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New Analytical Solutions for Coupled Stochastic Korteweg–de Vries Equations via Generalized Derivatives

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Abstract: In this paper, the coupled nonlinear KdV (CNKdV) equations are solved in a stochastic environment. Hermite transforms, generalized conformable derivative, and an algorithm that merges the white noise instruments and the $(\mathfrak{G}'/\mathfrak{G}^2)$-expansion technique are utilized to obtain white noise functional conformable solutions for these equations. New stochastic kinds of periodic and soliton solutions for these equations under conformable generalized derivatives are produced. Moreover, three-dimensional (3D) depictions are shown to illustrate that the monotonicity and symmetry of the obtained solutions can be controlled by giving a value of the conformable parameter. Furthermore, some remarks are presented to give a comparison between the obtained wave solutions and the wave solutions constructed under the conformable derivatives and Newton's derivatives.

Keywords: coupled nonlinear KdV equations; stochastic solutions; symmetry solutions; generalized derivatives; white noise

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1. Introduction

Nonlinear phenomena exist in numerous areas of natural sciences, such as applied mathematics and physics, computational fluid dynamics, and some other sciences [1]. These phenomena can be represented by nonlinear partial differential equations (PDEs) [2–5]. The importance of the field of nonlinear differential equations lies in the search for their exact solutions that in turn give an explanation for multiple phenomena. There is a significant class of nonlinear differential equations known as fractional differential equations, and there have been many studies and advancements, particularly in determining their exact solutions. Some models of differential equations contain random variables, called stochastic differential equations (SDEs) and deterministic differential equations (DDEs). Because of the fact that the random kinds are more natural than deterministic kinds, we confine our task to stochastic PDEs with recently generalized derivatives.

The concept of generalized derivatives or fractional derivatives is to extend traditional derivatives such as Newton's classical derivative [6–8]. Recently, Khalil et al. [9] gave a simple concept of fractional derivatives called conformable derivatives, which depend on the basic limit notion. Consequently, certain generalizations have been obtained in the literature such as the definitions of Zhao and Luo [10,11] and Hyder and Soliman [12].

The KdV equation is a world-class mathematical model for the characterization of long nonlinear weakly wave diffusion in scattered media. It has many applications in physics, among them: gravity waves in shallow water [13], bubbly fluid waves [14], ocean waves, and numerous others [15]. This wide range of applicability for the KdV equation is characterized by the common effect of the lowest-order, cubic, and long-wave dispersal terms. Moreover, one can obtain derivations of the KdV equation for various physical contexts [16–18]. In recent decades, the KdV equation has been developed and modified into the modified KdV equation (mKdV) [19], two-mode KdV equation (TKdV) [20], two-mode modified KdV equation (TmKdV) [19], coupled nonlinear KdV equations [21], the KdV equation in random environments [22], etc. All these modifications and developments of the KdV equation aimed to obtain more general solutions than their predecessors.

Given the importance of the exact solutions for SDEs and DDEs, searching for different techniques to find these solutions has become an increasingly appealing field in applied mathematics. So, new techniques are being sought to tackle more sophisticated problems. Hence, numerous new techniques have been proposed, for example, the Kudryashov technique and all of their modifications [23–26], the F-expansion technique [27], Jacobi elliptic function technique [28,29], $(\mathfrak{G}'/\mathfrak{G})$-expansion technique [30], and $(\mathfrak{G}'/\mathfrak{G}^2)$-expansion technique [31]. Moreover, there are ongoing advancements in techniques in terms of their modifications and generalizations for obtaining solutions for SDEs and DDEs, see [32–35] and references therein.

This work is devoted to inspecting diverse deterministic and stochastic accurate solutions for the CNKdV equations under recently generalized derivatives. Foremost, we present an algorithm for solving conformable stochastic partial differential equations (SPDEs). This algorithm merges the utilizing of the white noise instruments, the generalized conformable derivative, and the $(\mathfrak{G}'/\mathfrak{G}^2)$-expansion technique. Using the offered algorithm, we produce a family of accurate deterministic solutions for the CNKdV. Under certain restrictions and the reverse Hermite transform, we transform the solutions of the deterministic CNKdV to stochastic kinds of periodic and soliton solutions for the stochastic CNKdV equations. Moreover, 3D portrayals are displayed to point out that the monotonicity and symmetry of the obtaining solutions can be controlled by giving a value of the conformable parameter. According to the current literature, this work is a novel contribution, and the proposed technique for addressing nonlinear issues is straightforward and simple to execute. Additionally, some remarks are offered to provide a comparison between the obtained wave solutions and the wave solutions constructed under the conformable derivatives and Newton's derivatives.

2. Preliminaries

In this section, the definitions and attributes of the generalized conformable derivative are presented.

Definition 1 ([36])**.** *If we have a nonnegative continuous function $\delta : [c,d] \to \mathbb{R}$ with $\delta(s) \neq 0$ and $\delta'(s) \neq 0$ for each $s \in (c,d)$. For any function $g : [c,d] \to \mathbb{R}$, the generalized conformable derivative of order $\pi \in (0,1)$ is obtained by*

$$D_s^{\pi} g(s) = \lim_{\eta \to 0} \frac{g\left(s - \delta(s) + \delta(s)\exp\left(\frac{\eta}{(\delta(s))^{\pi}\delta'(s)}\right)\right) - g(s)}{\eta}, \tag{1}$$

assuming the limit exists.

Definition 2 ([36])**.** *Let g be a π-differentiable function on (a,s), and let $\lim_{s\to 0} D_s^{\pi} g(s)$ exist. Then, $D_s^{\pi} g(0) = \lim_{s\to 0} D_s^{\pi} g(s)$, and the generalized conformable integral of g is given by*

$$\mathfrak{I}^{\pi,a} g(s) = \int_a^s \frac{\delta'(u)g(u)}{(\delta(u))^{1-\pi}} du. \tag{2}$$

The following theorems outline the essential properties of the generalized conformable derivative.

Theorem 1 ([36]). *If we have two $\pi-$ differentiable functions h and g with $0 \le \pi \le 1.$, then*
(a) $D_s^\pi(C) = 0$, for all constant function $h(s) = C$,
(b) $D_s^\pi(s^p) = (ps^{p-1})(\frac{(\delta(s))^{1-\pi}}{\delta'(s)})$, $\quad p \in \mathbb{R}$.
(c) $D_s^\pi(c_1 h(s) + c_2 g(s)) = c_1 D_s^\pi(h(s)) + c_2 D_s^\pi(g(s))$,
(d) $D_s^\pi(hg)(s) = h(s)D_s^\pi(g(s)) + g(s)D_s^\pi(h(s))$,
(e) $D_s^\pi(\frac{h(s)}{g(s)}) = \frac{g(s)D_s^\pi(h(s)) - h(s)D_s^\pi(g(s))}{(g(s))^2}$, $g(s) \neq 0 \, \forall \, s > 0$,
(f) $D_s^\pi(h(s)) = (\frac{dh(s)}{ds})(\frac{(\delta(s))^{1-\pi}}{\delta'(s)})$.

Theorem 2 ([36]). *For $s > 0$, suppose that h is a π-differentiable function, and suppose that g is a differentiable function its domain equal to the range of h, and its range is $\mathbb{R}$. Then, $h(g(s))$ is π-differentiable function, and*

$$D_s^\pi(h(g(s))) = (\frac{dh}{dg})(\frac{dg}{ds})\Big(\frac{(\delta(s))^{1-\pi}}{\delta'(s)}\Big). \tag{3}$$

3. An Algorithm for Solving Conformable SPDEs

This section elucidates an algorithm for obtaining accurate wave solutions for conformable SPDEs under conformable generalized derivatives. This algorithm merges the white noise instruments, the features of conformable generalized derivatives, and the $(\mathfrak{G}'/\mathfrak{G}^2)$-expansion technique.

We assume the stochastic space of Kondratiev $(Z)^m_{-1}$ having the orthogonal basis $\{Q_\rho\}_{\rho \in P}$, such that $P = \Big\{\rho = (\rho_1, \rho_2, \rho_3, \ldots); \rho_i \in \mathbb{N} \text{ and } \sum_{i=1}^\infty \rho_i < \infty\Big\}$. We choose two elements $U, V \in (Z)^m_{-1}$; then, $U = \sum_\rho \vartheta_\rho Q_\rho$ and $V = \sum_\rho \varpi_\rho Q_\rho$, where ϑ_ρ, $\varpi_\rho \in \mathbb{R}^n$. We define the wick multiplication of U and V by:

$$U \diamond V = \sum_{\rho, \bar{\rho}} \sum_{j=1}^m \vartheta_\rho^j \varpi_{\bar{\rho}}^j Q_{\rho + \bar{\rho}}. \tag{4}$$

In addition, the U's Hermite transform has the ability to expand by

$$\mathcal{H}(U) = \widetilde{U}(\tau) = \sum_\rho \vartheta_\rho \tau^\rho \in \mathbb{C}^m, \tag{5}$$

where $\tau = (\tau_j)_{j \ge 1} \in \mathbb{C}^\mathbb{N}$ and $\tau^\rho = \Pi_{j=1}^\infty \tau^{\rho_j}$. Equations (4) and (5) can be used to derive the relationship between Wick multiplication and Hermite transform as the following:

$$\widetilde{U \diamond V}(\tau) = \widetilde{U}(\tau) \bullet \widetilde{V}(\tau), \tag{6}$$

where the operation "$\bullet$" is the bilinear multiplication in $\mathbb{C}^m$ that is determined by $(\tau_1, \tau_2, \tau_3, \ldots, \tau_m) \bullet (\bar{\tau}_1, \bar{\tau}_2, \bar{\tau}_3, \ldots, \bar{\tau}_m) = \sum_{j=1}^m \tau_j \bar{\tau}_j$, and $\widetilde{U}(\tau)$, $\widetilde{V}(\tau)$ are finite for all τ. In $\mathbb{C}^\mathbb{N}$, a zero neighborhood is defined for each $\chi < \infty$, $\Delta > 0$ as $\mathcal{N}(\chi, \Delta) = \{(\tau_1, \tau_2, \ldots) \in \mathbb{C}^\mathbb{N} : \sum_{\rho \neq 0} |\tau^\rho|^2 (2\mathbb{N})^{\chi\rho} < \Delta^2\}$. We choose $U \in (Z)_{-1}$ such that $U = \sum_\rho \vartheta_\rho Q_\rho$. Following that, the vector $\widetilde{U}(0) \in \mathbb{R}^m$ defines the generalized expectation of U. Moreover, we assume that $w : \mathcal{N}(\chi, \Delta) \to \mathbb{C}^n$ is an analytic function with real coefficients in its Taylor expansion around $\widetilde{U}(0)$. Then, the Wick form of w is known as $w^\diamond(U) = \mathcal{H}^{-1}(w \circ \widetilde{U})$.

