symmetry

Article

(γ,a)-Nabla Reverse Hardy–Hilbert-Type Inequalities on Time Scales

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Abstract: In this article, using a (γ,a)-nabla conformable integral on time scales, we study several novel Hilbert-type dynamic inequalities via nabla time scales calculus. Our results generalize various inequalities on time scales, unifying and extending several discrete inequalities and their corresponding continuous analogues. We say that symmetry plays an essential role in determining the correct methods with which to solve dynamic inequalities.

Keywords: Hölder inequality; Jensen inequality; time scales; nabla calculus

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1. Introduction

Hardy [1] proved the classical discrete double series inequality of the Hilbert type: If $\{a_m\}, \{b_n\} \geqslant 0$ and $\sum_{n=1}^{\infty} a_n^p < \infty$ and $\sum_{n=1}^{\infty} b_n^q < \infty$, then we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_n b_m}{m+n} \leqslant \frac{\pi}{\sin\frac{\pi}{p}}\left(\sum_{n=1}^{\infty} a_n^p\right)^{\frac{1}{p}}\left(\sum_{m=1}^{\infty} b_m^q\right)^{\frac{1}{q}}, \tag{1}$$

Unless all the sequence $\{a_m\}$ or $\{b_n\}$ is null. Hardy also established the following integral analogous of the inequality (1):

$$\int_0^{\infty}\int_0^{\infty}\frac{\vartheta(\xi)g(\zeta)}{\xi+\zeta}dxdy \leqslant \frac{\pi}{\sin\frac{\pi}{p}}\left(\int_0^{\infty}\vartheta^p(\xi)dx\right)^{\frac{1}{p}}\left(\int_0^{\infty}g^{p'}(\xi)dx\right)^{\frac{1}{p'}}, \tag{2}$$

Unless $\vartheta \equiv 0$ or $g \equiv 0$, where ϑ and g are measurable non-negative functions, such that $\int_0^{\infty}\vartheta^p(\xi)dx < \infty$ and $\int_0^{\infty} g^{p'}(\xi)dx < \infty$ and $p > 1$, $p' = p(p-1)$, with the best constant $\frac{\Pi}{\sin\frac{\pi}{p}}$, in (1) and (2).

In [2], Pachpatte established the following Hilbert-type integral inequalities under the following conditions: If $h \geqslant 1$, $l \geqslant 1$, and $f(\varsigma) \geqslant 0$, $g(\varsigma) \geqslant 0$, for $\varsigma \in (0,\xi)$ and $\tau \in (0,\zeta)$, where ξ and ζ are positive real numbers and define $\vartheta(s) = \int_0^s f(\varsigma)d\varsigma$ and $G(\varsigma) = \int_0^{\varsigma} g(\tau)d\tau$, for $s \in (0,\xi)$ and $\varsigma \in (0,\zeta)$, and Φ, Ψ are two real-valued non-negative, convex, and submultiplicative functions defined on $(0,\infty]$, then

$$\int_0^{\xi}\int_0^{\zeta}\frac{\vartheta^h(s)G^l(\chi)}{s+\chi}dsd\chi \leqslant \frac{1}{2}hl(\xi\zeta)^{\frac{1}{2}}\left(\int_0^{\xi}(\xi-s)\left(\vartheta^{h-1}(s)\vartheta(s)\right)^2 ds\right)^{\frac{1}{2}}$$
$$\times\left(\int_0^{\zeta}(\zeta-\chi)\left(G^{l-1}g(\chi)\right)^2 d\chi\right)^{\frac{1}{2}}, \tag{3}$$

and

$$\int_0^{\xi}\int_0^{\zeta}\frac{\Phi(\vartheta(s))\Psi(G(\chi))}{s+\chi}dsd\chi \leqslant L(\xi,\zeta)\left(\int_0^{\xi}(\xi-s)\left(p(s)\Phi\left(\frac{\vartheta(s)}{p(s)}\right)\right)^2 ds\right)^{\frac{1}{2}} \times\left(\int_0^{\zeta}(\zeta-\chi)\left(q(\chi)\Psi\left(\frac{g(\chi)}{q(\chi)}\right)\right)^2 d\chi\right)^{\frac{1}{2}} \tag{4}$$

where

$$L(\xi,\zeta)=\frac{1}{2}\left(\int_0^{\xi}\left(\frac{\Phi(P(s))}{P(s)}\right)^2 ds\right)^{\frac{1}{2}}\left(\int_0^{\zeta}\left(\frac{\Psi(Q(\chi))}{Q(\chi)}\right)^2 d\chi\right)^{\frac{1}{2}},$$

and

$$\int_0^{\xi}\int_0^{\zeta}\frac{P(s)Q(\chi)\Phi(\vartheta(s))\Psi(G(\chi))}{s+\chi}dsd\chi \leqslant \frac{1}{2}(\xi\zeta)^{\frac{1}{2}}\left(\int_0^{\xi}(\xi-s)\left(p(s)\Phi\left(\vartheta(s)\right)\right)^2 ds\right)^{\frac{1}{2}} \times\left(\int_0^{\zeta}(\zeta-\chi)\left(q(\chi)\Psi\left(g(\chi)\right)\right)^2 d\chi\right)^{\frac{1}{2}}. \tag{5}$$

For more results on Hilbert-type inequalities and other, please see [3–6]. See also [7–13]. As we know that time scale $\mathbb{T}$ is an arbitrary, non-empty closed subset of the set of real numbers $\mathbb{R}$, and the jump operators, forward and backward, are defined by $\sigma(\varsigma) := \inf\{\iota \in \mathbb{T} : \iota > \varsigma\}$, $\rho(\varsigma) := \sup\{\iota \in \mathbb{T} : \iota < \varsigma\}$. For more details, see [7]. The following relations are used.

(i) For $\mathbb{T} = \mathbb{R}$, then

$$\begin{gathered}\sigma(\varsigma)=\rho(\varsigma)=\varsigma, \quad \mu(\varsigma)=\nu(\varsigma)=0, \quad \vartheta^{\Delta}(\varsigma)=\vartheta^{\nabla}(\varsigma)=\vartheta'(\varsigma),\\ \int_a^b\vartheta(\varsigma)\Delta\varsigma=\int_a^b\vartheta(\varsigma)\nabla\varsigma=\int_a^b\vartheta(\varsigma)d\varsigma.\end{gathered} \tag{6}$$

(ii) For $\mathbb{T} = \mathbb{Z}$, then

$$\begin{gathered}\sigma(\varsigma)=\varsigma+1, \quad \rho(\varsigma)=\varsigma-1, \quad \mu(\varsigma)=\nu(\varsigma)=1,\\ \vartheta^{\Delta}(\varsigma)=\Delta\vartheta(\varsigma), \quad \vartheta^{\nabla}(\varsigma)=\nabla\vartheta(\varsigma),\\ \int_a^b\vartheta(\varsigma)\Delta\varsigma=\sum_{\varsigma=a}^{b-1}\vartheta(\varsigma), \quad \int_a^b\vartheta(\varsigma)\nabla\varsigma=\sum_{\varsigma=a+1}^{b}\vartheta(\varsigma).\end{gathered} \tag{7}$$

For more details on the conformable nabla calculus, see [3].

We suppose that CC_{ld} denotes the set of all ld-continuous functions $\vartheta(\xi,\zeta)$ in ξ and ζ and CC^1_{ld} is the set of all functions in CC_{ld} for which both the first partial derivative $\nabla^{\gamma}_{a_1}$ and the first partial derivative $\nabla^{\gamma}_{a_2}$ exist in CC_{ld}. Similarly, we can define CC^2_{ld}.

In order to obtain our result in this paper, we need the following lemmas.

Lemma 1 (Reversed Dynamic Hölder's Inequality)**.** *Suppose $u, v \in \mathbb{T}$ with $u < v$. Assume ϑ, $g \in CC^1_{ld}([u,v]_{\mathbb{T}} \times [u,v]_{\mathbb{T}}, \mathbb{R})$ be integrable functions and $\frac{1}{p}+\frac{1}{q}=1$ with $p<1$, then*

$$\int_u^v\int_u^v|\vartheta(r,\chi)g(r,\chi)|\nabla^{\gamma}_a r\nabla^{\gamma}_a\chi \geq \left[\int_u^v\int_u^v|\vartheta(r,\chi)|^p\nabla^{\gamma}_a r\nabla^{\gamma}_a\chi\right]^{\frac{1}{p}} \times\left[\int_u^v\int_u^v|g(r,\chi)|^q\nabla^{\gamma}_a r\nabla^{\gamma}_a\chi\right]^{\frac{1}{q}}. \tag{8}$$

Proof. This lemma is a direct extension of the (Lemma 9, [14]). □

Lemma 2 (Reversed Dynamic Jensen's inequality)**.** *Let $r, \chi \in \mathbb{R}$ and $-\infty \leqslant m, n \leqslant \infty$. If $\vartheta \in CC^1_{ld}(\mathbb{R},(m,n))$ and $\Phi : (m,n) \longrightarrow \mathbb{R}$ is concave, then*

$$\phi\left(\frac{\int_u^v \int_\omega^s \vartheta(r,\chi)\nabla^\gamma_{a_1} r \nabla^\gamma_{a_2}\chi}{\int_u^v \int_\omega^s \nabla^\gamma_{a_1} r \nabla^\gamma_{a_2}\chi}\right) \geq \frac{\int_u^v \int_\omega^s \phi(\vartheta(r,\chi))\nabla^\gamma_{a_1} r \nabla^\gamma_{a_2}\chi}{\int_u^v \int_\omega^s \nabla^\gamma_{a_1} r \nabla^\gamma_{a_2}\chi}. \tag{9}$$

Proof. This lemma is a direct extension of the (Lemma 10, [14]). □

Definition 1. *Φ is called a supermultiplicative function on $[0,\infty)$ if*

$$\Phi(\xi\zeta) \geqslant \Phi(\xi)\Phi(\zeta), \ \textit{for all} \ \xi,\zeta \geqslant 0. \tag{10}$$

In this paper, we establish a (γ,a)-nabla conformable integral inequality of Hardy–Hilbert type on a time scale. In special cases, we will recover some dynamic continuous and discrete inequalities known in the literature. Symmetry plays an essential role in determining the correct methods with which to solve dynamic inequalities.

Now, our main results will be presented.

