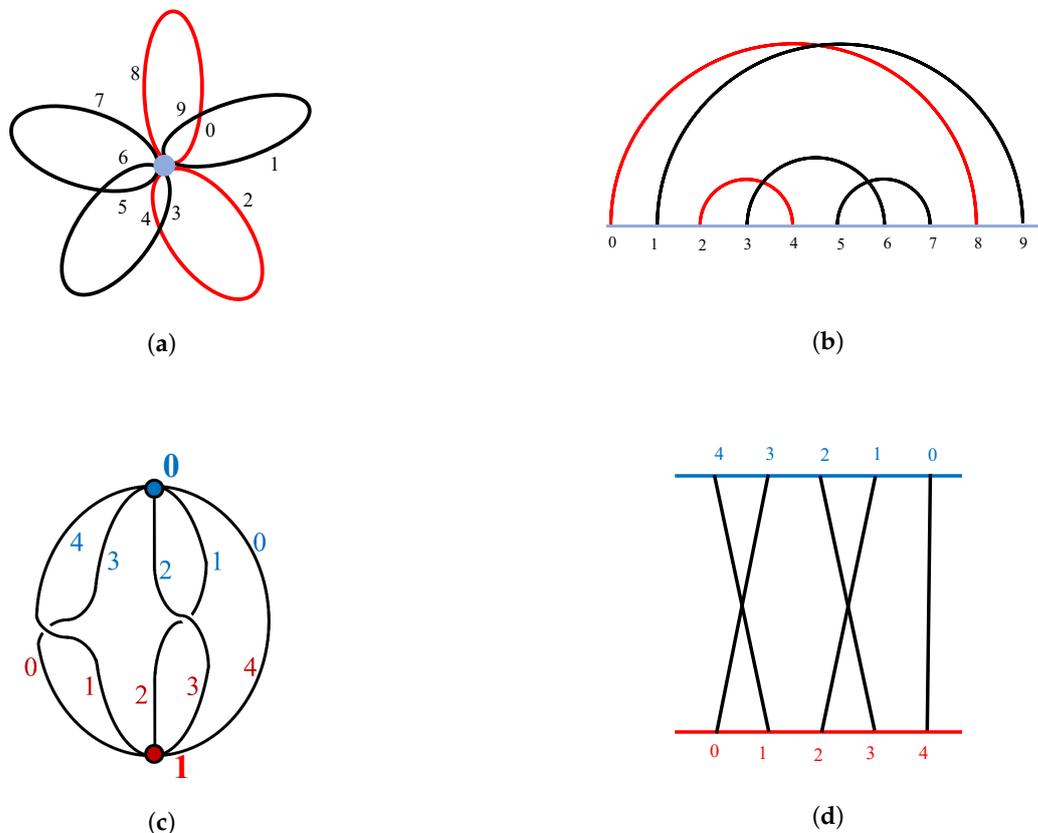


Figure 4. Bouquets and dipoles with fixed labels have no rotational symmetry, so can be identified with linear diagrams. (a) A bouquet with fixed labels. (b) Linear diagram for the bouquet. (c) A dipole with fixed labels. (d) Linear diagram for the dipole.

Because of the diverse applications, existing formulas are widely scattered in the literature, making them sometimes challenging to find. Thus, we include known results here as well, incorporating them into our overall framework and in some cases simplifying the proofs or formulas. We then complete the work of finding enumeration formulas for the remaining symmetries in the groups we consider, particularly providing enumeration formulas for bouquets with colored or directed edges and for orientable (as opposed to oriented) embeddings of dipoles. When the number of colors is two, the formulas for colored bouquets enumerate embeddings of bouquets in nonorientable surfaces. Greater numbers of colors can be used in applications to differentiate types of edges, for example, to differentiate different types of attachments in RNA secondary structures. The enumeration of dipolar cogs and other structures related to dipole embeddings also uses some ideas that may support the more challenging problem of enumerating dipole embeddings in nonorientable surfaces.

We have verified our counting formulas for small values of n (the number of edges) and, where appropriate, k (the number of colors) by explicit construction by computer of the objects being counted. Specifically, we verified formulas (D1)–(D8), (B1)–(B2), and (A1)–(A5) in Part II. Since all other formulas here are linear combinations of these, this also provides verification for our other results. Our programs use the same framework as the counting formulas. We consider our objects as orbits of easily generated basic objects under various group actions. While generating the basic objects in a fixed order, the programs test whether a given object is the earliest in its orbit under action of the group elements, thereby identifying a unique representative of each orbit.

2. Terminology and Notation

2.1. Embedding Concepts

The objects we are counting are embeddings of graphs in compact surfaces, or objects related to these. We assume that the reader is familiar with embeddings of graphs and their combinatorial representations; standard references are [37,38], and details for the *cogs* defined below may be found in [36]. Unless stated otherwise, all embeddings of graphs and digraphs in this paper are *cellular*, meaning that each face is homeomorphic to an open disk.

An embedding of a graph in an orientable surface can be described up to a homeomorphism by giving a *rotation scheme* specifying a *rotation*, that is, a cyclic ordering of the half-edges, at each vertex. A generic embedding (in either an orientable or nonorientable surface) of a graph can be described up to homeomorphism by a rotation scheme together with *edge signatures*, specifying whether each edge is *twisted* or *untwisted*. The representation using a rotation scheme is unique for oriented embeddings but not for orientable embeddings (see below for the distinction). The representation of a generic embedding using a rotation scheme and edge signatures is in general not unique.

An *oriented embedding* of a graph is an embedding in an orientable surface with a fixed clockwise orientation, up to graph isomorphism and orientation-preserving surface homeomorphism. Oriented embeddings are in one-to-one correspondence with rotation schemes (without edge signatures). The *reflection* of an oriented embedding is obtained by reversing the clockwise orientation of the surface. An *orientable embedding* of a graph is an embedding in an orientable surface where the clockwise orientation is not specified, up to graph isomorphism and surface homeomorphism. Equivalently, orientable embeddings are equivalence classes of oriented embeddings under reflection.

We make an explicit distinction between oriented and orientable embeddings. In many settings, the distinction between a surface being oriented and being orientable is either unnecessary or implicitly understood. However, for enumerative results, it is important to distinguish between these.

We also consider *cogs*, also known as *cyclically ordered graphs* or *rigid-vertex graphs*. These are graphs with a rotation at each vertex as above, but here two cogs are equivalent if there is a graph isomorphism between the two underlying graphs, which, at each vertex, either preserves the rotation at the vertex or reverses it. In other words, there is an

undirected cyclic ordering, defined only up to reversal, at each vertex, instead of the directed cyclic ordering in a rotation scheme. The edges of cogs have no signatures. Cogs are important because a cog represents an equivalence class of graph embeddings under partial Petrie duality (edge twisting) operations (see [36], Lemma 3.16(2)), or an equivalence class of orientable embeddings under vertex flips (rotation reversals).

A *digraph* is a graph with a direction (from one end-vertex to the other) specified for each edge. The directed edges are called *arcs*. A *directed embedding* of a digraph is an embedding where every facial walk is a directed walk. This is equivalent to the property that at each vertex the half-arcs in the rotation alternate in direction between outwards and inwards. Thus, for every vertex of a directed embedding of a digraph, the in-degree equals the out-degree, and there are no vertices of an odd degree.

Objects based on graphs or digraphs can be considered to be *vertex-labeled* if each vertex has a unique label that must be preserved by any symmetry operation (although edges can be permuted), or *vertex-unlabeled* if symmetry operations that permute vertices are allowed. In this paper, we deal with bouquets, where this distinction is irrelevant, and with dipoles, where we have only two vertices, which can be swapped in the vertex-unlabeled situation. We will say explicitly if an object derived from a dipole is vertex-labeled; otherwise it is assumed to be vertex-unlabeled.

2.2. Bouquets and Dipoles

A *bouquet* is a graph with exactly one vertex, and a *dipole* is a graph with exactly two vertices and no loops. These may be embedded in either orientable or nonorientable surfaces, or given related structures, such as a cog structure. We will consistently denote the number of edges by n .

In our work, we will label the half-edges around each vertex in an embedded bouquet or dipole to encode embedding information. Enumerating embeddings with fixed half-edge labels (see the discussion of ‘(colored) labeled bouquets’ and ‘labeled dipoles’ in Part II) is easy, since only the identity permutation leaves the fixed labels in their original position. Since fixed edge labels break cyclic symmetries, the diagrams may be ‘linearized’ at the vertices, that is, starting at the half-edges labeled 0 and opening up the vertices to lines, as in Figure 4. The number of bouquets labeled in this way is the same as the number of chord diagrams, which is the number of perfect matchings in a complete graph of order $2n$, namely $(2n - 1)!! = (2n - 1)(2n - 3)(2n - 5) \cdots 3 \cdot 1$. The number of labeled dipoles is just $n!$.

Since bouquets and dipoles with fixed labels have no symmetries and their counting formulas are easy and well known, we focus our attention on bouquets and dipoles where the labels may be permuted. Here, the permutations reveal the underlying structural symmetries of the embedded graph, and the enumeration formulas are much more complex.

We can convert an embedding of a bouquet (possibly with colors or edge directions) into a labeled bouquet, an abstract graph with labels on the half-edges. We label the half-edges of the embedding with elements of the cyclic group \mathbb{Z}_{2n} in order around the vertex. The two labels on each edge encode the position of that edge in the embedding. We can then simply consider the underlying abstract graph, with these half-edge labels. For example, the colored bouquet embedding represented by the rotation shown in Figure 5a is represented by the colored labeled bouquet in Figure 5b (drawn in the plane for convenience, although only the graph structure and labels matter). We can also draw a chord diagram, which can be thought of as an ‘inside-out’ drawing of the embedded bouquet. We expand the vertex of the bouquet into a large circle, and draw the edges inside this circle, as chords, instead of outside it. We transfer the label on each half-edge to the point at which that half-edge meets the circle, so that the points are labeled by elements of \mathbb{Z}_{2n} in cyclic order around the circle. The chords form a *perfect matching*, a partition of all of the vertices into pairs, in the complete graph whose vertices are the elements of \mathbb{Z}_{2n} . Figure 5c shows the chord diagram corresponding to the other two parts of Figure 5.

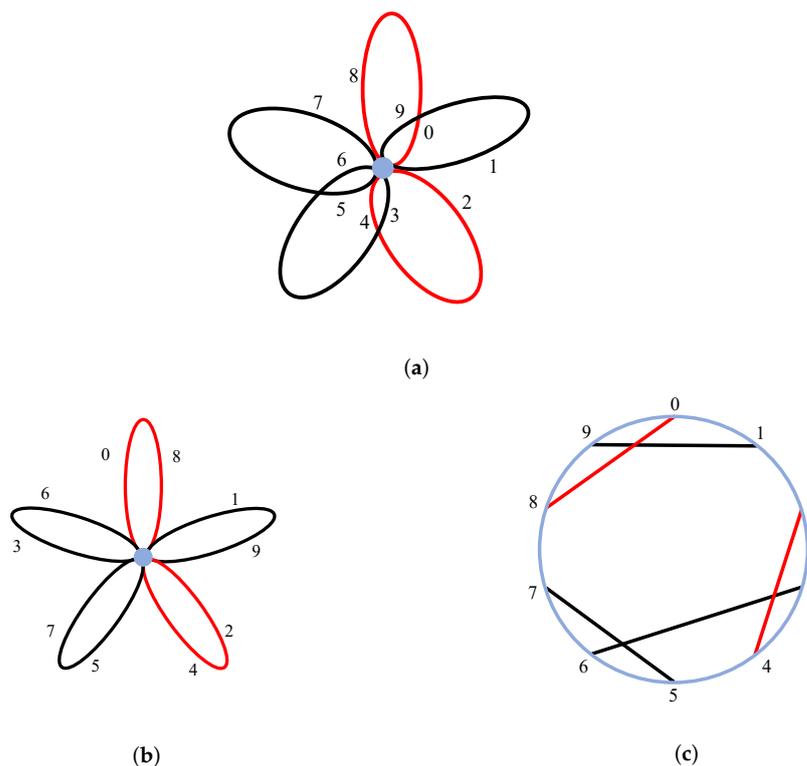


Figure 5. Colored labeled abstract bouquets or chord diagrams correspond to colored embedded bouquets labeled in rotational order. (a) A colored bouquet embedding with half-edges labeled in rotational order. (b) The colored labeled abstract bouquet. (c) The colored chord diagram.

We enumerate embeddings of bouquets up to the following symmetries. We consider embeddings of bouquets in an oriented surface up to rotational symmetry. Thus, two bouquets that can be superimposed by rotating one of them are considered the same, but reflection is not allowed. Allowing reflections (reversal of the orientation of the surface) means we are counting embeddings in an orientable surface, where we allow orientation-reversing homeomorphisms of the surface. We also consider the reflexible embeddings, preserved by reflection, and the chiral embeddings, which are not preserved by reflection.

One of our main contributions for bouquets is counting the embeddings of colored bouquets, which also allows us to count the generic (orientable and nonorientable) embeddings of bouquets, and hence, by simply subtracting, the nonorientable embeddings of bouquets. We also count the directed embeddings of *directed bouquets*, i.e., directed graphs with one vertex. In this setting, we again count embeddings with colors, which allows us to count generic and nonorientable embeddings.

We enumerate embeddings of dipoles up to the following symmetries. Since dipoles have two vertices, we can consider the vertices as being distinct, e.g., having two different labels, or not. For embeddings in oriented surfaces, we can rotate (cyclically shift) the labels around each vertex independently. For embeddings in orientable surfaces, we also allow reflection, i.e., reversal of the orientation of the surface. For dipoles, we also count cogs, both with and without distinguished vertices.

Our approach for counting embeddings of dipoles will be similar to that for counting embeddings of bouquets. We turn a dipole embedding into an object called a labeled dipole, by labeling the edges in rotational order around each vertex. See Figure 6, where the rotational information for the embedding in Figure 6a is represented by the correspondence between labels around the two vertices in Figure 6b. Again, while Figure 6b is drawn in a planar way for convenience, the details of the drawing are unimportant: only the graph structure and the labels matter. See also Section 8.1, where we provide a detailed illustrated example of how an embedding may be represented by more than one labeled dipole.

The difficulty in counting comes from the fact that different embeddings correspond to different numbers of basic labeled objects (for us, colored labeled bouquets or labeled dipoles). To use embeddings of bouquets as an example, the number of labeled bouquets or chord diagrams that correspond to a given bouquet embedding depends on the intrinsic symmetries of the chord diagrams, and also on which symmetries matter for the kind of embedding we are counting. Figure 7 shows a number of different symmetries. If we have an oriented embedding, where we only allow rotational symmetry, all rotations of Figure 7a give different chord diagrams, so there are 10 different chord diagrams for the corresponding embedding, but rotations of Figure 7b by 5 (corresponding to 180°) give the same chord diagram, so there are only 5 different chord diagrams for the corresponding embedding. If we are counting orientable embeddings rather than oriented embeddings, we also need to consider reflexive symmetry, as in Figure 7c. The other parts of Figure 7 show examples of other symmetries that can affect counting for different types of objects. The tool we use to handle this difficulty is Burnside’s lemma, Lemma 1.

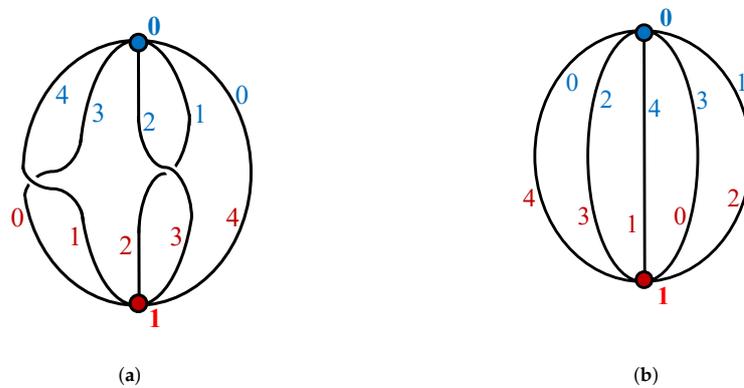


Figure 6. Labeled dipoles correspond to a dipole embedding. (a) Dipole embedding with half-edges labeled in rotational order. (b) Corresponding labeled dipole.

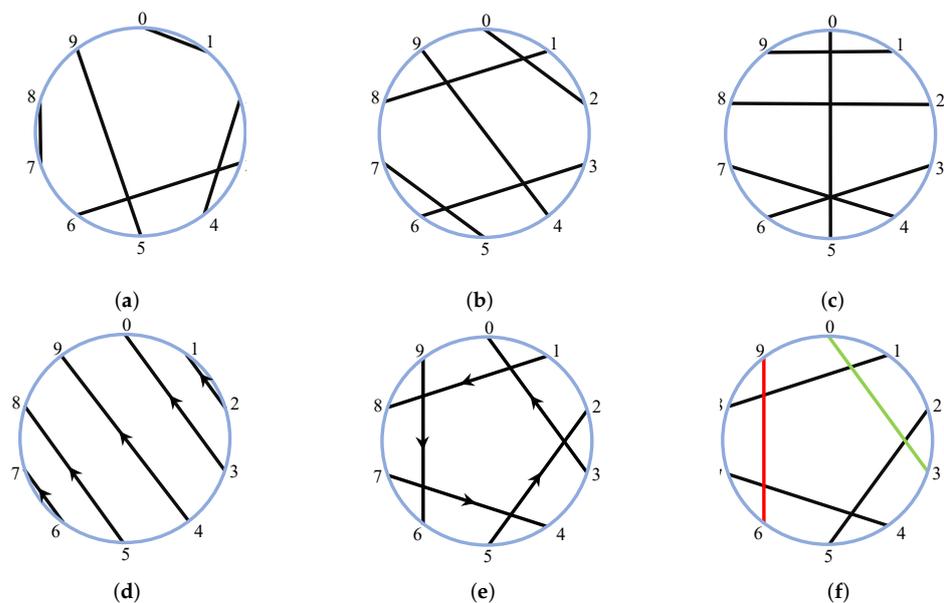


Figure 7. Chord diagram symmetries. Figures (d) and (e) correspond to directed bouquets, and Figure (f) to a colored bouquet. (a) Neither reflexive nor rotational symmetry. (b) 180° rotational symmetry; no reflexive symmetry. (c) Reflexive symmetry through 0–5; no rotational symmetry. (d) 180° rotation plus arc-reversal; reflection through 4–9; reflection through 1½ – 6½ plus arc-reversal. (e) Rotation by 72°; reflection through ½ – 5½ plus arc-reversal. (f) Neither reflexive nor rotational symmetry, although the underlying uncolored diagram has many symmetries.

2.3. Symmetry and Asymmetry under an Involution

In many of our results, we count a set of objects, and we have an involution (i.e., self-inverse permutation) that can be applied to these objects. If we count the number of equivalence classes of our objects under the involution, we can also easily count the number of symmetric objects (preserved by the involution), and the number of pairs of the remaining asymmetric objects, where the objects in each pair are swapped by the involution.

A common type of involution is some kind of reflection, and we would like to count the *reflexible* or *achiral* objects that are preserved by reflection, and the number of pairs of *chiral* objects that are asymmetric under (i.e., not preserved by) reflection. To give a specific example, oriented embeddings of a graph can be divided into reflexible embeddings and chiral pairs of embeddings, where the reflection operation is the reversal of the clockwise orientation of the surface.

The following general formulas apply for a given involution.

$$\begin{aligned} \#objects &= \#symmetric + 2\#(asymmetric\ pairs), \\ \#(involution\ classes) &= \#symmetric + \#(asymmetric\ pairs). \end{aligned}$$

These equations mean that any two of these four numbers determine the other two. In our results, generally we can count the number of objects and the number of involution classes of objects; we can then compute the number of symmetric objects and pairs of asymmetric objects as follows.

$$\#symmetric = 2\#(involution\ classes) - \#objects, \tag{1}$$

$$\#(asymmetric\ pairs) = \#objects - \#(involution\ classes). \tag{2}$$

We will apply these formulas many times.

