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New Obstructions to Warped Product Immersions in Complex Space Forms

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Abstract: In this paper, we obtain a geometric inequality for warped product pointwise semi-slant submanifolds of complex space forms endowed with a semi-symmetric metric connection and discuss the equality case of this inequality. We provide some applications concerning the minimality and compactness of such submanifolds.

Keywords: complex space form; semi-symmetric metric connection; warped product; pointwise semi-slant submanifold; minimal submanifold; geometric inequality

MSC: 53B05; 53B25; 53B20; 53C15; 53C40

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1. Introduction

In 1969, R.L. Bishop et al. [1] defined the notion of warped product manifolds, by generalizing the Riemannian product manifolds, in order to study manifolds of negative sectional curvature. Since then, it has remained a topic of research due to its usefulness and links to other fields, especially physics. Many research articles have been published in this area [2–8].

In differential geometry, one of the fundamental problems is the immersibility of a Riemannian manifold in a space form. According to the very famous Nash's embedding theorem, every Riemannian manifold can be isometrically immersed in some Euclidean spaces with sufficiently high codimensions. Starting from this theorem, B.-Y. Chen discovered a method to study intrinsic and extrinsic invariants of a submanifold and provided many applications. For example, for isometric warped product immersion $\varphi : M_1 \times_f M_2 \to \overline{M}(c)$ into a Riemannian space form, the following inequality holds ([2]):

$$\frac{\Delta f}{f} \leq \frac{(n_1 + n_2)^2}{4n_2} \|H\|^2 + n_1 c, \tag{1}$$

where $n_i = \dim M_i,\ i = 1, 2,\ \|H\|^2$ is the squared mean curvature of the immersion φ, and Δ is the Laplace operator of M_1. In the same paper [2], the author discussed the equality case of this inequality.

Motivated by the inequality (1), many researchers proved corresponding inequalities for different classes of submanifolds in various space forms. In these papers, the space forms are endowed with the Levi–Civita connection, which is torsion-free.

An important class of connections with non-vanishing torsion are the semi-symmetric connections. They have many applications in affine differential geometry, information geometry, etc.

In the present paper, we consider complex space forms endowed with semi-symmetric metric connections. We extend the above-mentioned result of Chen and prove a geometric inequality for warped product pointwise semi-slant submanifolds $M = M_1 \times_f M_2$ in a complex space form $\overline{M}(c)$ endowed with a semi-symmetric metric connection. We also discuss the equality case and provide several applications in the compact case and the minimal case, respectively.

2. Preliminaries

Let $(\overline{M}, J, g)$ be an almost Hermitian manifold, where J is an almost complex structure and g a Hermitian metric. Then, $\overline{M}$ is a *Kaehler manifold* if $(\overline{\nabla}_X J)Y = 0$, for all $X, Y \in T\overline{M}$, where $\overline{\nabla}$ is the Levi–Civita connection of the Riemannian metric g.

A *complex space form* $\overline{M}(c)$ is a Kaehler manifold of constant holomorphic sectional curvature c; its Riemannian curvature tensor $\overline{R}$ is given by

$$\begin{aligned}\overline{R}(X,Y)Z = \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y\} \\ + g(X,JZ)JY - g(Y,JZ)JX + 2g(X,JY)JZ\},\end{aligned} \tag{2}$$

for all $X, Y, Z \in T\overline{M}$.

Let $\overline{M}$ be an almost Hermitian manifold and M a submanifold of $\overline{M}$ with induced metric g. Let ∇ be the Levi–Civita connection on the tangent bundle TM and $\nabla^{\perp}$ the connection on the normal bundle $T^{\perp}M$ of M. Then, the Gauss and Weingarten formulae are

$$\begin{aligned}\overline{\nabla}_X Y &= \nabla_X Y + h(X,Y), \\ \overline{\nabla}_X N &= -A_N X + \nabla^{\perp}_X N,\end{aligned}$$

where $X, Y \in TM$, $N \in T^{\perp}M$ and h, A_N are the second fundamental form and the shape operator, respectively.

The relationship between the shape operator and the second fundamental form is

$$g(h(X,Y), N) = g(A_N X, Y),$$

for vector fields $X, Y \in TM$ and $N \in T^{\perp}M$.