Now, we describe our method for solving conformable SPDEs by using a combination of generalized conformable derivatives, white noise analysis tools, and $(\mathfrak{G}'/\mathfrak{G}^2)$-expansion techniques in the phases below:

In the first phase, we take the following stochastic equation as a result of a physical phenomenon:

$$q(X, r, s, D_s^{\pi}, D_r^{\pi}, D_s^{2\pi}, \dots) = 0 \text{ in } (Z)_{-1}^{m}, \tag{7}$$

where $r \in \mathbb{R}^+$, $s \in \mathbb{R}$, and X is the desirable stochastic wave.

In the second phase, Equation (7) is transformed using the Hermite transform, and the relation (6) is used to produce a conformable deterministic PDE of the form:

$$\widetilde{q}(x, r, s, D_s^{\pi}, D_r^{\pi}, D_s^{2\pi}, \dots, \tau) = 0 \text{ in } \mathbb{C}^m, \tag{8}$$

where $\tau \in \mathbb{C}^{\mathbb{N}}$ is the transformation variable, and $x(r, s, \tau) = \widetilde{X}(r, s, \tau)$ is the needed deterministic wave.

In the third phase, using the following transformation, the variables r and s can be merged into a single wave variable:

$$x(r, s, \tau) = x(\theta), \quad \theta(r, s, \tau) = \alpha\left(\frac{(\delta(r))^{\pi}}{\pi}\right) + \beta \int_0^s \frac{\delta'(u)\phi(u, \tau)}{\delta(u))^{1-\pi}} du, \tag{9}$$

where $\phi(u, \tau) \neq 0$ is a function to be recognized later, and α, β are arbitrary numbers satisfying $\alpha\beta \neq 0$. Then, Equation (8) can be transformed into an ordinary nonlinear differential equation by utilizing the transformation (9) as follows:

$$q(x, \theta, \alpha D_{\theta}^{\pi}, \beta D_{\theta}^{\pi}, \beta^2 D_{\theta}^{2\pi}, \dots, \tau) = 0. \tag{10}$$

In the fourth phase, according to the $(\frac{\mathfrak{G}'}{\mathfrak{G}^2})$-expansion approach (see [31,32,37]), we assume the solution of Equation (10) as the form:

$$x(\theta) = \mu_0(s) + \sum_{i}^{M}\left[\mu_i(s)\left(\frac{\mathfrak{G}'}{\mathfrak{G}^2}\right)^{i} + \bar{\mu}_i(s)\left(\frac{\mathfrak{G}'}{\mathfrak{G}^2}\right)^{-i}\right], \tag{11}$$

and

$$\left(\frac{\mathfrak{G}'}{\mathfrak{G}^2}\right)' = \xi_0 + \xi_1\left(\frac{\mathfrak{G}'}{\mathfrak{G}^2}\right)^2, \tag{12}$$

where M is a positive integer that can be calculated by balancing the highest derivative terms with the highest power nonlinear terms in Equation (10). We can create a system of algebraic nonlinear equations by adding Equations (11) and (12) into Equation (10), combining the coefficients of all powers of $(\frac{\mathfrak{G}'}{\mathfrak{G}^2})$ and setting them to zero. The values of the unknowns μ_0, μ_i, and $\bar{\mu}_i (i = 1, 2, \dots, M)$ are obtained using Mathematica algorithms for this algebraic system.

Equation (10) can be used to obtain the value of M in more detail, as shown below.

If the degree of x is defined as $\mathbf{D}[x(\theta)] = M$, then

$$\begin{cases} \mathbf{D}\left[\dfrac{d^u x}{d\theta^u}\right] = M + u, \\ \mathbf{D}\left[x^a\left(\dfrac{d^u x}{d\theta^u}\right)^b\right] = aM + b(M + u). \end{cases} \tag{13}$$

Finally, the requisite exact solutions will be achieved in the following conditions based on the general solution of Equation (12):

(i) At $\xi_0\xi_1 > 0$, then

$$\frac{\mathfrak{G}'}{\mathfrak{G}^2} = \sqrt{\frac{\xi_0}{\xi_1}}\left(\frac{R\cos\left(\sqrt{\xi_0\xi_1}\theta(r,s,\tau)\right) + S\sin\left(\sqrt{\xi_0\xi_1}\theta(r,s,\tau)\right)}{S\cos\left(\sqrt{\xi_0\xi_1}\theta(r,s,\tau)\right) - R\sin\left(\sqrt{\xi_0\xi_1}\theta(r,s,\tau)\right)}\right). \tag{14}$$

(ii) At $\xi_0\xi_1 < 0$, then

$$\frac{\mathfrak{G}'}{\mathfrak{G}^2} = -\frac{\sqrt{|\xi_0\xi_1|}}{\xi_1}\left(\frac{R\sinh\left(2\sqrt{|\xi_0\xi_1|}\theta(r,s,\tau)\right) + R\cosh\left(2\sqrt{|\xi_0\xi_1|}\theta(r,s,\tau)\right) + S}{R\sinh\left(2\sqrt{|\xi_0\xi_1|}\theta(r,s,\tau)\right) + R\cosh\left(2\sqrt{|\xi_0\xi_1|}\theta(r,s,\tau)\right) - S}\right). \tag{15}$$

where $S, R \in \mathbb{R} - \{0\}$.

4. Applications to the Stochastic CNKdV

In this section, we apply the above algorithm to inspect diverse deterministic and stochastic accurate solutions for the CNKdV equations under recently generalized derivatives. The CNKdV equations with generalized conformable derivatives take the form [38]:

$$\begin{cases} D_s^{\pi}x + k(s)xD_r^{\pi}x + l(s)yD_r^{\pi}y + m(s)D_r^{3\pi}y = 0, \\ D_s^{\pi}y + D_s^{3\pi}y - 3xD_r^{\pi}y = 0, \end{cases} \tag{16}$$

where $k(s)$, $l(s)$, and $m(s)$ are measurable and bounded functions on $\mathbb{R}_+$. As mentioned in [39], the abnormal movement of dual weakly nonlinear water waves with roughly simultaneous linear long-wave speeds may be modeled by Equation (16) when $\pi = 1$. In order to have more accuracy and controllability in the wave behavior, the conformable parameter is assumed to be in $(0,1]$. Moreover, if the issue is deemed in a stochastic medium, we can obtain stochastic CNKdV equations. In order to produce accurate solutions to the stochastic CNKdV equations, we only deem this issue in a white noise medium; that is, we will examine the next Wick-type stochastic CNKdV equations.

$$\begin{cases} D_s^{\pi}X + K(s)\diamond X\diamond D_r^{\pi}X + L(s)\diamond Y\diamond D_r^{\pi}Y + M(s)\diamond D_r^{3\pi}X = 0, \\ D_s^{\pi}Y + D_s^{3\pi}Y - 3X\diamond D_r^{\pi}Y = 0, \end{cases} \tag{17}$$

where $(r,s) \in \mathbb{R}\times\mathbb{R}_+$, $X(r,s)$, $Y(r,s)$ are the unknown stochastic distributions, $\pi \in (0,1]$, and $D_s^{\pi}X$, $D_r^{\pi}X$ are the π-order generalized derivatives with respect to r, s, respectively. Moreover, "$\diamond$" denotes the Wick multiplication on the Kondratiev stochastic distribution space $(Z)_{-1}$, and $K(s)$, $L(s)$, and $M(s)$ are $(Z)_{-1}$-valued functions.