2. Main Results

First, we suppose the following assumptions:

(S_1) $\mathbb{T}$ be time scales with $\chi_0, \xi_\ell, \zeta_\ell, s_\ell, \chi_\ell \in \mathbb{T}$, $(\ell = 1, \ldots, n)$.
(S_2) $\vartheta_\ell(s_\ell,\chi_\ell)$ are non-negative, nabla integrable functions defined as $[\chi_0,\xi_\ell)_\mathbb{T} \times [\chi_0,\zeta_\ell)_\mathbb{T}$ $(\ell = 1, \ldots, n)$.
(S_3) $\vartheta_\ell(s_\ell,\chi_\ell)$ have partial ∇^γ_a- derivatives $\vartheta_\ell^{\nabla^\gamma_{a_1}}(s_\ell,\chi_\ell)$ and $\vartheta_\ell^{\nabla^\gamma_{a_2}}(s_\ell,\chi_\ell)$ with respect to s_ℓ and χ_ℓ, respectively.
(S_4) All functions used in this section are integrable according to ∇^γ_a sense.
(S_5) $\vartheta_\ell(s_\ell,\chi_\ell) \in C^2_{ld}([\chi_0,\xi_\ell)_\mathbb{T} \times [\chi_0,\zeta_\ell)_\mathbb{T},[0,\infty))$ $(\ell = 1, \ldots, n)$.
(S_6) $p_\ell(\varsigma_\ell,\tau_\ell)$ are n positive nabla-integrable functions defined for $\varsigma_\ell \in (\chi_0,s_\ell)_\mathbb{T}$, $\tau_\ell \in (\chi_0,\chi_\ell)_\mathbb{T}$.
(S_7) $p_\ell(\varsigma_\ell)$ and $q_\ell(\tau_\ell)$ are positive nabla-integrable functions defined for $\varsigma_\ell \in (\chi_0,s_\ell)_\mathbb{T}$, $\tau_\ell \in (\chi_0,\chi_\ell)_\mathbb{T}$.
(S_8) Φ_ℓ and Ψ_ℓ, $(\ell = 1, \ldots, n)$ are n real-valued, non-negative concave and supermultiplicative functions defined on $(0,\infty)$.
(S_9) ξ_ℓ and ζ_ℓ are positive real numbers.
(S_{10}) $s_\ell \in [\chi_0,\xi_\ell)_\mathbb{T}$ and $\chi_\ell \in [\chi_0,\zeta_\ell)_\mathbb{T}$.
(S_{11}) $\vartheta_\ell(\chi_0,\chi_\ell) = \vartheta_\ell(s_\ell,\chi_0) = 0$, $(\ell = 1, \ldots, n)$.
(S_{12}) $\vartheta_\ell^{\nabla^\gamma_{a_1}\nabla^\gamma_{a_2}}(s_\ell,\chi_\ell) = \vartheta_\ell^{\nabla^\gamma_{a_2}\nabla^\gamma_{a_1}}(s_\ell,\chi_\ell)$.
(S_{13}) $P_\ell(s_\ell,\chi_\ell) = \int_{\chi_0}^{\chi_\ell}\int_{\chi_0}^{s_\ell} p_\ell(\varsigma_\ell)q_\ell(\tau_\ell)\nabla^\gamma_a\varsigma_\ell\nabla^\gamma_a\tau_\ell$.
(S_{14}) $\Theta_\ell(s_\ell,\chi_\ell) = \int_{\chi_0}^{s_\ell}\int_{\chi_0}^{\chi_\ell} \vartheta_\ell(\varsigma_\ell,\tau_\ell)\nabla^\gamma_a\varsigma_\ell\nabla^\gamma_a\tau_\ell$.
(S_{15}) $P_\ell(s_\ell,\chi_\ell) = \int_{\chi_0}^{s_\ell}\int_{\chi_0}^{\chi_\ell} p_\ell(\varsigma_\ell,\tau_\ell)\nabla^\gamma_a\varsigma_\ell\nabla^\gamma_a\tau_\ell$.
(S_{16}) $\Theta_\ell(s_\ell,\chi_\ell) = \frac{1}{P_\ell(\varsigma_\ell,\tau_\ell)}\int_{\chi_0}^{s_\ell}\int_{\chi_0}^{\chi_\ell} p_\ell(\varsigma_\ell,\tau_\ell)\vartheta_\ell(\varsigma_\ell,\tau_\ell)\nabla^\gamma_a\varsigma_\ell\nabla^\gamma_a\tau_\ell$.
(S_{17}) $\alpha_\ell \in (1,\infty)$, $\alpha'_\ell = 1-\alpha_\ell$, $\alpha = \sum_{\ell=1}^n \alpha_\ell$, and $\alpha' = \sum_{\ell=1}^n \alpha'_\ell = n-\alpha$, $(\ell = 1, \ldots, n)$.
(S_{18}) $0 < \beta_\ell < 1$.
(S_{19}) $h_\ell \geqslant 2$.
(S_{20}) $\sum_{\ell=1}^n \frac{1}{\alpha_\ell} = \frac{1}{\alpha}$.
(S_{21}) $h_\ell \geqslant 1$.
(S_{22}) $\vartheta_\ell(\varsigma_\ell) \in C^1_{ld}[\chi_0,\xi_\ell]_\mathbb{T}$, $(\ell = 1, \ldots, n)$.
(S_{23}) ξ_ℓ is a positive real number.
(S_{24}) $\Theta_\ell(s_\ell) = \int_{\chi_0}^{s_\ell} \vartheta_\ell(\varsigma_\ell)\nabla^\gamma_a\varsigma_\ell$.
(S_{25}) $s_\ell \in [\chi_0,\xi_\ell)_\mathbb{T}$.
(S_{26}) $p_\ell(\varsigma_\ell)$ are n positive functions.

$(S_{27}) P_\ell(s_\ell) = \int_{\chi_0}^{s_\ell} p_\ell(\varsigma_\ell) \nabla_a^\gamma \varsigma_\ell$.
$(S_{28}) \Theta_\ell(s_\ell) = \frac{1}{P_\ell(s_\ell)} \int_{\chi_0}^{s_\ell} p(\varsigma_\ell) \vartheta(\varsigma_\ell) \nabla_a^\gamma \varsigma_\ell$.
$(S_{29}) \vartheta_\ell(\chi_0) = 0$.

The first important inequality is stated in the following theorem:

Theorem 1. *Let S_1, S_4, S_5, S_{14}, S_6, S_{15} and S_8 be satisfied. Then, for S_{10}, S_{18} and S_{20}, we find that*

$$\int_{\chi_0}^{\xi_1} \int_{\chi_0}^{\zeta_1} \cdots \int_{\chi_0}^{\xi_n} \int_{\chi_0}^{\zeta_n} \frac{\prod_{\ell=1}^{n} \Phi_\ell(\Theta_\ell(s_\ell, \chi_\ell))}{\left(\alpha \sum_{\ell=1}^{n} \frac{1}{\alpha_\ell}(s_\ell - \chi_0)(\chi_\ell - \chi_0)\right)^{\frac{1}{\alpha}}} \nabla_a^\gamma s_n \nabla_a^\gamma \chi_n \ldots \nabla_a^\gamma s_1 \nabla_a^\gamma \chi_1 \quad (11)$$

$$\geqslant L(\xi_1 \zeta_1, \ldots, \xi_n y_n)$$

$$\times \prod_{\ell=1}^{n} \left(\int_{\chi_0}^{\xi_\ell} \int_{\chi_0}^{\zeta_\ell} (\rho(\xi_\ell) - s_\ell)(\rho(\zeta_\ell) - \chi_\ell) \left(p_\ell(s_\ell, \chi_\ell) \Phi_\ell\left(\frac{\vartheta_\ell(s_\ell, \chi_\ell)}{p_\ell(s_\ell, \chi_\ell)} \right) \right)^{\beta_\ell} \nabla_a^\gamma s_\ell \nabla_a^\gamma \chi_\ell \right)^{\frac{1}{\beta_\ell}}$$

where

$$L(\xi_1 \zeta_1, \ldots, \xi_n y_n) = \prod_{\ell=1}^{n} \left(\int_{\chi_0}^{\xi_\ell} \int_{\chi_0}^{\zeta_\ell} \left(\frac{\Phi_\ell(P_\ell(s_\ell, \chi_\ell))}{P_\ell(s_\ell, \chi_\ell)} \right)^{\alpha_\ell} \nabla_a^\gamma s_\ell \nabla_a^\gamma \chi_\ell \right)^{\frac{1}{\alpha_\ell}}.$$

Proof. From the hypotheses of Theorem 1, S_{14}, S_{15}, and S_8, it is easy to observe that

$$\begin{aligned} \Phi_\ell(\Theta_\ell(s_\ell, \chi_\ell)) &= \Phi_\ell\left(\frac{P_\ell(s_\ell, \chi_\ell) \int_{\chi_0}^{s_\ell} \int_{\chi_0}^{\chi_\ell} p_\ell(\varsigma_\ell, \tau_\ell) \left(\frac{\vartheta_\ell(\varsigma_\ell, \tau_\ell)}{p_\ell(\varsigma_\ell, \tau_\ell)} \right) \nabla_a^\gamma \varsigma_\ell \nabla_a^\gamma \tau_\ell}{\int_{\chi_0}^{s_\ell} \int_{\chi_0}^{\chi_\ell} p_\ell(\varsigma_\ell, \tau_\ell) \nabla_a^\gamma \varsigma_\ell \nabla_a^\gamma \tau_\ell} \right) \\ &\geqslant \Phi_\ell(P_\ell(s_\ell, \chi_\ell)) \Phi_\ell\left(\frac{\int_{\chi_0}^{s_\ell} \int_{\chi_0}^{\chi_\ell} p_\ell(\varsigma_\ell, \tau_\ell) \left(\frac{\vartheta_\ell(\varsigma_\ell, \tau_\ell)}{p_\ell(\varsigma_\ell, \tau_\ell)} \right) \nabla_a^\gamma \varsigma_\ell \nabla_a^\gamma \tau_\ell}{\int_{\chi_0}^{s_\ell} \int_{\chi_0}^{\chi_\ell} p_\ell(\varsigma_\ell, \tau_\ell) \nabla_a^\gamma \varsigma_\ell \nabla_a^\gamma \tau_\ell} \right). \quad (12) \end{aligned}$$

Using inverse Jensen dynamic inequality, we obtain that

$$\Phi_\ell(\Theta_\ell(s_\ell, \chi_\ell)) \geqslant \frac{\Phi_\ell(P_\ell(s_\ell, \chi_\ell))}{P_\ell(s_\ell, \chi_\ell)} \int_{\chi_0}^{s_\ell} \int_{\chi_0}^{\chi_\ell} p_\ell(\varsigma_\ell, \tau_\ell) \Phi_\ell\left(\frac{\vartheta_\ell(\varsigma_\ell, \tau_\ell)}{p_\ell(\varsigma_\ell, \tau_\ell)} \right) \nabla_a^\gamma \varsigma_\ell \nabla_a^\gamma \tau_\ell. \quad (13)$$

Applying inverse Hölder's inequality on the right-hand side of (13) with indices α_ℓ and β_ℓ, it is easy to observe that

$$\Phi_\ell(\Theta_\ell(s_\ell, \chi_\ell)) \geqslant \frac{\Phi_\ell(P_\ell(s_\ell, \chi_\ell))}{P_\ell(s_\ell, \chi_\ell)} [(s_\ell - \chi_0)(\chi_\ell - \chi_0)]^{\frac{1}{\alpha_\ell}} \left(\int_{\chi_0}^{s_\ell} \int_{\chi_0}^{\chi_\ell} \left(p_\ell(\varsigma_\ell, \tau_\ell) \Phi_\ell\left(\frac{\vartheta_\ell(\varsigma_\ell, \tau_\ell)}{p_\ell(\varsigma_\ell, \tau_\ell)} \right) \right)^{\beta_\ell} \nabla_a^\gamma \varsigma_\ell \nabla_a^\gamma \tau_\ell \right)^{\frac{1}{\beta_\ell}}. \quad (14)$$