Besides reflection, we have other types of involutory symmetry, If our objects are vertex-labeled dipoles and our involution is exchanging the two vertices, we refer to the equivalence classes as *vertex-unlabeled dipoles* or just *dipoles*. The symmetric ones are *vertex-interchangeable*, and the asymmetric ones come in *non-vertex-interchangeable* pairs. If our objects are directed graphs and the involution is reversing all of the arcs, we refer to the equivalence classes as *arc-reversal classes*. The symmetric ones are *arc-reversible* and the asymmetric ones come in *arc-irreversible* pairs.

Although it should be intuitively clear that our specific involutions make sense, we provide some technical details for the interested reader. Generally, we have a set of basic objects \mathcal{S} and a group Γ acting on \mathcal{S} . If we have a normal subgroup $\Delta \trianglelefteq \Gamma$, then elements of Γ act on \mathcal{S}/Δ , the orbits of \mathcal{S} under the action of Δ , which we consider as a set of derived objects. In particular, if Δ is a subgroup of Γ and $|\Gamma| = 2|\Delta|$, then $\Delta \trianglelefteq \Gamma$, and if $J \in \Gamma - \Delta$ is an involution in Γ then J acts as an involution on the set of derived objects \mathcal{S}/Δ . When we use the results of this subsection, the situation is always an instance of this general framework.

2.4. Standard Counting Functions

In our results, we often use Euler’s totient function $\varphi(n)$, which is the number of integers k that are relatively prime to n and satisfy $1 \leq k \leq n$. We use the notation $\gcd(a, b)$ for the greatest common divisor of a and b .

We also frequently use $\mu(n, j)$, the number of j -matchings (sets of j disjoint 2-subsets) of an n -set, which is

$$\mu(n, j) = \binom{n}{2j} \frac{(2j)!}{2^j j!} = \binom{n}{2j} (2j - 1)!! = \frac{n!}{(n - 2j)! 2^j j!}$$

where $(2j - 1)!! = (2j - 1)(2j - 3)(2j - 5) \dots 1$ is a double factorial. To keep our formulas simple, we often keep μ in our final expressions rather than replacing it with expressions involving factorials and powers.

Part II

The Enumeration Formulas

In this part, we summarize our results. For results that are already known, we provide references, and in particular, references to entries in the Online Encyclopedia of Integer Sequences [1]. If a result is not attributed, it is (as far as we know) new, although some results can be derived from known results using the approach in Section 2.3.

We begin with dipole results, then present the (undirected) bouquet results, and finally the directed bouquet results. In each section, we give a small number of basic quantities (technically *coset averages*, defined in Part III) and then all of the counting results are expressed as simple linear combinations of these quantities. The formulas are indexed by (D1), (D2), ... for the dipoles, (B1), (B2), ... for the bouquets, and (A1), (A2), ... for directed bouquets. These indices correspond to the proofs of the formulas given in Sections 8.3, 9.3, and 10.3, respectively. For each enumeration formula, we give the sequence of values for n (the number of edges) with $0 \leq n \leq 12$. For colored objects, we just provide the values for k (number of colors) equal to 1, i.e., in the uncolored situation.

The formulas we give below are generally valid only for $n \geq 1$. For $n = 0$, there is always one trivial object with no edges, which is preserved under all symmetries. Therefore, the numbers for $n = 0$ are either 1 (for objects under some equivalence relation, or objects symmetric under some involution) or 0 (for pairs of objects asymmetric under an involution).

3. Dipole Results

3.1. Dipole Coset Averages

There are five basic quantities for counting dipole embeddings and related objects, which we denote as $\delta^T(n)$ for various T . The superscripts T are mnemonics for the identity (I), reflections (R_0, R_1), cyclic shifts (S_0, S_1), and vertex exchanges (X), as detailed in Section 8.1 of Part III. Since a dipole has two vertices with n edges between them, we can think of each edge as consisting of two half-edges, each incident with one of the vertices. By labeling the half-edges with elements of \mathbb{Z}_n , we can identify the labeled dipoles with permutations that map the labels on the half-edges incident with one vertex to those incident with the other. Our results can then be proved by applications of Burnside's lemma based on groups acting by reflection, rotation, and so forth on these labeled dipoles. Consequently, our results can also be interpreted as counting results for permutations, or permutation matrices, under these various equivalence relations. Full details are given in Section 8.

As we show in Part III, all the following functions have integer values, given positive integer inputs. For $n = 0$, the value of all of these should be taken to be 1; the following formulas apply for $n \geq 1$.

$$\delta^I(n) = \frac{1}{n} \sum_{(d,g): dg=n} \varphi(d)^2 (g-1)! d^{g-1};$$

$$\delta^R(n) = \begin{cases} \left(\frac{n-1}{2}\right)! 2^{(n-1)/2} & \text{for odd } n, \\ (n+2) \left(\frac{n}{2}-1\right)! 2^{n/2-3} & \text{for even } n; \end{cases}$$

$$\delta^{R_1}(n) = \begin{cases} 1 & \text{if } n = 1 \text{ or } 2, \\ 0 & \text{if } n \geq 3 \text{ is odd,} \\ \frac{1}{n} \left(\frac{n}{2}\right)! 2^{n/2-1} & \text{if } n \geq 4 \text{ is even;} \end{cases}$$

$$\delta^X(n) = \frac{1}{n} \sum_{\substack{(d,g): dg=n \\ d, g \text{ even}}} \varphi(d) \mu(g, g/2) d^{g/2} + \frac{1}{n} \sum_{\substack{(d,g): dg=n \\ d \text{ odd}}} \varphi(d) \sum_{j=0}^{\lfloor g/2 \rfloor} \mu(g, j) d^j$$

(the first term here is nonzero only if $n \equiv 0 \pmod{4}$);

$$\delta^{R_1 X}(n) = \begin{cases} \mu(n/2, n/4) 2^{n/4-1} & \text{if } n \equiv 0 \pmod{4}, \\ \mu((n-1)/2, (n-1)/4) 2^{(n-1)/4} & \text{if } n \equiv 1 \pmod{4}, \\ \mu((n-2)/2, (n-2)/4) 2^{(n-2)/4} & \text{if } n \equiv 2 \pmod{4}, \\ 0 & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

The values of $\delta^I(n)$ and $\delta^{R_1}(n)$ occur in the OEIS [1]; see (D1) and (D3/D5:S) below.

3.2. Counting Basic Dipole Objects

As mentioned above, embedded dipoles can be represented using permutations. Full details are provided in Section 8.1 of Part III. Therefore, most of the results that we mention have natural interpretations in terms of permutations, or equivalently *permutation matrices*, with all entries 0, except for one 1 in each row and each column.

For permutations, specifically elements of the symmetric group $\text{Sym}(\mathbb{Z}_n)$, we consider operations that act by cyclically shifting the input variable (S_0), cyclically shifting the output variable (S_1), reversing the input variable (R_0), reversing the output variable (R_1), simultaneously reversing the input and output variables (R), and inverting the permutation (X). Formal definitions of these are provided in Section 8.1. For formulas (D1)–(D6), we briefly indicate what they count in terms of equivalence classes of permutations under these operations.

Table 1 summarizes our basic results on embeddings of dipoles. Further explanation is given below and technical details (such as the meaning of the ‘Group’ column in the table) can be found in Section 8. The results obtained by applying Section 2.3 do not have individual entries in the table but are discussed in Section 3.3.

Table 1. Basic counting results for n -edge dipole embeddings and related objects.

Item	Objects Counted	Coefficient of $f(n)$					Group (see Section 8)
		δ^I	δ^R	δ^{R_1}	δ^X	$\delta^{R_1 X}$	
(D1)	vertex-labeled oriented dipole embeddings	1					$\langle S_0, S_1 \rangle$
(D2)	oriented dipole embeddings	$\frac{1}{2}$			$\frac{1}{2}$		$\langle S_0, S_1, X \rangle$ (conjugate to $\langle S_0, S_1, RX \rangle$)
(D3)	vertex-labeled orientable dipole embeddings	$\frac{1}{2}$	$\frac{1}{2}$				$\langle S_0, S_1, R \rangle$
(D4)	orientable dipole embeddings	$\frac{1}{4}$	$\frac{1}{4}$		$\frac{1}{2}$		$\langle S_0, S_1, R, X \rangle$
(D5)	vertex-labeled dipolar cogs	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$			$\langle S_0, S_1, R_0, R_1 \rangle$
(D6)	dipolar cogs	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\langle S_0, S_1, R_0, R_1, X \rangle$
(D7)	equivalence classes of permutations under S_0, S_1 and R_0 (see Section 8)	$\frac{1}{2}$		$\frac{1}{2}$			$\langle S_0, S_1, R_0 \rangle$ (conjugate to $\langle S_0, S_1, R_1 \rangle$)
(D8)	permutation matrices under cyclic shifts and rotations	$\frac{1}{4}$	$\frac{1}{4}$			$\frac{1}{2}$	$\langle S_0, S_1, R_1 X \rangle$

All items (Di/Dj:S) or (Di/Dj:AP) are obtained by applying Section 2.3.

- **(D1)** The number of vertex-labeled oriented n -edge dipole embeddings is $\delta^I(n)$.

This also counts elements of $\text{Sym}(\mathbb{Z}_n)$ equivalent under S_0 and S_1 . It appears in the OEIS [1] as A002619.

Values for $0 \leq n \leq 12$: 1, 1, 1, 2, 3, 8, 24, 108, 640, 4492, 36336, 329900, 3326788.

- **(D2)** The number of oriented n -edge dipole embeddings is $\frac{1}{2}(\delta^I(n) + \delta^X(n))$.

This also counts elements of $\text{Sym}(\mathbb{Z}_n)$ equivalent under S_0 , S_1 and X . This sequence was found by Rieper [3] (Theorem 5.10), and also by Feng, Kwak, and Zhou [7] (Theorem 4.1).

Values for $0 \leq n \leq 12$: 1, 1, 1, 2, 3, 7, 19, 71, 369, 2393, 18644, 166573, 1669243.

- **(D3)** The number of vertex-labeled orientable n -edge dipole embeddings is $\frac{1}{2}(\delta^I(n) + \delta^R(n))$.

This also counts elements of $\text{Sym}(\mathbb{Z}_n)$ equivalent under S_0 , S_1 , and R .

Values for $0 \leq n \leq 12$: 1, 1, 1, 2, 3, 8, 20, 78, 380, 2438, 18744, 166870, 1670114.

- **(D4)** The number of orientable n -edge dipole embeddings is $\frac{1}{4}(\delta^I(n) + \delta^R(n) + 2\delta^X(n))$.

This also counts elements of $\text{Sym}(\mathbb{Z}_n)$ equivalent under S_0 , S_1 , R , and X .

Values for $0 \leq n \leq 12$: 1, 1, 1, 2, 3, 7, 17, 56, 239, 1366, 9848, 85058, 840906.

- **(D5)** The number of vertex-labeled n -edge dipolar cogs is $\frac{1}{4}(\delta^I(n) + \delta^R(n) + 2\delta^{R_1}(n))$.

This also counts elements of $\text{Sym}(\mathbb{Z}_n)$ equivalent under S_0 , S_1 , R_0 , and R_1 . It appears in the OEIS [1] as A000940 (with some initial terms missing).

Values for $0 \leq n \leq 12$: 1, 1, 1, 1, 2, 4, 12, 39, 202, 1219, 9468, 83435, 836017.

- **(D6)** The number of n -edge dipolar cogs is $\frac{1}{8}(\delta^I(n) + \delta^R(n) + 2\delta^{R_1}(n) + 2\delta^X(n) + 2\delta^{R_1X}(n))$.

This also counts elements of $\text{Sym}(\mathbb{Z}_n)$ equivalent under S_0 , S_1 , R_0 , R_1 , and X . It appears in the OEIS [1] as A006841.

Values for $0 \leq n \leq 12$: 1, 1, 1, 1, 2, 4, 10, 28, 127, 686, 4975, 42529, 420948.

Our final items in this subsection do not have natural interpretations in terms of dipole embeddings, but can be expressed in terms of permutations or permutation matrices.

- **(D7)** The number of equivalence classes of permutations in $\text{Sym}(\mathbb{Z}_n)$ under cyclic shifts of input variable (S_0), cyclic shifts of output variable (S_1), and reversal of input variable only (R_0) is $\frac{1}{2}(\delta^I(n) + \delta^{R_1}(n))$.

This also counts elements of $\text{Sym}(\mathbb{Z}_n)$ equivalent under S_0 , S_1 , and R_1 . It appears in the OEIS [1] as A000939.

Values for $0 \leq n \leq 12$: 1, 1, 1, 1, 2, 4, 14, 54, 332, 2246, 18264, 164950, 1664354.

- **(D8)** The number of equivalence classes of $n \times n$ permutation matrices under cyclic shifts of the row set, cyclic shifts of the column set, and rotation of the matrix by multiples of 90° is $\frac{1}{4}(\delta^I(n) + \delta^R(n) + 2\delta^{R_1X}(n))$.

Values for $0 \leq n \leq 12$: 1, 1, 1, 1, 2, 5, 11, 39, 193, 1225, 9378, 83435, 835087.

3.3. Symmetric and Asymmetric Dipole Objects

Here, we provide some formulas obtained by applying Equations (1) and (2) of Section 2.3. Given an involution that acts on objects counted by (D_i) and creates equivalence classes counted by (D_j) , item (D_i/D_j) states the formulas for the numbers of symmetric objects, (S) or $(D_i/D_j:S)$, and asymmetric pairs of objects, (AP) or $(D_i/D_j:AP)$.

First, we consider situations where our involution is ‘reflection’ in the sense of reversal of the surface orientation.

- **(D1/D3)** For vertex-labeled oriented n -edge dipole embeddings, the number of reflexible ones (S) is $2(D3) - (D1) = \delta^R(n)$, and the number of chiral pairs (AP) is $(D1) - (D3) = \frac{1}{2}(\delta^I(n) - \delta^R(n))$.

S values for $0 \leq n \leq 12$: 1, 1, 1, 2, 3, 8, 16, 48, 120, 384, 1152, 3840, 13440.

AP values for $0 \leq n \leq 12$: 0, 0, 0, 0, 0, 0, 4, 30, 260, 2054, 17592, 163030, 1656674.

• **(D2/D4)** For oriented n -edge dipole embeddings the number of reflexible ones (S) is $2(D4) - (D2) = \frac{1}{2}(\delta^R(n) + \delta^X(n))$, and the number of chiral pairs (AP) is $(D2) - (D4) = \frac{1}{4}(\delta^I(n) - \delta^R(n))$ (which is half of (D1/D3:AP)).

The number of reflexible ones (S) was found by Feng, Kwak, and Zhou [8] (Theorem 5.1).

S values for $0 \leq n \leq 12$: 1, 1, 1, 2, 3, 7, 15, 41, 109, 339, 1052, 3543, 12569.

AP values for $0 \leq n \leq 12$: 0, 0, 0, 0, 0, 0, 2, 15, 130, 1027, 8796, 81515, 828337.

Next we consider situations where our involution is exchanging vertices.

• **(D1/D2)** For oriented n -edge dipole embeddings the number of vertex-interchangeable ones (S) is $2(D2) - (D1) = \delta^X(n)$, and the number of non-vertex-interchangeable pairs (AP) is $(D1) - (D2) = \frac{1}{2}(\delta^I(n) - \delta^X(n))$.

S values for $0 \leq n \leq 12$: 1, 1, 1, 2, 3, 6, 14, 34, 98, 294, 952, 3246, 11698.

AP values for $0 \leq n \leq 12$: 0, 0, 0, 0, 0, 1, 5, 37, 271, 2099, 17692, 163327, 1657545.

• **(D3/D4)** For orientable n -edge dipole embeddings, the number of vertex-interchangeable ones (S) is $2(D4) - (D3) = \delta^X(n)$ (equal to (D1/D2:S), see values above), and the number of non-vertex-interchangeable pairs (AP) is $(D3) - (D4) = \frac{1}{4}(\delta^I(n) + \delta^R(n) - 2\delta^X(n))$.

AP values for $0 \leq n \leq 12$: 0, 0, 0, 0, 0, 1, 3, 22, 141, 1072, 8896, 81812, 829208.

• **(D5/D6)** For n -edge dipolar cogs the number of vertex-interchangeable ones (S) is $2(D6) - (D5) = \frac{1}{2}(\delta^X(n) + \delta^{R_1X}(n))$, and the number of non-vertex-interchangeable pairs (AP) is $(D5) - (D6) = \frac{1}{8}(\delta^I(n) + \delta^R(n) + 2\delta^{R_1}(n) - 2\delta^X(n) - 2\delta^{R_1X}(n))$.

S values for $0 \leq n \leq 12$: 1, 1, 1, 1, 2, 4, 8, 17, 52, 153, 482, 1623, 5879.

AP values for $0 \leq n \leq 12$: 0, 0, 0, 0, 0, 0, 2, 11, 75, 533, 4493, 40906, 415069.

For orientable embeddings of dipoles, reversing the cyclic ordering at either vertex is equivalent to taking the *Petrie dual* of the embedding, which twists all the edges (see [36], Section 1.3). We have two situations where our involution is the Petrie duality.

• **(D3/D5)** For vertex-labeled orientable n -edge dipole embeddings, the number of Petrie-self-dual ones (S) is $2(D5) - (D3) = \delta^{R_1}(n)$, and the number of non-Petrie-self-dual pairs (AP) is $(D3) - (D5) = \frac{1}{4}(\delta^I(n) + \delta^R(n) - 2\delta^{R_1}(n))$.

The number of Petrie-self-dual ones (S) for even positive n appears in the OEIS [1] as A002866.

S values for $0 \leq n \leq 12$: 1, 1, 1, 0, 1, 0, 4, 0, 24, 0, 192, 0, 1920.

AP values for $0 \leq n \leq 12$: 0, 0, 0, 1, 1, 4, 8, 39, 178, 1219, 9276, 83435, 834097.

• **(D4/D6)** For orientable n -edge dipole embeddings the number of Petrie-self-dual ones (S) is $2(D6) - (D4) = \frac{1}{2}(\delta^{R_1}(n) + \delta^{R_1X}(n))$, and the number of non-Petrie-self-dual pairs (AP) is $(D6) - (D4) = \frac{1}{8}(\delta^I(n) + \delta^R(n) + 2\delta^X(n) - 2\delta^{R_1}(n) - 2\delta^{R_1X}(n))$.

S values for $0 \leq n \leq 12$: 1, 1, 1, 0, 1, 1, 3, 0, 15, 6, 102, 0, 990.

AP values for $0 \leq n \leq 12$: 0, 0, 0, 1, 1, 3, 7, 28, 112, 680, 4873, 42529, 419958.

The next two situations involve formula (D7). To discuss these, it is most natural to reinterpret (D1), (D5) and (D7) as counting Hamilton cycles in a directed or undirected complete graph whose vertices are unlabeled but have a directed or undirected cyclic ordering. In particular, (D1) counts directed Hamilton cycles on a set of unlabeled vertices with a (directed) cyclic ordering, (D5) counts undirected Hamilton cycles on a set of unlabeled vertices with a (directed) cyclic ordering and also directed Hamilton cycles on a

set of unlabeled vertices with an undirected cyclic ordering, and (D7) counts undirected hamilton cycles on a set of unlabeled vertices with an undirected cyclic ordering.

In this setting, there are two natural involutions: reversal of a directed cyclic ordering, which may be considered a reflection, and arc-reversal of a directed cycle. The quantities we obtain can be interpreted in terms of either of these involutions.

- **(D1/D7)** For directed Hamilton cycles on a set of (unlabeled) vertices with a (directed) cyclic ordering, the number of reflexible ones (S) is $2(D7) - (D1) = \delta^{R_1}(n)$ (equal to (D3/D5:S), see values above), and the number of chiral pairs (AP) is $(D1) - (D7) = \frac{1}{2}(\delta^I(n) - \delta^{R_1}(n))$.