4.1. Deterministic Solutions of Equation (17)

In this section, we examine accurate deterministic solutions for Equation (17) using the algorithm suggested in Section 3. On applying the Hermite transform to Equation (17), we obtain the next deterministic CNKdV equations.

$$\begin{cases} D_s^{\pi}\widetilde{X}(r,s,\tau) + \widetilde{K}(s,\tau)\widetilde{X}(r,s,\tau)D_r^{\pi}\widetilde{X}(r,s,\tau) + \widetilde{L}(s,\tau)\widetilde{Y}(r,s,\tau)D_r^{\pi}\widetilde{Y}(r,s,\tau) \\ \qquad +\widetilde{M}(s,\tau)D_r^{3\pi}\widetilde{X}(r,s,\tau) = 0, \\ D_s^{\pi}\widetilde{Y}(r,s,\tau) + D_s^{3\pi}\widetilde{Y}(r,s,\tau) - 3\widetilde{X}(r,s,\tau)D_r^{\pi}\widetilde{Y}(r,s,\tau) = 0, \end{cases} \tag{18}$$

where $\tau = (\tau_1, \tau_2, \ldots) \in \left(\mathbb{C}^{\mathbb{N}}\right)_c$ is the Hermite transform variable. To extract the deterministic solutions of Equation (18) in a traveling waveform, we suggest the transformations: $\widetilde{K}(s,\tau) = k(s,\tau)$, $\widetilde{L}(s,\tau) = l(s,\tau)$, $\widetilde{M}(s,\tau) = m(s,\tau)$, $\widetilde{X}(r,s,\tau) = x(r,s,\tau) = x(\theta(r,s,\tau))$, and $\widetilde{Y}(r,s,\tau) = y(r,s,\tau) = y(\theta(r,s,\tau))$ with

$$\theta(r,s,\tau) = \alpha\left(\frac{(\delta(r))^{\pi}}{\pi}\right) + \beta\int_0^s \frac{\delta'(u)\phi(u,\tau)}{\delta(u))^{1-\pi}}du, \tag{19}$$

where $\phi(u,\tau) \neq 0$ is a function to be recognized later, and α, β are arbitrary numbers satisfying $\alpha\beta \neq 0$. Therefore, Equation (18) can be converted to the system

$$\begin{cases} \beta\phi\dfrac{dx}{d\theta} + \alpha kx\dfrac{dx}{d\theta} + \alpha ly\dfrac{dy}{d\theta} + \alpha^3 m\dfrac{d^3x}{d\theta^3} = 0, \\ \beta\phi\dfrac{dy}{d\theta} + \alpha^3\dfrac{d^3y}{d\theta^3} - 3\alpha x\dfrac{dy}{d\theta} = 0. \end{cases} \tag{20}$$

According to Equations (11) and (13) and the degree balancing procedure, the solution of Equation (18) can be assumed as the form:

$$\begin{cases} x(\theta) = \mu_0(s,\tau) + \mu_1(s,\tau)\left(\frac{\mathfrak{G}'}{\mathfrak{G}^2}\right) + \mu_2(s,\tau)\left(\frac{\mathfrak{G}'}{\mathfrak{G}^2}\right)^2 + \bar{\mu}_1(s,\tau)\left(\frac{\mathfrak{G}'}{\mathfrak{G}^2}\right)^{-1} + \bar{\mu}_2(s,\tau)\left(\frac{\mathfrak{G}'}{\mathfrak{G}^2}\right)^{-2}, \\ y(\theta) = \nu_0(s,\tau) + \nu_1(s,\tau)\left(\frac{\mathfrak{G}'}{\mathfrak{G}^2}\right) + \nu_2(s,\tau)\left(\frac{\mathfrak{G}'}{\mathfrak{G}^2}\right)^2 + \bar{\nu}_1(s,\tau)\left(\frac{\mathfrak{G}'}{\mathfrak{G}^2}\right)^{-1} + \bar{\nu}_2(s,z)\left(\frac{\mathfrak{G}'}{\mathfrak{G}^2}\right)^{-2}, \end{cases} \tag{21}$$

where $\left(\frac{\mathfrak{G}'}{\mathfrak{G}^2}\right)$ fulfills the nonlinear Equation (12).

By inserting Equation (21) together with Equation (12) into Equation (20), combining the coefficients of $\left(\frac{\mathfrak{G}'}{\mathfrak{G}^2}\right)^i$ and putting them to zero, we will obtain a system of algebraic nonlinear equations in the unknowns $\mu_i, \nu_i (i = 0,1,2,3,4)$, and ϕ of the form

$$\begin{aligned} &\beta\phi(\xi_0(\mu_1+\nu_1) - \xi_1(\bar{\mu}_1+\bar{\nu}_1)) + \alpha k(\mu_0\mu_1 + \mu_2\bar{\mu}_1) + \alpha(\xi_0(l\nu_0\nu_1 + l\nu_2\bar{\nu}_1 - 3\mu_0\nu_1 \\ &+ \quad 3\mu_2\bar{\nu}_1 - 6\bar{\mu}_1\nu_2) - \xi_1(l\nu_0\bar{\nu}_1 + l\nu_1\bar{\nu}_2 - 3\mu_0\bar{\nu}_1 - 6\mu_1\bar{\nu}_2 + 3\bar{\mu}_2\nu_1)) \\ &+ \quad 2\alpha^3\xi_0\xi_1(\xi_0(\mu_1 m + \nu_1) - \xi_1(\bar{\mu}_1 m + \bar{\nu}_1)) = 0, \end{aligned} \tag{22}$$

$$\begin{aligned} &2\beta\xi_0\phi(\mu_2+\nu_2) + \alpha k\left(2\mu_0\mu_2 + \mu_1^2\right) + \alpha\xi_0\left(2l\nu_0\nu_2 + l\nu_1^2 - 6\mu_0\nu_2 - 3\mu_1\nu_1\right) \\ &+ \quad \alpha\xi_1\left(3\mu_1\bar{\nu}_1 + 6\mu_2\bar{\nu}_2 - 3\bar{\mu}_1\nu_1 - 6\bar{\mu}_2\nu_2 + 16\alpha^2\xi_0^2(\mu_2 m + \nu_2)\right) = 0, \end{aligned} \tag{23}$$

$$\begin{aligned} &\beta\xi_1\phi(\mu_1+\nu_1) + 3\alpha k\mu_1\mu_2 + \alpha\xi_1(l\nu_0\nu_1 + l\nu_2\bar{\nu}_1 - 3\mu_0\nu_1 + 3\mu_2\bar{\nu}_1 - 6\bar{\mu}_1\nu_2) \\ &- \quad 3\alpha\xi_0(-l\nu_1\nu_2 + 2\mu_1\nu_2 + \mu_2\nu_1) + 8\alpha^3\xi_0\xi_1^2(\mu_1 m + \nu_1) = 0, \end{aligned} \tag{24}$$

$$\begin{aligned} &2\beta\xi_1\phi(\mu_2+\nu_2) + 2\alpha k\mu_2^2 + \alpha\xi_1\left(2l\nu_0\nu_2 + l\nu_1^2 - 6\mu_0\nu_2 - 3\mu_1\nu_1\right) \\ &+ \quad 2\alpha\nu_2\xi_0(l\nu_2 - 3\mu_2) + 40\alpha^3\xi_0\xi_1^2(\mu_2 m + \nu_2) = 0, \end{aligned} \tag{25}$$

$$3\alpha\xi_1\left(l\nu_1\nu_2 - 2\mu_1\nu_2 - \mu_2\nu_1 + 2\alpha^2\xi_1^2(\mu_1 m + \nu_1)\right) = 0, \tag{26}$$

$$2\alpha\nu_2\xi_1(l\nu_2 - 3\mu_2) + 24\alpha^3\xi_1^3(\mu_2 m + \nu_2) = 0, \tag{27}$$

$$\begin{aligned} &3\alpha\xi_0(\mu_1\bar{\nu}_1 + 2\mu_2\bar{\nu}_2 - \bar{\mu}_1\nu_1 - 2\bar{\mu}_2\nu_2) - 2\beta\xi_1\phi(\bar{\mu}_2+\bar{\nu}_2) \\ &+ \quad \alpha\xi_1\left(-l\left(2\nu_0\bar{\nu}_2 + \bar{\nu}_1^2\right)6\mu_0\bar{\nu}_2 + 3\bar{\mu}_1\bar{\nu}_1\right) - 16\alpha^3\xi_0\xi_1^2(\bar{\mu}_2 m + \bar{\nu}_2) = 0, \end{aligned} \tag{28}$$

$$\begin{aligned} &-\beta\xi_0\phi(\bar{\mu}_1+\bar{\nu}_1) - \alpha(k(\mu_0\bar{\mu}_1 + \mu_1\bar{\mu}_2) + \xi_0(l\nu_0\bar{\nu}_1 + l\nu_1\bar{\nu}_2 - 3\mu_0\bar{\nu}_1 \\ &- \quad 6\mu_1\bar{\nu}_2 3\bar{\mu}_2\nu_1) + \xi_1\left(3l\bar{\nu}_1\bar{\nu}_2 - 6\bar{\mu}_1\bar{\nu}_2 - 3\bar{\mu}_2\bar{\nu}_1 + 8\alpha^2\xi_0^2(\bar{\mu}_1 m + \bar{\nu}_1)\right)) = 0, \end{aligned} \tag{29}$$

$$\begin{aligned} &-2\beta\xi_0\phi(\bar{\mu}_2+\bar{\nu}_2) + \alpha(-k)\left(2\mu_0\bar{\mu}_2 + \bar{\mu}_1^2\right) + \alpha\xi_0(-l\left(2\nu_0\bar{\nu}_2 + \bar{\nu}_1^2\right) \\ &+ \quad 6\mu_0\bar{\nu}_2 + 3\bar{\mu}_1\bar{\nu}_1) - 2\alpha\xi_1\left(\bar{\nu}_2(l\bar{\nu}_2 - 3\bar{\mu}_2) + 20\alpha^2\xi_0^2(\bar{\mu}_2 m + \bar{\nu}_2)\right) = 0, \end{aligned} \tag{30}$$

$$3\alpha\xi_0\left(-l\bar{\nu}_1\bar{\nu}_2 + 2\bar{\mu}_1\bar{\nu}_2 + \bar{\mu}_2\bar{\nu}_1 - 2\alpha^2\xi_0^2(\bar{\mu}_1 m + \bar{\nu}_1)\right) - 3\alpha k\bar{\mu}_1\bar{\mu}_2 = 0, \tag{31}$$

$$-2\alpha k\bar{\mu}_2^2 - 2\alpha\bar{\nu}_2\xi_0(l\bar{\nu}_2 - 3\bar{\mu}_2) - 24\alpha^3\xi_0^3(\bar{\mu}_2 m + \bar{\nu}_2) = 0. \tag{32}$$

Treating the system (22)–(32) through Mathematica, we have the following families of solutions.