Using the following inequality on the term $[(s_\ell - \chi_0)(\chi_\ell - \chi_0)]^{\frac{1}{\alpha_\ell}}$,

$$\prod_{\ell=1}^{n} m_\ell^{\frac{1}{\alpha_\ell}} \geqslant \left(\alpha \sum_{\ell=1}^{n} \frac{1}{\alpha_\ell} m_\ell \right)^{\frac{1}{\alpha}}, \quad (15)$$

we get that

$$\frac{\prod_{\ell=1}^{n} \Phi_\ell(\Theta_\ell(s_\ell, \chi_\ell))}{\left(\alpha \sum_{\ell=1}^{n} \frac{1}{\alpha_\ell}(s_\ell - \chi_0)(\chi_\ell - \chi_0)\right)^{\frac{1}{\alpha}}} \quad (16)$$

$$\geqslant \prod_{\ell=1}^{n} \frac{\Phi_\ell(P_\ell(s_\ell, \chi_\ell))}{P_\ell(s_\ell, \chi_\ell)} \left(\int_{\chi_0}^{s_\ell} \int_{\chi_0}^{\chi_\ell} \left(p_\ell(\varsigma_\ell, \tau_\ell) \Phi_\ell\left(\frac{\vartheta_\ell(\varsigma_\ell, \tau_\ell)}{p_\ell(\varsigma_\ell, \tau_\ell)} \right) \right)^{\frac{1}{\beta_\ell}} \nabla_a^\gamma \varsigma_\ell \nabla_a^\gamma \tau_\ell \right)^{\frac{1}{\beta_\ell}}.$$

Integrating both sides of (16) over s_ℓ, χ_ℓ from χ_0 to ξ_ℓ, ζ_ℓ $(\ell = 1, \ldots, n)$, we obtain that

$$\int_{\chi_0}^{\xi_1}\int_{\chi_0}^{\zeta_1}\cdots\int_{\chi_0}^{\xi_n}\int_{\chi_0}^{\zeta_n} \frac{\prod_{\ell=1}^{n}\Phi_\ell(\Theta_\ell(s_\ell,\chi_\ell))}{\left(\alpha\sum_{\ell=1}^{n}\frac{1}{\alpha_\ell}(s_\ell-\chi_0)(\chi_\ell-\chi_0)\right)^{\frac{1}{\alpha}}}\nabla_a^\gamma s_n\nabla_a^\gamma\chi_n\ldots\nabla_a^\gamma s_1\nabla_a^\gamma\chi_1$$

$$\geqslant \prod_{\ell=1}^{n}\int_{\chi_0}^{\xi_\ell}\int_{\chi_0}^{\zeta_\ell}\frac{\Phi_\ell(P_\ell(s_\ell,\chi_\ell))}{P_\ell(s_\ell,\chi_\ell)}\left(\int_{\chi_0}^{s_\ell}\int_{\chi_0}^{\chi_\ell}\left(p_\ell(\varsigma_\ell,\tau_\ell)\Phi_\ell\left(\frac{\vartheta_\ell(\varsigma_\ell,\tau_\ell)}{p_\ell(\varsigma_\ell,\tau_\ell)}\right)\right)^{\beta_\ell}\nabla_a^\gamma\varsigma_\ell\nabla_a^\gamma\tau_\ell\right)^{\frac{1}{\beta_\ell}}\nabla_a^\gamma s_\ell\nabla_a^\gamma\chi_\ell. \tag{17}$$

Applying inverse Hölder's inequality on the right-hand side of (17) with indices α_ℓ and β_ℓ, it is easy to observe that

$$\int_{\chi_0}^{\xi_1}\int_{\chi_0}^{\zeta_1}\cdots\int_{\chi_0}^{\xi_n}\int_{\chi_0}^{\zeta_n} \frac{\prod_{\ell=1}^{n}\Phi_\ell(\Theta_\ell(s_\ell,\chi_\ell))}{\left(\alpha\sum_{\ell=1}^{n}\frac{1}{\alpha_\ell}(s_\ell-\chi_0)(\chi_\ell-\chi_0)\right)^{\frac{1}{\alpha}}}\nabla_a^\gamma s_n\nabla_a^\gamma\chi_n\ldots\nabla_a^\gamma s_1\nabla_a^\gamma\chi_1$$

$$\geqslant \prod_{\ell=1}^{n}\left(\int_{\chi_0}^{\xi_\ell}\int_{\chi_0}^{\zeta_\ell}\left(\frac{\Phi_\ell(P_\ell(s_\ell,\chi_\ell))}{P_\ell(s_\ell,\chi_\ell)}\right)^{\alpha_\ell}\nabla_a^\gamma s_\ell\nabla_a^\gamma\chi_\ell\right)^{\frac{1}{\alpha_\ell}} \tag{18}$$

$$\times\prod_{\ell=1}^{n}\left(\int_{\chi_0}^{\xi_\ell}\int_{\chi_0}^{\zeta_\ell}\left(\int_{\chi_0}^{s_\ell}\int_{\chi_0}^{\chi_\ell}\left(p_\ell(\varsigma_\ell,\tau_\ell)\Phi_\ell\left(\frac{\vartheta_\ell(\varsigma_\ell,\tau_\ell)}{p_\ell(\varsigma_\ell,\tau_\ell)}\right)\right)^{\beta_\ell}\nabla_a^\gamma\varsigma_\ell\nabla_a^\gamma\tau_\ell\right)\nabla_a^\gamma s_\ell\nabla_a^\gamma\chi_\ell\right)^{\frac{1}{\beta_\ell}}.$$

Using Fubini's theorem, we observe that

$$\begin{aligned}&\int_{\chi_0}^{\xi_1}\int_{\chi_0}^{\zeta_1}\cdots\int_{\chi_0}^{\xi_n}\int_{\chi_0}^{\zeta_n} \frac{\prod_{\ell=1}^{n}\Phi_\ell(\Theta_\ell(s_\ell,\chi_\ell))}{\left(\alpha\sum_{\ell=1}^{n}\frac{1}{\alpha_\ell}(s_\ell-\chi_0)(\chi_\ell-\chi_0)\right)^{\frac{1}{\alpha}}}\nabla_a^\gamma s_n\nabla_a^\gamma\chi_n\ldots\nabla_a^\gamma s_1\nabla_a^\gamma\chi_1\\
&\geqslant L(\xi_1\zeta_1,\ldots,\xi_n y_n)\\
&\times\prod_{\ell=1}^{n}\left(\int_{\chi_0}^{\xi_\ell}\int_{\chi_0}^{\zeta_\ell}(\xi_\ell-s_\ell)(\zeta_\ell-\chi_\ell)\left(p_\ell(s_\ell,\chi_\ell)\Phi_\ell\left(\frac{\vartheta_\ell(s_\ell,\chi_\ell)}{p_\ell(s_\ell,\chi_\ell)}\right)\right)^{\beta_\ell}\nabla_a^\gamma s_\ell\nabla_a^\gamma\chi_\ell\right)^{\frac{1}{\beta_\ell}}.\end{aligned}$$

Using the fact $\xi_\ell \geqslant \rho(\xi_\ell)$, and $\zeta_\ell \geqslant \rho(\zeta_\ell)$,, we get that

$$\begin{aligned}&\int_{\chi_0}^{\xi_1}\int_{\chi_0}^{\zeta_1}\cdots\int_{\chi_0}^{\xi_n}\int_{\chi_0}^{\zeta_n} \frac{\prod_{\ell=1}^{n}\Phi_\ell(\Theta_\ell(s_\ell,\chi_\ell))}{\left(\alpha\sum_{\ell=1}^{n}\frac{1}{\alpha_\ell}(s_\ell-\chi_0)(\chi_\ell-\chi_0)\right)^{\frac{1}{\alpha}}}\nabla_a^\gamma s_n\nabla_a^\gamma\chi_n\ldots\nabla_a^\gamma s_1\nabla_a^\gamma\chi_1\\
&\geqslant L(\xi_1\zeta_1,\ldots,\xi_n y_n)\\
&\times\prod_{\ell=1}^{n}\left(\int_{\chi_0}^{\xi_\ell}\int_{\chi_0}^{\zeta_\ell}(\rho(\xi_\ell)-s_\ell)(\rho(\zeta_\ell)-\chi_\ell)\left(p_\ell(s_\ell,\chi_\ell)\Phi_\ell\left(\frac{\vartheta_\ell(s_\ell,\chi_\ell)}{p_\ell(s_\ell,\chi_\ell)}\right)\right)^{\beta_\ell}\nabla_a^\gamma s_\ell\nabla_a^\gamma\chi_\ell\right)^{\frac{1}{\beta_\ell}}.\end{aligned}$$

This completes the proof. □

Remark 1. *In Theorem 1, if $\mathbb{T}=\mathbb{R}$, $\gamma=1$, we get the result due to Zhao et al. [8] (Theorem 2).*

As a special case of Theorem 1, when $\mathbb{T}=\mathbb{Z}$, $\gamma=1$, we have $\rho(n)=n-1$, and we get the following result.

Corollary 1. *Let $\{a_{s_\ell,\chi_\ell,m_{s_\ell},m_{\chi_\ell}}\}$ and $\{p_{s_\ell,\chi_\ell,m_{s_\ell},m_{\chi_\ell}}\}$, $(\ell=1,\ldots,n)$ be n sequences of non-negative numbers defined for $m_{s_\ell}=1,\ldots,k_{s_\ell}$, and $m_{\chi_\ell}=1,\ldots,k_{\chi_\ell}$, and define*

$$A_{s_\ell,\chi_\ell,m_{s_\ell},m_{\chi_\ell}} = \sum_{m_{\varsigma_\ell}}^{m_{s_\ell}} \sum_{m_{\eta_\ell}}^{m_{\chi_\ell}} a_{s_\ell,\chi_\ell,m_{\varsigma_\ell},m_{\eta_\ell}}$$

$$P_{s_\ell,\chi_\ell,m_{s_\ell},m_{\chi_\ell}} = \sum_{m_{\varsigma_\ell}}^{m_{s_\ell}} \sum_{m_{\eta_\ell}}^{m_{\chi_\ell}} p_{s_\ell,\chi_\ell,m_{\varsigma_\ell},m_{\eta_\ell}}. \tag{19}$$