Let $\overline{R}$ and R be the Riemannian curvature tensors of $\overline{M}$ and M, respectively. We use the notation $R(X,Y,Z,W) = g(R(X,Y)Z, W)$, for any $X, Y, Z, W \in TM$. Then, the Gauss equation is given by

$$\begin{aligned}\overline{R}(X,Y,Z,W) = R(X,Y,Z,W) \\ + g(h(X,Z), h(Y,W)) - g(h(X,W), h(Y,Z)),\end{aligned} \tag{3}$$

for any $X, Y, Z, W \in TM$.

The notion of a semi-symmetric linear connection was introduced by Friedmann and Schouten [9]. Let $(\overline{M}, g)$ be a Riemannian manifold with a Riemannian metric g. A linear connection $\tilde{\nabla}$ on $\overline{M}$ is called a *semi-symmetric connection* if its torsion tensor T

$$T(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y]$$

satisfies

$$T(X,Y) = \omega(Y)X - \omega(X)Y,$$

for any vector fields X, Y on $\overline{M}$, where ω is a 1-form. Denote by P its dual vector field, i.e., $\omega(X) = g(X, P)$. If a semi-symmetric connection satisfies

$$\tilde{\nabla} g = 0,$$

then it is said to be a *semi-symmetric metric connection* $\tilde{\nabla}$.

Further, with respect to a semi-symmetric metric connection $\tilde{\nabla}$ on $\overline{M}$, the curvature tensor $\tilde{R}$ is given by

$$\begin{aligned}\tilde{R}(X,Y,Z,W) &= \overline{R}(X,Y,Z,W) - \alpha(Y,Z)g(X,W) \\ &+ \alpha(X,Z)g(Y,W) - \alpha(X,W)g(Y,W) + \alpha(Y,W)g(X,Z),\end{aligned} \tag{4}$$

for any $X, Y, Z, W \in T\overline{M}$, where α is the $(0,2)$-tensor field defined by

$$\alpha(X,Y) = (\overline{\nabla}_X \omega)Y - \omega(X)\omega(Y) + \frac{1}{2}\omega(P)g(X,Y), \tag{5}$$

for all $X, Y \in T\overline{M}$.

Let M be an n-dimensional submanifold of a complex space form $\overline{M}(c)$ of complex dimension m. Then, we decompose

$$JX = TX + FX,$$

where TX and FX are the tangential and normal components of JX, respectively, for any $X \in TM$.

The submanifold is called *anti-invariant* if $T = 0$.

The submanifold is called *invariant* if $F = 0$.

Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of T_pM, $p \in M$. One is denoted by

$$\|T\|^2 = \sum_{i,j=1}^{n} g^2(Je_i, e_j). \tag{6}$$

Let $p \in M$ and $\pi \subset T_pM$ be a plane section.

If $\{e_1, \ldots, e_n\}$ is an orthonormal basis of T_pM and $\{e_{n+1}, \ldots, e_{2m}\}$ an orthonormal basis of $T_p^{\perp}M$, then the sectional curvature $K(\pi)$ is defined by $K(\pi) = g(R(X,Y)X,Y)$, where $X, Y \in \pi$ are orthonormal, and the scalar curvature τ at p by

$$\tau(p) = \sum_{1 \le i < j \le n} K(e_i \wedge e_j). \tag{7}$$

The mean curvature vector field H of M is

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i).$$

A submanifold is minimal if the mean curvature vector H vanishes identically, that is, $H = 0$.

We recall the definition of pointwise slant submanifolds.

Definition 1 ([10])**.** *Let $\overline{M}$ be an almost Hermitian manifold. Then a submanifold M of $\overline{M}$ is called a pointwise slant submanifold if for each point $p \in M$ and any non-zero vector $X \in T_pM$ the angle $\theta(X)$ between JX and T_pM is independent of the choice of X.*

In [10], Chen and Garay obtained a necessary and sufficient condition for a submanifold to be a pointwise slant submanifold. They proved that a submanifold M of an almost Hermitian manifold $\tilde{M}$ is *pointwise slant* if and only if

$$T^2 = -(\cos^2\theta)I, \tag{8}$$

for a real-valued function θ defined on M, where I is the identity transformation of the tangent bundle TM of M.

On the other hand, Chen and the forth author generalized the above concept and defined pointwise semi-slant submanifolds as follows [6].

Definition 2. *Let $\overline{M}$ be an almost Hermitian manifold. Then a submanifold M of $\overline{M}$ is called a pointwise semi-slant submanifold if a pair of orthogonal distributions $\mathfrak{D}_1$ and $\mathfrak{D}_2$ exist such that*

(i) *TM admits the orthogonal direct decomposition $TM = \mathfrak{D}_1 \oplus \mathfrak{D}_2$;*
(ii) *$\mathfrak{D}_1$ is invariant;*
(iii) *$\mathfrak{D}_2$ is pointwise slant with a slant function θ.*

The submanifold M is called a proper pointwise semi-slant submanifold if both distributions are non-trivial. Denote their dimensions by $2d_1$ and $2d_2$.