Family I.

$$\begin{cases} \mu_0 = -\dfrac{-12\alpha^3\xi_0^2\xi_1(3+lm)}{kl},\ \mu_1 = \mu_2 = \bar{\mu}_1 = 0,\ \bar{\mu}_2 = \dfrac{12\alpha^2\xi_0^3 m}{k}, \\ \nu_0 = \dfrac{-4\alpha^2\xi_0\xi_1(9\xi_0(lm+3) + (9-2l(m-l))k)}{kl^2},\ \nu_1 = 0,\ \nu_2 = \dfrac{-12\alpha^2\xi_1^2}{l}, \\ \bar{\nu}_1 = \bar{\nu}_2 = 0,\ \phi = \dfrac{-8\alpha^3\xi_0\xi_1 m}{\beta}. \end{cases} \tag{33}$$

Family II.

$$\begin{cases} \mu_0 = \dfrac{-9(\bar{\mu}_1^2k^2 - 8\alpha^4\xi_0^5\xi_1 m^2)}{4\alpha^2\xi_0^2k^2lm},\ \mu_1 = \mu_2 = 0,\ \bar{\mu}_1 = \bar{\mu}_1,\ \bar{\mu}_2 = \dfrac{-2\alpha^2\xi_0^3 m}{k}, \\ \nu_0 = \dfrac{-9\bar{\mu}_1^2(3\xi_0+k)k^2 + 8\alpha^4\xi_0^4\xi_1(4(m-l)k^2l + 27\xi_0^2 m)m}{4\alpha^2\xi_0^2k^2l^2m},\ \nu_1 = \dfrac{3\bar{\mu}_1\xi_1}{\xi_0 l}, \\ \nu_2 = \bar{\nu}_1 = \bar{\nu}_2 = 0,\ \phi = \dfrac{9\bar{\mu}_1^2k - 32\alpha^4\xi_0^4\xi_1 lm^2}{4\alpha\beta\xi_0^2 lm}. \end{cases} \tag{34}$$

Family III.

$$\begin{cases} \mu_0 = 0,\ \mu_1 = 0,\ \mu_2 = \dfrac{\alpha^2\xi_1^2}{9}(lm-12),\ \bar{\mu}_1 = \dfrac{5\bar{\nu}_1 l}{lm+12},\ \bar{\mu}_2 = 0, \\ \nu_0 = 0,\ \nu_1 = 0,\ \nu_2 = \alpha^2\xi_1^2 m,\ \bar{\nu}_1 = \bar{\nu}_1,\ \bar{\nu}_2 = \dfrac{-12\alpha^2\xi_0^2}{l}, \\ \phi = \dfrac{\bar{\nu}_1^2\xi_1(lm-3)l^2 - 8\alpha^4\xi_0^3\xi_1^2(lm+36)(lm+12)}{24\alpha\beta\xi_0^2\xi_1(lm+12)}. \end{cases} \tag{35}$$

where, $\bar{\mu}_1$ and $\bar{\nu}_1$ are arbitrary functions.

Substituting the above produced functions into (21) and considering Equations (14) and (15), we acquire two sets of accurate solutions for Equation (18) as follows.

Set I.

(A) If $\xi_0\xi_1 > 0$, we have the periodic solution:

$$\begin{aligned} x_1(r,s,\tau) &= \frac{-12\alpha^3\xi_0^2\xi_1(3+l(s,\tau)m(s,\tau))}{k(s,\tau)l(s,\tau)} + \frac{12\alpha^2\xi_0^2\xi_1 m(s,\tau)}{k(s,\tau)} \\ &\times \left(\frac{S\cos\left(\sqrt{\xi_0\xi_1}\theta_1(r,s,\tau)\right) - R\sin\left(\sqrt{\xi_0\xi_1}\theta_1(r,s,\tau)\right)}{R\cos\left(\sqrt{\xi_0\xi_1}\theta_1(r,s,\tau)\right) + S\sin\left(\sqrt{\xi_0\xi_1}\theta_1(r,s,\tau)\right)}\right)^2, \end{aligned} \tag{36}$$

$$\begin{aligned} y_1(r,s,\tau) &= \frac{-4\alpha^2\xi_0\xi_1(9\xi_0(l(s,\tau)m(s,\tau)+3) + (9-2(m(s,\tau)-l(s,\tau)))k(s,\tau)l(s,\tau))}{k(s,\tau)L^2(s)} \\ &- \frac{12\alpha^2\xi_0\xi_1}{l(s,\tau)}\left(\frac{R\cos\left(\sqrt{\xi_0\xi_1}\theta_1(r,s,\tau)\right) + S\sin\left(\sqrt{\xi_0\xi_1}\theta_1(r,s,\tau)\right)}{S\cos\left(\sqrt{\xi_0\xi_1}\theta_1(r,s,\tau)\right) - R\sin\left(\sqrt{\xi_0\xi_1}\theta_1(r,s,\tau)\right)}\right)^2. \end{aligned} \tag{37}$$

(B) If $\xi_0\xi_1 < 0$, we obtain the soliton solution:

$$\begin{aligned} x_2(r,s,\tau) &= \frac{-12\alpha^3\xi_0^2\xi_1(3+l(s,\tau)m(s,\tau))}{k(s,\tau)l(s,\tau)} + \frac{12\alpha^2\xi_0|\xi_0\xi_1|m(s,\tau)}{k(s,\tau)} \\ &\times \left(\frac{R\sinh\Big(2\sqrt{|\xi_0\xi_1|}\theta_1(r,s,\tau)\Big) + R\cosh\Big(2\sqrt{|\xi_0\xi_1|}\theta_1(r,s,\tau)\Big) - S}{R\sinh\Big(\sqrt{2|\xi_0\xi_1|}\theta_1(r,s,\tau)\Big) + R\cosh\Big(\sqrt{2|\xi_0\xi_1|}\theta_1(r,s,\tau)\Big) + S}\right)^2, \end{aligned} \tag{38}$$

$$\begin{aligned} y_2(r,s,\tau) = {}& \frac{-4\alpha^2\xi_0\xi_1(9\xi_0(l(s,\tau)m(s,\tau)+3) + (9-2(m(s,\tau)-l(s,\tau)))k(s,\tau)l(s,\tau))}{k(s,\tau)l^2(s,\tau)} \\ - {}& \frac{12\alpha^2|\xi_0|^{3/2}|\xi_1|}{\xi_1 l(s,\tau)}\left(\frac{R\sinh\Big(\sqrt{2|\xi_0\xi_1|}\theta_1(r,s,\tau)\Big) + R\cosh\Big(\sqrt{2|\xi_0\xi_1|}\theta_1(r,s,\tau)\Big) + S}{R\sinh\Big(\sqrt{2|\xi_0\xi_1|}\theta_1(r,s,\tau)\Big) + R\cosh\Big(\sqrt{2|\xi_0\xi_1|}\theta_1(r,s,\tau)\Big) - S}\right)^2. \end{aligned} \tag{39}$$

where

$$\theta_1(r,s,\tau) = \alpha\left(\frac{(\delta(r))^\pi}{\pi}\right) - 8\alpha^3\xi_0\xi_1\int_0^s \frac{m(u,\tau)\delta'(u)}{(\delta(u))^{1-\pi}}du. \tag{40}$$

Set II.
(A) If $\xi_0\xi_1 > 0$, we obtain the periodic solution:

$$\begin{aligned} x_3(r,s,\tau) &= \frac{-9(\bar{\mu}_1^2k^2(s,\tau) - 8\alpha^4\xi_0^5\xi_1 M^2(s))}{4\alpha^2\xi_0^2k^2l(s,\tau)m(s,\tau)} - \frac{2\alpha^2\xi_0^2\xi_1 m(s,\tau)}{k(s,\tau)} \\ &\times \left(\frac{S\cos\big(\sqrt{\xi_0\xi_1}\theta_2(r,s,\tau)\big) - R\sin\big(\sqrt{\xi_0\xi_1}\theta_2(r,s,\tau)\big)}{R\cos\big(\sqrt{\xi_0\xi_1}\theta_2(r,s,\tau)\big) + S\sin\big(\sqrt{\xi_0\xi_1}\theta_2(r,s,\tau)\big)}\right)^2, \end{aligned} \tag{41}$$

$$\begin{aligned} y_3(r,s,\tau) &= \frac{\begin{pmatrix} -9\bar{\mu}_1^2(3\xi_0 + k(s,\tau))k^2(s,\tau) + 8\alpha^4\xi_0^4\xi_1(4(m(s,\tau) \\ -l(s,\tau))k^2(s,\tau)l(s,\tau) + 27\xi_0^2m(s,\tau))m(s,\tau) \end{pmatrix}}{4\alpha^2\xi_0^2k^2(s,\tau)l^2(s,\tau)m(s,\tau)} \\ &+ \sqrt{\frac{\xi_1}{\xi_0}}\left(\frac{3\bar{\mu}_1}{l(s,\tau)}\right)\left(\frac{R\cos\big(\sqrt{\xi_0\xi_1}\theta_2(r,s,\tau)\big) + S\sin\big(\sqrt{\xi_0\xi_1}\theta_2(r,s,\tau)\big)}{S\cos\big(\sqrt{\xi_0\xi_1}\theta_2(r,s,\tau)\big) - R\sin\big(\sqrt{\xi_0\xi_1}\theta_2(r,s,\tau)\big)}\right). \end{aligned} \tag{42}$$