Then,

$$\sum_{m_{s_1}}^{k_{s_1}} \sum_{m_{\chi_1}}^{k_{\chi_1}} \cdots \sum_{m_{s_n}}^{k_{s_n}} \sum_{m_{\chi_n}}^{k_{\chi_n}} \frac{\prod_{\ell=1}^{n} \Phi_\ell(A_{s_\ell,\chi_\ell,m_{s_\ell},m_{\chi_\ell}})}{\left(\alpha \sum_{\ell=1}^{n} \frac{1}{\alpha_\ell}(m_{s_\ell} m_{\chi_\ell})\right)^{\frac{1}{\alpha}}}$$
$$\geqslant C(k_{s_1}k_{\chi_1},\ldots,k_{s_n}k_{\chi_n})$$
$$\times \prod_{\ell=1}^{n} \left(\sum_{m_{s_\ell}}^{k_{s_\ell}} \sum_{m_{\chi_\ell}}^{k_{\chi_\ell}} (k_{s_\ell} - (m_{s_\ell} - 1))(k_{\chi_\ell} - (m_{\chi_\ell} - 1)) \left(P_{s_\ell,\chi_\ell,m_{s_\ell},m_{\chi_\ell}} \Phi_\ell\left(\frac{a_{s_\ell,\chi_\ell,m_{s_\ell},m_{\chi_\ell}}}{P_{s_\ell,\chi_\ell,m_{s_\ell},m_{\chi_\ell}}} \right) \right)^{\beta_\ell} \right)^{\frac{1}{\beta_\ell}}$$

where

$$C(k_{s_1}k_{\chi_1},\ldots,k_{s_n}k_{\chi_n}) = \prod_{\ell=1}^{n} \left(\sum_{m_{s_\ell}}^{k_{s_\ell}} \sum_{m_{\chi_\ell}}^{k_{\chi_\ell}} \left(\frac{\Phi_\ell(P_{s_\ell,\chi_\ell,m_{s_\ell},m_{\chi_\ell}})}{P_{s_\ell,\chi_\ell,m_{s_\ell},m_{\chi_\ell}}} \right) \right)^{\beta_\ell} \Bigg)^{\frac{1}{\beta_\ell}}.$$

Remark 2. *Let $\vartheta_\ell(\varsigma_\ell,\tau_\ell)$, $p_\ell(\varsigma_\ell,\tau_\ell)$, $P_\ell(\varsigma_\ell,\tau_\ell)$, and $\vartheta_\ell(\varsigma_\ell,\tau_\ell)$ change to $\vartheta_\ell(\varsigma_\ell)$, $p_\ell(\varsigma_\ell)$, $P_\ell(s_\ell)$ and $\vartheta_\ell(s_\ell)$, respectively, and with suitable changes, we have the following new corollary:*

Corollary 2. *Let S_{22}, S_{23}, S_{24}, S_{26}, S_{27} be satisfied. Then, for S_{18}, S_{20} and S_{25} we find that*

$$\int_{\chi_0}^{\xi_1} \cdots \int_{\chi_0}^{\xi_n} \frac{\prod_{\ell=1}^{n} \Phi_\ell(\Theta_\ell(s_\ell)}{\left(\alpha \sum_{\ell=1}^{n} \frac{1}{\alpha_\ell}(s_\ell - \chi_0)\right)^{\frac{1}{\alpha}}} \nabla_a^\gamma s_1 \ldots \nabla_a^\gamma s_n \tag{20}$$
$$\geqslant L^*(\xi_1,\ldots,\xi_n) \prod_{\ell=1}^{n} \left(\int_{\chi_0}^{\xi_\ell} (\rho(\xi_\ell) - s_\ell) \left(p_\ell(s_\ell) \Phi_\ell\left(\frac{\vartheta_\ell(s_\ell)}{p_\ell(s_\ell)} \right) \right)^{\beta_\ell} \nabla_a^\gamma s_\ell \right)^{\frac{1}{\beta_\ell}}$$

where

$$L^*(\xi_1,\ldots,\xi_n) = \prod_{\ell=1}^{n} \left(\int_{\chi_0}^{\xi_\ell} \left(\frac{\Phi_\ell(P_\ell(s_\ell))}{P_\ell(s_\ell)} \right)^{\alpha_\ell} \nabla_a^\gamma s_\ell \right)^{\frac{1}{\alpha_\ell}}.$$

Remark 3. *In Corollary 2, if we take $n = 2$, $\beta_\ell = \frac{1}{2}$, then the Inequality (21) changes to*

$$\int_{\chi_0}^{\xi_1} \int_{\chi_0}^{\xi_1} \frac{\Phi_1(\Theta_1(s_1))\Phi_2(\Theta_2(s_2))}{((s_1 - \chi_0) + (s_2 - \chi_0))^{-2}} \nabla_a^\gamma s_1 \nabla_a^\gamma s_2 \geqslant L^{**}(\xi_1,\xi_2) \Bigg(\int_{\chi_0}^{\xi_1} (\rho(\xi_1) - s_1)$$
$$\left(p_1(s_1) \Phi\left(\frac{\vartheta_1(s_1)}{p_1(s_1)} \right) \right)^2 \nabla_a^\gamma s_1 \Bigg)^{\frac{1}{2}} \tag{21}$$
$$\times \left(\int_{\chi_0}^{\xi_2} (\rho(\xi_2) - s_2) \left(p_2(s_2) \Psi\left(\frac{\vartheta_2(s_2)}{p_2(s_2)} \right) \right)^2 \nabla_a^\gamma s_2 \right)^{\frac{1}{2}}$$

where

$$L^{**}(\xi_1,\xi_2) = 4 \left(\int_{\chi_0}^{\xi_1} \left(\frac{\Phi_1(P_1(s_1))}{P_1(s_1)} \right)^{-1} \nabla_a^\gamma s_1 \right)^{-1} \left(\int_{\chi_0}^{\xi_2} \left(\frac{\Phi_2(P_2(s_2))}{P_2(s_2)} \right)^{-1} \nabla_a^\gamma s_2 \right)^{-1}$$

.

Remark 4. *In Corollary 3, if we take $\mathbb{T}=\mathbb{R}$, $\gamma=1$, then the inequality (22) changes to*

$$\begin{aligned}&\int_0^{\xi_1}\int_0^{\zeta_1}\frac{\Phi_1(\Theta_1(s_1))\Phi_2(\Theta_2(s_2))}{(s_1+s_2)^{-2}}ds_1ds_2\\ &\geqslant L^{**}(\xi_1,\xi_2)\left(\int_0^{\xi_1}(\xi_1-s_1)\left(p_1(s_1)\Phi\left(\frac{\vartheta_1(s_1)}{p_1(s_1)}\right)\right)^2ds_1\right)^{\frac{1}{2}}\\ &\times\left(\int_0^{\xi_2}(\xi_2-s_2)\left(p_2(s_2)\Psi\left(\frac{\vartheta_2(s_2)}{p_2(s_2)}\right)\right)^2ds_2\right)^{\frac{1}{2}}\end{aligned}\tag{22}$$

where

$$L^{**}(\xi_1,\xi_2)=4\left(\int_0^{\xi_1}\left(\frac{\Phi_1(P_1(s_1))}{P_1(s_1)}\right)^{-1}ds_1\right)^{-1}\left(\int_0^{\xi_2}\left(\frac{\Phi_2(P_2(s_2))}{P_2(s_2)}\right)^{-1}ds_2\right)^{-1}.$$

This is an inverse of the Inequality (4), which was proved by Pachpatte [15].

Corollary 3. *In Corollary 2, if we take $\beta_\ell=\frac{n-1}{n}$ the Inequality (21) becomes*

$$\begin{aligned}&\int_{\chi_0}^{\xi_1}\cdots\int_{\chi_0}^{\xi_n}\frac{\prod_{\ell=1}^n\Phi_\ell(\Theta_\ell(s_\ell)}{\left(\sum_{\ell=1}^n(s_\ell-\chi_0)\right)^{\frac{-n}{n-1}}}\nabla_a^\gamma s_n\ldots\nabla_a^\gamma s_1\\ &\geqslant L^*(\xi_1,\ldots,\xi_n)\prod_{\ell=1}^n\left(\int_{\chi_0}^{\xi_\ell}(\rho(\xi_\ell)-s_\ell)\left(p_\ell(s_\ell)\Phi_\ell\left(\frac{\vartheta_\ell(s_\ell)}{p_\ell(s_\ell)}\right)\right)^{\frac{n-1}{n}}\nabla_a^\gamma s_\ell\right)^{\frac{n}{n-1}}\end{aligned}$$

where

$$L^*(\xi_1,\ldots,\xi_n)=n^{\frac{n}{n-1}}\prod_{\ell=1}^n\left(\int_{\chi_0}^{\xi_\ell}\left(\frac{\Phi_\ell(P_\ell(s_\ell))}{P_\ell(s_\ell)}\right)^{-(n-1)}\nabla_a^\gamma s_\ell\right)^{\frac{-1}{n-1}}.$$

Theorem 2. *Let S_1, S_4, S_5, S_6, S_9, S_{15}, and S_{16} be satisfied. Then for S_{10}, S_{18} and S_{20}, we have that*

$$\begin{aligned}&\int_{\chi_0}^{\xi_1}\int_{\chi_0}^{\zeta_1}\cdots\int_{\chi_0}^{\xi_n}\int_{\chi_0}^{\zeta_n}\frac{\prod_{\ell=1}^nP_\ell(s_\ell,\chi_\ell)\Phi_\ell(\Theta_\ell(s_\ell,\chi_\ell))}{\left(\alpha\sum_{\ell=1}^n\frac{1}{\alpha_\ell}(s_\ell-\chi_0)(\chi_\ell-\chi_0)\right)^{\frac{1}{\alpha}}}\nabla_a^\gamma s_n\nabla_a^\gamma\chi_n\ldots\nabla_a^\gamma s_1\nabla_a^\gamma\chi_1\\ &\geqslant\prod_{\ell=1}^n\Big[(\xi_\ell-\chi_0)(\zeta_\ell-\chi_0)\Big]^{\frac{1}{\alpha_\ell}}\left(\int_{\chi_0}^{\xi_\ell}\int_{\chi_0}^{\zeta_\ell}(\rho(\xi_\ell)-s_\ell)(\rho(\zeta_\ell)-\chi_\ell)(p_\ell(s_\ell,\chi_\ell)\Phi_\ell(\vartheta_\ell(s_\ell,\chi_\ell)))^{\beta_\ell}\nabla_a^\gamma s_\ell\nabla_a^\gamma\chi_\ell\right)^{\frac{1}{\beta_\ell}}.\end{aligned}\tag{23}$$

Proof. From the hypotheses of Theorem 2, and using inverse Jensen dynamic inequality, we have

$$\begin{aligned}\Phi_\ell(\vartheta_\ell(s_\ell,\chi_\ell)) &= \Phi_\ell\left(\frac{1}{P_\ell(s_\ell,\chi_\ell)}\int_{\chi_0}^{s_\ell}\int_{\chi_0}^{\chi_\ell}p_\ell(\varsigma_\ell,\tau_\ell)\Theta_\ell(\varsigma_\ell,\tau_\ell)\nabla_a^\gamma\varsigma_\ell\nabla_a^\gamma\tau_\ell\right)\\ &\geqslant\frac{1}{P_\ell(s_\ell,\chi_\ell)}\int_{\chi_0}^{s_\ell}\int_{\chi_0}^{\chi_\ell}p_\ell(\sigma_\ell,\tau_\ell)\Phi_\ell(\vartheta_\ell(\varsigma_\ell,\tau_\ell))\nabla_a^\gamma\varsigma_\ell\nabla_a^\gamma\tau_\ell.\end{aligned}\tag{24}$$

Applying inverse Hölder's inequality on the right-hand side of (24) with indices α_ℓ and β_ℓ,, it is easy to observe that

$$\begin{aligned}\Phi_\ell(\Theta_\ell(s_\ell,\chi_\ell))\geqslant\frac{1}{P_\ell(s_\ell,\chi_\ell)}\Big[(s_\ell-\chi_0)(\chi_\ell-\chi_0)\Big]^{\frac{1}{\alpha_\ell}}\Bigg(\int_{\chi_0}^{s_\ell}\int_{\chi_0}^{\chi_\ell}(p_\ell(\varsigma_\ell,\tau_\ell)\Phi_\ell\\ (\vartheta_\ell(\varsigma_\ell,\tau_\ell)))^{\beta_\ell}\nabla_a^\gamma\varsigma_\ell\nabla_a^\gamma\tau_\ell\Bigg)^{\frac{1}{\beta_\ell}}.\end{aligned}$$