We may also interpret (S) as the number of arc-reversible ones and (AP) as the number of arc-irreversible pairs.

AP values for $0 \leq n \leq 12$: 0, 0, 0, 1, 1, 4, 10, 54, 308, 2246, 18072, 164950, 1662434.

- **(D7/D5)** For undirected Hamilton cycles on a set of (unlabeled) vertices with a (directed) cyclic ordering, the number of reflexible ones (S) is $2(D5) - (D7) = \frac{1}{2}(\delta^R(n) + \delta^{R_1}(n))$, and the number of chiral pairs (AP) is $(D7) - (D5) = \frac{1}{4}(\delta^I(n) - \delta^R(n))$ (equal to (D2/D4:AP), see values above).

We may also interpret these in terms of directed hamilton cycles on a set of (unlabeled) vertices with an undirected cyclic ordering: (S) is the number of arc-reversible ones, and (AP) is the number of arc-irreversible pairs.

S values for $0 \leq n \leq 12$: 1, 1, 1, 1, 2, 4, 10, 24, 72, 192, 672, 1920, 7680.

Our final two situations involve formula (D8). To discuss these, we re-interpret formulas (D3) and (D6) in terms of permutation matrices. Formula (D3) counts the equivalence classes of $n \times n$ permutation matrices under cyclic shifts of the row set, cyclic shifts of the column set, and rotation by 180° . Formula (D6) counts the equivalence classes of $n \times n$ permutation matrices under cyclic shifts of the row set, cyclic shifts of the column set, rotations of multiples of 90° , and transposition.

- **(D3/D8)** For equivalence classes of permutation matrices under cyclic shifts of row and column sets and rotations of 180° , the number invariant under rotations of 90° (S) is $2(D8) - (D3) = \delta^{R_1^X}(n)$, and the number of pairs that are swapped by rotations of 90° (AP) is $(D3) - (D8) = \frac{1}{4}(\delta^I(n) + \delta^R(n) - 2\delta^{R_1^X}(n))$.

S values for $0 \leq n \leq 12$: 1, 1, 1, 0, 1, 2, 2, 0, 6, 12, 12, 0, 60.

AP values for $0 \leq n \leq 12$: 0, 0, 0, 1, 1, 3, 9, 39, 187, 1213, 9366, 83435, 835027.

- **(D8/D6)** For equivalence classes of permutation matrices under cyclic shifts of row and column sets and rotations of multiples of 90° , the number that is symmetric (invariant under transposition) (S) is $2(D6) - (D8) = \frac{1}{2}(\delta^{R_1}(n) + \delta^X(n))$, and the number of asymmetric pairs (swapped by transposition) (AP) is $(D8) - (D6) = \frac{1}{8}(\delta^I(n) + \delta^R(n) + 2\delta^{R_1^X}(n) - 2\delta^{R_1}(n) - 2\delta^X(n))$.

S values for $0 \leq n \leq 12$: 1, 1, 1, 1, 2, 3, 9, 17, 61, 147, 572, 1623, 6809.

AP values for $0 \leq n \leq 12$: 0, 0, 0, 0, 0, 1, 1, 11, 66, 539, 4403, 40906, 414139.

4. Bouquet Results

4.1. Bouquet Coset Averages

To obtain the results for embedded bouquets on n vertices, B_n , we transform them into colored labeled bouquets, again thinking of each edge as consisting of two half-edges, and then into colored perfect matchings in the complete graph with vertex set \mathbb{Z}_{2n} . Perfect matchings on a cyclically ordered set, such as \mathbb{Z}_{2n} , are often represented by *chord diagrams*, so our results may also be interpreted as results for colored chord diagrams. See Figures 1 and 5. The colored perfect matchings corresponding to a given embedding are not in general unique, so we apply Burnside's lemma to groups of symmetries acting on the set of such matchings. Full details are given in Section 9.

There are two basic quantities for counting bouquet embeddings and related objects. As we show in Part III, both of these basic quantities given below have integer values, given positive integer inputs. Here again we use mnemonic superscripts for the identity (I), reflection (R), and cyclic shifts (S). For $n = 0$ and an arbitrary value of k , both of these quantities should be taken to be 1; the following formulas apply for $n \geq 1$.

$$\beta^I(n, k) = \frac{1}{2n} \sum_{\substack{(d, g): dg=2n \\ d \text{ odd}}} \varphi(d) \mu(g, g/2) d^{g/2} k^{g/2} + \frac{1}{2n} \sum_{\substack{(d, g): dg=2n \\ d \text{ even}}} \varphi(d) \sum_{j=0}^{\lfloor g/2 \rfloor} \mu(g, j) d^j k^{g-j};$$

$$\beta^R(n, k) = \frac{1}{2} \left(\sum_{j=0}^{\lfloor n/2 \rfloor} \mu(n, j) 2^j k^{n-j} + \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \mu(n-1, j) 2^j k^{n-j} \right).$$

The values of $\beta^I(n, 1)$ and $\beta^R(n, 1)$ appear in the OEIS [1]; see (B1) and (B1/B2:S) below.

4.2. Counting Basic Bouquet Objects

Table 2 summarizes our basic results on embeddings of bouquets. Further explanation is given below. Results obtained by applying Section 2.3 do not have individual entries in the table, but are discussed in Section 4.3.

Table 2. Basic counting results for k -colored embeddings of n -edge bouquets.

Item	Objects counted	Coefficient of $f(n, k)$		Group (see Section 9)
		β^I	β^R	
(B1)	oriented embeddings of bouquets	1		$\langle S \rangle$
(B2)	orientable embeddings of bouquets	$\frac{1}{2}$	$\frac{1}{2}$	$\langle S, R \rangle$
Item	Objects counted	How computed (see Section 9)		
(B3)	orientable or nonorientable embeddings of bouquets	replace k by $2k$ in (B2)		
(B4)	nonorientable embeddings of bouquets	(B3)–(B2)		

All items (Bi/Bj:S) or (Bi/Bj:AP) are obtained by applying Section 2.3.

- **(B1)** The number of oriented embeddings of k -colored n -edge bouquets is $\beta^I(n, k)$.
 This is also the number of k -colored n -chord diagrams up to rotations (cyclic shifts). This sequence for $k = 1$ appears in the OEIS [1] as A007769. It was also presented by Feng, Kwak, and Zhou [7] (Theorem 3.2), and by Orbanić et al. [9] (formula $\gamma_{\overline{S} \overline{D} \overline{P}}$), specifically in the context of embeddings of bouquets. This sequence for $k = 2$ appears in Orbanić et al. [9] (formula $\gamma_{\overline{S} \overline{D} \overline{P}}$) as counting signed (i.e., 2-colored) oriented embeddings of bouquets.
Values for $k = 1$ and $0 \leq n \leq 12$: 1, 1, 2, 5, 18, 105, 902, 9749, 127072, 1915951, 32743182, 624999093, 13176573910.
- **(B2)** The number of orientable embeddings of k -colored n -edge bouquets is $\frac{1}{2}(\beta^I(n, k) + \beta^R(n, k))$.
 This is also the number of k -colored n -chord diagrams up to rotations and reflections. This sequence for $k = 1$ appears in the OEIS [1] as A054499. It was also presented by Orbanić et al. [9] (formula $\delta_{\overline{S} \overline{D} \overline{P}}$), specifically in the context of embeddings of bouquets.
Values for $k = 1$ and $0 \leq n \leq 12$: 1, 1, 2, 5, 17, 79, 554, 5283, 65346, 966156, 16411700, 312700297, 6589356711.
- **(B3)** The number of generic (orientable or nonorientable) embeddings of k -colored n -edge bouquets is $\frac{1}{2}(\beta^I(n, 2k) + \beta^R(n, 2k))$ For $k = 1$, this gives

$$\frac{1}{2}(\beta^I(n, 2) + \beta^R(n, 2))$$

as the number of generic embeddings of n -edge bouquets.

This sequence for $k = 1$ was found by Kim and Park [6] (Theorem 3.2). Their approach involves subdividing a bouquet B_n to give a graph with n triangles meeting at a common vertex. It was also found by Orbanić et al. [9] (formula $\delta_{S\overline{D}\overline{P}}$).

Values for $k = 1$ and $0 \leq n \leq 12$: 1, 2, 6, 26, 173, 1844, 29570, 628680, 16286084, 490560202, 16764409276, 639992710196, 26985505589784.

• **(B4)** The number of nonorientable embeddings of k -colored n -edge bouquets is $(B3) - (B2) = \frac{1}{2}(\beta^I(n, 2k) + \beta^R(n, 2k) - \beta^I(n, k) - \beta^R(n, k))$. For $k = 1$, this gives

$$\frac{1}{2}(\beta^I(n, 2) + \beta^R(n, 2) - \beta^I(n, 1) - \beta^R(n, 1))$$

as the number of nonorientable embeddings of n -edge bouquets.

Values for $k = 1$ and $0 \leq n \leq 12$: 0, 1, 4, 21, 156, 1765, 29016, 623397, 16220738, 489594046, 16747997576, 639680009899, 26978916233073.

4.3. Symmetric and Asymmetric Bouquet Objects

Again we provide some applications of Equations (1) and (2) of Section 2.3. Given an involution (here, just reflection) that acts on objects counted by (Bi) and creates equivalence classes counted by (Bj) , item (Bi/Bj) states the formulas for the numbers of symmetric objects, (S) or $(Bi/Bj:S)$, and asymmetric pairs of objects, (AP) or $(Bi/Bj:AP)$.

• **(B1/B2)** For n -edge bouquet embeddings, the number of reflexible ones (S) is $2(B2) - (B1) = \beta^R(n, k)$, and the number of chiral pairs (AP) is $(B1) - (B2) = \frac{1}{2}(\beta^I(n, k) - \beta^R(n, k))$.

The number of reflexible ones (S) for $k = 1$ appears in the OEIS [1] as A018191. It was also presented by Feng, Kwak, and Zhou [8] (Theorem 4.2), specifically in the context of embeddings of bouquets. The number of chiral pairs (AP) for $k = 1$ occurs in the OEIS [1] as A054938.

S values for $k = 1$ and $0 \leq n \leq 12$: 1, 1, 2, 5, 16, 53, 206, 817, 3620, 16361, 80218, 401501, 2139512.

AP values for $k = 1$ and $0 \leq n \leq 12$: 0, 0, 0, 0, 1, 26, 348, 4466, 61726, 949795, 16331482, 312298796, 6587217199.

5. Directed Bouquet Results

5.1. Directed Bouquet Coset Averages

There are four basic quantities for counting the directed embeddings of directed bouquets and related objects. To obtain these quantities, we transform directed embeddings of directed bouquets into labeled directed bouquets and then into perfect matchings in the complete graph with vertex set \mathbb{Z}_{2n} and signed vertices. We apply Burnside’s lemma to groups of symmetries acting on the set of matchings. Full details are given in Section 10.

As we show in Part III, all of the following have integer values, given positive integer inputs. For $n = 0$ and an arbitrary value of k , all of these quantities should be taken to be 1; the following formulas apply for $n \geq 1$.

$$\alpha^I(n, k) = \frac{1}{n} \sum_{(d,g):dg=n} \phi(d) g! d^g k^g;$$

$$\alpha^R(n, k) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \left(\frac{n-1}{2}\right)! 2^{(n-1)/2} k^{(n+1)/2} & \text{if } n \text{ is odd;} \end{cases}$$

$$\alpha^F(n, k) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{1}{n} \sum_{\substack{(d,g):dg=n \\ g \text{ odd}}} \phi(2d) \sum_{j=0}^{\lfloor g/2 \rfloor} \mu(g, j) d^j k^{g-j} & \text{if } n \text{ is odd;} \end{cases}$$

$$\alpha^{RF}(n, k) = \sum_{j=1}^{\lfloor n/2 \rfloor} \mu(n, j) k^{n-j}.$$

The values of $\alpha^I(n, 1)$, $\alpha^R(n, 1)$, and $\alpha^{RF}(n, 1)$ appear in the OEIS; see (A1), (A1/A2:S), and (A1/A5:S) below.

5.2. Counting Basic Directed Bouquet Objects

Table 3 summarizes our basic results on the embeddings of bouquets. Further explanation is given below. Results obtained by applying Section 2.3 do not have individual entries in the table, but are discussed in Section 5.3.

Table 3. Basic counting results for k -colored directed embeddings of n -arc directed bouquets.

Item	Objects Counted	Coefficient of $f(n, k)$				Group (see Section 10)
		α^I	α^R	α^F	α^{RF}	
(A1)	oriented directed embeddings of directed bouquets	1				$\langle S \rangle$
(A2)	orientable directed embeddings of directed bouquets	$\frac{1}{2}$	$\frac{1}{2}$			$\langle S, R \rangle$
(A3)	arc-reversal classes of oriented directed embeddings of directed bouquets	$\frac{1}{2}$		$\frac{1}{2}$		$\langle S, F \rangle$
(A4)	arc-reversal classes of orientable directed embeddings of directed bouquets	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\langle S, R, F \rangle$
(A5)	classes of oriented directed embeddings of directed bouquets under simultaneous reflection and arc-reversal	$\frac{1}{2}$			$\frac{1}{2}$	$\langle S, RF \rangle$
Item	Objects Counted	How Computed (see Section 10)				
(A6)	orientable or nonorientable directed embeddings of directed bouquets	replace k by $2k$ in (A2)				
(A7)	nonorientable directed embeddings of directed bouquets	(A6)–(A2)				
(A8)	arc-reversal classes of orientable or nonorientable directed embeddings of directed bouquets	replace k by $2k$ in (A4)				
(A9)	arc-reversal classes of nonorientable directed embeddings of directed bouquets	(A8)–(A4)				

All items (Ai/Aj:S) or (Ai/Aj:AP) are obtained by applying Section 2.3.

- **(A1)** The number of oriented k -colored directed embeddings of n -arc directed bouquets is $\alpha^I(n, k)$.

This sequence for $k = 1$ appears in the OEIS [1] as A061417. It was also presented by Chen, Gao, and Huang [10] (Theorem 3.3) specifically in the context of directed embeddings of directed bouquets.

Values for $k = 1$ and $0 \leq n \leq 12$: 1, 1, 2, 4, 10, 28, 136, 726, 5100, 40362, 363288, 3628810, 39921044.

- **(A2)** The number of orientable k -colored directed embeddings of n -arc directed bouquets is $\frac{1}{2}(\alpha^I(n, k) + \alpha^R(n, k))$.

Values for $k = 1$ and $0 \leq n \leq 12$: 1, 1, 1, 3, 5, 18, 68, 387, 2550, 20373, 181644, 1816325, 19960522.

- **(A3)** The number of arc-reversal classes of oriented k -colored directed embeddings of n -arc directed bouquets is $\frac{1}{2}(\alpha^I(n, k) + \alpha^F(n, k))$.

Values for $k = 1$ and $0 \leq n \leq 12$: 1, 1, 1, 3, 5, 17, 68, 380, 2550, 20328, 181644, 1816028, 19960522.

• **(A4)** The number of arc-reversal classes of orientable k -colored directed embeddings of n -arc directed bouquets is $\frac{1}{4}(\alpha^I(n, k) + \alpha^R(n, k) + \alpha^F(n, k) + \alpha^{RF}(n, k))$.

Values for $k = 1$ and $0 \leq n \leq 12$: 1, 1, 1, 3, 5, 17, 53, 260, 1466, 10915, 93196, 917898, 10015299.

• **(A5)** The number of classes of oriented k -colored directed embeddings of n -arc directed bouquets under simultaneous reflection and arc reversal is $\frac{1}{2}(\alpha^I(n, k) + \alpha^{RF}(n, k))$.

Values for $k = 1$ and $0 \leq n \leq 12$: 1, 1, 2, 4, 10, 27, 106, 479, 2932, 21491, 186392, 1832253, 20030598.

• **(A6)** The number of generic (orientable or nonorientable) k -colored directed embeddings of n -arc directed bouquets is $\frac{1}{2}(\alpha^I(n, 2k) + \alpha^R(n, 2k))$. For $k = 1$ this gives

$$\frac{1}{2}(\alpha^I(n, 2) + \alpha^R(n, 2))$$

as the number of generic directed embeddings of directed bouquets.

Values for $k = 1$ and $0 \leq n \leq 12$: 1, 2, 3, 14, 54, 420, 3886, 46470, 645524, 10328214, 185800748, 3716014090, 81749732156.

• **(A7)** The number of nonorientable k -colored directed embeddings of n -arc directed bouquets is $(A6) - (A2) = \frac{1}{2}(\alpha^I(n, 2k) + \alpha^R(n, 2k) - \alpha^I(n, k) - \alpha^R(n, k))$. For $k = 1$, this gives

$$\frac{1}{2}(\alpha^I(n, 2) + \alpha^R(n, 2) - \alpha^I(n, 1) - \alpha^R(n, 1))$$

as the number of nonorientable directed embeddings of directed bouquets.

Values for $k = 1$ and $0 \leq n \leq 12$: 0, 1, 2, 11, 49, 402, 3818, 46083, 642974, 10307841, 185619104, 3714197765, 81729771634.

• **(A8)** The number of arc-reversal classes of generic (orientable or nonorientable) k -colored directed embeddings of n -arc directed bouquets is $\frac{1}{4}(\alpha^I(n, 2k) + \alpha^R(n, 2k) + \alpha^F(n, 2k) + \alpha^{RF}(n, 2k))$. For $k = 1$ this gives

$$\frac{1}{4}(\alpha^I(n, 2) + \alpha^R(n, 2) + \alpha^F(n, 2) + \alpha^{RF}(n, 2))$$

as the number of arc-reversal classes of generic directed embeddings of directed bouquets.

Values for $k = 1$ and $0 \leq n \leq 12$: 1, 2, 3, 14, 46, 304, 2289, 25096, 330862, 5211052, 93130670, 1859431284, 40882543694.

• **(A9)** The number of arc-reversal classes of nonorientable k -colored directed embeddings of n -arc directed bouquets is $(A8) - (A4) = \frac{1}{4}(\alpha^I(n, 2k) + \alpha^R(n, 2k) + \alpha^F(n, 2k) + \alpha^{RF}(n, 2k) - \alpha^I(n, k) - \alpha^R(n, k) - \alpha^F(n, k) - \alpha^{RF}(n, k))$. For $k = 1$ this gives

$$\frac{1}{4}(\alpha^I(n, 2) + \alpha^R(n, 2) + \alpha^F(n, 2) + \alpha^{RF}(n, 2) - \alpha^I(n, 1) - \alpha^R(n, 1) - \alpha^F(n, 1) - \alpha^{RF}(n, 1))$$

as the number of arc-reversal classes of nonorientable directed embeddings of directed bouquets.

Values for $k = 1$ and $0 \leq n \leq 12$: 0, 1, 2, 11, 41, 287, 2236, 24836, 329396, 5200137, 93037474, 1858513386, 40872528395.

5.3. Symmetric and Asymmetric Directed Bouquet Objects

Again we provide some applications of Equations (1) and (2) of Section 2.3. Given an involution that acts on objects counted by (A_i) and creates equivalence classes counted

by (A_j), item (A_i/A_j) states the formulas for the numbers of symmetric objects, (S) or (A_i/A_j:S), and asymmetric pairs of objects, (AP) or (A_i/A_j:AP).

First, we consider situations where our involution is a ‘reflection’ in the sense of the reversal of the surface orientation.

- **(A1/A2)** For oriented k -colored directed embeddings of n -arc directed bouquets, the number of reflexible ones (S) is $2(A2) - (A1) = \alpha^R(n, k)$, and the number of chiral pairs (AP) is $(A1) - (A2) = \frac{1}{2}(\alpha^I(n, k) - \alpha^R(n, k))$.

The number of reflexible ones for $k = 1$ and odd n appears in the OEIS [1] as A000165.

S values for $k = 1$ and $0 \leq n \leq 12$: 1, 1, 0, 2, 0, 8, 0, 48, 0, 384, 0, 3840, 0.

AP values for $k = 1$ and $0 \leq n \leq 12$: 0, 0, 1, 1, 5, 10, 68, 339, 2550, 19989, 181644, 1812485, 19960522.