(B) If $\xi_0\xi_1 < 0$, we have the soliton solution:

$$\begin{aligned} x_4(r,s,\tau) &= \frac{-9(\bar{\mu}_1^2k^2(s,\tau) - 8\alpha^4\xi_0^5\xi_1 m^2(s,\tau))}{4\alpha^2\xi_0^2k^2l(s,\tau)m(s,\tau)} - \frac{2\alpha^2\xi_0|\xi_0\xi_1|m(s,\tau)}{k(s,\tau)} \\ &\times \left(\frac{R\sinh\Big(2\sqrt{|\xi_0\xi_1|}\theta_2(r,s,\tau)\Big) + R\cosh\Big(2\sqrt{|\xi_0\xi_1|}\theta_2(r,s,\tau)\Big) - S}{R\sinh\Big(\sqrt{2|\xi_0\xi_1|}\theta_2(r,s,\tau)\Big) + R\cosh\Big(\sqrt{2|\xi_0\xi_1|}\theta_2(r,s,\tau)\Big) + S}\right)^2, \end{aligned} \tag{43}$$

$$\begin{aligned} y_4(r,s,\tau) &= \frac{\begin{pmatrix} -9\bar{\mu}_1^2(3\xi_0 + k(s,\tau))k^2(s,\tau) + 8\alpha^4\xi_0^4\xi_1(4(m(s,\tau) \\ -l(s,\tau))k^2(s,\tau)l(s,\tau) + 27\xi_0^2m(s,\tau))m(s,\tau) \end{pmatrix}}{4\alpha^2\xi_0^2k^2(s,\tau)l^2(s,\tau)m(s,\tau)} + \frac{|\xi_0\xi_1|}{\xi_0}\left(\frac{3\bar{\mu}_1}{l(s,\tau)}\right) \\ &\times \left(\frac{R\sinh\Big(\sqrt{2|\xi_0\xi_1|}\theta_2(r,s,\tau)\Big) + R\cosh\Big(\sqrt{2|\xi_0\xi_1|}\theta_2(r,s,\tau)\Big) + S}{R\sinh\Big(\sqrt{2|\xi_0\xi_1|}\theta_2(r,s,\tau)\Big) + R\cosh\Big(\sqrt{2|\xi_0\xi_1|}\theta_2(r,s,\tau)\Big) - S}\right), \end{aligned} \tag{44}$$

where

$$\theta_2(r,s,\tau) = \alpha\left(\frac{(\delta(r))^\pi}{\pi}\right) + \frac{1}{4\alpha\xi_0^2}\int_0^s \frac{(9\bar{\mu}_1^2k(u,\tau) - 32\alpha^4\xi_0^4\xi_1 l(u,\tau)m^2(u,\tau))\delta'(u)}{l(u,\tau)m(u,\tau)(\delta(u))^{1-\pi}}du. \tag{45}$$

Set III.
(A) If $\xi_0\xi_1 > 0$, we acquire the periodic solution:

$$x_5(r,s,\tau) = \frac{\alpha^2\xi_0\xi_1}{9}(l(s,\tau)m(s,\tau)-12)\left(\frac{R\cos(\sqrt{\xi_0\xi_1}\theta_3(r,s,\tau))+S\sin(\sqrt{\xi_0\xi_1}\theta_3(r,s,\tau))}{S\cos(\sqrt{\xi_0\xi_1}\theta_3(r,s,\tau))-R\sin(\sqrt{\xi_0\xi_1}\theta_2(r,s,\tau))}\right)^2$$
$$+\sqrt{\frac{\xi_1}{\xi_0}}\left(\frac{5\bar{v}_1 l(s,\tau)}{l(s,\tau)m(s,\tau)+12}\right)\left(\frac{S\cos(\sqrt{\xi_0\xi_1}\theta_3(r,s,\tau))-R\sin(\sqrt{\xi_0\xi_1}\theta_3(r,s,\tau))}{R\cos(\sqrt{\xi_0\xi_1}\theta_3(r,s,\tau))+S\sin(\sqrt{\xi_0\xi_1}\theta_3(r,s,\tau))}\right), \tag{46}$$

$$\begin{aligned} y_5(r,s,\tau) &= \alpha^2\xi_1 m(s,\tau)\left(\frac{R\cos(\sqrt{\xi_0\xi_1}\theta_3(r,s,\tau))+S\sin(\sqrt{\xi_0\xi_1}\theta_3(r,s,\tau))}{S\cos(\sqrt{\xi_0\xi_1}\theta_3(r,s,\tau))-R\sin(\sqrt{\xi_0\xi_1}\theta_3(r,s,\tau))}\right)^2 \\ &- \frac{12\alpha^2|\xi_0|\xi_1}{|\xi_1|}\left(\frac{S\cos(\sqrt{\xi_0\xi_1}\theta_3(r,s,\tau))-R\sin(\sqrt{\xi_0\xi_1}\theta_3(r,s,\tau))}{R\cos(\sqrt{\xi_0\xi_1}\theta_3(r,s,\tau))+S\sin(\sqrt{\xi_0\xi_1}\theta_2(r,s,\tau))}\right)^2. \end{aligned} \tag{47}$$

(B) If $\xi_0\xi_1 < 0$, we obtain the soliton solution:

$$\begin{aligned} x_6(r,s,\tau) &= \frac{\alpha^2|\xi_0\xi_1|}{9}(l(s,\tau)m(s,\tau)-12) \\ &\times \left(\frac{R\sinh\left(\sqrt{2|\xi_0\xi_1|}\theta_3(r,s,\tau)\right)+R\cosh\left(\sqrt{2|\xi_0\xi_1|}\theta_3(r,s,\tau)\right)+S}{R\sinh\left(\sqrt{2|\xi_0\xi_1|}\theta_3(r,s,\tau)\right)+R\cosh\left(\sqrt{2|\xi_0\xi_1|}\theta_3(r,s,\tau)\right)-S}\right)^2 \\ &- \frac{\xi_1}{\sqrt{|\xi_0\xi_1|}}\left(\frac{5\bar{v}_1 l(s,\tau)}{l(s,\tau)m(s,\tau)+12}\right) \\ &\times \left(\frac{R\sinh\left(2\sqrt{|\xi_0\xi_1|}\theta_3(r,s,\tau)\right)+R\cosh\left(2\sqrt{|\xi_0\xi_1|}\theta_3(r,s,\tau)\right)-S}{R\sinh\left(\sqrt{2|\xi_0\xi_1|}\theta_3(r,s,\tau)\right)+R\cosh\left(\sqrt{2|\xi_0\xi_1|}\theta_3(r,s,\tau)\right)+S}\right), \end{aligned} \tag{48}$$

$$\begin{aligned} y_6(r,s,\tau) &= \alpha^2|\xi_0\xi_1|m(s,\tau) \\ &\times \left(\frac{R\sinh\left(\sqrt{2|\xi_0\xi_1|}\theta_3(r,s,\tau)\right)+R\cosh\left(\sqrt{2|\xi_0\xi_1|}\theta_3(r,s,\tau)\right)+S}{R\sinh\left(\sqrt{2|\xi_0\xi_1|}\theta_3(r,s,\tau)\right)+R\cosh\left(\sqrt{2|\xi_0\xi_1|}\theta_3(r,s,\tau)\right)-S}\right)^2 \\ &- \frac{12\alpha^2|\xi_0|\xi_1}{|\xi_1|} \\ &\times \left(\frac{R\sinh\left(2\sqrt{|\xi_0\xi_1|}\theta_3(r,s,\tau)\right)+R\cosh\left(2\sqrt{|\xi_0\xi_1|}\theta_3(r,s,\tau)\right)-S}{R\sinh\left(\sqrt{2|\xi_0\xi_1|}\theta_3(r,s,\tau)\right)+R\cosh\left(\sqrt{2|\xi_0\xi_1|}\theta_3(r,s,\tau)\right)+S}\right)^2, \end{aligned} \tag{49}$$

where

$$\theta_3(r,s,\tau) = \alpha\left(\frac{(\delta(r))^\pi}{\pi}\right) + \frac{1}{24\alpha\xi_0^2\xi_1} \tag{50}$$
$$\times\int_0^s \frac{\left(\begin{array}{c}(\bar{v}_1^2\xi_1(l(u,\tau)m(u,\tau)-3)l^2(u,\tau)-8\alpha^4\xi_0^3\xi_1^2(l(u,\tau)m(u,\tau)\\ +36)(l(u,\tau)m(u,\tau)+12))\delta'(u)\end{array}\right)}{(l(u,\tau)m(u,\tau)+12)(\delta(u))^{1-\pi}}du. \tag{51}$$

Obviously, by applying other cases to the algebraic system (22)–(32), one may obtain distinct deterministic wave solutions of Equation (18). We have only exhibited the above three sets of solutions to highlight the effectiveness of our algorithm.