Using the Inequality (15), on the term $\left[(s_\ell-\chi_0)(\chi_\ell-\chi_0)\right]^{\frac{1}{\alpha_\ell}}$, we get that

$$\frac{P_\ell(s_\ell,\chi_\ell)\Phi_\ell(\Theta_\ell(s_\ell,\chi_\ell))}{\left(\alpha\sum_{\ell=1}^{n}\frac{1}{\alpha_\ell}(s_\ell-\chi_0)(\chi_\ell-\chi_0)\right)^{\frac{1}{\alpha}}}\geqslant \left(\int_{\chi_0}^{s_\ell}\int_{\chi_0}^{\chi_\ell}\left(p_\ell(\varsigma_\ell,\tau_\ell)\Phi_\ell(\vartheta_\ell(\varsigma_\ell,\tau_\ell))\right)^{\beta_\ell}\nabla_a^\gamma\varsigma_\ell\nabla_a^\gamma\tau_\ell\right)^{\frac{1}{\beta_\ell}} \tag{25}$$

Integrating both sides of (25) over s_ℓ, χ_ℓ from χ_0 to ξ_ℓ, ζ_ℓ $(\ell=1,\ldots,n)$, we get that

$$\int_{\chi_0}^{\xi_1}\int_{\chi_0}^{\zeta_1}\cdots\int_{\chi_0}^{\xi_n}\int_{\chi_0}^{\zeta_n}\frac{\prod_{\ell=1}^{n}P_\ell(s_\ell,\chi_\ell)\Phi_\ell(\Theta_\ell(s_\ell,\chi_\ell))}{\left(\alpha\sum_{\ell=1}^{n}\frac{1}{\alpha_\ell}(s_\ell-\chi_0)(\chi_\ell-\chi_0)\right)^{\frac{1}{\alpha}}}\nabla_a^\gamma s_n\nabla_a^\gamma\chi_n\ldots\nabla_a^\gamma s_1\nabla_a^\gamma\chi_1$$
$$\geqslant\prod_{\ell=1}^{n}\int_{\chi_0}^{\xi_\ell}\int_{\chi_0}^{\zeta_\ell}\left(\int_{\chi_0}^{s_\ell}\int_{\chi_0}^{\chi_\ell}\left(p_\ell(\varsigma_\ell,\tau_\ell)\Phi_\ell(\vartheta_\ell(\varsigma_\ell,\tau_\ell))\right)^{\beta_\ell}\nabla_a^\gamma\sigma_\ell\nabla_a^\gamma\tau_\ell\right)^{\frac{1}{\beta_\ell}}.$$

Applying inverse Hölder's inequality on the right hand side of (26) with indices α_ℓ and β_ℓ, it is easy to observe that

$$\int_{\chi_0}^{\xi_1}\int_{\chi_0}^{\zeta_1}\cdots\int_{\chi_0}^{\xi_n}\int_{\chi_0}^{\zeta_n}\frac{\prod_{\ell=1}^{n}P_\ell(s_\ell,\chi_\ell)\Phi_\ell(\Theta_\ell(s_\ell,\chi_\ell))}{\left(\alpha\sum_{\ell=1}^{n}\frac{1}{\alpha_\ell}(s_\ell-\chi_0)(\chi_\ell-\chi_0)\right)^{\frac{1}{\alpha}}}\nabla_a^\gamma s_n\nabla_a^\gamma\chi_n\ldots\nabla_a^\gamma s_1\nabla_a^\gamma\chi_1$$
$$\geqslant\prod_{\ell=1}^{n}\left[(\xi_\ell-\chi_0)(\zeta_\ell-\chi_0)\right]^{\frac{1}{\alpha_\ell}}\left(\int_{\chi_0}^{\xi_\ell}\int_{\chi_0}^{\zeta_\ell}\int_{\chi_0}^{s_\ell}\int_{\chi_0}^{\chi_\ell}\left(p_\ell(\varsigma_\ell,\tau_\ell)\Phi_\ell(\vartheta_\ell(\varsigma_\ell,\tau_\ell))\right)^{\beta_\ell}\nabla_a^\gamma\varsigma_\ell\nabla_a^\gamma\tau_\ell\nabla_a^\gamma s_\ell\nabla_a^\gamma\chi_\ell\right)^{\frac{1}{\beta_\ell}}.$$

Using Fubini's theorem, we observe that

$$\int_{\chi_0}^{\xi_1}\int_{\chi_0}^{\zeta_1}\cdots\int_{\chi_0}^{\xi_n}\int_{\chi_0}^{\zeta_n}\frac{\prod_{\ell=1}^{n}P_\ell(s_\ell,\chi_\ell)\Phi_\ell(\Theta_\ell(s_\ell,\chi_\ell))}{\left(\alpha\sum_{\ell=1}^{n}\frac{1}{\alpha_\ell}(s_\ell-\chi_0)(\chi_\ell-\chi_0)\right)^{\frac{1}{\alpha}}}\nabla_a^\gamma s_n\nabla_a^\gamma\chi_n\ldots\nabla_a^\gamma s_1\nabla_a^\gamma\chi_1$$
$$\geqslant\prod_{\ell=1}^{n}\left[(\xi_\ell-\chi_0)(\zeta_\ell-\chi_0)\right]^{\frac{1}{\alpha_\ell}}\left(\int_{\chi_0}^{\xi_\ell}\int_{\chi_0}^{\zeta_\ell}(\xi_\ell-s_\ell)(\zeta_\ell-\chi_\ell)\left(p_\ell(s_\ell,\chi_\ell)\Phi_\ell(\vartheta_\ell(s_\ell,\chi_\ell))\right)^{\beta_\ell}\nabla_a^\gamma s_\ell\nabla_a^\gamma\chi_\ell\right)^{\frac{1}{\beta_\ell}}.$$

By using the fact $\xi_\ell\geqslant\rho(\xi_\ell)$, and $\zeta_\ell\geqslant\rho(\zeta_\ell)$, we get that

$$\int_{\chi_0}^{\xi_1}\int_{\chi_0}^{\zeta_1}\cdots\int_{\chi_0}^{\xi_n}\int_{\chi_0}^{\zeta_n}\frac{\prod_{\ell=1}^{n}P_\ell(s_\ell,\chi_\ell)\Phi_\ell(\Theta_\ell(s_\ell,\chi_\ell))}{\left(\alpha\sum_{\ell=1}^{n}\frac{1}{\alpha_\ell}(s_\ell-\chi_0)(\chi_\ell-\chi_0)\right)^{\frac{1}{\alpha}}}\nabla_a^\gamma s_n\nabla_a^\gamma\chi_n\ldots\nabla_a^\gamma s_1\nabla_a^\gamma\chi_1$$
$$\geqslant\prod_{\ell=1}^{n}\left[(\xi_\ell-\chi_0)(\zeta_\ell-\chi_0)\right]^{\frac{1}{\alpha_\ell}}\left(\int_{\chi_0}^{\xi_\ell}\int_{\chi_0}^{\zeta_\ell}(\rho(\xi_\ell)-s_\ell)(\rho(\zeta_\ell)-\chi_\ell)\left(p_\ell(s_\ell,\chi_\ell)\Phi_\ell(\vartheta_\ell(s_\ell,\chi_\ell))\right)^{\beta_\ell}\nabla_a^\gamma s_\ell\nabla_a^\gamma\chi_\ell\right)^{\frac{1}{\beta_\ell}}$$

This completes the proof. □

Remark 5. *In Theorem 2, if $\mathbb{T}=\mathbb{R}$, we get the result due to Zhao et al. [8] (Theorem 3).*

As a special case of Theorem 2, when $\mathbb{T} = \mathbb{Z}$, $\gamma = 1$, we have $\rho(n) = n - 1$, we get the following result.

Corollary 4. *Let $\{a_{s_\ell,\chi_\ell,m_{s_\ell},m_{\chi_\ell}}\}$ and $\{p_{s_\ell,\chi_\ell,m_{s_\ell},m_{\chi_\ell}}\}$, $(\ell = 1, \ldots, n)$ be n sequences of non-negative numbers defined for $m_{s_\ell} = 1, \ldots, k_{s_\ell}$, and $m_{\chi_\ell} = 1, \ldots, k_{\chi_\ell}$, and define*

$$\begin{aligned} A_{s_\ell,\chi_\ell,m_{s_\ell},m_{\chi_\ell}} &= \frac{1}{P_{s_\ell,\chi_\ell,m_{s_\ell},m_{\chi_\ell}}} \sum_{m_{\varsigma_\ell}}^{m_{s_\ell}} \sum_{m_{\eta_\ell}}^{m_{\chi_\ell}} a_{s_\ell,\chi_\ell,m_{\varsigma_\ell},m_{\eta_\ell}} p_{s_\ell,\chi_\ell,m_{\varsigma_\ell},m_{\eta_\ell}}, \\ P_{s_\ell,\chi_\ell,m_{s_\ell},m_{\chi_\ell}} &= \sum_{m_{\varsigma_\ell}}^{m_{s_\ell}} \sum_{m_{\eta_\ell}}^{m_{\chi_\ell}} p_{s_\ell,\chi_\ell,m_{\varsigma_\ell},m_{\eta_\ell}}. \end{aligned} \tag{26}$$

Then,

$$\begin{aligned} &\sum_{m_{s_1}}^{k_{s_1}} \sum_{m_{\chi_1}}^{k_{\chi_1}} \cdots \sum_{m_{s_n}}^{k_{s_n}} \sum_{m_{\chi_n}}^{k_{\chi_n}} \frac{\prod_{\ell=1}^{n} P_{s_\ell,\chi_\ell,m_{s_\ell},m_{\chi_\ell}} \Phi_\ell(A_{s_\ell,\chi_\ell,m_{s_\ell},m_{\chi_\ell}})}{\left(\alpha \sum_{\ell=1}^{n} \frac{1}{\alpha_\ell}(m_{s_\ell} m_{\chi_\ell})\right)^{\frac{1}{\alpha}}} \\ &\geqslant \prod_{\ell=1}^{n} (k_{s_\ell} k_{\chi_\ell})^{\frac{1}{\alpha_\ell}} \left(\sum_{m_{s_\ell}}^{k_{s_\ell}} \sum_{m_{\chi_\ell}}^{k_{\chi_\ell}} (k_{s_\ell} - (m_{s_\ell} - 1))(k_{\chi_\ell} - (m_{\chi_\ell} - 1)) \right. \\ &\left. \left(p_{s_\ell,\chi_\ell,m_{s_\ell},m_{\chi_\ell}} \Phi_\ell\left(a_{s_\ell,\chi_\ell,m_{s_\ell},m_{\chi_\ell}} \right) \right)^{\beta_\ell} \right)^{\frac{1}{\beta_\ell}}. \end{aligned}$$