Let M be a proper pointwise semi-slant submanifold of a complex space form $\overline{M}(c)$. We set the following

$$\begin{aligned} &e_1, e_2 = Te_1, \cdots, e_{2d_1-1}, e_{2d_1} = Te_{2d_1-1}, \\ &e_{2d_1+1}, e_{2d_1+2} = \sec\theta Te_{2d_1+1}, \cdots, e_{2d_1+2d_2-1}, e_{2d_1+2d_2} = \sec\theta Te_{d_1-1}. \end{aligned}$$

Then, we obtain

$$g^2(Je_i, e_{i+1}) = \begin{cases} 1, & \forall\, i = 1, 3, \cdots, 2d_1 - 1, \\ \cos^2\theta, & \forall\, i = 2d_1 + 1, \cdots, 2d_1 + 2d_2 - 1. \end{cases}$$

Hence, we have

$$\|T\|^2 = \sum_{i,j=1}^{n} g^2(Je_i, e_j) = n_1 + n_2 \cos^2\theta, \tag{9}$$

where $n_1 = 2d_1$ and $n_2 = 2d_2$.

Further, we state an algebraic lemma due to Chen.

Lemma 1 ([11]). *Let $n \geq 2$ and $a_1, \cdots, a_n, b$ real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + b\right).$$

Then $2a_1a_2 \geq b$ and the equality holds if and only if

$$a_1 + a_2 = a_3 = \cdots = a_n.$$

Finally, we conclude this section with the following relation between sectional curvature and the Laplacian of the warping function for warped products. Let $\{e_1, e_2, \ldots, e_n, e_{n+1}, \ldots, e_{2m}\}$ be a local orthonormal frame such that $e_1, e_2, \ldots, e_{n_1}$ are tangent to M_1, $e_{n_1+1}, \ldots, e_n$ are tangent to M_2 and e_{n+1} is parallel to the mean curvature vector H. Then,

$$\sum_{1 \leq i \leq n_1} \sum_{n_1+1 \leq j \leq n} K(e_i \wedge e_j) = n_2 \frac{\Delta f}{f} = n_2\Big(\|\nabla(\ln f)\|^2 - \Delta(\ln f)\Big), \tag{10}$$

where $\nabla(\ln f)$ is the gradient vector on M_1.

3. An Inequality for Warped Product Pointwise Semi-Slant Submanifolds

The following theorem is the main result of this article; it gives an estimate of the squared mean curvature in terms of the warping function.

Theorem 1. *Let $\overline{M}(c)$ be a complex space form endowed with a semi-symmetric metric connection $\tilde{\nabla}$ and $M = M_1 \times_f M_2$ a warped product pointwise semi-slant submanifold of $\overline{M}(c)$. Then we have*

$$n_2 \frac{\Delta f}{f} \leq \frac{(n_1+n_2)^2}{4}\|H\|^2 + \frac{c}{4} n_1 n_2 + \varpi - \frac{3c}{8}\left(n_1 + n_2 \cos^2\theta\right), \tag{11}$$

where ϖ denotes the trace of α, θ is the slant function on M_2 and $\dim M_i = n_i$, $i = 1, 2$.

The equality case holds if and only if M is a mixed totally geodesic submanifold and $n_1 H_1 = n_2 H_2$, where H_1 and H_2 are the partial mean curvature vectors corresponding to M_1 and M_2, respectively.

Proof. From (2)–(4) we have

$$\begin{aligned} &R(X,Y,Z,W) + g(h(X,Z),h(Y,W)) - g(h(X,W),h(Y,Z)) \\ &\quad = \frac{c}{4}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\ &\quad + \frac{c}{4}\{g(X,JZ)g(JY,W) - g(Y,JZ)g(JX,W) + 2g(X,JY)g(JZ,W)\} \\ &\quad - \alpha(Y,Z)g(X,W) + \alpha(X,Z)g(Y,W) - \alpha(X,W)g(Y,W) + \alpha(Y,W)g(X,Z). \end{aligned} \tag{12}$$