4.2. *Stochastic Solutions of Equation* (17)

Based on the characteristics of trigonometric and hyperbolic functions, we can allocate an open bounded region $\mathcal{B} \subset \mathbb{R}\times\mathbb{R}_+$, $\chi < \infty$ and $\Delta > 0$ such that the solution $\{x(r,s,\tau), y(r,s,\tau)\}$ and its derivatives, which are included in Equation (18), are analytic on $\mathcal{N}(\chi,\Delta)$ for all $(r,s)\in\mathcal{B}$, continuous on $\mathcal{B}$ for all $\tau\in\mathcal{N}(\chi,\Delta)$, and uniformly bounded on $\mathcal{B}\times\mathcal{N}(\chi,\Delta)$. On taking the inverse Hermite transform for the wave variables (40), (45), and (50), we have their stochastic Wick versions as follows:

$$\Theta_1(r,s) = \alpha\left(\frac{(\delta(r))^\pi}{\pi}\right) - 8\alpha^3\xi_0\xi_1\int_0^s \frac{M(u)\delta'(u)}{(\delta(u))^{1-\pi}}du, \tag{52}$$

$$\Theta_2(r,s) = \alpha\left(\frac{(\delta(r))^{\pi}}{\pi}\right) + \frac{1}{4\alpha\xi_0^2}\int_0^s \frac{(9\bar{\mu}_1^2 K(u) - 32\alpha^4\xi_0^4\xi_1 L(u)\diamond M^{\diamond 2}(u))\delta'(u)}{L(u)\diamond M(u)(\delta(u))^{1-\pi}}du, \quad (53)$$

$$\begin{aligned}\Theta_3(r,s) &= \alpha\left(\frac{(\delta(r))^{\pi}}{\pi}\right)\\ &+ \frac{1}{24\alpha\xi_0^2\xi_1}\int_0^s \frac{\begin{pmatrix}(\bar{v}_1^2\xi_1(L(u)\diamond M(u)-3)L^{\diamond 2}(u) - 8\alpha^4\xi_0^3\xi_1^2(L(u)\diamond M(u)\\ +36)\diamond(L(u)\diamond M(u)+12))\delta'(u)\end{pmatrix}}{(L(u)\diamond M(u)+12)(\delta(u))^{1-\pi}}du. \quad (54)\end{aligned}$$

Moreover, by Theorem 4.1.1 of [40], there exists $X(r,s)\in(Z)_{-1}$ such that $x(r,s,\tau) = \widetilde{X}(r,s,\tau)$ for all $(r,s,\tau)\in\mathcal{B}\times\mathcal{N}(\chi,\Delta)$. As a result, we infer several stochastic wave solutions of Equation (17) via applying the inverse Hermite transform to the solutions (36)–(50) of Equation (18).

4.3. Stochastic Wave Solutions of a Periodic Kind

If $(\xi_0\xi_1 > 0)$, we have the following periodic wave solutions of Equation (17).

$$\begin{aligned}X_1(r,s) &= \mathcal{H}^{-1}(x_1(r,s,\tau)) = \frac{-12\alpha^3\xi_0^2\xi_1(3+L(s)\diamond M(s))}{K(s)\diamond L(s)} + \frac{12\alpha^2\xi_0^2\xi_1 M(s)}{K(s)}\\ &\diamond\left(\frac{S\cos^{\diamond}(\sqrt{\xi_0\xi_1}\Theta_1(r,s)) - R\sin^{\diamond}(\sqrt{\xi_0\xi_1}\Theta_1(r,s))}{R\cos^{\diamond}(\sqrt{\xi_0\xi_1}\Theta_1(r,s)) + S\sin^{\diamond}(\sqrt{\xi_0\xi_1}\Theta_1(r,s))}\right)^{\diamond 2}, \quad (55)\end{aligned}$$

$$\begin{aligned}Y_1(r,s) &= \mathcal{H}^{-1}(y_1(r,s,\tau))\\ &= \frac{-4\alpha^2\xi_0\xi_1(9\xi_0(L(s)\diamond M(s)+3) + (9-2(M(s)-L(s)))\diamond K(s)\diamond L(s))}{K(s)\diamond L^{\diamond 2}(s)}\\ &- \frac{12\alpha^2\xi_0\xi_1}{L(s)}\diamond\left(\frac{R\cos^{\diamond}(\sqrt{\xi_0\xi_1}\Theta_1(r,s)) + S\sin^{\diamond}(\sqrt{\xi_0\xi_1}\Theta_1(r,s))}{S\cos^{\diamond}(\sqrt{\xi_0\xi_1}\Theta_1(r,s)) - R\sin^{\diamond}(\sqrt{\xi_0\xi_1}\Theta_1(r,s))}\right)^{\diamond 2}, \quad (56)\end{aligned}$$

$$\begin{aligned}X_2(r,s) &= \mathcal{H}^{-1}(x_3(r,s,\tau)) = \frac{-9(\bar{\mu}_1^2 K^{\diamond 2}(s) - 8\alpha^4\xi_0^5\xi_1 M^{\diamond 2}(s))}{4\alpha^2\xi_0^2 K^{\diamond 2}\diamond L(s)\diamond M(s)} - \frac{2\alpha^2\xi_0^2\xi_1 M(s)}{K(s)}\\ &\times\left(\frac{S\cos^{\diamond}(\sqrt{\xi_0\xi_1}\Theta_2(r,s)) - R\sin^{\diamond}(\sqrt{\xi_0\xi_1}\Theta_2(r,s))}{R\cos^{\diamond}(\sqrt{\xi_0\xi_1}\Theta_2(r,s)) + S\sin^{\diamond}(\sqrt{\xi_0\xi_1}\Theta_2(r,s))}\right)^{\diamond 2}, \quad (57)\end{aligned}$$

$$\begin{aligned}Y_2(r,s) &= \mathcal{H}^{-1}(y_3(r,s,\tau)) = \frac{\begin{pmatrix}-9\bar{\mu}_1^2(3\xi_0 + K(s))\diamond K^{\diamond 2}(s) + 8\alpha^4\xi_0^4\xi_1(4(M(s)\\ -L(s))\diamond K^{\diamond 2}(s)\diamond L(s) + 27\xi_0^2 M(s))\diamond M(s)\end{pmatrix}}{4\alpha^2\xi_0^2 K^{\diamond 2}(s)\diamond L^{\diamond 2}(s)\diamond M(s)}\\ &+ \sqrt{\frac{\xi_1}{\xi_0}}\left(\frac{3\bar{\mu}_1}{L(s)}\right)\left(\frac{R\cos^{\diamond}(\sqrt{\xi_0\xi_1}\Theta_2(r,s)) + S\sin^{\diamond}(\sqrt{\xi_0\xi_1}\Theta_2(r,s))}{S\cos^{\diamond}(\sqrt{\xi_0\xi_1}\Theta_2(r,s)) - R\sin^{\diamond}(\sqrt{\xi_0\xi_1}\Theta_2(r,s))}\right), \quad (58)\end{aligned}$$

$$\begin{aligned}X_3(r,s) &= \mathcal{H}^{-1}(x_5(r,s,\tau)) = \frac{\alpha^2\xi_0\xi_1}{9}(L(s)\diamond M(s) - 12)\\ &\diamond\left(\frac{R\cos^{\diamond}(\sqrt{\xi_0\xi_1}\Theta_3(r,s)) + S\sin^{\diamond}(\sqrt{\xi_0\xi_1}\Theta_3(r,s))}{S\cos^{\diamond}(\sqrt{\xi_0\xi_1}\Theta_3(r,s)) - R\sin^{\diamond}(\sqrt{\xi_0\xi_1}\Theta_3(r,s))}\right)^{\diamond 2}\\ &+ \sqrt{\frac{\xi_1}{\xi_0}}\left(\frac{5\bar{v}_1 L(s)}{L(s)\diamond M(s)+12}\right)\\ &\diamond\left(\frac{S\cos^{\diamond}(\sqrt{\xi_0\xi_1}\Theta_3(r,s)) - R\sin^{\diamond}(\sqrt{\xi_0\xi_1}\Theta_3(r,s))}{R\cos^{\diamond}(\sqrt{\xi_0\xi_1}\Theta_3(r,s)) + S\sin^{\diamond}(\sqrt{\xi_0\xi_1}\Theta_3(r,s))}\right), \quad (59)\end{aligned}$$

$$\begin{aligned}Y_3(r,s,\tau) &= \mathcal{H}^{-1}(y_5(r,s,\tau)) = \alpha^2\xi_1 M(s)\\ &\diamond\left(\frac{R\cos^{\diamond}(\sqrt{\xi_0\xi_1}\Theta_3(r,s)) + S\sin^{\diamond}(\sqrt{\xi_0\xi_1}\Theta_3(r,s))}{S\cos^{\diamond}(\sqrt{\xi_0\xi_1}\Theta_3(r,s)) - R\sin^{\diamond}(\sqrt{\xi_0\xi_1}\Theta_3(r,s))}\right)^{\diamond 2}\\ &- \frac{12\alpha^2|\xi_0|\xi_1}{|\xi_1|}\left(\frac{S\cos(\sqrt{\xi_0\xi_1}\Theta_3(r,s)) - R\sin(\sqrt{\xi_0\xi_1}\Theta_3(r,s))}{R\cos(\sqrt{\xi_0\xi_1}\Theta_3(r,s)) + S\sin(\sqrt{\xi_0\xi_1}\Theta_3(r,s))}\right)^{2}, \quad (60)\end{aligned}$$

where $\Theta_1(r,s)$, $\Theta_2(r,s)$, and $\Theta_3(r,s)$ are determined by (52), (53), and (54), respectively.