Remark 6. *Let $\vartheta_\ell(\varsigma_\ell, \tau_\ell)$, $p_\ell(\varsigma_\ell, \tau_\ell)$, $P_\ell(\varsigma_\ell, \tau_\ell)$ be defined as above and*

$$\Theta_\ell(s_\ell, \chi_\ell) = \frac{1}{P_\ell(s_\ell, \chi_\ell)} \int_{\chi_0}^{s_\ell} \int_0^{\chi_\ell} p_\ell(\varsigma_\ell, \tau_\ell) \vartheta_\ell(\varsigma_\ell, \tau_\ell) \nabla_a^\gamma \varsigma_\ell \nabla_a^\gamma \tau_\ell$$

changes to $\vartheta_\ell(\varsigma_\ell)$, $p_\ell(\varsigma_\ell)$, $P_\ell(s_\ell)$, and

$$\Theta_\ell(s_\ell) = \frac{1}{P_\ell(s_\ell)} \int_{\chi_0}^{s_\ell} p_\ell(\varsigma_\ell) \vartheta_\ell(\varsigma_\ell) \nabla_a^\gamma \varsigma_\ell.$$

respectively, and with suitable changes, we have the following new corollary:

Corollary 5. *Let S_{22}, S_{23}, S_{26}, S_{27} and S_{28} be satisfied. Then, for S_{18}, S_{20} and S_{25}, we find that*

$$\begin{aligned} &\int_{\chi_0}^{\xi_1} \cdots \int_{\chi_0}^{\xi_n} \frac{\prod_{\ell=1}^{n} P_\ell(s_\ell) \Phi_\ell(\Theta_\ell(s_\ell)}{\left(\alpha \sum_{\ell=1}^{n} \frac{1}{\alpha_\ell}(s_\ell - \chi_0)\right)^{\frac{1}{\alpha}}} \nabla_a^\gamma s_n \ldots \nabla_a^\gamma s_1 \\ &\geqslant \prod_{\ell=1}^{n} (\xi_\ell - \chi_0)^{\frac{1}{\alpha_\ell}} \left(\int_{\chi_0}^{\xi_\ell} (\rho(\xi_\ell) - s_\ell) \left(p_\ell(s_\ell) \Phi_\ell(\vartheta_\ell(s_\ell)) \right)^{\beta_\ell} \nabla_a^\gamma s_\ell \right)^{\frac{1}{\beta_\ell}}. \end{aligned} \tag{27}$$

Corollary 6. *In Corollary 5, if we take $n = 2$, $\beta_\ell = \frac{1}{2}$, then the inequality (21) changes to*

$$\begin{aligned} &\int_{\chi_0}^{\xi_1} \int_{\chi_0}^{\xi_1} \frac{P_1(s_1) P_2(s_2) \Phi_1(\Theta_1(s_1)) \Phi_2(\Theta_2(s_2))}{((s_1 - \chi_0) + (s_2 - \chi_0))^{-2}} \nabla_a^\gamma s_1 \nabla_a^\gamma s_2 \geqslant 4\left[(\xi_1 - \chi_0)(\xi_2 - \chi_0)\right]^{-1} \\ &\times \left(\int_{\chi_0}^{\xi_1} (\rho(\xi_1) - s_1) \left(p_1(s_1) \Phi_1(\vartheta_1(s_1)) \right)^2 \nabla_a^\gamma s_1 \right)^{\frac{1}{2}} \left(\int_{\chi_0}^{\xi_2} (\rho(\xi_2) - s_2) \left(p_2(s_2) \right. \right. \\ &\left. \left. \Phi_2\left(\vartheta_2(s_2) \right) \right)^2 \nabla_a^\gamma s_2 \right)^{\frac{1}{2}}. \end{aligned} \tag{28}$$

Remark 7. *In Corollary 6, if we take $\mathbb{T} = \mathbb{R}$, $\gamma = 1$, then Inequality (28) changes to*

$$\begin{aligned}\int_0^{\xi_1}\int_0^{\xi_1}\frac{P_1(s_1)P_2(s_2)\Phi_1(\Theta_1(s_1))\Phi_2(\Theta_2(s_2))}{(s_1+s_2)^{-2}}ds_1ds_2 \geqslant 4\big[\xi_1\xi_2\big]^{-1}\\ \times\bigg(\int_0^{\xi_1}(\xi_1-s_1)\Big(p_1(s_1)\Phi_1(\vartheta_1(s_1))\Big)^2ds_1\bigg)^{\frac{1}{2}}\\ \bigg(\int_0^{\xi_2}(\xi_2-s_2)\Big(p_2(s_2)\Phi_2\Big(\vartheta_2(s_2)\Big)\Big)^2ds_2\bigg)^{\frac{1}{2}}.\end{aligned} \tag{29}$$

This is an inverse of Inequality (5), which was proved by Pachpatte [15].

Corollary 7. *In Corollary 6, let $p_1(s_1) = p_2(s_2) = 1$, then $P_1(s_1) = s_1$, $P_2(s_2) = s_2$. Therefore, Inequality (28) changes to*

$$\begin{aligned}\int_{\chi_0}^{\xi_1}\int_{\chi_0}^{\xi_1}\frac{\Phi_1(\Theta_1(s_1))\Phi_2(\Theta_2(s_2))}{(s_1s_2)^{-1}\big((s_1-\chi_0)+(s_2-\chi_0)\big)^{-2}}\nabla_a^\gamma s_1\nabla_a^\gamma s_2 \geqslant 4\big[(\xi_1-\chi_0)(\xi_2-\chi_0)\big]^{-1}\\ \times\bigg(\int_{\chi_0}^{\xi_1}(\rho(\xi_1)-s_1)\Big(\Phi_1(\vartheta_1(s_1))\Big)^2\nabla_a^\gamma s_1\bigg)^{\frac{1}{2}}\\ \bigg(\int_{\chi_0}^{\xi_2}(\rho(\xi_2)-s_2)\Big(\Phi_2\Big(\vartheta_2(s_2)\Big)\Big)^2\nabla_a^\gamma s_2\bigg)^{\frac{1}{2}}.\end{aligned} \tag{30}$$

Remark 8. *In Corollary 7, if we take $\mathbb{T} = \mathbb{R}$, then, the Inequality (30) changes to*

$$\begin{aligned}\int_0^{\xi_1}\int_0^{\xi_1}\frac{\Phi_1(\Theta_1(s_1))\Phi_2(\Theta_2(s_2))}{(s_1s_2)^{-1}(s_1+s_2)^{-2}}ds_1ds_2 \geqslant 4\big[\xi_1\xi_2\big]^{-1}\\ \times\bigg(\int_0^{\xi_1}(\xi_1-s_1)\Big(\Phi_1(\vartheta_1(s_1))\Big)^2ds_1\bigg)^{\frac{1}{2}}\bigg(\int_0^{\xi_2}(\xi_2-s_2)\Big(\Phi_2\Big(\vartheta_2(s_2)\Big)\Big)^2ds_2\bigg)^{\frac{1}{2}}.\end{aligned}$$

This is an inverse inequality of the following inequality, which was proved by Pachpatte [8].

$$\begin{aligned}\int_0^{\xi}\int_0^{\zeta}\frac{\Phi(\Theta(s))\Psi(G(\chi))}{(s\chi)^{-1}(s+\chi)}dsd\chi \leqslant \frac{1}{2}[\xi\zeta]^{\frac{1}{2}}\\ \times\bigg(\int_0^{\xi}(\xi-s_1)\Big(\Phi(\vartheta(s))\Big)^2ds\bigg)^{\frac{1}{2}}\bigg(\int_0^{\zeta}(\zeta-\chi)\Big(\Psi\Big(g(\chi)\Big)\Big)^2d\chi\bigg)^{\frac{1}{2}}.\end{aligned}$$

Corollary 8. *In Corollary 5, if we take $\beta_\ell = \frac{n-1}{n}$ $(\ell = 1,\dots,n)$ Inequality (27)*

$$\begin{aligned}\int_{\chi_0}^{\xi_1}\cdots\int_{\chi_0}^{\xi_n}\frac{\prod_{\ell=1}^n P_\ell(s_\ell)\Phi_\ell(\Theta_\ell(s_\ell)}{\Big(\sum_{\ell=1}^n(s_\ell-\chi_0)\Big)^{\frac{-n}{n-1}}}\nabla_a^\gamma s_n\dots\nabla_a^\gamma s_1\\ \geqslant n^{\frac{n}{n-1}}\prod_{\ell=1}^n(\xi_\ell-\chi_0)^{\frac{-1}{n-1}}\bigg(\int_{\chi_0}^{\xi_\ell}(\rho(\xi_\ell)-s_\ell)\Big(p_\ell(s_\ell)\Phi_\ell(\vartheta_\ell(s_\ell))\Big)^{\frac{n-1}{n}}\nabla_a^\gamma s_\ell\bigg)^{\frac{n}{n-1}}.\end{aligned}$$

Theorem 3. *Let S_1, S_4, S_2, S_9, S_{11}, S_7, S_{13}, S_3, S_{12}, S_8 and S_{17} be satisfied. Then, for S_{10} we have that*

$$\begin{aligned}\int_{\chi_0}^{\xi_1}\int_{\chi_0}^{\zeta_1}\cdots\int_{\chi_0}^{\xi_n}\int_{\chi_0}^{\zeta_n}\prod_{\ell=1}^n\frac{\Phi_\ell(\vartheta_\ell(s_\ell,\chi_\ell))}{\Big(\frac{1}{\alpha'}\sum_{\ell=1}^n\alpha'_\ell(s_\ell-\chi_0)(\chi_\ell-\chi_0)\Big)^{\alpha'}}\nabla_a^\gamma s_n\nabla_a^\gamma\chi_n\dots\nabla_a^\gamma s_1\nabla_a^\gamma\chi_1\\ \geqslant G(\xi_1\zeta_1,\dots,\xi_n y_n)\\ \times\prod_{\ell=1}^n\bigg(\int_{\chi_0}^{\xi_\ell}\int_{\chi_0}^{\zeta_\ell}(\rho(\xi_\ell)-s_\ell)(\rho(\zeta_\ell)-\chi_\ell)\Big(p_\ell(s_\ell)q_\ell(\chi_\ell)\Phi_\ell\Big(\frac{\vartheta_\ell^{\nabla_{a_2}^\gamma\nabla_{a_1}^\gamma}(s_\ell,\chi_\ell)}{p_\ell(s_\ell)q_\ell(\chi_\ell)}\Big)\Big)^{\frac{1}{\alpha_\ell}}\\ \nabla_a^\gamma s_\ell\nabla_a^\gamma\chi_\ell\bigg)^{\alpha_\ell},\end{aligned} \tag{31}$$

where

$$G(\xi_1\zeta_1,\ldots,\xi_n y_n)=\prod_{\ell=1}^{n}\left(\int_{\chi_0}^{\xi_\ell}\int_{\chi_0}^{\zeta_\ell}\left(\frac{\Phi_\ell(P_\ell(s_\ell,\chi_\ell))}{P_\ell(s_\ell,\chi_\ell)}\right)^{\frac{1}{\alpha'_\ell}}\nabla_a^\gamma s_\ell\nabla_a^\gamma\chi_\ell\right)^{\alpha'_\ell}.$$