- **(A3/A4)** For the arc-reversal classes of k -colored directed embeddings of n -arc directed bouquets, the number of reflexible ones (S) is $2(A4) - (A3) = \frac{1}{2}(\alpha^R(n, k) + \alpha^{RF}(n, k))$, and the number of chiral pairs (AP) is $(A3) - (A4) = \frac{1}{4}(\alpha^I(n, k) + \alpha^F(n, k) - \alpha^R(n, k) - \alpha^{RF}(n, k))$.

S values for $k = 1$ and $0 \leq n \leq 12$: 1, 1, 1, 3, 5, 17, 38, 140, 382, 1502, 4748, 19768, 70076.

AP values for $k = 1$ and $0 \leq n \leq 12$: 0, 0, 0, 0, 0, 0, 15, 120, 1084, 9413, 88448, 898130, 9945223.

Next, we consider situations where our involution is arc-reversal.

- **(A1/A3)** For oriented k -colored directed embeddings of n -arc directed bouquets, the number of arc-reversible ones (S) is $2(A3) - (A1) = \alpha^F(n, k)$, and the number of arc-irreversible pairs (AP) is $(A1) - (A3) = \frac{1}{2}(\alpha^I(n, k) - \alpha^F(n, k))$.

S values for $k = 1$ and $0 \leq n \leq 12$: 1, 1, 0, 2, 0, 6, 0, 34, 0, 294, 0, 3246, 0.

AP values for $k = 1$ and $0 \leq n \leq 12$: 0, 0, 1, 1, 5, 11, 68, 346, 2550, 20034, 181644, 1812782, 19960522.

- **(A2/A4)** For orientable k -colored directed embeddings of n -arc directed bouquets, the number of arc-reversible ones (S) is $2(A4) - (A2) = \frac{1}{2}(\alpha^F(n, k) + \alpha^{RF}(n, k))$, and the number of arc-irreversible pairs (AP) is $(A2) - (A4) = \frac{1}{4}(\alpha^I(n, k) + \alpha^R(n, k) - \alpha^F(n, k) - \alpha^{RF}(n, k))$.

S values for $k = 1$ and $0 \leq n \leq 12$: 1, 1, 1, 3, 5, 16, 38, 133, 382, 1457, 4748, 19471, 70076.

AP values for $k = 1$ and $0 \leq n \leq 12$: 0, 0, 0, 0, 0, 1, 15, 127, 1084, 9458, 88448, 898427, 9945223.

- **(A6/A8)** For generic k -colored directed embeddings of n -arc directed bouquets. the number of arc-reversible ones (S) is $2(A8) - (A6) = \frac{1}{2}(\alpha^F(n, 2k) + \alpha^{RF}(n, 2k))$, and the number of arc-irreversible pairs (AP) is $(A6) - (A8) = \frac{1}{4}(\alpha^I(n, 2k) + \alpha^R(n, 2k) - \alpha^F(n, 2k) - \alpha^{RF}(n, 2k))$.

S values for $k = 1$ and $0 \leq n \leq 12$: 1, 2, 3, 14, 38, 188, 692, 3722, 16200, 93890, 460592, 2848478, 15355232.

AP values for $k = 1$ and $0 \leq n \leq 12$: 0, 0, 0, 0, 8, 116, 1597, 21374, 314662, 5117162, 92670078, 1856582806, 40867188462.

- **(A7/A9)** For nonorientable k -colored directed embeddings of n -arc directed bouquets, the number of arc-reversible ones (S) is $2(A9) - (A7) = \frac{1}{2}(\alpha^F(n, 2k) + \alpha^{RF}(n, 2k) - \alpha^F(n, k) - \alpha^{RF}(n, k))$, and the number of arc-irreversible pairs (AP) is $(A7) - (A9) = \frac{1}{4}(\alpha^I(n, 2k) + \alpha^R(n, 2k) - \alpha^F(n, 2k) - \alpha^{RF}(n, 2k) - \alpha^I(n, k) - \alpha^R(n, k) + \alpha^F(n, k) + \alpha^{RF}(n, k))$.

S values for $k = 1$ and $0 \leq n \leq 12$: 0, 1, 2, 11, 33, 172, 654, 3589, 15818, 92433, 455844, 2829007, 15285156.

AP values for $k = 1$ and $0 \leq n \leq 12$: 0, 0, 0, 0, 8, 115, 1582, 21247, 313578, 5107704, 92581630, 1855684379, 40857243239.

Finally we consider situations involving simultaneous reflection and arc reversal.

- **(A1/A5)** For oriented k -colored directed embeddings of n -arc directed bouquets, the number symmetric under simultaneous reflection and arc reversal (S) is $2(A5) - (A1) = \alpha^{RF}(n, k)$, and the number of asymmetric pairs (AP) is $(A1) - (A5) = \frac{1}{2}(\alpha^I(n, k) - \alpha^{RF}(n, k))$.

The number of symmetric ones (S) for $k = 1$ appears in the OEIS [1] as A000085; it is the total number of matchings in an n -vertex complete graph.

S values for $k = 1$ and $0 \leq n \leq 12$: 1, 1, 2, 4, 10, 26, 76, 232, 764, 2620, 9496, 35696, 140152.

AP values for $k = 1$ and $0 \leq n \leq 12$: 0, 0, 0, 0, 0, 1, 30, 247, 2168, 18871, 176896, 1796557, 19890446.

- **(A5/A4)** Let \mathcal{S} be the set of equivalence classes of oriented k -colored directed embeddings of n -arc directed bouquets under simultaneous reflection and arc reversal. Considering elements of \mathcal{S} up to reflection is the same as considering elements of \mathcal{S} up to arc-reversal. The number of reflexible (or arc-reversible) elements of \mathcal{S} (S) is $2(A4) - (A5) = \frac{1}{2}(\alpha^R(n, k) + \alpha^F(n, k))$, and the number of chiral (or arc-irreversible) pairs (AP) is $(A5) - (A4) = \frac{1}{4}(\alpha^I(n, k) + \alpha^{RF}(n, k) - \alpha^R(n, k) - \alpha^F(n, k))$.

S values for $k = 1$ and $0 \leq n \leq 12$: 1, 1, 0, 2, 0, 7, 0, 41, 0, 339, 0, 3543, 0.

AP values for $k = 1$ and $0 \leq n \leq 12$: 0, 0, 1, 1, 5, 10, 53, 219, 1466, 10576, 93196, 914355, 10015299.

6. Related Work and Future Directions

There are a number of problems that follow naturally from the work in this paper. For bouquets and directed bouquets, we found the number of generic (orientable or nonorientable) and nonorientable embeddings. It would also be natural to count equivalence classes of generic embeddings under the Petrie duality. For dipoles, we have not yet counted generic and nonorientable embeddings, and counting those would be a natural next step, building on some of the ideas we used in counting dipolar cogs. We already counted Petrie duality classes of orientable dipole embeddings, but it would also be interesting to do this for generic dipole embeddings. Chen, Gao, and Huang [10] (Theorem 2.4) counted the directed embeddings of Eulerian directed dipoles in oriented surfaces, and it should be possible to obtain related results similar to the results in Sections 3 and 5. All of these problems seem approachable by extending the techniques used here.

It would be straightforward to extend our results on edge-colored bouquets and arc-colored directed bouquets to set up generating functions with variables marking individual colors. Our proofs in Sections 9 and 10 below already contain most of the analyses needed to find the relevant cycle indices for applying the Redfield–Pólya theory.

Orbanić et al. [9] investigated embeddings of one-vertex graphs in two directions that we did not pursue in this paper. First, they considered *pre-maps*, which are embeddings of *pre-graphs*, graphs where *semi-edges* that have only one incidence with a vertex are allowed. Second, they considered embeddings of directed bouquets that are not necessarily directed embeddings: the faces need not be bounded by directed walks. It would be natural to find ways to extend these results to dipoles.

Another open problem is to enumerate the looped dipoles of Figure 3c, which is equivalent to counting upper embeddable graphs with two faces. A similar problem is to count ‘pointed’ graphs, that is, graphs with one distinguished vertex that is incident to every edge. This vertex is then incident to a collection of loops and digons as in Figure 3d. Such graphs are dual to edge-outer embeddable graphs. Because these problems involve multiple graphs for a given number of edges, they seem more difficult than the problems discussed in the first paragraph above. However, the formulas and approaches given here can likely serve as a foundation for further work on looped dipoles and pointed graphs. Moreover, counting embeddings of pre-graphs with two vertices but no loops may serve as a stepping stone from dipoles to looped dipoles.

Part III

Technical Details

This part of the paper provides the technical details and formal proofs of the results in Part II.

7. Basic Counting Results

We begin with some basic counting results we will use. The first is very well known.

Theorem 1 (Burnside's Lemma (stated earlier by Cauchy and Frobenius)). *Suppose Γ is a group acting on a set S . Then the number of orbits of the action, i.e., the number of equivalence classes under the symmetries provided by Γ , is*

$$\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} |\text{Fix}(\gamma)|,$$

where $\text{Fix}(\gamma)$ is the set of elements of S fixed by γ .

In several places, we need to count how many permutations τ have $\tau^2 = \alpha$ for a given permutation α . We will use the following lemma. Recall that $\mu(n, j)$ is the number of j -matchings of an n -set.

Lemma 1. *For integers $\ell \geq 1$ and $m \geq 0$ define*

$$q(\ell, m) = \begin{cases} 0 & \text{if } \ell \text{ is even and } m \text{ is odd,} \\ \mu(m, m/2) \ell^{m/2} & \text{if } \ell \text{ is even and } m \text{ is even,} \\ \sum_{j=0}^{\lfloor m/2 \rfloor} \mu(m, j) \ell^j & \text{if } \ell \text{ is odd.} \end{cases}$$

Notice that $q(\ell, m) = 1$ if $m = 0$. Let α be a permutation of an n -set with a_ℓ cycles of length ℓ for each ℓ , $1 \leq \ell \leq n$. Then the number of permutations τ with $\tau^2 = \alpha$ is $\prod_{\ell=1}^n q(\ell, a_\ell)$.

Proof. We need to consider how the cycles of α come from the cycles of τ . A cycle of even length 2ℓ in τ yields two cycles of length ℓ in α . A cycle of odd length in τ yields a cycle of the same odd length in α . Therefore, when ℓ is even each cycle of length ℓ in α must be paired with another such cycle and come from a cycle of length 2ℓ in τ . When ℓ is odd each cycle of length ℓ in α can either come from a cycle of length ℓ in τ , or be paired with another cycle and come from a cycle of length 2ℓ in τ .

If α has an odd number $m = a_\ell$ of cycles of even length ℓ , we cannot pair them all up, so the number of possibilities for the corresponding cycles in τ is 0, which is $q(\ell, m) = q(\ell, a_\ell)$.

If α has an even number $m = a_\ell$ of cycles of even length ℓ , then there are $\mu(m, m/2)$ ways to pair these up, and for each pairing there are ℓ ways to interleave two cycles of length ℓ to obtain a cycle of length 2ℓ . Therefore, the number of possibilities in τ is $\mu(m, m/2) \ell^{m/2}$, which is $q(\ell, m) = q(\ell, a_\ell)$.

If α has $m = a_\ell$ cycles of odd length ℓ , then for each j with $0 \leq j \leq \lfloor m/2 \rfloor$ we can pair up $2j$ of these cycles in $\mu(m, j)$ ways, interleave each of the j pairs in ℓ ways, and use the unique square root of each of the $m - 2j$ unpaired cycles, giving $\mu(m, j) \ell^j$ possibilities. Summing over all j gives that the number of possibilities for the corresponding cycles in τ is $\sum_{j=0}^{\lfloor m/2 \rfloor} \mu(m, j) \ell^j$, which is $q(\ell, m) = q(\ell, a_\ell)$.

Now multiplying the number of possibilities for each ℓ gives the result. \square

We use the following easy argument in several places so it is convenient to summarize it here for reference.

Observation 1. Suppose we have sets A, B with $|A| = |B| = 2k$, and partitions \mathcal{A}, \mathcal{B} of A and B , respectively, into pairs (2-subsets). Let $\psi : A \rightarrow B$ be a bijection that preserves pairs, i.e., such that $\psi(A') \in \mathcal{B}$ for all $A' \in \mathcal{A}$. There are $k!$ choices for the bijection that ψ establishes a between \mathcal{A} and \mathcal{B} , and 2 ways for ψ to map the elements of each A' to the elements of $\psi(A')$. Hence the number of possible maps ψ is $k! 2^k$.

8. Proofs for Dipole Formulas

8.1. Labeled Dipoles and Symmetry Operations

In this section, we prove the counting results from Section 3 regarding embeddings of dipoles and related objects. Recall that a dipole D_n has two vertices and n edges, each edge having both vertices as its ends (so there are multiple edges but no loops). We think of each edge as consisting of two half-edges, each incident with one of the vertices.

Our results on dipoles will be proved by elementary techniques (straightforward applications of Burnside’s Lemma) based on groups acting on a set of objects that we will call labeled dipoles. As we will see, labeled dipoles are in one-to-one correspondence with elements of the symmetric group $\text{Sym}(\mathbb{Z}_n)$, so our results can also be interpreted as counting results for permutations, or permutation matrices, under various equivalence relations.

A *labeled dipole* is a dipole D where the vertices receive distinct labels 0 and 1, and for each vertex the half-edges incident with that vertex receive distinct labels from $\mathbb{Z}_n = \{0, 1, 2, \dots, n - 1\}$, where $n = |E(D)|$. An example is given in Figure 6b. We let \mathcal{D}_n denote the set of n -edge labeled dipoles. For $j \in \{0, 1\}$, we refer to the vertex labeled j as *vertex j* , and the half-edges incident to vertex j as *j -half-edges*. For each edge e , we let $\lambda_j(e)$ denote the label of the j -half-edge of e .

A labeled dipole D is completely described by the set of ordered pairs $\{(\lambda_0(e), \lambda_1(e)) \mid e \in E(D)\}$. There is, therefore, a one-to-one correspondence between n -edge labeled dipoles and sets $P = \{(a_0, b_0), (a_1, b_1), \dots, (a_{n-1}, b_{n-1})\}$ with $\{a_0, a_1, \dots, a_{n-1}\} = \{b_0, b_1, \dots, b_{n-1}\} = \mathbb{Z}_n$. Let \mathcal{P}_n denote the collection of all such sets P . Recall that a function is formally defined as a set of ordered pairs whose first components are distinct. Therefore, P may be regarded as the formal representation of a function $\pi : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ with $\pi(a_i) = b_i$ for $0 \leq i \leq n - 1$. It is easy to see that π is a bijection. Thus, π is a permutation of \mathbb{Z}_n , an element of the symmetric group $\text{Sym}(\mathbb{Z}_n)$. Thus, we have natural bijections between $\mathcal{D}_n, \mathcal{P}_n$ and $\text{Sym}(\mathbb{Z}_n)$.

For example, the labeled dipole in Figure 6b corresponds to $\{(0, 4), (1, 2), (2, 3), (3, 0), (4, 1)\} \in \mathcal{P}_5$ and the permutation $\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 0 & 1 \end{bmatrix} \in \text{Sym}(\mathbb{Z}_5)$.

Each object of types (D1)–(D8) above can be turned into a labeled dipole (or permutation of \mathbb{Z}_n) in a natural way. However, some choices are involved in doing this, so each object can be turned into several different labeled dipoles (or permutations), which we wish to characterize as being equivalent under certain symmetries. The symmetries always include cyclically shifting the labels of the 0-half-edges and cyclically shifting the labels of the 1-half-edges, and may also include reversing the labels of the 0-half-edges or the 1-half-edges, or swapping the vertex labels.

Our symmetries can be defined in terms of how they act on ordered pairs $(a, b) \in \mathbb{Z}_n \times \mathbb{Z}_n$. However, to simplify some proofs, and to avoid treating $n = 1$ and 2 as special cases, we will define them more generally, as permutations of the set $\mathbb{R}_n \times \mathbb{R}_n$, where $\mathbb{R}_n = \mathbb{R}/n\mathbb{Z}$ is the additive group of the real numbers modulo n , whose underlying set can be identified with the real interval $[0, n)$. Note that \mathbb{R}_n contains \mathbb{Z}_n as a subgroup. Let I denote the identity of $\text{Sym}(\mathbb{R}_n \times \mathbb{R}_n)$, and define $S_0, S_1, R_0, R_1, X \in \text{Sym}(\mathbb{R}_n \times \mathbb{R}_n)$ as follows:

$$\begin{aligned} S_0(a, b) &= (a + 1, b); & S_1(a, b) &= (a, b + 1); \\ R_0(a, b) &= (-a, b); & R_1(a, b) &= (a, -b); & \text{and} \\ X(a, b) &= (b, a). \end{aligned}$$

Let $\Sigma^d = \langle S_0, S_1 \rangle$ and $\Omega^d = \langle S_0, S_1, R_0, R_1, X \rangle$ (the superscript ‘d’ indicating groups acting on dipoles), so that $\Sigma^d \leq \Omega^d \leq \text{Sym}(\mathbb{R}_n \times \mathbb{R}_n)$.

Consider the induced action of $\text{Sym}(\mathbb{R}_n \times \mathbb{R}_n)$ on $P \subseteq \mathbb{R}_n \times \mathbb{R}_n$ by $T(P) = \{T(a, b) \mid (a, b) \in P\}$. If $T \in \{S_0, S_1, R_0, R_1, X\}$, then both T and T^{-1} map $\mathbb{Z}_n \times \mathbb{Z}_n$ to itself, and moreover both T and T^{-1} map \mathcal{P}_n to itself under the induced subset action. It follows that these statements also hold for all $T \in \Omega^d$. So we have an action of Ω^d on \mathcal{P}_n , and using the natural bijections between \mathcal{P}_n and \mathcal{D}_n or $\text{Sym}(\mathbb{Z}_n)$, we also obtain actions of Ω^d on those sets.

The actions of S_0 and S_1 on \mathcal{D}_n correspond to shifting the labels of the 0- and 1-half-edges, respectively; R_0 and R_1 correspond to reversing (specifically, negating) the labels of the 0- and 1-half-edges, respectively, and X corresponds to swapping the vertex labels.

We note that ‘reflections’ of labels can be regarded as elements of a dihedral group generated by cyclic shifts (rotations) and one reflection. In particular, for labels around vertex i (assumed to be equally spaced around a circle), $S_i^h R_i$ maps $j \mapsto h - j$, which is a reflection about an axis passing through the center of the circle and opposite points $h/2$ and $h/2 + n/2$ on the circle.

In some contexts, the ‘reversal’ of labels might naturally be regarded as the map on \mathbb{Z}_n with $0 \mapsto n - 1, 1 \mapsto n - 2, \dots, n - 1 \mapsto 0$. This maps $j \mapsto n - 1 - j$, and is just a reflection about an axis through $n - 1/2$ or $(n - 1)/2$, and for labels around the vertex, i is represented by the transformation $V_i = S_i^{n-1} R_i$. However, using R_0 and R_1 instead of V_0 and V_1 generally simplifies the calculations. Since we will always include S_0 and S_1 in our groups, and since $\langle S_j, R_j \rangle = \langle S_j, V_j \rangle$ for $j \in \{0, 1\}$, working with R_0 and R_1 is equivalent to working with V_0 and V_1 , respectively.