4.4. Stochastic Wave Solutions of a Soliton Kind

If $(\xi_0\xi_1 < 0)$, we have the following soliton wave solutions of Equation (17).

$$\begin{aligned} X_4(r,s) &= \mathcal{H}^{-1}(x_2(r,s,\tau)) = \frac{-12\alpha^3\xi_0^2\xi_1(3+L(s)\diamond M(s))}{K(s)\diamond L(s)} + \frac{12\alpha^2\xi_0|\xi_0\xi_1|M(s)}{K(s)} \\ &\diamond \left(\frac{R\sinh^\diamond\left(2\sqrt{|\xi_0\xi_1|}\Theta_1(r,s)\right)+R\cosh^\diamond\left(2\sqrt{|\xi_0\xi_1|}\Theta_1(r,s)\right)-S}{R\sinh^\diamond\left(\sqrt{2|\xi_0\xi_1|}\Theta_1(r,s)\right)+R\cosh^\diamond\left(\sqrt{2|\xi_0\xi_1|}\Theta_1(r,s)\right)+S}\right)^{\diamond 2}, \end{aligned} \tag{61}$$

$$\begin{aligned} Y_4(r,s,\tau) &= \mathcal{H}^{-1}(y_2(r,s,\tau)) \\ &= \frac{-4\alpha^2\xi_0\xi_1(9\xi_0(L(s)\diamond M(s)+3)+(9-2(M(s)-L(s)))\diamond K(s)\diamond L(s))}{K(s)\diamond L^{\diamond 2}(s)} \\ &- \frac{12\alpha^2|\xi_0|^{3/2}|\xi_1|}{\xi_1 L(s)} \\ &\diamond \left(\frac{R\sinh^\diamond\left(\sqrt{2|\xi_0\xi_1|}\Theta_1(r,s)\right)+R\cosh^\diamond\left(\sqrt{2|\xi_0\xi_1|}\Theta_1(r,s)\right)+S}{R\sinh^\diamond\left(\sqrt{2|\xi_0\xi_1|}\Theta_1(r,s)\right)+R\cosh^\diamond\left(\sqrt{2|\xi_0\xi_1|}\Theta_1(r,s)\right)-S}\right)^{\diamond 2}, \end{aligned} \tag{62}$$

$$\begin{aligned} X_5(r,s) &= \mathcal{H}^{-1}(x_4(r,s,\tau)) = \frac{-9(\bar{\mu}_1^2K^{\diamond 2}(s)-8\alpha^4\xi_0^5\xi_1M^{\diamond 2}(s))}{4\alpha^2\xi_0^2K^{\diamond 2}(s)\diamond L(s)\diamond M(s)} - \frac{2\alpha^2\xi_0|\xi_0\xi_1|M(s)}{K(s)} \\ &\times \left(\frac{R\sinh^\diamond\left(2\sqrt{|\xi_0\xi_1|}\Theta_2(r,s)\right)+R\cosh^\diamond\left(2\sqrt{|\xi_0\xi_1|}\Theta_2(r,s)\right)-S}{R\sinh^\diamond\left(\sqrt{2|\xi_0\xi_1|}\Theta_2(r,s)\right)+R\cosh^\diamond\left(\sqrt{2|\xi_0\xi_1|}\Theta_2(r,s)\right)+S}\right)^{\diamond 2}, \end{aligned} \tag{63}$$

$$\begin{aligned} Y_5(r,s) &= \mathcal{H}^{-1}(y_4(r,s,\tau)) \\ &= \frac{\begin{pmatrix} -9\bar{\mu}_1^2(3\xi_0+K(s))\diamond K^{\diamond 2}(s)+8\alpha^4\xi_0^4\xi_1(4(M(s) \\ -L(s))K^{\diamond 2}(s)\diamond L(s)+27\xi_0^2M(s))\diamond M(s) \end{pmatrix}}{4\alpha^2\xi_0^2K^{\diamond 2}(s)\diamond L^{\diamond 2}(s)\diamond M(s)} + \frac{|\xi_0\xi_1|}{\xi_0}\left(\frac{3\bar{\mu}_1}{L(s)}\right) \\ &\diamond \left(\frac{R\sinh^\diamond\left(\sqrt{2|\xi_0\xi_1|}\Theta_2(r,s)\right)+R\cosh^\diamond\left(\sqrt{2|\xi_0\xi_1|}\Theta_2(r,s)\right)+S}{R\sinh^\diamond\left(\sqrt{2|\xi_0\xi_1|}\Theta_2(r,s)\right)+R\cosh^\diamond\left(\sqrt{2|\xi_0\xi_1|}\Theta_2(r,s)\right)-S}\right), \end{aligned} \tag{64}$$

$$\begin{aligned} X_6(r,s) &= \mathcal{H}^{-1}(x_6(r,s,\tau)) = \frac{\alpha^2|\xi_0\xi_1|}{9}(L(s)\diamond M(s)-12) \\ &\diamond \left(\frac{R\sinh^\diamond\left(\sqrt{2|\xi_0\xi_1|}\Theta_3(r,s)\right)+R\cosh^\diamond\left(\sqrt{2|\xi_0\xi_1|}\Theta_3(r,s)\right)+S}{R\sinh^\diamond\left(\sqrt{2|\xi_0\xi_1|}\Theta_3(r,s)\right)+R\cosh^\diamond\left(\sqrt{2|\xi_0\xi_1|}\Theta_3(r,s)\right)-S}\right)^{\diamond 2} \\ &- \frac{\xi_1}{\sqrt{|\xi_0\xi_1|}}\left(\frac{5\bar{\nu}_1L(s)}{L(s)\diamond M(s)+12}\right) \\ &\diamond \left(\frac{R\sinh^\diamond\left(2\sqrt{|\xi_0\xi_1|}\Theta_3(r,s)\right)+R\cosh^\diamond\left(2\sqrt{|\xi_0\xi_1|}\Theta_3(r,s)\right)-S}{R\sinh^\diamond\left(\sqrt{2|\xi_0\xi_1|}\Theta_3(r,s)\right)+R\cosh^\diamond\left(\sqrt{2|\xi_0\xi_1|}\Theta_3(r,s)\right)+S}\right), \end{aligned} \tag{65}$$

$$\begin{aligned} Y_6(r,s) &= \mathcal{H}^{-1}(y_6(r,s,\tau)) = \alpha^2|\xi_0\xi_1|M(s) \\ &\diamond \left(\frac{R\sinh^\diamond\left(\sqrt{2|\xi_0\xi_1|}\Theta_3(r,s)\right)+R\cosh^\diamond\left(\sqrt{2|\xi_0\xi_1|}\Theta_3(r,s)\right)+S}{R\sinh^\diamond\left(\sqrt{2|\xi_0\xi_1|}\Theta_3(r,s)\right)+R\cosh^\diamond\left(\sqrt{2|\xi_0\xi_1|}\Theta_3(r,s)\right)-S}\right)^{\diamond 2} \\ &- \frac{12\alpha^2|\xi_0|\xi_1}{|\xi_1|} \\ &\times \left(\frac{R\sinh^\diamond\left(2\sqrt{|\xi_0\xi_1|}\Theta_3(r,s)\right)+R\cosh^\diamond\left(2\sqrt{|\xi_0\xi_1|}\Theta_3(r,s)\right)-S}{R\sinh^\diamond\left(\sqrt{2|\xi_0\xi_1|}\Theta_3(r,s)\right)+R\cosh^\diamond\left(\sqrt{2|\xi_0\xi_1|}\Theta_3(r,s)\right)+S}\right)^{\diamond 2}, \end{aligned} \tag{66}$$

where $\Theta_1(r,s)$, $\Theta_2(r,s)$, and $\Theta_3(r,s)$ are specified by (52), (53), and (54), respectively.

5. Graphical Interpretation and Ultimate Remarks

The graphical interpretations for the obtained solutions and some ultimate remarks are included in this section.

It is clear that a change in the forms of the given functions $K(s)$, $L(s)$, and $M(s)$ leads to a change in the solutions we obtained in Equations (55) and (66). So, we can obtain different solutions of Equation (17) via assuming different shapes for the functions $K(s)$, $L(s)$, and $M(s)$. Below, we detail this actuality for the solution $\{X_1, Y_1\}$. For the solutions $\{X_2, Y_2\} - \{X_6, Y_6\}$, the procedures are the same. We presume that $L(s) = \lambda_1 K(s)$, $M(s) = \lambda_2 K(s)$, and $K(s) = r(s) + \frac{\lambda_3(\delta(s))^{1-\pi}}{\delta'(s)}\mathbb{I}(s)$, where λ_1, λ_2, and λ_3 are constants, $r(s)$ is an π-integrable function on $\mathbb{R}_+$, and $\mathbb{I}(s)$ is the white Gaussian noise, which is the first derivative of the single parameter Brownian motion $\mathbb{J}(s)$. The white Gaussian noise has the Hermite transform $\widetilde{\mathbb{I}}(s,\tau) = \sum_{i=1}^{\infty} \tau_i \int_0^s \epsilon(u)du$ [40]. Hence, the utilization of the Hermite transform $\mathbb{I}(s,\tau)$, the equality $\exp^{\diamond}(\mathbb{J}(s)) = \exp\left(\mathbb{J}(s) - \frac{s^2}{2}\right)$ [40], and Equations (52), (55), and (56) leads us to the next stochastic functional solution for Equation (17) of white noise and the Brownian motion kind.