Proof. From the hypotheses of Theorem 3, we obtain

$$\vartheta_\ell(s_\ell,\chi_\ell)=\int_{\chi_0}^{s_\ell}\int_{\chi_0}^{\chi_\ell}\vartheta_\ell^{\nabla_{a_2}^\gamma\nabla_{a_1}^\gamma}(\varsigma_\ell,\tau_\ell)\nabla_a^\gamma\varsigma_\ell\nabla_a^\gamma\tau_\ell. \tag{32}$$

From (32) and S_8, it is easy to observe that

$$\Phi_\ell(\vartheta_\ell(s_\ell,\chi_\ell))=\Phi_\ell\left(\frac{P_\ell(s_\ell,\chi_\ell)\int_{\chi_0}^{s_\ell}\int_{\chi_0}^{\chi_\ell}p_\ell(\varsigma_\ell)q_\ell(\tau_\ell)\big(\frac{\vartheta_\ell^{\nabla_{a_2}^\gamma\nabla_{a_1}^\gamma}(\varsigma_\ell,\tau_\ell)}{p_\ell(\varsigma_\ell)q_\ell(\tau_\ell)}\big)\nabla_a^\gamma\varsigma_\ell\nabla_a^\gamma\tau_\ell}{\int_{\chi_0}^{s_\ell}\int_{\chi_0}^{\chi_\ell}p_\ell(\varsigma_\ell)q_\ell(\tau_\ell)\nabla_a^\gamma\varsigma_\ell\nabla_a^\gamma\tau_\ell}\right)$$

$$\geqslant\Phi_\ell(P_\ell(s_\ell,\chi_\ell))\Phi_\ell\left(\frac{\int_{\chi_0}^{s_\ell}\int_{\chi_0}^{\chi_\ell}p_\ell(\varsigma_\ell)q_\ell(\tau_\ell)\big(\frac{\vartheta_\ell^{\nabla_{a_2}^\gamma\nabla_{a_1}^\gamma}(\varsigma_\ell,\tau_\ell)}{p_\ell(\varsigma_\ell)q_\ell(\tau_\ell)}\big)\nabla_a^\gamma\varsigma_\ell\nabla_a^\gamma\tau_\ell}{\int_{\chi_0}^{s_\ell}\int_{\chi_0}^{\chi_\ell}p_\ell(\varsigma_\ell)q_\ell(\tau_\ell)\nabla_a^\gamma\varsigma_\ell\nabla_a^\gamma\tau_\ell}\right). \tag{33}$$

Using inverse Jensen's dynamic inequality, we get that

$$\begin{aligned}\Phi_\ell(\vartheta_\ell(s_\ell,\chi_\ell))\geqslant&\frac{\Phi_\ell(P_\ell(s_\ell,\chi_\ell))}{P_\ell(s_\ell,\chi_\ell)}\int_{\chi_0}^{s_\ell}\int_{\chi_0}^{\chi_\ell}p_\ell(\varsigma_\ell)q_\ell(\tau_\ell)\\&\Phi_\ell\left(\frac{\vartheta_\ell^{\nabla_{a_2}^\gamma\nabla_{a_1}^\gamma}(\varsigma_\ell,\tau_\ell)}{p_\ell(\varsigma_\ell)q_\ell(\tau_\ell)}\right)\nabla_a^\gamma\varsigma_\ell\nabla_a^\gamma\tau_\ell.\end{aligned} \tag{34}$$

Applying inverse Hölder's inequality on the right-hand side of (34) with indices $1/\alpha_\ell$ and $1/\alpha'_\ell$, we obtain

$$\Phi_\ell(\vartheta_\ell(s_\ell,\chi_\ell))\geqslant\frac{\Phi_\ell(P_\ell(s_\ell,\chi_\ell))}{P_\ell(s_\ell,\chi_\ell)}\left[(s_\ell-\chi_0)(\chi_\ell-\chi_0)\right]^{\alpha'_\ell}$$

$$\times\left(\int_{\chi_0}^{s_\ell}\int_{\chi_0}^{\chi_\ell}\left(p_\ell(\varsigma_\ell)q_\ell(\tau_\ell)\Phi_\ell\left(\frac{\vartheta_\ell^{\nabla_{a_2}^\gamma\nabla_{a_1}^\gamma}(\varsigma_\ell,\tau_\ell)}{p_\ell(\varsigma_\ell)q_\ell(\tau_\ell)}\right)\right)^{\frac{1}{\alpha_\ell}}\nabla_a^\gamma\varsigma_\ell\nabla_a^\gamma\tau_\ell\right)^{\alpha_\ell}. \tag{35}$$

Using the following inequality on the term $\left[(s_\ell-\chi_0)(\chi_\ell-\chi_0)\right]^{\alpha'_\ell}$, where $\alpha'_\ell<0$ and $\lambda_\ell>0$.

$$\prod_{\ell=1}^{n}\lambda_\ell^{\alpha'_\ell}\geqslant\left(\frac{1}{\alpha'}\big(\sum_{\ell=1}^{n}\alpha'_\ell\lambda_\ell\big)\right)^{\alpha'}. \tag{36}$$

We obtain that

$$\prod_{\ell=1}^{n}\Phi_\ell(\vartheta_\ell(s_\ell,\chi_\ell))\geqslant\prod_{\ell=1}^{n}\frac{\Phi_\ell(P_\ell(s_\ell,\chi_\ell))}{P_\ell(s_\ell,\chi_\ell)}\left(\frac{1}{\alpha'}\sum_{\ell=1}^{n}\alpha'_\ell(s_\ell-\chi_0)(\chi_\ell-\chi_0)\right)^{\alpha'}$$

$$\times\left(\int_{\chi_0}^{s_\ell}\int_{\chi_0}^{\chi_\ell}\left(p_\ell(\varsigma_\ell)q_\ell(\tau_\ell)\Phi_\ell\left(\frac{\vartheta_\ell^{\nabla_{a_2}^\gamma\nabla_{a_1}^\gamma}(\varsigma_\ell,\tau_\ell)}{p_\ell(\varsigma_\ell)q_\ell(\tau_\ell)}\right)\right)^{\frac{1}{\alpha_\ell}}\nabla_a^\gamma\varsigma_\ell\nabla_a^\gamma\tau_\ell\right)^{\alpha_\ell}. \tag{37}$$

From (37), we find that

$$\prod_{\ell=1}^{n}\frac{\Phi_\ell(\vartheta_\ell(s_\ell,\chi_\ell))}{\left(\frac{1}{\alpha'}\sum_{\ell=1}^{n}\alpha'_\ell(s_\ell-\chi_0)(\chi_\ell-\chi_0)\right)^{\alpha'}}$$

$$\geqslant \prod_{\ell=1}^{n} \frac{\Phi_\ell(P_\ell(s_\ell,\chi_\ell))}{P_\ell(s_\ell,\chi_\ell)} \left(\int_{\chi_0}^{s_\ell} \int_{\chi_0}^{\chi_\ell} \left(p_\ell(\varsigma_\ell) q_\ell(\tau_\ell) \Phi_\ell\left(\frac{\vartheta_\ell^{\nabla_{a_2}^\gamma \nabla_{a_1}^\gamma}(\varsigma_\ell,\tau_\ell)}{p_\ell(\varsigma_\ell) q_\ell(\tau_\ell)} \right) \right)^{\frac{1}{\alpha_\ell}} \nabla_a^\gamma \varsigma_\ell \nabla_a^\gamma \tau_\ell \right)^{\alpha_\ell}. \tag{38}$$

Integrating both sides of (38) over s_ℓ, χ_ℓ from χ_0 to ξ_ℓ, ζ_ℓ $(\ell = 1, \dots, n)$, we find that

$$\int_{\chi_0}^{\xi_1} \int_{\chi_0}^{\zeta_1} \cdots \int_{\chi_0}^{\xi_n} \int_{\chi_0}^{\zeta_n} \prod_{\ell=1}^{n} \frac{\Phi_\ell(\vartheta_\ell(s_\ell,\chi_\ell))}{\left(\frac{1}{\alpha'} \sum_{\ell=1}^{n} \alpha'_\ell (s_\ell - \chi_0)(\chi_\ell - \chi_0) \right)^{\alpha'}} \nabla_a^\gamma s_n \nabla_a^\gamma \chi_n \dots \nabla_a^\gamma s_1 \nabla_a^\gamma \chi_1$$

$$\geqslant \prod_{\ell=1}^{n} \int_{\chi_0}^{\xi_\ell} \int_{\chi_0}^{\zeta_\ell} \frac{\Phi_\ell(P_\ell(s_\ell,\chi_\ell))}{P_\ell(s_\ell,\chi_\ell)} \left(\int_{\chi_0}^{s_\ell} \int_{\chi_0}^{\chi_\ell} \left(p_\ell(\varsigma_\ell) q_\ell(\tau_\ell) \Phi_\ell\left(\frac{\vartheta_\ell^{\nabla_{a_2}^\gamma \nabla_{a_1}^\gamma}(\varsigma_\ell,\tau_\ell)}{p_\ell(\varsigma_\ell) q_\ell(\tau_\ell)} \right) \right)^{\frac{1}{\alpha_\ell}} \nabla_a^\gamma \varsigma_\ell \nabla_a^\gamma \tau_\ell \right)^{\alpha_\ell} \nabla_a^\gamma s_\ell \nabla_a^\gamma \chi_\ell \tag{39}$$

Applying inverse Hölder's inequality on the right-hand side of (39) with indices $1/\alpha_\ell$ and $1/\alpha'_\ell$,, we obtain

$$\begin{aligned} &\int_{\chi_0}^{\xi_1} \int_{\chi_0}^{\zeta_1} \cdots \int_{\chi_0}^{\xi_n} \int_{\chi_0}^{\zeta_n} \prod_{\ell=1}^{n} \frac{\Phi_\ell(\vartheta_\ell(s_\ell,\chi_\ell))}{\left(\frac{1}{\alpha'} \sum_{\ell=1}^{n} \alpha'_\ell (s_\ell - \chi_0)(\chi_\ell - \chi_0) \right)^{\alpha'}} \nabla_a^\gamma s_n \nabla_a^\gamma \chi_n \dots \nabla_a^\gamma s_1 \nabla_a^\gamma \chi_1 \\ &\geqslant \prod_{\ell=1}^{n} \left(\int_{\chi_0}^{\xi_\ell} \int_{\chi_0}^{\zeta_\ell} \left(\frac{\Phi_\ell(P_\ell(s_\ell,\chi_\ell))}{P_\ell(s_\ell,\chi_\ell)} \right)^{\frac{1}{\alpha'_\ell}} \nabla_a^\gamma s_\ell \nabla_a^\gamma \chi_\ell \right)^{\alpha'_\ell} \\ &\times \prod_{\ell=1}^{n} \left(\int_{\chi_0}^{\xi_\ell} \int_{\chi_0}^{\zeta_\ell} \left(\int_{\chi_0}^{s_\ell} \int_{\chi_0}^{\chi_\ell} \left(p_\ell(\varsigma_\ell) q_\ell(\tau_\ell) \Phi_\ell\left(\frac{\vartheta_\ell^{\nabla_{a_2}^\gamma \nabla_{a_1}^\gamma}(\varsigma_\ell,\tau_\ell)}{p_\ell(\varsigma_\ell) q_\ell(\tau_\ell)} \right) \right)^{\frac{1}{\alpha_\ell}} \nabla_a^\gamma \varsigma_\ell \nabla_a^\gamma \tau_\ell \right) \nabla_a^\gamma s_\ell \nabla_a^\gamma \chi_\ell \right)^{\alpha_\ell} \end{aligned} \tag{40}$$