The actions on $\text{Sym}(\mathbb{Z}_n)$ are as follows. Let $\sigma, \rho \in \text{Sym}(\mathbb{Z}_n)$ be defined by $\sigma(i) = i + 1$ and $\rho(i) = -i$, and let π be an arbitrary element of $\text{Sym}(\mathbb{Z}_n)$. First, S_0 represents a cyclic shift in the input variable of π : instead of $i \mapsto \pi(i)$, we have $i + 1 \mapsto \pi(i)$, or $j \mapsto \pi(j - 1) = \pi\sigma^{-1}(j)$. Next, S_1 represents a cyclic shift in the output variable: instead of $i \mapsto \pi(i)$, we have $i \mapsto \pi(i) + 1 = \sigma\pi(i)$. Now, R_0 represents reversal (negation) of the input variable: instead of $i \mapsto \pi(i)$, we have $-i \mapsto \pi(i)$, or $j \mapsto \pi(-j) = \pi\rho(j)$. Next, R_1 represents reversal of the output variable: instead of $i \mapsto \pi(i)$, we have $i \mapsto -\pi(i) = \rho\pi(i)$. Finally, X corresponds to swapping the input and output variables, which inverts the permutation: instead of $i \mapsto \pi(i)$, we have $\pi(i) \mapsto i$, or $j \mapsto \pi^{-1}(j)$. To summarize,

$$S_0(\pi) = \pi\sigma^{-1}, \quad S_1(\pi) = \sigma\pi, \quad R_0(\pi) = \pi\rho, \quad R_1(\pi) = \rho\pi, \quad \text{and} \quad X(\pi) = \pi^{-1}.$$

Figure 8 illustrates how we may obtain two equivalent labeled dipoles from the oriented embedding of a vertex-unlabeled dipole in Figure 8a, and describe the relationship using the symmetries we have just defined. The permutation $\pi_1 = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 0 & 1 \end{bmatrix}$ corresponds to the labeling L1 in Figure 8b and may be read from the diagram in Figure 8d. Similarly, the permutation $\pi_2 = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 3 & 0 & 1 & 2 & 4 \end{bmatrix}$ corresponds to the labeling L2 in Figure 8c and may be read from the diagram in Figure 8e. The permutation π_1 is related to the permutation π_2 by exchanging the vertex labels (rotating the diagram by 180°), then incrementing the edge labels on the top by 2 and those on the bottom by 3. Formally, this is

$$\pi_2 = S_0^2 S_1^3 X(\pi_1) \quad \text{or} \quad \pi_2 = \sigma^3 \pi_1^{-1} \sigma^{-2}.$$

The following theorem provides some relationships between the generators of Ω^d .

Proposition 1. *Let n be a positive integer, and consider $S_0, S_1, R_0, R_1, X \in \text{Sym}(\mathbb{R}_n \times \mathbb{R}_n)$ as defined above. Let I be the identity of $\text{Sym}(\mathbb{R}_n \times \mathbb{R}_n)$.*

- (a) Then (composing functions right to left),
 - (1) $S_0^n = S_1^n = R_0^2 = R_1^2 = X^2 = I$;
 - (2) each of S_0 or R_0 commutes with each of S_1 or R_1 ;

- (3) $R_0S_0 = S_0^{-1}R_0$, and $R_1S_1 = S_1^{-1}R_1$;
- (4) $XS_0 = S_1X$, and $XR_0 = R_1X$.

Moreover, if $R = R_0R_1$ so that $R(a, b) = (-a, -b)$ then

- (5) $R^2 = I$, $RS_0 = S_0^{-1}R$, $RS_1 = S_1^{-1}R$, and $XR = RX$.
- (b) Every element of $\Omega^d = \langle S_0, S_1, R_0, R_1, X \rangle$ can be written uniquely as $S_0^h S_1^k R_0^p R_1^q X^s$ for some $h, k \in \mathbb{Z}_n$ and $p, q, s \in \{0, 1\}$. Hence $|\Omega^d| = 8n^2$.

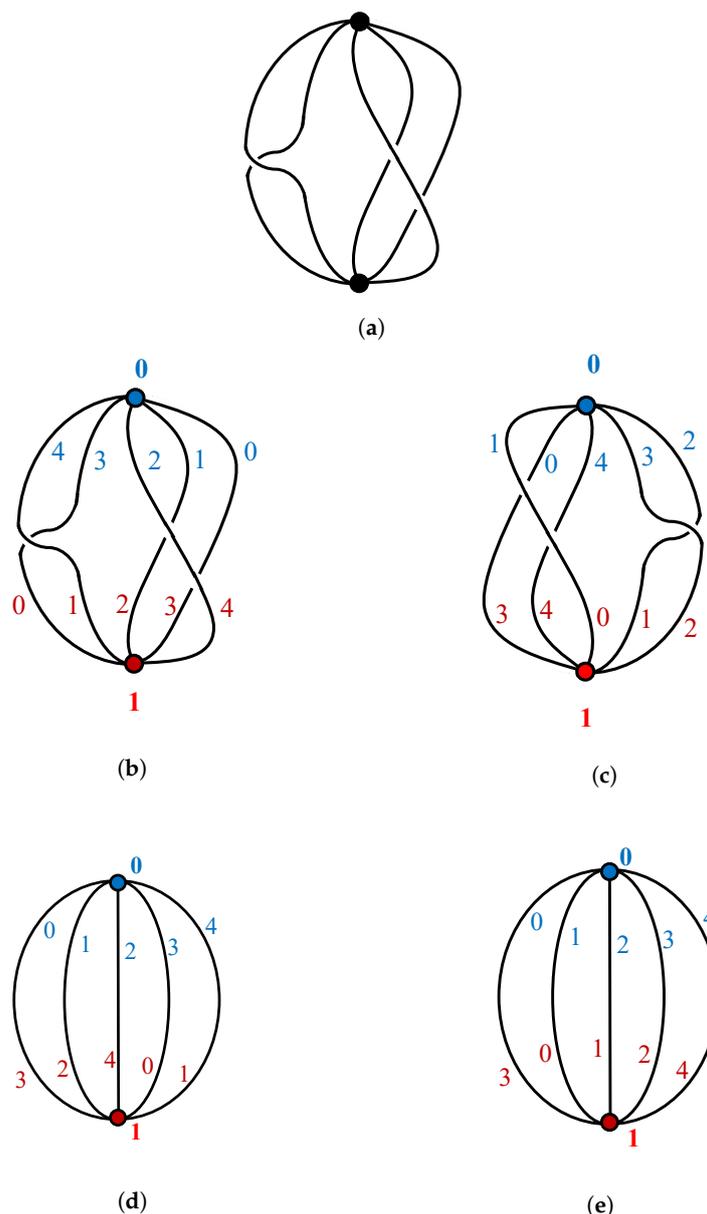


Figure 8. There are many ways to label the vertices and edges of an embedded dipole, but any labelings of the same embedded dipole will correspond to permutations related by the symmetry operations. (a) An unlabeled dipole embedded in an oriented surface. (b) L1: An arbitrary vertex and edge labeling. (c) L2: A different arbitrary vertex and edge labeling. (d) Extracting the permutation for L1. (e) Extracting the permutation for L2.

Proof. (a) All of these relations can be checked easily from the definitions.
 (b) To obtain that each $T \in \Gamma$ can be expressed as $S_0^h S_1^k R_0^p R_1^q X^s$ for some $h, k \in \mathbb{Z}_n$ and $p, q, s \in \{0, 1\}$ we can just apply the relations in (a) to put any word W in our generators into this form. First move any X to the end of the word, to write $W = W_1 X^s$ with $s \in \{0, 1\}$.

Then move any R_0 or R_1 to the end of W_1 , then rearrange them to write $W_1 = W_2 R_0^p R_1^q$. Finally W_2 can be rearranged to have the form $W_2 = S_0^h S_1^k$.

Now consider the effect of $T = S_0^h S_1^k R_0^p R_1^q X^s$ on the single point $(0.1, 0.2) \in \mathbb{R}_n \times \mathbb{R}_n$. If $s = 0$ then $T(0.1, 0.2) = ((-1)^p 0.1 + h, (-1)^q 0.2 + k)$ and if $s = 1$ then $T(0.1, 0.2) = ((-1)^p 0.2 + h, (-1)^q 0.1 + k)$. It is easy to determine the values of s then p, q, h, k from $T(0.1, 0.2)$ and therefore T cannot have a different representation. (This is one place where it is helpful to consider permutations of $\mathbb{R}_n \times \mathbb{R}_n$, not just $\mathbb{Z}_n \times \mathbb{Z}_n$.)

Since there are n choices for each of h and k and 2 choices for each of p, q , and s , we obtain $|\Omega^d| = 8n^2$. \square

8.2. Dipole Coset Averages

We are going to consider the action of $\Omega^d = \langle S_0, S_1, R_0, R_1, X \rangle \leq \text{Sym}(\mathbb{R}_n \times \mathbb{R}_n)$ and some of its subgroups on \mathcal{P}_n , and the corresponding actions on $\text{Sym}(\mathbb{Z}_n)$ and \mathcal{D}_n . We know that $|\Omega^d| = 8n^2$ even if $n = 1$ or 2. When $n = 1$ or 2 the action of Ω^d on \mathcal{D}_n is not faithful (two different elements of Ω^d may act in the same way), but this does not matter for counting arguments using Burnside’s lemma.

For determining fixed points, we use the action of Ω^d or its subgroups on $\text{Sym}(\mathbb{Z}_n)$. From Theorem 1 and results in the last subsection, we know that the action of an element of Ω^d on $\pi \in \text{Sym}(\mathbb{Z}_n)$ can be described as

$$S_0^h S_1^k R_0^p R_1^q X^s(\pi) = \sigma^k \rho^q \pi^{(-1)^s} \rho^p \sigma^{-h}.$$

The action of Ω^d itself will allow us to count dipolar cogs, problem (D6) in our list above.

The subgroups of Ω^d that we consider will always contain S_0 and S_1 , and hence will always have $\Sigma^d = \langle S_0, S_1 \rangle$ as a subgroup. From Theorem 1, we know that $\Sigma^d = \{S_0^h S_1^k \mid h, k \in \mathbb{Z}_n\}$, with n^2 elements. The action of Σ^d on \mathcal{D}_n or $\text{Sym}(\mathbb{Z}_n)$ will allow us to count vertex-labeled oriented dipole embeddings, problem (D1) in our list above.

Any group of symmetries containing Σ^d can be considered a union of right cosets $\Sigma^d T$ of Σ^d . In Ω^d , there are eight right cosets $\Sigma^d T$, for $T \in \{R_0^p R_1^q X^s \mid p, q, s \in \{0, 1\}\} = \{I, X, R, RX, R_0, R_1, R_0 X, R_1 X\}$, where $R = R_0 R_1$. Therefore, in applying Burnside’s lemma, we can use expressions giving the average number of fixed points for $\Sigma^d T$,

$$\delta^T(n) = \frac{1}{n^2} \sum_{\gamma \in \Sigma^d T} |\text{Fix}(\gamma)| = \frac{1}{n^2} \sum_{h \in \mathbb{Z}_n} \sum_{k \in \mathbb{Z}_n} |\text{Fix}(S_0^h S_1^k T)|$$

where we consider the action on $\text{Sym}(\mathbb{Z}_n)$. In this subsection, we compute all eight corresponding values $\delta^T(n)$. We see that there are five distinct values for these coset averages.

For all of our dipole counting problems, the relevant group Γ satisfies $\Sigma^d \leq \Gamma \leq \Omega^d$, and the cosets of Σ^d in Γ are a subset of the cosets of Σ^d in Ω^d . So the coset averages $\delta^T(n)$ computed for Ω^d will also provide solutions for the problems involving the other groups Γ .

We can reduce one counting problem to another if two cosets are related by conjugacy, as follows.

Lemma 2. Suppose $T_1, T_2, U \in \Omega^d$ and $U^{-1} \Sigma^d T_1 U = \Sigma^d T_2$. Then $\delta^{T_1}(n) = \delta^{T_2}(n)$.

Proof. The map $\gamma \mapsto \gamma' = U^{-1} \gamma U$ is a bijection from $\Sigma^d T_1$ to $\Sigma^d T_2$. Moreover, $\pi \in \text{Fix}(\gamma)$ if and only if $U^{-1} \pi \in \text{Fix}(\gamma')$. So there is a bijection between $\Sigma^d T_1$ and $\Sigma^d T_2$ that preserves the number of fixed points of each element, and hence $\delta^{T_1}(n) = \delta^{T_2}(n)$. \square

Computing $\delta^I(n)$. To compute $\delta^I(n)$, we consider the fixed points of elements of Σ^d . Suppose there is $\pi \in \text{Fix}(S_0^h S_1^k)$. Then $\pi = \sigma^k \pi \sigma^{-h}$, so $\pi \sigma^h = \sigma^k \pi$, and thus $\pi(i + h) = \pi(i) + k$ for all $i \in \mathbb{Z}_n$. Hence, by induction,

$$\pi(i + th) = \pi(i) + tk \quad \text{for all } t \geq 1 \text{ and } i \in \mathbb{Z}_n. \tag{3}$$

Let $g = \gcd(h, n)$ and $d = n/g$, and $j = \gcd(k, n)$ and $e = n/j$. Then d and e are the smallest positive integers such that $n \mid dh$ and $n \mid ek$, respectively. If $d < e$ we have $\pi(i) = \pi(i + dh) = \pi(i) + dk \neq \pi(i)$, and if $e < d$ we have $\pi(i) \neq \pi(i + eh) = \pi(i) + ek = \pi(i)$, both of which are contradictions. Hence $d = e$ and so $g = j$, i.e., $\gcd(h, n) = \gcd(k, n)$; otherwise no such π exists.

Now if $\gcd(h, n) = \gcd(k, n) = g$ then we can write $h = ag$, $k = bg$, and $n = dg$, where $\gcd(a, d) = \gcd(b, d) = 1$. For a given g , there are $\varphi(d)$ possible values of a (and hence of h), and the same number of values of b (and hence of k).

Consider one of the $\varphi(d)^2$ pairs (h, k) corresponding to a given g and d . Now $h\mathbb{Z}_n = k\mathbb{Z}_n = g\mathbb{Z}_n$, which has $d = n/g$ elements. By Equation (3), once we determine $\pi(i)$ we determine π for every element of the coset $i + h\mathbb{Z}_n = i + g\mathbb{Z}_n$, and those values exhaust all the elements of the coset $\pi(i) + k\mathbb{Z}_n = \pi(i) + g\mathbb{Z}_n$. Thus, $\pi \in \text{Fix}(S_0^h S_1^k)$ is determined by its values $\pi(0), \pi(1), \dots, \pi(g - 1)$, and each of these values lies in a distinct coset $j + g\mathbb{Z}_n$ for $0 \leq j \leq g - 1$, each of which has d elements. We may therefore assign the cosets to $\pi(0), \pi(1), \dots, \pi(g - 1)$ in $g!$ ways, and then pick one of d values in each coset, giving $g! d^g$ choices of π .

Therefore,

$$\delta^I(n) = \frac{1}{n^2} \sum_{h \in \mathbb{Z}_n} \sum_{k \in \mathbb{Z}_n} |\text{Fix}(S_0^h S_1^k)| = \frac{1}{n^2} \sum_{(d,g): dg=n} \varphi(d)^2 g! d^g = \frac{1}{n} \sum_{(d,g): dg=n} \varphi(d)^2 (g - 1)! d^{g-1}.$$

Computing $\delta^R(n)$. Suppose $\gamma = S_0^h S_1^k R = S_0^h S_1^k R_0 R_1$ and $\pi \in \text{Fix}(\gamma)$. Then $\pi = \sigma^k \rho \pi \rho \sigma^{-h}$, from which $\pi(i) = k - \pi(-(i - h)) = k - \pi(h - i)$ and hence $\pi(i) + \pi(h - i) = k$ for all $i \in \mathbb{Z}_n$. Our analysis of this equation will depend on whether n is odd or even.

Suppose first that n is odd. Then the sets $\{i, h - i\}$ partition \mathbb{Z}_n into $(n - 1)/2$ pairs and one singleton $\{h/2\}$ (since n is odd, $h/2$ is well-defined for all $h \in \mathbb{Z}_n$). We must have $\pi(h/2) + \pi(h - h/2) = k$, i.e., $2\pi(h/2) = k$, so $\pi(h/2) = k/2$ is determined. For $i \neq h/2$, the pair $\pi(\{i, h - i\}) = \{\pi(i), \pi(h - i)\}$ is a pair $\{j, k - j\}$ with $j \neq k/2$. Since π maps $(n - 1)/2$ pairs to another $(n - 1)/2$ pairs, by Observation 1, the number of possible π is $((n - 1)/2)! 2^{(n-1)/2}$. This is constant for all h and k . Therefore, we have

$$\delta^R(n) = \left(\frac{n - 1}{2}\right)! 2^{(n-1)/2} \quad \text{for odd } n.$$

Suppose now that n is even. Then the way in which the sets $\{i, h - i\}$ partition \mathbb{Z}_n depends on whether h is odd or even. If h is odd, the partition has $n/2$ pairs. If h is even, define $h/2$ by treating $h \in [0, n - 1]$ as a real number rather than an element of \mathbb{Z}_n ; then there are two singletons $\{h/2\}$ and $\{h/2 + n/2\}$, and $n/2 - 1$ pairs. Similarly, the sets $\{j, k - j\}$ partition \mathbb{Z}_n in ways that depend on whether k is odd or even. Since each set $\{i, h - i\}$ must map to a set $\{j, k - j\}$ of the same size, either both h and k are odd, or both h and k are even. If both are odd then π maps $n/2$ pairs to another $n/2$ pairs, so there are $(n/2)! 2^{n/2}$ possible π for these $n^2/4$ choices of (h, k) . If both h and k are even, there are 2 ways to match up the singletons $\{h/2\}$ and $\{h/2 + n/2\}$ with the singletons $\{k/2\}$ and $\{k/2 + n/2\}$, and $(n/2 - 1)! 2^{n/2-1}$ ways for π to preserve the pairings of the remaining elements. So there are $(n/2 - 1)! 2^{n/2}$ possible π for these $n^2/4$ choices of (h, k) . Thus,

$$\begin{aligned} \delta^R(n) &= \frac{1}{n^2} \left(\frac{n^2}{4} \left(\frac{n}{2}\right)! 2^{n/2} + \frac{n^2}{4} \left(\frac{n}{2} - 1\right)! 2^{n/2} \right) \\ &= (n + 2) \left(\frac{n}{2} - 1\right)! 2^{n/2-3} \quad \text{for even } n. \end{aligned}$$

Since R_1 affects the output of a permutation, and R_0 affects the input, the effect of R_1 is easier to analyze than the effect of R_0 . Therefore, we will consider R_1 before R_0 , and $R_1 X$ before $R_0 X$.

Computing $\delta^{R_1}(n)$. Suppose $\gamma = S_0^h S_1^k R_1$ and $\pi \in \text{Fix}(\gamma)$. Then $\pi = \sigma^k \rho \pi \sigma^{-h}$, so $\pi(i) = k - \pi(i - h)$ and hence $\pi(i) + \pi(i - h) = k$ for all $i \in \mathbb{Z}_n$. Then $\pi(i - h) + \pi(i - 2h) = k$ for all i so $\pi(i - 2h) = \pi(i)$ for all i . This can only happen if $2h = 0$ which means $h = 0$, or n is even and $h = n/2$.

If $h = 0$, then the only way we can have $\pi(i) + \pi(i - h) = 2\pi(i) = k$ for all i is if $n = 1$ or 2 and $k = 0$. There are one such π for $n = 1$ and two such π for $n = 2$.