We consider the orthonormal frame defined in the previous section. For $X = W = e_i$, $Y = Z = e_j$, $i, j = 1, ..., n$, summing after $1 \leq i, j \leq n$, one obtains

$$\begin{aligned} 2\tau = {} & \frac{c}{4} n(n-1) - \|h\|^2 \\ & + \frac{c}{4}\left(3n_1 + 3n_2\cos^2\theta\right) + 2(n-1)\varpi + n^2\|H\|^2. \end{aligned} \tag{13}$$

We denote

$$\delta = 2\tau - \frac{c}{4} n(n-1) - \frac{c}{4}\left(3n_1 + 3n_2\cos^2\theta\right) - 2(n-1)\varpi - \frac{n^2}{2}\|H\|^2. \tag{14}$$

Then, from (13) and (14) we derive that

$$n^2\|H\|^2 = 2(\delta + \|h\|^2), \tag{15}$$

which can be written as

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = 2\left\{\delta + \sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^r)^2\right\}, \tag{16}$$

where $h_{ij}^r = g(h(e_i, e_j), e_r)$, $i, j = 1, ..., n; r = n+1, ..., 2m$.

If we put $a_1 = h_{11}^{n+1}$, $a_2 = \sum_{i=2}^{n_1} h_{ii}^{n+1}$ and $a_3 = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}$, the above equation is reduced to

$$\begin{aligned} \left(\sum_{i=1}^{3} a_i\right)^2 = 2\Bigg\{ & \delta + \sum_{i=1}^{3} a_i^2 + \sum_{1\leq i\neq j\leq n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^r)^2 \\ & - \sum_{2\leq j\neq k_{\leq} n_1} h_{jj}^{n+1}h_{kk}^{n+1} - \sum_{n_1+1\leq s\neq t_{\leq} n} h_{ss}^{n+1}h_{tt}^{n+1}\Bigg\}. \end{aligned} \tag{17}$$

It follows that a_1, a_2, a_3 satisfy Lemma 1 for $n = 3$; then, $2a_1a_2 \geq b$, with equality holding if and only if $a_1 + a_2 = a_3$. This means

$$\sum_{1\leq j<k\leq n_1} h_{jj}^{n+1}h_{kk}^{n+1} + \sum_{n_1+1\leq s<t\leq n} h_{ss}^{n+1}h_{tt}^{n+1} \geq \frac{\delta}{2} + \sum_{1\leq\alpha<\beta\leq n}(h_{\alpha\beta}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{2m}\sum_{\alpha,\beta=1}^{n}(h_{\alpha\beta}^{r})^2. \tag{18}$$

We have the equality if and only if

$$\sum_{i=1}^{n_1} h_{ii}^{n+1} = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}. \tag{19}$$

From the Gauss equation we obtain

$$n_2\frac{\Delta f}{f} = \tau - \sum_{1\leq j<k\leq n_1} K(e_j \wedge e_k) - \sum_{n_1+1\leq s<t\leq n} K(e_s \wedge e_t). \tag{20}$$

Combining (10), (18) and (20), we derive

$$\begin{aligned} n_2\frac{\Delta f}{f} &= \tau - \frac{c}{8}n_1(n_1-1) - \frac{3c}{4}n_1 - (n_1-1)\varpi \\ &- \sum_{r=n+1}^{2m}\sum_{1\leq j<k\leq n_1}(h_{jj}^r h_{kk}^r - (h_{jk}^r)^2) \\ &- \frac{c}{8}n_2(n_2-1) - \frac{3c}{4}n_2\cos^2\theta - (n_2-1)\varpi \\ &- \sum_{r=n+1}^{2m}\sum_{n_1+1\leq s<t\leq n}(h_{ss}^r h_{tt}^r - (h_{st}^r)^2). \end{aligned} \tag{21}$$

Taking into account (18) and (21), we find

$$\begin{aligned} n_2\frac{\Delta f}{f} &\leq \tau - \frac{c}{8}n(n-1) + \frac{c}{4}n_1n_2 \\ &- (n-2)\varpi - \frac{3c}{4}\left(n_1 + n_2\cos^2\theta\right) - \frac{\delta}{2}. \end{aligned} \tag{22}$$

Using (14) in the previous inequality, we derive

$$n_2\frac{\Delta f}{f} \leq \frac{n^2}{4}\|H\|^2 + \frac{c}{4}n_1n_2 + \varpi - \frac{3c}{8}\left(n_1 + n_2\cos^2\theta\right), \tag{23}$$

which represents the inequality to prove.

For the equality case, from (19) it follows that $n_1H_1 = n_2H_2$.