By using Fubini's theorem, we observe that

$$\begin{aligned} &\int_{\chi_0}^{\xi_1} \int_{\chi_0}^{\zeta_1} \cdots \int_{\chi_0}^{\xi_n} \int_{\chi_0}^{\zeta_n} \prod_{\ell=1}^{n} \frac{\Phi_\ell(\vartheta_\ell(s_\ell,\chi_\ell))}{\left(\frac{1}{\alpha'} \sum_{\ell=1}^{n} \alpha'_\ell (s_\ell - \chi_0)(\chi_\ell - \chi_0) \right)^{\alpha'}} \nabla_a^\gamma s_n \nabla_a^\gamma \chi_n \dots \nabla_a^\gamma s_1 \nabla_a^\gamma \chi_1 \\ &\geqslant G(\xi_1\zeta_1, \dots, \xi_n y_n) \\ &\times \prod_{\ell=1}^{n} \left(\int_{\chi_0}^{\xi_\ell} \int_{\chi_0}^{\zeta_\ell} (\xi_\ell - s_\ell)(\zeta_\ell - \chi_\ell) \left(p_\ell(s_\ell) q_\ell(\chi_\ell) \Phi_\ell\left(\frac{\vartheta_\ell^{\nabla_{a_2}^\gamma \nabla_{a_1}^\gamma}(s_\ell,\chi_\ell)}{p_\ell(s_\ell) q_\ell(\chi_\ell)} \right) \right)^{\frac{1}{\alpha_\ell}} \nabla_a^\gamma s_\ell \nabla_a^\gamma \chi_\ell \right)^{\alpha_\ell}. \end{aligned} \tag{41}$$

Using the fact $\xi_\ell \geqslant \rho(\xi_\ell)$, and $\zeta_\ell \geqslant \rho(\zeta_\ell)$, we get that

$$\begin{aligned} &\int_{\chi_0}^{\xi_1} \int_{\chi_0}^{\zeta_1} \cdots \int_{\chi_0}^{\xi_n} \int_{\chi_0}^{\zeta_n} \prod_{\ell=1}^{n} \frac{\Phi_\ell(\vartheta_\ell(s_\ell,\chi_\ell))}{\left(\frac{1}{\alpha'} \sum_{\ell=1}^{n} \alpha'_\ell (s_\ell - \chi_0)(\chi_\ell - \chi_0) \right)^{\alpha'}} \nabla_a^\gamma s_n \nabla_a^\gamma \chi_n \dots \nabla_a^\gamma s_1 \nabla_a^\gamma \chi_1 \\ &\geqslant G(\xi_1\zeta_1, \dots, \xi_n y_n) \\ &\times \prod_{\ell=1}^{n} \left(\int_{\chi_0}^{\xi_\ell} \int_{\chi_0}^{\zeta_\ell} (\rho(\xi_\ell) - s_\ell)(\rho(\zeta_\ell) - \chi_\ell) \left(p_\ell(s_\ell) q_\ell(\chi_\ell) \Phi_\ell\left(\frac{\vartheta_\ell^{\nabla_{a_2}^\gamma \nabla_{a_1}^\gamma}(s_\ell,\chi_\ell)}{p_\ell(s_\ell) q_\ell(\chi_\ell)} \right) \right)^{\frac{1}{\alpha_\ell}} \nabla_a^\gamma s_\ell \nabla_a^\gamma \chi_\ell \right)^{\alpha_\ell}. \end{aligned}$$

This completes the proof. □

Remark 9. *In Theorem 3, if $\mathbb{T} = \mathbb{Z}$, $\gamma = 1$, we get the result due to Zhao et al. [9] (Theorem 1.5).*

Remark 10. *In Theorem 3, if we take $\mathbb{T} = \mathbb{R}$, $\gamma = 1$, we get the result due to Zhao et al. [9] (Theorem 1.6).*

Remark 11. *Let S_1, S_2, S_9, S_{11}, S_7, S_{13}, S_3 and S_{12} be satisfied and let Φ_ℓ, α_ℓ, α'_ℓ, α, and α' be the same as Theorem 3. Similar to proof of Theorem 3, we have*

$$\int_{\chi_0}^{\xi_1}\int_{\chi_0}^{\zeta_1}\cdots\int_{\chi_0}^{\xi_n}\int_{\chi_0}^{\zeta_n}\prod_{\ell=1}^{n}\frac{\Phi_\ell(\vartheta_\ell(s_\ell,\chi_\ell))}{\left(\frac{1}{\alpha'}\sum_{\ell=1}^{n}\alpha'_\ell(s_\ell-\chi_0)(\chi_\ell-\chi_0)\right)^{\alpha'}}\nabla_a^\gamma s_n\nabla_a^\gamma\chi_n\ldots\nabla_a^\gamma s_1\nabla_a^\gamma\chi_1$$
$$\leqslant G^*(\xi_1\zeta_1,\ldots,\xi_n y_n)$$
$$\times\prod_{\ell=1}^{n}\left(\int_{\chi_0}^{\xi_\ell}\int_{\chi_0}^{\zeta_\ell}(\sigma(\xi_\ell)-s_\ell)(\sigma(\zeta_\ell)-\chi_\ell)\left(p_\ell(s_\ell)q_\ell(\chi_\ell)\Phi_\ell\left(\frac{\vartheta_\ell^{\nabla_{a_2}^\gamma\nabla_{a_1}^\gamma}(s_\ell,\chi_\ell)}{p_\ell(s_\ell)q_\ell(\chi_\ell)}\right)\right)^{\frac{1}{\alpha_\ell}}\nabla_a^\gamma s_\ell\nabla_a^\gamma\chi_\ell\right)^{\alpha_\ell}.$$

where

$$G^*(\xi_1\zeta_1,\ldots,\xi_n y_n)=\frac{1}{(\alpha')^{\alpha'}}\prod_{\ell=1}^{n}\left(\int_{\chi_0}^{\xi_\ell}\int_{\chi_0}^{\zeta_\ell}\left(\frac{\Phi_\ell(P_\ell(s_\ell,\chi_\ell))}{P_\ell(s_\ell,\chi_\ell)}\right)^{\frac{1}{\alpha'_\ell}}\nabla_a^\gamma s_\ell\nabla_a^\gamma\chi_\ell\right)^{\alpha'_\ell}.$$

This is an inverse form of Inequality (31).

Corollary 9. *Let S_{22}, S_{23}, S_{25}, S_{26}, S_{27}, S_{29}, S_{17} and S_8 be satisfied. Then, we have that*

$$\int_{\chi_0}^{\xi_1}\cdots\int_{\chi_0}^{\xi_n}\prod_{\ell=1}^{n}\frac{\Phi_\ell(\vartheta_\ell(s_\ell))}{\left(\frac{1}{\alpha'}\sum_{\ell=1}^{n}\alpha'_\ell(s_\ell-\chi_0)\right)^{\alpha'}}\nabla_a^\gamma s_n\ldots\nabla_a^\gamma s_1 \tag{42}$$
$$\geqslant G^{**}(\xi_1,\ldots,\xi_n)$$
$$\times\prod_{\ell=1}^{n}\left(\int_{\chi_0}^{\xi_\ell}(\rho(\xi_\ell)-s_\ell)\left(p_\ell(s_\ell)\Phi_\ell\left(\frac{\vartheta_\ell^{\nabla_a^\gamma}(s_\ell)}{p_\ell(s_\ell)}\right)\right)^{\frac{1}{\alpha_\ell}}\nabla_a^\gamma s_\ell\right)^{\alpha_\ell}.$$

where

$$G^{**}(\xi_1,\ldots,\xi_n)=\prod_{\ell=1}^{n}\left(\int_{\chi_0}^{\xi_\ell}\left(\frac{\Phi_\ell(P_\ell(s_\ell))}{P_\ell(s_\ell)}\right)^{\frac{1}{\alpha'_\ell}}\nabla_a^\gamma s_\ell\right)^{\alpha'_\ell}.$$

Remark 12. *In Corollary 9, if we take $\mathbb{T} = \mathbb{Z}$, $\gamma = 1$, we get an inverse form of inequality due to Handley et al. [16].*

Remark 13. *In Corollary 9, if we take $\mathbb{T} = \mathbb{R}$, $\gamma = 1$, we get an inverse form of inequality due to Handley et al. [16].*

Remark 14. *In Inequality (42), taking $n = 2$, $\alpha_1 = \alpha_2 = 2$, then $\alpha'_1 = \alpha'_2 = -1$, we have*

$$\int_{\chi_0}^{\xi_1}\int_{\chi_0}^{\xi_2}\prod_{\ell=1}^{n}\frac{\Phi_1(\vartheta_1(s_1))\Phi_1(\vartheta_2(s_2))}{\left((s_1-\chi_0)+(s_2-\chi_0)\right)^{-2}}\nabla_a^\gamma s_1\nabla_a^\gamma s_2 \tag{43}$$
$$\geqslant D(\xi_1,\xi_2)\left(\int_{\chi_0}^{\xi_1}(\rho(\xi_1)-s_1)\left(p_1(s_1)\Phi_1\left(\frac{\vartheta_1^{\nabla_a^\gamma}(s_1)}{p_1(s_1)}\right)\right)^{\frac{1}{2}}\nabla_a^\gamma s_1\right)^2$$
$$\times\left(\int_{\chi_0}^{\xi_2}(\rho(\xi_2)-s_2)\left(p_2(s_2)\Phi_2\left(\frac{\vartheta_2^{\nabla_a^\gamma}(s_2)}{p_2(s_2)}\right)\right)^{\frac{1}{2}}\nabla_a^\gamma s_2\right)^2.$$

where

$$D(\xi_1, \xi_2) = 4\left(\int_{\chi_0}^{\xi_1} \left(\frac{\Phi_1(P_1(s_1))}{P_2(s_1)} \right)^{-1} \nabla_a^\gamma s_1 \right)^{-1} \left(\int_{\chi_0}^{\xi_2} \left(\frac{\Phi_2(P_2(s_2))}{P_2(s_2)} \right)^{-1} \nabla_a^\gamma s_2 \right)^{-1}.$$

Remark 15. *If we take $\mathbb{T} = \mathbb{Z}$, $\gamma = 1$, Inequality (43) is an inverse of inequality due to Pachpatte [2].*

Remark 16. *If we take $\mathbb{T} = \mathbb{R}$, $\gamma = 1$, Inequality (43) is an inverse of inequality due to Pachpatte [2].*

3. Conclusions

In this article, we presented some investigations of the (γ, a)-nabla Hilbert inequality on time scales. Some dynamic integral and discrete inequalities, known in the literature, are generalized as special cases of our results. We obtained the discrete and the continuous inequalities as special cases of our main results. In future work, I will ask if it is possible to generalize these results using a q-difference operator.

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