If n is even and $h = n/2$, then π must map each of the $n/2$ pairs $\{i, i + n/2\}$ to a pair $\{j, k - j\}$. For all sets $\{j, k - j\}$ to be pairs, k must be odd. For each of the $n/2$ odd values of k , Observation 1 tells us there are $(n/2)! 2^{n/2}$ possible π . Thus, taking into account the special cases when $n = 1$ or 2 ,

$$\delta^{R_1}(n) = \begin{cases} 1 & \text{if } n = 1 \text{ or } 2, \\ 0 & \text{if } n \geq 3 \text{ is odd,} \\ \frac{1}{n} \binom{n}{2}! 2^{n/2-1} & \text{if } n \geq 4 \text{ is even.} \end{cases}$$

Computing $\delta^{R_0}(n)$. By Theorem 1, we can see that $R_0 = XR_1X$, and that $\Sigma^d X = X\Sigma^d$. Therefore, $\Sigma^d R_0 = \Sigma^d XR_1X = X\Sigma^d R_1X = X^{-1}\Sigma^d R_1X$. Hence, by Lemma 2, we have

$$\delta^{R_0}(n) = \delta^{R_1}(n).$$

Computing $\delta^X(n)$. Suppose $\gamma = S_0^h S_1^k X \in \Sigma^d X$ and $\pi \in \text{Fix}(\gamma)$. Then $\pi = \sigma^k \pi^{-1} \sigma^{-h}$, from which $(\pi \sigma^h)^2 = \sigma^{h+k}$. Thus, the number of fixed points π is the same as the number of $\tau = \pi \sigma^h$ that satisfy $\tau^2 = \sigma^{h+k} = \sigma^\ell$, where we let $\ell = h + k$. Each value of ℓ occurs for n pairs (h, k) . For a given ℓ , there are $g = \text{gcd}(\ell, n)$ cycles of σ^ℓ , each with length $d = n/g$. Therefore, by Lemma 1, there are $q(d, g)$ possible τ and hence $q(d, g)$ possible π . Writing $\ell = cg$ and $n = dg$, we see that $\text{gcd}(c, d) = 1$, so there are $\varphi(d)$ possible values of c , and hence of ℓ , for a given g and d . Thus, applying Lemma 1,

$$\begin{aligned} \delta^X(n) &= \frac{1}{n^2} \sum_{h \in \mathbb{Z}_n} \sum_{k \in \mathbb{Z}_n} |\text{Fix}(S_0^h S_1^k X)| = \frac{1}{n^2} \sum_{(d,g): dg=n} n \varphi(d) q(d, g) \\ &= \frac{1}{n} \sum_{\substack{(d,g): dg=n \\ d, g \text{ even}}} \varphi(d) \mu(g, g/2) d^{g/2} + \frac{1}{n} \sum_{\substack{(d,g): dg=n \\ d \text{ odd}}} \varphi(d) \sum_{j=0}^{\lfloor g/2 \rfloor} \mu(g, j) d^j. \end{aligned}$$

Note that the first term here is nonzero only if $n \equiv 0 \pmod{4}$.

Computing $\delta^{R_X}(n)$. By Theorem 1, we can see that $R_X = R_0 R_1 X = R_0 X R_0$, and also that $\Sigma^d R_0 = R_0 \Sigma^d$. Therefore, $\Sigma^d R_X = \Sigma^d R_0 X R_0 = R_0 \Sigma^d X R_0 = R_0^{-1} \Sigma^d X R_0$. Hence, by Lemma 2, we have

$$\delta^{R_X}(n) = \delta^X(n).$$

Computing $\delta^{R_1 X}(n)$. Suppose $\gamma = S_0^h S_1^k R_1 X$ and $\pi \in \text{Fix}(\gamma)$. Then $\pi = \sigma^k \rho \pi^{-1} \sigma^{-h}$, from which $(\sigma^h \pi)^2 = \sigma^{h+k} \rho$. Thus, the number of fixed points π is the same as the number of $\tau = \sigma^h \pi$ that satisfy $\tau^2 = \sigma^{h+k} \rho = \sigma^\ell \rho = \alpha$, where we let $\ell = h + k$. Each value of ℓ occurs for n pairs (h, k) . Now $\alpha(i) = \sigma^\ell \rho(i) = \ell - i$, so α is always an involution.

If n is odd then for all n values of ℓ , then the number of 2-cycles of α is $(n - 1)/2$ and α has one 1-cycle, namely $(\ell/2)$. Applying Lemma 1, we obtain

$$\begin{aligned} \delta^{R_1 X}(n) &= \frac{1}{n^2} n^2 q(1, 1) q(2, (n - 1)/2) = q(2, (n - 1)/2) \\ &= \begin{cases} 0 & \text{if } (n - 1)/2 \text{ is odd, i.e., } n \equiv 3 \pmod{4}, \\ \mu((n - 1)/2, (n - 1)/4) 2^{(n-1)/4} & \text{if } (n - 1)/2 \text{ is even, i.e., } n \equiv 1 \pmod{4}. \end{cases} \end{aligned}$$

If n is even, then the number of 2-cycles of α is $n/2$ for the $n/2$ odd values of ℓ . For the $n/2$ even values of ℓ , the number of 2-cycles of α is $n/2 - 1$, and there are also two 1-cycles, namely $(\ell/2)$ and $(\ell/2 + n/2)$. Applying Lemma 1, we obtain

$$\begin{aligned} \delta^{R_1 X}(n) &= \frac{1}{n^2} \left(\frac{n^2}{2} q(2, n/2) + \frac{n^2}{2} q(1, 2) q(2, n/2 - 1) \right) \\ &= \begin{cases} \frac{1}{2} \mu(n/2, n/4) 2^{n/4} & \text{if } n/2 \text{ is even, i.e., } n \equiv 0 \pmod{4}, \\ \frac{1}{2} 2 \mu(n/2 - 1, \frac{1}{2}(n/2 - 1)) 2^{\frac{1}{2}(n/2 - 1)} & \text{if } n/2 - 1 \text{ is even, i.e., } n \equiv 2 \pmod{4}, \end{cases} \\ &= \begin{cases} \mu(n/2, n/4) 2^{n/4 - 1} & \text{if } n \equiv 0 \pmod{4}, \\ \mu((n - 2)/2, (n - 2)/4) 2^{(n - 2)/4} & \text{if } n \equiv 2 \pmod{4}, \end{cases} \end{aligned}$$

Computing $\delta^{R_0 X}(n)$. By Theorem 1, we can see that $R_0 X = X R_1$, and that $\Sigma^d X = X \Sigma^d$. Therefore, $\Sigma^d R_0 X = \Sigma^d X R_1 = X \Sigma^d R_1 = X (\Sigma^d R_1 X) X^{-1}$. Hence, by Lemma 2 we have

$$\delta^{R_0 X}(n) = \delta^{R_1 X}(n).$$

8.3. Counting Dipole Embeddings and Related Objects

In this subsection, we prove counting formulas (D1)–(D8) from Section 3, which are summarized in Table 1. We consider \mathcal{D}_n , or equivalently $\text{Sym}(\mathbb{Z}_n)$, under the action of various groups Γ with $\Sigma^d \leq \Gamma \leq \Omega^d$. Each such group can be written as a union of cosets of Σ^d , i.e., as $\Gamma = \Sigma^d T_1 \cup \Sigma^d T_2 \cup \dots \cup \Sigma^d T_k$ for some $T_i \in \langle R_0, R_1, X \rangle = \{R_0^p R_1^q X^s \mid p, q, s \in \{0, 1\}\}$. Therefore, by Burnside’s lemma, the number of equivalence classes is

$$\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} |\text{Fix}(\gamma)| = \frac{1}{kn^2} \sum_{i=1}^k \sum_{\gamma \in \Sigma^d T_i} |\text{Fix}(\gamma)| = \frac{1}{k} \sum_{i=1}^k \delta^{T_i}(n). \tag{4}$$

Counting (D1) vertex-labeled oriented dipole embeddings. Given a vertex-labeled oriented dipole embedding Φ , we can transform it into a labeled dipole by choosing a half-edge incident with vertex 0 to label 0, and then labeling the other half-edges incident with vertex 0 in ascending clockwise order, and similarly for vertex 1. However, our choices of which half-edge incident with vertex 0 to label 0, and which half-edge incident with vertex 1 to label 0, are arbitrary. So other labeled dipoles for Φ can be obtained by applying arbitrary cyclic shifts S_0^h and S_1^k .

Thus, a vertex-labeled oriented dipole embedding may be regarded as an equivalence class of labeled dipoles under the action of $\Sigma^d = \langle S_0, S_1 \rangle = \Sigma^d I$, and so the number of equivalence classes is just $\delta^I(n)$.

Counting (D2) oriented dipole embeddings. To count oriented dipole embeddings where the vertices are unlabeled, we include vertex swaps in our allowed symmetries, so the group of symmetries is $\langle S_0, S_1, X \rangle$, which from Theorem 1 we know is $\{S_0^h S_1^k X^s \mid h, k \in \mathbb{Z}_n, s \in \{0, 1\}\} = \Sigma^d \cup \Sigma^d X$. The number of equivalence classes is therefore $\frac{1}{2}(\delta^I(n) + \delta^X(n))$.

Counting (D3) vertex-labeled orientable dipole embeddings. To count orientable dipole embeddings, we must allow for the orientation of the surface to be reversed. This means that when we choose the two half-edges incident with vertices 0 and 1 to give label 0, we then label the other half-edges in the reverse order, at both vertices. This means we are applying the action of $R = R_0 R_1$. The group of symmetries in the vertex-labeled case is therefore $\langle S_0, S_1, R \rangle$, which by applying Theorem 1, can be written as $\{S_0^h S_1^k R^r \mid h, k \in \mathbb{Z}_n, r \in \{0, 1\}\} = \Sigma^d \cup \Sigma^d R$. The number of equivalence classes is therefore $\frac{1}{2}(\delta^I(n) + \delta^R(n))$.

Counting (D4) orientable dipole embeddings. To count orientable dipole embeddings where the vertices are unlabeled, we add X to the group of symmetries, giving $\langle S_0, S_1, R, X \rangle$.

By Theorem 1, this can be written as $\{S_0^h S_1^k R^r X^s \mid h, k \in \mathbb{Z}_n, r, s \in \{0, 1\}\} = \Sigma^d \cup \Sigma^d R \cup \Sigma^d X \cup \Sigma^d R X$. The number of equivalence classes is therefore $\frac{1}{4}(\delta^I(n) + \delta^R(n) + \delta^X(n) + \delta^{RX}(n))$. Since $\delta^{RX}(n) = \delta^X(n)$, this simplifies to $\frac{1}{4}(\delta^I(n) + \delta^R(n) + 2\delta^X(n))$.

Counting (D5) vertex-labeled dipolar cogs. To count cogs, we must allow the cyclic ordering at each vertex to be reversed independently. This means that we can apply the actions of both R_0 and R_1 . The group of symmetries in the vertex-labeled case is therefore $\langle S_0, S_1, R_0, R_1 \rangle$, which by Theorem 1 can be written as $\{S_0^h S_1^k R_0^p R_1^q \mid h, k \in \mathbb{Z}_n, p, q \in \{0, 1\}\} = \Sigma^d \cup \Sigma^d R_0 \cup \Sigma^d R_1 \cup \Sigma^d R_0 R_1$. The number of equivalence classes is therefore $\frac{1}{4}(\delta^I(n) + \delta^{R_0}(n) + \delta^{R_1}(n) + \delta^{R_0 R_1}(n))$. Since $\delta^{R_0}(n) = \delta^{R_1}(n)$ and $R_0 R_1 = R$, this simplifies to $\frac{1}{4}(\delta^I(n) + \delta^R(n) + 2\delta^{R_1}(n))$.

Counting (D6) dipolar cogs. Again, when the vertices are unlabeled, we add X to the group of symmetries, which is therefore $\Omega^d = \langle S_0, S_1, R_0, R_1, X \rangle$. By Theorem 1, this can be written as $\{S_0^h S_1^k R_0^p R_1^q X^s \mid h, k \in \mathbb{Z}_n, p, q, s \in \{0, 1\}\} = \Sigma^d \cup \Sigma^d R_0 \cup \Sigma^d R_1 \cup \Sigma^d R_0 R_1 \cup \Sigma^d X \cup \Sigma^d R_0 X \cup \Sigma^d R_1 X \cup \Sigma^d R_0 R_1 X$. The number of equivalence classes is therefore

$$\frac{1}{8}(\delta^I(n) + \delta^{R_0}(n) + \delta^{R_1}(n) + \delta^{R_0 R_1}(n) + \delta^X(n) + \delta^{R_0 X}(n) + \delta^{R_1 X}(n) + \delta^{R_0 R_1 X}(n)).$$

Since $\delta^{R_0}(n) = \delta^{R_1}(n)$, $\delta^{R_0 X}(n) = \delta^{R_1 X}(n)$, $R_0 R_1 = R$ and $\delta^{R_0 R_1 X}(n) = \delta^{RX}(n) = \delta^X(n)$, this simplifies to $\frac{1}{8}(\delta^I(n) + \delta^R(n) + 2\delta^{R_1}(n) + 2\delta^X(n) + 2\delta^{R_1 X}(n))$.

Counting (D7) equivalence classes of permutations under cyclic shifts and reversal of input variables only (or output variables only). There are two subgroups of Ω^d that do not have very natural interpretations in terms of dipole embeddings, but can be considered as groups of symmetries of $\text{Sym}(\mathbb{Z}_n)$. These are where we allow cyclic shifts of both input and output variables, and the reversal of just one set of variables (input or output, but not both). These give the groups $\Gamma_7 = \langle S_0, S_1, R_1 \rangle$ and $\Gamma'_7 = \langle S_0, S_1, R_0 \rangle$. Since $\Gamma'_7 = X^{-1} \Gamma_7 X$, the number of equivalence classes for both groups will be the same, by applying Lemma 2. So we just consider $\Gamma_7 = \langle S_0, S_1, R_1 \rangle$, which by Theorem 1 can be written as $\{S_0^h S_1^k R_1^q \mid h, k \in \mathbb{Z}_n, q \in \{0, 1\}\} = \Sigma^d \cup \Sigma^d R_1$. The number of equivalence classes is therefore $\frac{1}{2}(\delta^I(n) + \delta^{R_1}(n))$.

Counting (D8) equivalence classes of permutation matrices under cyclic shifts of the row set, cyclic shifts of the column set, and rotations by multiples of 90° . For permutation matrices, a cyclic shift of the row set corresponds to S_0 , a cyclic shift of the column set corresponds to S_1 , and a rotation by 90° clockwise corresponds to $V_1 X = S_1^{n-1} R_1 X$ (a transposition, X , followed by reversal of the column set, $V_1 = S_1^{n-1} R_1$). Thus, the group of symmetries is $\langle S_0, S_1, R_1 X \rangle$. By Theorem 1 we see that $(R_1 X)^2 = R$, $(R_1 X)^3 = R_0 X$, and $(R_1 X)^4 = I$. So the group can be written as $\Sigma^d \cup \Sigma^d R \cup \Sigma^d R_0 X \cup \Sigma^d R_1 X$. The number of equivalence classes is therefore $\frac{1}{4}(\delta^I(n) + \delta^R(n) + \delta^{R_0 X}(n) + \delta^{R_1 X}(n))$. Since $\delta^{R_0 X}(n) = \delta^{R_1 X}(n)$, this simplifies to $\frac{1}{4}(\delta^I(n) + \delta^R(n) + 2\delta^{R_1 X}(n))$.

There are 10 groups Γ that satisfy $\Sigma^d \leq \Gamma \leq \Omega^d$. These correspond to the subgroups of the quotient group Ω^d / Σ^d , which is an 8-element dihedral group. Thinking of Ω^d / Σ^d as symmetries of a square gives a natural correspondence with operations on permutation matrices.

We counted the equivalence classes of \mathcal{D}_n (or \mathcal{P}_n or $\text{Sym}(\mathbb{Z}_n)$) under the action of Γ for eight of these groups in (D1)–(D8). Actually, as we noted above, (D7) handles two of these groups that are conjugate in Ω^d . Item (D2) also handles two of these groups, because the groups $\Gamma_2 = \langle S_0, S_1, X \rangle$ and $\Gamma'_2 = \langle S_0, S_1, R X \rangle$ are conjugate in Ω^d , with $\Gamma'_2 = R_0^{-1} \Gamma_2 R_0$. Thus, we covered all 10 groups. Since $X R_0 X = R_1$, the groups $\langle S_0, S_1, R_0, X \rangle$ and $\langle S_0, S_1, R_1, X \rangle$ are just the full group $\Omega^d = \langle S_0, S_1, R_0, R_1, X \rangle$, and therefore we do not need to consider separate counting questions involving these groups.

Observation 2. The above results tell us that all of $\delta^I(n)$, $\delta^R(n)$, $\delta^{R_1}(n)$, $\delta^X(n)$, and $\delta^{R_1X}(n)$ are integers, as follows. By (D1), we know that $\delta^I(n)$ is the number of equivalence classes under Σ^d , so it is an integer. By (D2), (D3), and (D7), we also know that $\delta^I(n) + \delta^X(n)$, $\delta^I(n) + \delta^R(n)$, and $\delta^I(n) + \delta^{R_1}(n)$ are even integers. Therefore, $\delta^X(n)$, $\delta^R(n)$, and $\delta^{R_1}(n)$ are integers. Finally, we know by (D6) that $\delta^I(n) + \delta^R(n) + 2\delta^{R_1}(n) + 2\delta^X(n) + 2\delta^{R_1}(n) + 2\delta^{R_1X}(n)$ is divisible by 8, so it is even, and all of $\delta^I(n) + \delta^R(n)$, $2\delta^{R_1}(n)$, and $2\delta^X(n)$ are even. Thus, $2\delta^{R_1X}(n)$ is even, and $\delta^{R_1X}(n)$ is an integer.

9. Proofs for Bouquet Formulas

9.1. Colored Labeled Bouquets and Symmetry Operations

In this section, we prove counting formulas (B1)–(B4) from Section 4 regarding the embeddings of colored bouquets. Recall that a bouquet B_n has one vertex and n loops. Again, we think of each edge as consisting of two half-edges. By ‘colored’, we mean that each edge receives an arbitrary color from a set of k colors. By using results in the case $k = 2$, we are able to count nonorientable embeddings of bouquets, where previous counting results for embeddings of bouquets only considered orientable embeddings. Our results will be proved by elementary techniques based on groups acting on a set of objects that we will call colored labeled bouquets.

A *colored labeled bouquet* is a bouquet B where the half-edges receive distinct labels from $\mathbb{Z}_{2n} = \{0, 1, 2, \dots, 2n - 1\}$, where $n = |E(B)|$, and where each edge receives a color from $\mathbb{Z}_k = \{0, 1, 2, \dots, k - 1\}$. We let $\mathcal{B}_{n,k}$ denote the set of n -edge k -colored bouquets.

A colored labeled bouquet B is completely described by a perfect matching M in the complete graph $K(\mathbb{Z}_{2n})$ with vertex set \mathbb{Z}_{2n} , plus a coloring function $\psi : M \rightarrow \mathbb{Z}_k$. For each $e \in E(B)$, the perfect matching M contains an edge $\{a_e, b_e\}$ (describing an edge as an unordered pair of vertices), where a_e and b_e are the labels of the two half-edges of e , and ψ assigns the color of $e \in E(B)$ to $\{a_e, b_e\} \in M$. There is a one-to-one correspondence between $\mathcal{B}_{n,k}$ and $\mathcal{M}_{n,k}$, the set of k -colored perfect matchings (M, ψ) in $K(\mathbb{Z}_{2n})$. Perfect matchings on a cyclically ordered set, such as \mathbb{Z}_{2n} , are often represented by *chord diagrams*, so our results may also be interpreted as results for colored chord diagrams. See Figure 1.

Each oriented or orientable embedding of a colored bouquet can be turned into a colored labeled bouquet (or k -colored perfect matching) in a natural way. We refer the reader back to Figure 5 for an example. However, the colored labeled bouquet is not, in general, unique, and we wish to characterize the possible bouquets by equivalence under certain symmetry operations. The symmetries always include cyclically shifting the labels of the half-edges, but may also include reversing the labels.

Our symmetries can be defined in terms of their effect on elements of \mathbb{Z}_{2n} , but to avoid treating $n = 1$ and 2 as special cases, and to simplify some proofs, we define them more generally as permutations of $\mathbb{R}_{2n} = \mathbb{R}/2n\mathbb{Z}$, which contains \mathbb{Z}_{2n} as a subgroup. Let I denote the identity of $\text{Sym}(\mathbb{R}_{2n})$, and define $S, R \in \text{Sym}(\mathbb{R}_{2n})$ as follows:

$S(a) = a + 1$; and $R(a) = -a$.
 Let $\Sigma^b = \langle S \rangle$ and $\Omega^b = \langle S, R \rangle$ (the superscript ‘b’ indicating bouquets), so that $\Sigma^b \leq \Omega^b \leq \text{Sym}(\mathbb{R}_{2n})$.