$$\begin{aligned} X_7(r,s) &= \frac{-12\alpha^3\xi_0^2\xi_1(3+\lambda_1\lambda_2K^2(s))}{\lambda_1K^2(s)} \\ &+ 12\lambda_2\alpha^2\xi_0^2\xi_1K(s)\left(\frac{S\cos\left(\sqrt{\xi_0\xi_1}\Theta_1^*(r,s)\right) - R\sin\left(\sqrt{\xi_0\xi_1}\Theta_1^*(r,s)\right)}{R\cos\left(\sqrt{\xi_0\xi_1}\Theta_1^*(r,s)\right) + S\sin\left(\sqrt{\xi_0\xi_1}\Theta_1^*(r,s)\right)}\right)^2, \end{aligned} \tag{67}$$

$$\begin{aligned} Y_7(r,s) &= \frac{-4\alpha^2\xi_0\xi_1(9\xi_0(\lambda_1\lambda_2K^2(s)+3)+\lambda_1(9-2(\lambda_2-\lambda_1)K(s))K^2(s))}{\lambda_1^3K^3(s)} \\ &- \frac{12\alpha^2\xi_0\xi_1}{\lambda_1K(s)}\left(\frac{R\cos\left(\sqrt{\xi_0\xi_1}\Theta_1^*(r,s)\right) + S\sin\left(\sqrt{\xi_0\xi_1}\Theta_1^*(r,s)\right)}{S\cos\left(\sqrt{\xi_0\xi_1}\Theta_1^*(r,s)\right) - R\sin\left(\sqrt{\xi_0\xi_1}\Theta_1^*(r,s)\right)}\right)^2, \end{aligned} \tag{68}$$

where

$$\Theta_1^*(r,s) = \alpha\left(\frac{(\delta(r))^{\pi}}{\pi}\right) - 8\alpha^3\lambda_2\xi_0\xi_1\left[\int_0^s \frac{r(u)\delta'(u)}{(\delta(u))^{1-\pi}}du + \lambda_3\left(\mathbb{J}(s) - \frac{s^2}{2}\right)\right]. \tag{69}$$

For $\pi = 0.5$ and 1, the graphical interpretation of the solution (67)–(69) is presented in Figures 1 and 2, when $R = S = \lambda_1 = 2\lambda_2 = \lambda_3 = \xi_0 = \xi_1 = 4\alpha = 1$, and $\delta(s) = r(s) = \sin^2(s)$. In the absence of the stochastic influence ($\mathbb{J}(s) = 0$), the evolutionary conduct of the solution (67)–(69) is shown in Figure 1. Figure 2 illustrates the stochastic evolutionary conduct of the solution (67)–(69) under the influence of the stochastic forces $\mathbb{J}(s) = \tan(0.7s) \times \text{Random}[0,1]$ and $\mathbb{I}(s) = 0.7\sec^2(0.7s) \times \text{Random}[0,1]$. According to Figures 1 and 2, one can conclude that the traveling wave amplitudes are unpredictable due to the stochastic driving components. Moreover, the aggregate impact of the generalized conformable derivatives may be depicted as a specific type of velocities whose directions are determined by the conformable parameter π.

Remark 1. *Many branches of nonlinear sciences rely heavily on wave solutions of nonlinear PDEs. The wave solutions of periodic and soliton kinds are the most often used wave solutions. In Equations* (55)–(66)*, we have extracted new stochastic versions of periodic and soliton solutions for the stochastic CNKdV equations under recently generalized derivatives. The utility of periodic solutions appears in a variety of physical scopes, including collisionless plasmas, diffusion-advection systems impulsive systems, and more [41–43]. Moreover, the involvement of soliton solutions may also be seen in several physical areas, such as optical fibers, self-reinforcing systems, nuclear physics, controllability, and so on [44,45]. In actuality, the employed conformable derivatives lead to a particular rate of change relying on the conformable order π. So, depending on space, time, and the conformable order π, the acquired soliton and periodic solutions in* (55)–(66) *allow us to witness the slowdown or speedup in the monotonicity and symmetry of the researched stochastic nonlinear waves.*

Remark 2. *It is clear that the extracted results are comprehensive and include many published results as special cases. For instances, if $\delta(s) = s$, the produced solutions* (55)–(66) *can be diminished to a new set of wave solutions for the stochastic CNKdV equations with the standard conformable derivatives of Khalil et al. [9]. Moreover, If $\pi = 1$, the solutions* (55)–(66) *turn into a novel gathering of wave solutions for the stochastic CNKdV equations with traditional Newton's derivatives [46].*

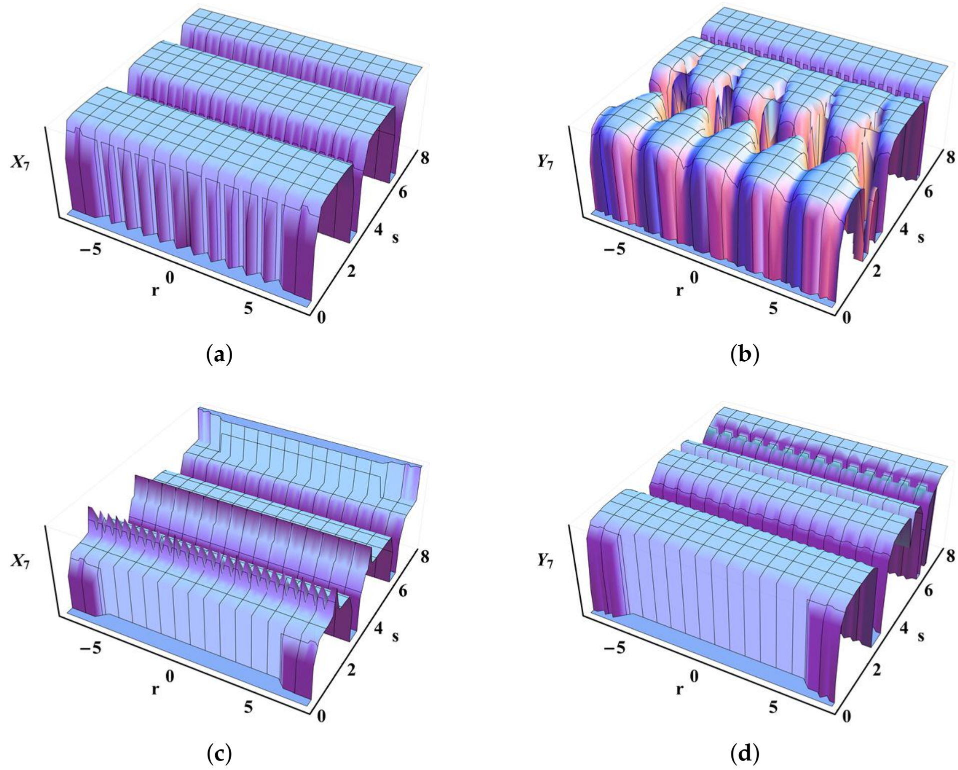

Figure 1. In the absence of the noise influence, (**a**,**b**) are 3D graphs for the solution (67)–(69) when $\pi = 0.5$, while, (**c**,**d**) are 3D graphs of the solution (67)–(69) when $\pi = 1$.

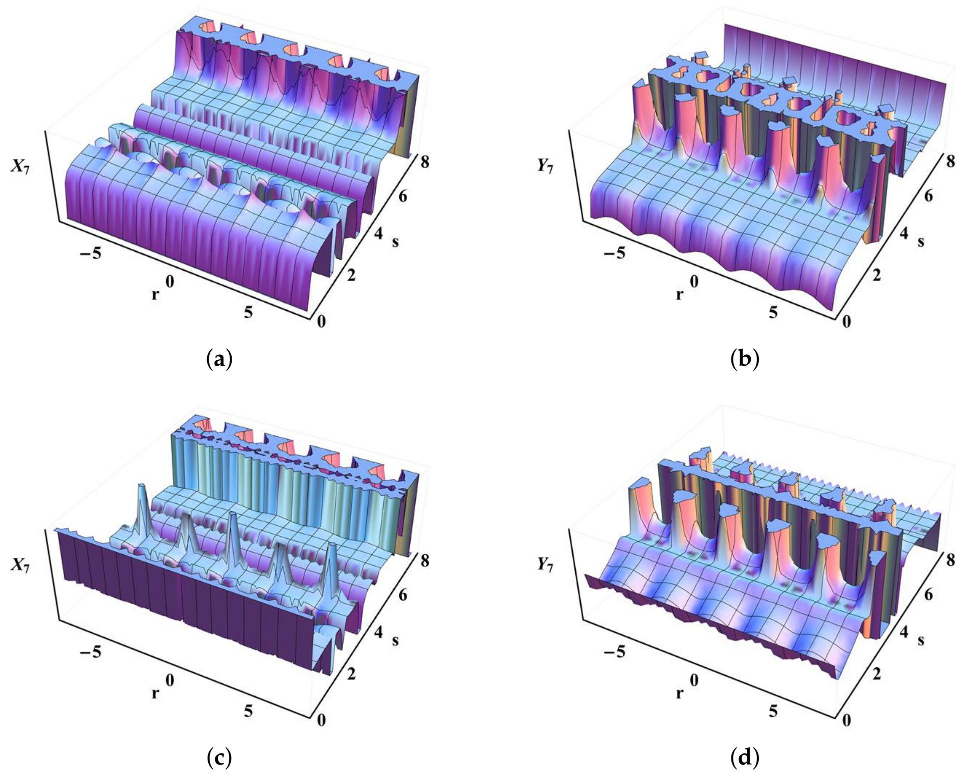

Figure 2. Under the noise influence, (**a**,**b**) are 3D graphs for the solution (67)–(69) when $\pi = 0.5$, while, (**c**,**d**) are 3D graphs of the solution (67)–(69) when $\pi = 1$.

6. Conclusions

In this article, we established deterministic and stochastic accurate solutions for the CNKdV equations with recently generalized conformable derivatives. New stochastic versions of periodic and soliton solutions for the these equations under conformable generalized derivatives were extracted. Some graphical interpretations and ultimate remarks were given to illustrate the stochastic evolutionary conduct of the solutions under the influence of the stochastic forces. It is notable that the presented solutions show the aggregate impact of the generalized conformable derivatives as specific type of velocities whose directions are determined by a conformable parameter. Moreover, we showed that the acceleration of the monotonicity and symmetry for the stochastic nonlinear wave solutions can be controlled via the conformable order π. Further, we gave a comparison between the wave solutions for the stochastic CNKdV equations under conformable generalized derivatives and the wave solutions constructed under the conformable derivatives of Khalil et al. [9] and Newton's derivatives [46].

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