Moreover, from (18) we obtain

$$h_{\alpha\beta}^r = 0, \ \ \forall\, 1 \leq \alpha \leq n_1, \ \ n_1+1 \leq \beta \leq n, \ \ n+1 \leq r \leq 2m, \tag{24}$$

i.e., M is a mixed totally geodesic submanifold.

The converse is trivial, and the proof is achieved. □

In the following subsections, we derive certain consequences of Theorem 1.

3.1. The Compact Case

It is well-known that if M is a compact oriented Riemannian manifold without boundary, one has

$$\int_M \Delta f dV = 0, \tag{25}$$

where f is a smooth function on M and dV denotes the volume element of M.

As a consequence of Theorem 1, we prove the following result.

Theorem 2. *Let $M = M_1 \times_f M_2$ be a compact oriented warped product pointwise semi-slant submanifold in a complex space form $\overline{M}(c)$ endowed with a semi-symmetric metric connection. Then M is a Riemannian product if and only if*

$$\|H\|^2 \geq \frac{3c}{2(n_1+n_2)^2}\Big(n_1 + n_2\cos^2\theta\Big) - \frac{n_1 n_2}{(n_1+n_2)^2}c - \frac{4\varpi}{(n_1+n_2)^2}. \tag{26}$$

Proof. By using (10), the inequality (11) reduces to

$$\begin{aligned} &n_2(\Delta(\ln f) - \|\nabla(\ln f)\|^2) \\ &\qquad \leq \frac{n^2}{4}\|H\|^2 + \frac{c}{4}n_1 n_2 + \varpi - \frac{3c}{8}\Big(n_1 + n_2\cos^2\theta\Big). \end{aligned} \tag{27}$$

Let assume that M is a Riemannian product, i.e., f is constant on M. Then, we obtain (26).

Conversely, suppose that the inequality (26) holds; then, integrating (27) and using (25), we obtain

$$0 \leq \int_M (n_2\|\nabla(\ln f)\|^2) dV \leq 0,$$

from where $\nabla(\ln f) = 0$, which implies that f is a constant function on M. □

3.2. The Minimal Case

In this subsection, we obtain obstructions to the minimality of warped product pointwise semi-slant submanifolds in a complex space form endowed with a semi-symmetric metric connection.

An immediate consequence of Theorem 1 is the following:

Theorem 3. *Let $M = M_1 \times_f M_2$ be a warped product pointwise semi-slant submanifold in a complex space form $\overline{M}(c)$ endowed with a semi-symmetric metric connection. If there exists a point $p \in M$ such that*

$$n_2\frac{\Delta f}{f} > \frac{c}{4}n_1 n_2 + \varpi - \frac{3c}{8}\Big(n_1 + n_2\cos^2\theta\Big),$$

at p, then M cannot be minimal.

As with special cases of Theorem 3, we state the following corollaries.

Corollary 1. *There does not exist any minimal warped product pointwise semi-slant submanifold $M = M_1 \times_f M_2$ in a complex space form $\overline{M}(c)$ endowed with a semi-symmetric metric connection if*

$$n_2\frac{\Delta f}{f} > \frac{c}{4}n_1 n_2 + \varpi - \frac{3c}{8}\Big(n_1 + n_2\cos^2\theta\Big).$$

Corollary 2. *Let $M = M_T \times_f M_\perp$ be a warped product CR-submanifold of a complex space form $\overline{M}(c)$ endowed with a semi-symmetric metric connection, where M_T and $M_\perp$ are holomorphic and totally real submanifolds of $\overline{M}$, respectively. If the following inequality*

$$n_2 \frac{\Delta f}{f} > \frac{c}{4} n_1 n_2 + \varpi - \frac{3cn_1}{8}$$

holds at a point $p \in M$, then M cannot be a minimal submanifold.

Also from Theorem 2, we obtain

Corollary 3. *Let $M = M_1 \times_f M_2$ be a compact oriented warped product pointwise semi-slant submanifold in a complex space form $\overline{M}(c)$ endowed with a semi-symmetric metric connection. If*

$$c < \frac{3\left(n_1 + n_2 \cos^2\theta\right)}{n_1 n_2} - \frac{(n_1 + n_2)^2}{n_1 n_2}\varpi,$$

then M cannot be a minimal submanifold of $\overline{M}$.

4. Conclusions

By using the above methods, one can obtain corresponding inequalities for other classes of warped product submanifolds in complex space forms endowed with semi-symmetric connections.

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