If $T = S$ or R then it is clear that applying T , or T^{-1} , to all the labels in a colored labeled bouquet, without changing any colors, produces a new colored labeled bouquet, so that T and T^{-1} permute $\mathcal{B}_{n,k}$. Correspondingly, we may apply T , or T^{-1} , to the vertices of $K(\mathbb{Z}_{2n})$, and define $T(M) = \{\{T(a), T(b)\} \mid \{a, b\} \in M\}$, with the edge $T(\{a, b\}) = \{T(a), T(b)\}$ receiving the color assigned to $\{a, b\}$. In this way T and T^{-1} permute $\mathcal{M}_{n,k}$. We can then extend this to all $T \in \Omega^b$, so we have an action of Ω^b on $\mathcal{B}_{n,k}$ and an equivalent action on $\mathcal{M}_{n,k}$.

The basic properties of S, R and Ω^b are as follows.

Proposition 2. *Let n be a positive integer, and consider $S, R \in \text{Sym}(\mathbb{R}_{2n})$ as defined above. Let I be the identity of $\text{Sym}(\mathbb{R}_{2n})$.*

- (a) *Then (composing functions right to left) $S^{2n} = R^2 = I$ and $RS = S^{-1}R$.*

- (b) Every element of $\Omega^b = \langle S, R \rangle$ can be written uniquely as $S^h R^r$ where $h \in \mathbb{Z}_{2n}$ and $r \in \{0, 1\}$. Hence, $|\Omega^b| = 4n$.

We omit the proof, which is straightforward.

9.2. Bouquet Coset Averages

We are going to consider the action of $\Omega^b = \langle S, R \rangle \leq \text{Sym}(\mathbb{R}_{2n})$ (a dihedral group) and its subgroup $\Sigma^b = \langle S \rangle$ (a cyclic group) on $\mathcal{B}_{n,k}$ and $\mathcal{M}_{n,k}$. When $n = 1$ or 2 the action is not faithful (for example, if $n = 2$ then S^2 acts in the same way as I) but this does not matter.

For determining fixed points for Burnside's lemma, we use the action of Ω^b or Σ^b on $\mathcal{M}_{n,k}$. As in Section 8, we will consider average numbers of fixed points for the right cosets of Σ^b , which are just Σ^b itself and $\Sigma^b R$. Each coset has size $|\Sigma^b| = 2n$, so for $T = I$ or R we define

$$\beta^T(n, k) = \frac{1}{2n} \sum_{\gamma \in \Sigma^b T} |\text{Fix}(\gamma)| = \frac{1}{2n} \sum_{h \in \mathbb{Z}_{2n}} |\text{Fix}(S^h T)|$$

where we are considering the action on $\mathcal{M}_{n,k}$.

Before computing the coset averages, we consider what a fixed point $(M, \psi) \in \mathcal{M}_{n,k}$ of $T \in \Omega^b$ must look like in general. Suppose $\{a, b\} \in M$. Then we also have $\{T(a), T(b)\} \in M$ and hence we have $\{T^t(a), T^t(b)\} \in M$ for all integers t . For $t \geq 0$, this follows by induction. For $t < 0$ it follows because something is a fixed point of T if and only if it is a fixed point of T^{-1} . The edges $\{T^t(a), T^t(b)\} \in M$ for all integers t form an edge-orbit of $K(\mathbb{Z}_{2n})$ under the action of $\langle T \rangle$. Since ψ is also fixed by T , all edges in an edge-orbit must have the same color.

The points $T^t(a)$ form a vertex-orbit of the action of $\langle T \rangle$ on $K(\mathbb{Z}_{2n})$, as do the points $T^t(b)$. If the two vertex-orbits are the same, we must have an orbit of even size, because M matches up the vertices in the vertex-orbit. If the two vertex-orbits are different, they must have the same size, because M matches the vertices of one vertex-orbit to the vertices of the other.

Thus, if $(M, \psi) \in \mathcal{M}_{n,k}$ is a fixed point of T , then the edges of M can be partitioned into edge-orbits under the action of $\langle T \rangle$. Each edge-orbit either matches the vertices of a single vertex-orbit O of even size, or matches the vertices of one vertex-orbit O_1 to the vertices of a paired vertex-orbit O_2 of the same size, say s . In the latter case, if we fix $a \in O_1$ then there are s possible vertices $b \in O_2$ for which we could have $\{a, b\} \in M$, and once we choose this edge the rest of the edge-orbit is determined. So (assuming no special restrictions apply), there are s possible edge-orbits matching two paired vertex-orbits. All edges of M in the same edge-orbit receive the same color under ψ .

Computing $\beta^I(n, k)$. To compute $\beta^I(n, k)$ we consider fixed points of elements of Σ^b , which have the form S^h . Suppose there is $(M, \psi) \in \text{Fix}(S^h)$. If we let $g = \gcd(h, 2n)$ then $h\mathbb{Z}_{2n} = g\mathbb{Z}_{2n}$. The number of elements of $g\mathbb{Z}_{2n}$ is $d = 2n/g$. For a given g there are $\varphi(d)$ possible values of h .

The vertex-orbits of $K(\mathbb{Z}_{2n})$ under the action of $\langle S^h \rangle$ are just the cosets $a + h\mathbb{Z}_{2n} = a + g\mathbb{Z}_{2n}$. So each edge-orbit must match up vertices in the same coset, or must match one coset to another.

Suppose an edge-orbit matches vertices in the same coset $a + g\mathbb{Z}_{2n}$. Then we have an edge $\{a, a + ug\}$ for some integer u , where $ug \neq 0$. We also have an edge $\{a + ug, a + 2ug\}$, so we must have $a = a + 2ug$, so that $2ug = 0$. Since $ug \neq 0$, we must have $ug = n$. Since $n \in g\mathbb{Z}_{2n}$, the number of elements in $g\mathbb{Z}_{2n}$, which is d , must be even.

So suppose d is even such that $n \in g\mathbb{Z}_{2n}$. We can construct fixed points of S^h as follows. Choose j with $0 \leq j \leq \lfloor d/2 \rfloor$, and pair up j pairs of the g cosets, which may be done in $\mu(g, j)$ ways. If $a + g\mathbb{Z}_{2n}$ is paired with $b + g\mathbb{Z}_{2n}$, which both have size d , then there are d possible choices of edge-orbit, so there are d^j possible edge-orbits for all the paired cosets. If $a + g\mathbb{Z}_{2n}$ is not one of the paired cosets, then from the above, each $a' \in a + g\mathbb{Z}_{2n}$ must be joined by M to $a' + n$, so there is only one possibility for the edges in the unpaired cosets.

There are j edge-orbits between paired cosets, and $g - 2j$ edge-orbits inside unpaired cosets. So there are $g - j$ edge-orbits and hence k^{g-j} choices of ψ .

The total number of fixed points for a particular value of h with an even value of d is therefore

$$\sum_{j=0}^{\lfloor g/2 \rfloor} \mu(g, j) d^j k^{g-j}.$$

Now suppose d is odd, which means that $g = 2n/d$ is even. From the above, we cannot have edges inside a single coset $a + g\mathbb{Z}_{2n}$, so all g cosets are paired up. This means we just have the case $j = g/2$ from above, so the number of fixed points for a particular value of h with an odd value of d is $\mu(g, g/2) d^{g/2} k^{g/2}$.

Putting these together,

$$\begin{aligned} \beta^I(n, k) &= \frac{1}{2n} \sum_{h \in \mathbb{Z}_{2n}} |\text{Fix}(S^h)| \\ &= \frac{1}{2n} \sum_{\substack{(d,g): dg=2n \\ d \text{ odd}}} \varphi(d) \mu(g, g/2) d^{g/2} k^{g/2} + \frac{1}{2n} \sum_{\substack{(d,g): dg=2n \\ d \text{ even}}} \varphi(d) \sum_{j=0}^{\lfloor g/2 \rfloor} \mu(g, j) d^j k^{g-j}. \end{aligned}$$

Computing $\beta^R(n, k)$. Suppose $\gamma = S^h R \in \Sigma^b R$, and $(M, \psi) \in \text{Fix}(\gamma)$. For $i \in \mathbb{Z}_{2n}$, $S^h R(i) = h - i$, so the vertex-orbits of $\langle S^h R \rangle$ are sets $\{i, h - i\}$. Such a vertex-orbit may have size 1 if $i = h - i$, which happens when h is even and $i = h/2$ or $h/2 + n$. Otherwise the vertex-orbits have size 2.

Suppose h is odd. Then there are n vertex-orbits, all of size 2. Thus, all edge-orbits in M either match the two vertices in a single vertex-orbit, or match one vertex-orbit to a paired vertex-orbit. To construct a fixed point of $S^h R$ we choose j with $0 \leq j \leq \lfloor n/2 \rfloor$, and pair up j pairs of vertex-orbits of size 2, which may be done in $\mu(n, j)$ ways. There are 2 choices of edge-orbit for each pair of vertex-orbits, for 2^j total choices. The unpaired vertex-orbits have one edge of M joining their two vertices, which can only be done in one way.

There are j edge-orbits between paired cosets, and $n - 2j$ edge-orbits inside unpaired cosets, so there are $n - j$ edge-orbits and hence k^{n-j} choices of ψ .

The total number of fixed points for one of the n odd values of h is therefore

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \mu(n, j) 2^j k^{n-j}.$$

Suppose now that h is even. Then there are two vertex-orbits $\{h/2\}$ and $\{h/2 + n\}$ of size 1, and $(n - 1)$ vertex-orbits of size 2. There must be an edge of M joining the two vertex-orbits of size 1, and there are k choices for the color of this edge. Then we can apply the same analysis as above, replacing n by $n - 1$, to the vertex-orbits of size 2. So the total number of fixed points for one of the n even values of h is

$$k \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \mu(n - 1, j) 2^j k^{n-1-j} = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \mu(n - 1, j) 2^j k^{n-j}.$$

Putting everything together, we have

$$\begin{aligned} \beta^R(n, k) &= \frac{1}{2n} \left(n \sum_{j=0}^{\lfloor n/2 \rfloor} \mu(n, j) 2^j k^{n-j} + n \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \mu(n - 1, j) 2^j k^{n-j} \right) \\ &= \frac{1}{2} \left(\sum_{j=0}^{\lfloor n/2 \rfloor} \mu(n, j) 2^j k^{n-j} + \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \mu(n - 1, j) 2^j k^{n-j} \right). \end{aligned}$$

9.3. Counting Colored Bouquet Embeddings and Related Objects

In this subsection, we verify the results summarized in Table 2. We consider $\mathcal{B}_{n,k}$, or equivalently $\mathcal{M}_{n,k}$, under the action of either $\Gamma = \Sigma^b$ or $\Gamma = \Omega^b = \Sigma^b \cup \Sigma^b R$. In a similar way to Equation (4), we can just take the average over all cosets of Σ^b in Γ of the average number of fixed points in a coset.

Counting (B1) oriented embeddings of colored bouquets. Given an oriented embedding Φ of a colored bouquet, we can transform it into a colored labeled bouquet by choosing a half-edge to label 0, and then labeling the other half-edge in ascending clockwise order. This is illustrated by Figure 5. However, our choice of which half-edge to give label 0 is arbitrary. So other colored labeled bouquets for Φ can be obtained by applying an arbitrary cyclic shift S^h .

Thus, an oriented embedding of a colored bouquet may be regarded as an equivalence class of colored labeled bouquets under the action of $\Sigma^b = \langle S \rangle$, and so the number of equivalence classes is just $\beta^I(n, k)$.

This can also be regarded as the number of colored chord diagrams equivalent under cyclic shifts.

Counting (B2) orientable embeddings of colored bouquets. To count orientable embeddings of colored bouquets, we must allow for the orientation of the surface being reversed. This means that when we choose the half-edge to label 0, we can then label the other half-edges in one of two cyclic orders. These two labelings are related by the transformation R . So the group of symmetries we must consider is $\Omega^b = \langle S, R \rangle = \Sigma^b \cup \Sigma^b R$. The number of equivalence classes is therefore $\frac{1}{2}(\beta^I(n, k) + \beta^R(n, k))$.

Counting (B3) generic (orientable or nonorientable) embeddings of colored bouquets. Generic embeddings are described by a rotation scheme together with edge signatures, which describe whether an edge should be considered twisted or not. For graphs in general this representation is not unique. However, when a graph has only one vertex the representation is unique, and so a generic embedding can be regarded as a rotation scheme (which may be reversed without changing the embedding) together with edge signatures, which are just a 2-coloring of the edges. So generic embeddings of bouquets are in one-to-one correspondence with orientable embeddings of 2-colored bouquets. More generally, generic embeddings of k -colored bouquets are in one-to-one correspondence with orientable embeddings of $2k$ -colored bouquets. Therefore, the number of generic embeddings of k -colored bouquets is $\frac{1}{2}(\beta^I(n, 2k) + \beta^R(n, 2k))$.

Counting (B4) nonorientable embeddings of colored bouquets. The number of nonorientable embeddings is just the number of generic embeddings minus the number of orientable embeddings. So the number of nonorientable embeddings of k -colored bouquets is $(B3) - (B2) = \frac{1}{2}(\beta^I(n, 2k) + \beta^R(n, 2k) - \beta^I(n, k) - \beta^R(n, k))$.

Observation 3. From (B1) and (B2), we see that $\beta^I(n, k)$ and $\beta^R(n, k)$ are both integers.

10. Proofs for Directed Bouquets

10.1. Colored Signed Labeled Bouquets and Symmetry Operations

In this section, we prove counting Formulas (A1)–(A9) from Section 5 for directed embeddings of colored directed bouquets and related objects. Recall that a directed bouquet \vec{B}_n is a digraph with one vertex and n directed loops. We think of each arc (directed edge) as consisting of an outward half-arc and an inward half-arc. A directed embedding of a directed bouquet requires that the directions on the half-arcs alternate when going around the cyclic order at the vertex. By ‘colored’, we mean that each arc receives an arbitrary color from a set of k colors.

We also consider equivalence classes of digraphs under the operation of reversing the direction of all of the arcs. We call such an equivalence class an *arc-reversal class*. If we have

a directed embedding of a digraph, reversing all the arcs preserves the fact that we have a directed embedding, so we can also consider arc-reversal classes of directed embeddings.

As with embeddings of bouquets, by using the results in the case $k = 2$, we are able to count nonorientable directed embeddings of directed bouquets. Our results will be proved by elementary techniques based on groups acting on a set of objects that we will call colored signed labeled bouquets.

A *colored signed labeled bouquet* is a bouquet B where each of the half-edges receives the following: a distinct label from $\mathbb{Z}_{2n} = \{0, 1, 2, \dots, 2n - 1\}$, where $n = |E(B)|$; either a $+$ or a $-$ sign so that all even half-edges have one sign and all odd half-edges have the opposite sign so that every edge has both an even half-edge and an odd half-edge; and a color from $\mathbb{Z}_k = \{0, 1, 2, \dots, k - 1\}$. The signs indicate how to convert B into a directed bouquet, namely by directing each edge from its positive half-edge to its negative half-edge. Requiring the even half-edges to have one sign and the odd half-edges to have the other assures the alternation of signs required for a directed embedding of the bouquet. We let $\mathcal{A}_{n,k}$ denote the set of n -edge k -colored signed labeled bouquets.

A colored signed labeled bouquet is completely described by a triple (N, ψ, ε) . Here N is a perfect matching in the complete bipartite graph $K(2\mathbb{Z}_{2n}, 1 + 2\mathbb{Z}_{2n})$, whose vertices are partitioned into the set $2\mathbb{Z}_{2n}$ of even numbers and the set $1 + 2\mathbb{Z}_{2n}$ of odd numbers. The function $\psi : N \rightarrow \mathbb{Z}_k$ assigns one of k colors to each edge in N . Finally, $\varepsilon \in \{+, -\}$ is the sign assigned to all the even vertices (elements of $2\mathbb{Z}_{2n}$), while $-\varepsilon$ is the sign assigned to all the odd vertices (elements of $1 + 2\mathbb{Z}_{2n}$). The vertices of the graph correspond to the half-edges in the bouquet as described in detail in Section 9.1, and the edges of N correspond to the edges of the bouquet. We let $\mathcal{N}_{n,k}$ denote the set of triples (N, ψ, ε) .

We will use the same symmetries as in Section 9.1, namely $S, R \in \text{Sym}(\mathbb{R}_{2n})$ where $\mathbb{R}_{2n} = \mathbb{R}/2n\mathbb{Z}$. Thus, if $T \in \langle S, R \rangle$ then we have an action of T on $V(K(2\mathbb{Z}_{2n}, 1 + 2\mathbb{Z}_{2n})) = \mathbb{Z}_{2n} \subseteq \mathbb{R}_{2n}$, which we use to define $T(N, \psi, \varepsilon) = (T(N), \psi', \varepsilon')$ where $T(N) = \{\{T(a), T(b)\} \mid \{a, b\} \in N\}$, with $\psi'(\{T(a), T(b)\}) = \psi(\{a, b\})$, and ε' is ε if T preserves the parity of elements of \mathbb{Z}_{2n} and $-\varepsilon$ otherwise. The reader may think of T as relabeling the vertices without changing edge colors or vertex signs.

However, we have a third basic symmetry F , which flips the sign of all vertices. Clearly $F^2 = I$ and F commutes with elements of $\langle S, R \rangle$, so our overall group of symmetries is $\Omega^a = \langle S, R \rangle \times \langle F \rangle$. We regard F as acting as the identity on both vertices and edges of $K(2\mathbb{Z}_{2n}, 1 + 2\mathbb{Z}_{2n})$.

10.2. Directed Bouquet Coset Averages

We are going to consider the action of $\Omega^a = \langle S, R \rangle \times \langle F \rangle$ and some of its subgroups on $\mathcal{A}_{n,k}$ and $\mathcal{N}_{n,k}$. For small n the action is not faithful but this does not matter. All the groups Γ that we will consider contain $\Sigma^a = \langle S \rangle$ as a subgroup, so that $\Sigma^a \leq \Gamma \leq \Omega^a$, and we can write Γ as a union of cosets of Σ^a in Ω^a . There are four such cosets $\Sigma^a, \Sigma^a R, \Sigma^a F, \Sigma^a RF$, and for each $T \in \{I, R, F, RF\}$ we will compute the coset average

$$\alpha^T(n, k) = \frac{1}{2n} \sum_{\gamma \in \Sigma^a T} |\text{Fix}(\gamma)| = \frac{1}{2n} \sum_{h \in \mathbb{Z}_{2n}} |\text{Fix}(S^h T)|$$

for the action on $\mathcal{N}_{n,k}$.

The analysis of fixed points here will be similar to that in Section 9.2, using vertex-orbits and edge-orbits of the action of elements of $\langle S, R \rangle$. However, since edges now must join vertices of opposite sign (i.e., opposite parity) we have additional restrictions on the edge-orbits.

Computing $\alpha^I(n, k)$. To compute $\alpha^I(n, k)$ we consider fixed points of elements of Σ^a , which have the form S^h . If h is odd, S^h will move vertices to vertices of opposite sign, and hence will not preserve a triple (N, ψ, ε) . Therefore, we can only have fixed points if h is even. If h is even, we know the signs are preserved, and there are 2 choices of ε , so we just need to determine when (N, ψ) is fixed.

Therefore, we may suppose that $h = 2j$ is even. The vertex-orbits of \mathbb{Z}_{2n} under the action of $\langle S^h \rangle$ are just the cosets $a + 2j\mathbb{Z}_{2n} = a + 2g\mathbb{Z}_{2n}$, where $g = \gcd(j, n)$. Each coset has size $d = 2n/2g = n/g$. There are $\phi(d)$ possible values of j for each given g and d with $gd = n$. Since $2g$ is even, all elements of each coset have the same parity and hence the same sign, and there are g even cosets and g odd cosets. We can therefore construct a fixed point of S^h by matching even cosets to odd cosets in $g!$ ways, choosing an edge-orbit for each pair of cosets in d^g ways, and choosing a coloring of the edge-orbits in k^g ways. Thus, the number of fixed (N, ψ) for S^h is $g!d^gk^g$.

Adding over all possible even h , and remembering that ε can be chosen in 2 ways, gives

$$\alpha^I(n, k) = \frac{1}{2n} 2 \sum_{(d,g):dg=n} \phi(d) g! d^g k^g = \frac{1}{n} \sum_{(d,g):dg=n} \phi(d) g! d^g k^g.$$

Computing $\alpha^R(n, k)$. Suppose $\gamma = S^hR \in \Sigma^aR$ and $(N, \psi, \varepsilon) \in \text{Fix}(\gamma)$. Since R preserves the parity, hence the sign, of elements of \mathbb{Z}_{2n} , again h must be even to have a fixed point. If h is even, we know S^hR preserves signs, and there are 2 choices of ε , so we just need to determine when (N, ψ) is fixed.

Therefore, we may suppose that $h = 2j$ is even. For $i \in \mathbb{Z}_{2n}$, we have $S^hR(i) = h - i$, and the vertex-orbits of S^hR are $\{j\}$ and $\{j + n\}$ of size 1, and $n - 1$ vertex-orbits of size 2. The two vertex-orbits of size 1 must be matched by N to each other. This is only possible if one is even and the other odd, which means n must be odd; if n is even, we have no fixed points.

Thus, we may assume n is odd so that there is one even and one odd vertex-orbit of size 1, and $(n - 1)/2$ even and $(n - 1)/2$ odd vertex-orbits of size 2. We can therefore construct a fixed point of S^hR by matching the two vertex-orbits of size 1 in a unique way, matching even vertex-orbits of size 2 to odd vertex-orbits of size 2 in $((n - 1)/2)!$ ways, choosing edge-orbits for these pairs in $2^{(n-1)/2}$ ways, and then coloring the $1 + (n - 1)/2 = (n + 1)/2$ edge-orbits in $k^{(n+1)/2}$ ways.

Adding over all n possible even values of h , and remembering that ε can be chosen in 2 ways, gives

$$\alpha^R(n, k) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{1}{2n} 2n \left(\frac{n-1}{2}\right)! 2^{(n-1)/2} k^{(n+1)/2} = \left(\frac{n-1}{2}\right)! 2^{(n-1)/2} k^{(n+1)/2} & \text{if } n \text{ is odd.} \end{cases}$$

Computing $\alpha^F(n, k)$. Suppose $\gamma = S^hF \in \Sigma^aF$ and $(N, \psi, \varepsilon) \in \text{Fix}(\gamma)$. Now γ flips signs, so we have the opposite situation to when $\gamma = S^h$: we only obtain fixed points when h is odd. When h is odd, we know S^hF preserves signs, and there are 2 choices of ε , so we just need to determine when (N, ψ) is fixed.

Therefore, we may suppose that h is odd. The vertex-orbits of \mathbb{Z}_{2n} under the action of $\langle S^hF \rangle$ are just the cosets $a + h\mathbb{Z}_{2n} = a + g\mathbb{Z}_{2n}$, where $g = \gcd(h, 2n)$ is odd, so $g = \gcd(h, n) \mid n$. Each coset has size $2d$ where $d = n/g$, and consists of d even and d odd elements. For a given g and d , there are $\phi(2d)$ possible values of h .

An edge-orbit of N can match two cosets, or match a coset to itself. Since there are g cosets and g is odd, at least one coset must be matched to itself. However, if a coset is matched to itself, then each element a' must be matched by N to $a' + n$ (as in the analysis of $\alpha^I(n, k)$ in Section 9.2). Since a' and $a' + n$ must have different signs, n must be odd, and hence when n is even, there are no fixed points. If a coset $a + g\mathbb{Z}_{2n}$ is matched to a different coset $b + g\mathbb{Z}_{2n}$, then there are d choices for the edge $\{a, b + ug\} \in N$, which determines the other edges in this edge-orbit.

Therefore, when n is odd the (N, ψ) fixed by γ can be constructed as follows. Choose j with $0 \leq j \leq \lfloor g/2 \rfloor$, match j pairs of cosets in $\mu(g, j)$ ways, choose the edge-orbits for these cosets in d^j ways, then the other $g - 2j$ edge-orbits which match cosets to themselves are

uniquely determined, and the $j + (g - 2j) = g - j$ edge-orbits can be colored in k^{g-j} ways. Thus, the number of fixed (N, ψ) for $S^h F$ is

$$\sum_{j=1}^{\lfloor g/2 \rfloor} \mu(g, j) d^j k^{g-j}.$$

Adding over all possible odd h , and remembering that ε can be chosen in 2 ways, gives

$$\alpha^F(n, k) = \frac{1}{2n} 2 \sum_{\substack{(d,g): dg=n \\ g \text{ odd}}} \varphi(2d) \sum_{j=0}^{\lfloor g/2 \rfloor} \mu(g, j) d^j k^{g-j} \quad \text{if } n \text{ is odd.}$$

Simplifying gives the general expression

$$\alpha^F(n, k) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{1}{n} \sum_{\substack{(d,g): dg=n \\ g \text{ odd}}} \varphi(2d) \sum_{j=0}^{\lfloor g/2 \rfloor} \mu(g, j) d^j k^{g-j} & \text{if } n \text{ is odd.} \end{cases}$$

Computing $\alpha^{RF}(n, k)$. Suppose $\gamma = S^h RF \in \Sigma^a RF$ and $(N, \psi, \varepsilon) \in \text{Fix}(\gamma)$. Since γ flips signs and R preserves signs, we only obtain fixed points when h is odd. When h is odd, we know $S^h RF$ preserves signs, and there are 2 choices of ε , so we just need to determine when (N, ψ) is fixed.

Therefore, we may suppose that h is odd. The vertex-orbits of \mathbb{Z}_{2n} under the action of $S^h RF$ are just the n pairs $\{i, h - i\}$, each of which contains one even element and one odd element. An edge-orbit of N can match two vertex-orbits, but also match a vertex-orbit to itself.

So (N, ψ) fixed by γ can be constructed as follows. Choose j with $0 \leq j \leq \lfloor n/2 \rfloor$, match j pairs of vertex-orbits in $\mu(g, j)$ ways, and match the $n - 2j$ remaining vertex-orbits to themselves. We have only one choice for the edge-orbit for a pair of matched vertex-orbits or for the edge-orbit matching a vertex-orbit to itself. We can color the $j + (n - 2j) = n - j$ edge-orbits in k^{n-j} ways. Thus, the number of (N, ψ) fixed by $S^h RF$ is

$$\sum_{j=1}^{\lfloor n/2 \rfloor} \mu(n, j) k^{n-j}.$$

Adding over the n possible odd values of h , and remembering that ε can be chosen in 2 ways, gives

$$\alpha^{RF}(n, k) = \frac{1}{2n} 2n \sum_{j=1}^{\lfloor n/2 \rfloor} \mu(n, j) k^{n-j} = \sum_{j=1}^{\lfloor n/2 \rfloor} \mu(n, j) k^{n-j}.$$

10.3. Counting Directed Embeddings of Colored Bouquets and Related Objects

In this subsection, we verify the results summarized in Table 3. We consider $\mathcal{A}_{n,k}$, or equivalently $\mathcal{N}_{n,k}$, under the action of a group Γ with $\Sigma^a \leq \Gamma \leq \Omega^a$. In a similar way to the previous sections, we can just take the average over all cosets of Σ^a in Γ of the average number of fixed points for a coset.

Counting (A1) oriented colored directed embeddings of directed bouquets. Given an oriented directed embedding Φ of a colored directed bouquet, we can transform it into a colored signed labeled bouquet by choosing a half-arc to label 0, labeling the other half-arcs in ascending clockwise order, and then transforming the arc directions into positive signs

on outward half-arcs and negative signs on inward half-arcs. However, our choice of which half-arc to label 0 is arbitrary. So other colored signed labeled bouquets for Φ can be obtained by applying an arbitrary cyclic shift S^h .

Thus, an oriented colored directed embeddings of a directed bouquet may be regarded as an equivalence class of colored signed labeled bouquets under the action of $\Sigma^a = \langle S \rangle$, and so the number of equivalence classes is just $\alpha^I(n, k)$.

Counting (A2) orientable colored directed embeddings of directed bouquets. To count orientable directed embeddings, we must allow for the orientation of the surface being reversed. As with (B2) earlier, we have to add the transformation R to our group of symmetries, giving $\langle S, R \rangle = \Sigma^a \cup \Sigma^a R$. The number of equivalence classes is therefore $\frac{1}{2}(\alpha^I(n, k) + \alpha^R(n, k))$.

Counting (A3) arc reversal classes of oriented colored directed embeddings of directed bouquets. To allow for arc reversal, we must add the transformation F to our group of symmetries, giving $\langle S, F \rangle = \Sigma^a \cup \Sigma^a F$. The number of equivalence classes is therefore $\frac{1}{2}(\alpha^I(n, k) + \alpha^F(n, k))$.

Counting (A4) arc reversal classes of orientable colored directed embeddings of directed bouquets. To allow for both surface orientation reversal and arc reversal, we must add both R and F to our group of symmetries, giving $\langle S, R, F \rangle = \Sigma^a \cup \Sigma^a R \cup \Sigma^a F \cup \Sigma^a RF$. The number of equivalence classes is therefore $\frac{1}{4}(\alpha^I(n, k) + \alpha^R(n, k) + \alpha^F(n, k) + \alpha^{RF}(n, k))$.

Counting (A5) simultaneous reflection and arc reversal classes of oriented colored directed embeddings of directed bouquets. To allow for simultaneous reflection and arc reversal, we must add the transformation RF to our group of symmetries, giving $\langle S, RF \rangle = \Sigma^a \cup \Sigma^a RF$. The number of equivalence classes is therefore $\frac{1}{2}(\alpha^I(n, k) + \alpha^{RF}(n, k))$.

There are five groups Γ that satisfy $\Sigma^a \leq \Gamma \leq \Omega^a$. These correspond to the subgroups of the quotient group Ω^a/Σ^a , which is a 4-element dihedral group (or Klein group). We have counted equivalence classes of $\mathcal{A}_{n,k}$ (or $\mathcal{N}_{n,k}$) under the action of Γ for all five groups in (A1)–(A5).

Counting (A6) generic (orientable or nonorientable) colored directed embeddings of directed bouquets. Similar to (B3), generic k -colored directed embeddings of directed bouquets are in one-to-one correspondence with $2k$ -colored orientable directed embeddings of directed bouquets. So this is (A2) with k replaced by $2k$, namely $\frac{1}{2}(\alpha^I(n, 2k) + \alpha^R(n, 2k))$.

Counting (A7) nonorientable colored directed embeddings of directed bouquets. Similar to (B4), this is (A6) – (A2) = $\frac{1}{2}(\alpha^I(n, 2k) + \alpha^R(n, 2k) - \alpha^I(n, k) - \alpha^R(n, k))$.

Counting (A8) arc-reversal classes of generic (orientable or nonorientable) colored directed embeddings of directed bouquets. Similar to (B3) and (A6), generic k -colored arc-reversible directed embeddings of directed bouquets are in one-to-one correspondence with the corresponding $2k$ -colored orientable objects. So this is (A4) with k replaced by $2k$, namely $\frac{1}{4}(\alpha^I(n, 2k) + \alpha^R(n, 2k) + \alpha^F(n, 2k) + \alpha^{RF}(n, 2k))$.

Counting (A9) arc-reversal classes of nonorientable colored directed embeddings of directed bouquets. Similar to (B4) and (A7), this is (A8) – (A4) = $\frac{1}{4}(\alpha^I(n, 2k) + \alpha^R(n, 2k) + \alpha^F(n, 2k) + \alpha^{RF}(n, 2k) - \alpha^I(n, k) - \alpha^R(n, k) - \alpha^F(n, k) - \alpha^{RF}(n, k))$.

Observation 4. From (A1), (A2), (A3), and (A5) we see that $\alpha^I(n, k)$, $\alpha^R(n, k)$, $\alpha^F(n, k)$, and $\alpha^{RF}(n, k)$ are all integers.

Author Contributions: Both authors contributed equally to this manuscript. All authors have read and agreed to the published version of the manuscript.

Funding: The work of M. N. Ellingham on this paper was supported by Simons Foundation award no. 429625.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We thank the anonymous referees for very helpful feedback, including directing us to several pertinent references.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences. 2022. Available online: <http://oeis.org> (accessed on 25 May 2022).
2. Mull, B.P.; Rieper, R.G.; White, A.T. Enumerating 2-cell imbeddings of connected graphs. *Proc. Am. Math. Soc.* **1988**, *103*, 321–330. [[CrossRef](#)]
3. Rieper, R.G. The Enumeration of Graph Imbeddings. Ph.D. Dissertation, Western Michigan University, Kalamazoo, MI, USA, 1990.
4. Kwak, J.H.; Lee, J. Enumeration of graph embeddings. *Discret. Math.* **1994**, *135*, 129–151. [[CrossRef](#)]
5. Mull, B.P. Enumerating the orientable 2-cell imbeddings of complete bipartite graphs. *J. Graph Theory* **1999**, *30*, 77–90. [[CrossRef](#)]
6. Kim, J.H.; Park, Y.K. Incongruent embeddings of a bouquet into surfaces. *Bull. Austral. Math. Soc.* **2000**, *61*, 89–96. [[CrossRef](#)]
7. Feng, Y.Q.; Kwak, J.H.; Zhou, J.X. Congruence classes of orientable 2-cell embeddings of bouquets of circles and dipoles. *Electron. J. Combin.* **2010**, *17*, R41. [[CrossRef](#)]
8. Feng, Y.; Kwak, J.H.; Zhou, J. Enumerating reflexible 2-cell embeddings of connected graphs. *Sci. China Math.* **2013**, *56*, 933–950. [[CrossRef](#)]
9. Orbanic, A.; Petkovšek, M.; Pisanski, T.; Potočnik, P. A note on enumeration of one-vertex maps. *Ars Math. Contemp.* **2010**, *3*, 1–12. [[CrossRef](#)]
10. Chen, Y.; Gao, X.; Huang, Y. Enumerating unlabelled embeddings of digraphs. *Canad. Math. Bull.* **2018**, *61*, 55–69. [[CrossRef](#)]
11. Stahl, S. Average genus of classes of graph embeddings. *Congr. Numer.* **1983**, *40*, 375–388.
12. Gross, J.L.; Furst, M.L. Hierarchy for imbedding-distribution invariants of a graph. *J. Graph Theory* **1987**, *11*, 205–220. [[CrossRef](#)]
13. Furst, M.L.; Gross, J.L.; Statman, R. Genus distributions for two classes of graphs. *J. Comb. Theory Ser. B* **1989**, *46*, 22–36. [[CrossRef](#)]
14. Gross, J.L.; Robbins, D.P.; Tucker, T.W. Genus distributions for bouquets of circles. *J. Comb. Theory Ser. B* **1989**, *47*, 292–306. [[CrossRef](#)]
15. Kwak, J.H.; Shim, S.H. Total embedding distributions for bouquets of circles. *Discret. Math.* **2002**, *248*, 93–108. [[CrossRef](#)]
16. Krzywinski, M.I.; Schein, J.E.; Birol, I.; Connors, J.; Gascoyne, R.; Horsman, D.; Marra, S.J.J.M.A. Circos: An information aesthetic for comparative genomics. *Genome Res.* **2009**, *19*, 1639–1645. [[CrossRef](#)]
17. Bon, M.; Vernizzi, G.; Orland, H.; Zee, A. Topological Classification of RNA Structures. *J. Mol. Biol.* **2008**, *379*, 900–911. [[CrossRef](#)]
18. Penner, R.C. Moduli spaces and macromolecules. *Bull. Am. Math. Soc.* **2016**, *53*, 217–268. [[CrossRef](#)]
19. Zając, S.; Geary, C.; Andersen, E.S.; Dabrowski-Tumanski, P.; Sułkowska, J.I.; Sulkowski, P. Genus trace reveals the topological complexity and domain structure of biomolecules. *Sci. Rep.* **2018**, *8*, 17537. [[CrossRef](#)]
20. Andersen, J.; Mattes, J.; Reshetikhin, N. The Poisson structure on the moduli space of flat connections and chord diagrams. *Topology* **1996**, *35*, 1069–1083. [[CrossRef](#)]
21. Touchard, J. Sur un problème de configurations et sur les fractions continues. *Canad. J. Math.* **1952**, *4*, 2–25. [[CrossRef](#)]
22. Riordan, J. The distribution of crossings of chords joining pairs of $2n$ points on a circle. *Math. Comp.* **1975**, *29*, 215–222. [[CrossRef](#)]
23. Bar-Natan, D. On the Vassiliev knot invariants. *Topology* **1995**, *34*, 423–472. [[CrossRef](#)]
24. Chmutov, S.; Duzhin, S.; Mostovoy, J. *Introduction to Vassiliev Knot Invariants*; Cambridge University Press: Cambridge, UK, 2012; pp. xvi+504. [[CrossRef](#)]
25. nLab Chord Diagram Page. Available online: <https://ncatlab.org/nlab/show/chord+diagram> (accessed on 25 April 2022).
26. Ellis-Monaghan, J.; Pangborn, G.; Seeman, N.; Blakeley, S.; Disher, C.; Falcigno, M.; Healy, B.; Morse, A.; Singh, B.; Westland, M. Design Tools For Reporter Strands and DNA Origami Scaffold Strands. *Theoret. Comput. Sci.* **2017**, *671*, 69–78. [[CrossRef](#)]
27. Ellis-Monaghan, J.; McDowell, A.; Moffatt, I.; Pangborn, G. DNA origami and the complexity of Eulerian circuits with turning costs. *Nat. Comput.* **2015**, *14*, 491–503. [[CrossRef](#)]
28. Veneziano, R.; Ratanalert, S.; Zhang, K.; Zhang, F.; Yan, H.; Chiu, W.; Bathe, M. Designer Nanoscale DNA Assemblies Programmed from the Top Down. *Science* **2016**, *352*, 1534–1534. [[CrossRef](#)]
29. Ellingham, M.N.; Ellis-Monaghan, J.A. Edge-outer graph embedding and the complexity of the DNA reporter strand problem. *Theoret. Comput. Sci.* **2019**, *785*, 117–127. [[CrossRef](#)]
30. Wu, G.; Jonoska, N.; Seeman, N.C. Construction of a DNA nano-object directly demonstrates computation. *Biosystems* **2009**, *98*, 80–84. [[CrossRef](#)]
31. Pelesko, J. *Self Assembly: The Science of Things That Put Themselves Together*; Taylor & Francis Group, LLC: Abingdon, UK, 2007.
32. Rothmund, P.W. Folding DNA to create nanoscale shapes and patterns. *Nature* **2006**, *440*, 297–302. [[CrossRef](#)]
33. Seeman, N.C. *Structural DNA Nanotechnology*; Cambridge University Press: Cambridge, UK, 2016.

34. Xuong, N.H. How to determine the maximum genus of a graph. *J. Comb. Theory Ser. B* **1979**, *26*, 217–225. [[CrossRef](#)]
35. Xuong, N.H. Upper-embeddable graphs and related topics. *J. Comb. Theory Ser. B* **1979**, *26*, 226–232. [[CrossRef](#)]
36. Ellis-Monaghan, J.; Moffatt, I. *Graphs on Surfaces: Twisted Duality, Polynomials and Knots*; SpringerBriefs in Mathematics; Springer: Berlin/Heidelberg, Germany, 2013.
37. Gross, J.L.; Tucker, T.W. *Topological Graph Theory*; Dover Publications, Inc.: Mineola, NY, USA, 2001; pp. xvi+361.
38. Mohar, B.; Thomassen, C. *Graphs on Surfaces*; Johns Hopkins Studies in the Mathematical Sciences; Johns Hopkins University Press: Baltimore, MD, USA, 2001; pp. xii+291.