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Study of Weak Solutions for Degenerate Parabolic Inequalities with Nonlocal Nonlinearities

Yan Dong

Department of Basic, Shaanxi Railway Institute, Weinan 714000, China; dongyan840214@126.com

Abstract: This paper studies a class of variational inequalities with degenerate parabolic operators and symmetric structure, which is an extension of the parabolic equation in a bounded domain. By solving a series of penalty problems, the existence and uniqueness of the solutions in the weak sense are proved by the energy method and a limit process.

Keywords: nonlocal parabolic variational inequality; weak solution; penalty problem; existence; uniqueness

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1. Introduction

In this paper, the author studied parabolic problems with nonlocal nonlinearity of the following type:

$$\begin{cases} \min\{Lu, u-u_0\} = 0, & (x,t) \in Q_T, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T], \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1}$$

where $Q_T = \Omega \times (0,T]$, $\Omega \subset R_N$ $(N \geq 2)$ is a bounded domain with appropriately smooth boundary $\partial\Omega$, and u_0 satisfies

$$0 \leq u_0 \in H_0^1(\Omega) \cap L_\infty(\Omega). \tag{2}$$

$a(\cdot)$ is a given function which satisfies $a(u) = \left(\int_\Omega u^2(x,t)\mathrm{d}x\right)^\gamma$ with $\gamma \in R$. Lu is a degenerate parabolic operator, which satisfies

$$Lu = \partial_t u - a(u)\Delta u - f(x,t), \gamma > 0.$$

Here, $\nabla u = (\partial_{x_1}u, \partial_{x_2}u, \cdots, \partial_{x_N}u)$, $|\nabla u|^{p(x,t)} = \left(\sum_{i=1}^{N} |\partial_{x_i}u|^2\right)^{\frac{p(x,t)}{2}}$. The problem (1) can be decomposed into two symmetric cases: if $u(x,t) = u_0(x)$ for any $(x,t) \in \Omega_T$, then $Lu > 0$ in Ω_T. On the contrary, if $u(x,t) > u_0(x)$ for any $(x,t) \in \Omega_T$, $Lu = 0$ in Ω_T. In applications, Problem (1) arises in the model of American option pricing in the Black–Scholes models. The author refers to [1–5] for the financial background of parabolic inequalities. Among them, the most interesting research topic is to construct different types of variational parabolic inequalities and analyze the existence and numerical method for their solutions (see, for example, refs. [3,4] and the references therein).

In the recent years, the study of variational and hemivariational inequalities has been considered extensively in the variety of numerical analysis (for details, see [6,7]) and mathematical theory analysis (see, for example, refs. [8–11] and the references therein). In 2014, the authors in [8] discussed the problem

$$\begin{cases} u_t - Lu - F(u,x,t) \geq 0 \text{ in } Q_T \\ u(x,t) \geq u_0(x) \text{ in } \Omega \\ (u_t - Lu - F(u,x,t)) \cdot (u(x,t) - u_0(x)) = 0 \text{ in } Q_T \\ u(x,0) = u_0(x) \text{ in } \Omega \\ u(x,t) = 0 \text{ on } \partial\Omega \end{cases}$$

with the second order elliptic operator

$$Lu = \operatorname{div}(a(x,t)\nabla u) + b(x,t)\nabla u + c(x,t)u.$$

They proved the existence and uniqueness of a solution to this problem with some restrictions on u_0, F, and L. Later, the authors in [9,10] extended the relative conclusions with the assumption that $a(u)$ is a constant, $\gamma = 0$, and $p(x) = 2$. The authors also discussed the existence and numerical algorithm of the proposed solution.

To the best of our knowledge, the existence and uniqueness of this problem with nonlocal nonlinearities are rarely studied. We cannot easily put the method in [10,12] for the case that Lu is the common second order elliptic operator.

The aim of this paper is to study the existence and uniqueness of solutions for a degenerate parabolic variational inequality problem with nonlocal nonlinearities. The innovation of this paper is to study the variational inequality based on parabolic operator L with nonlocal nonlinearity $a(u)$. Following a similar way in [8], the existence and uniqueness of the solutions in the weak sense are proved by solving a series of penalty problems.

The outline of this paper is as follows: in Section 2, we give the definition of the weak solution to problem and show the existence and uniqueness. In Section 3, we give some estimates of the penalty problem (approximating problem). Section 4 proves the existence and uniqueness of the solution given in Section 2.

2. The Main Results of Weak Solutions

In this section, we first recall some useful definitions and known results, which can be found in [13–18]. Denote

$$L^p(\Omega) = \{u | u \,\text{is measurable real} - \text{valued function}, \int_\Omega |u|^p \mathrm{d}x < \infty\},$$

and its norm is defined by

$$|u|_p = \inf\left\{ \lambda > 0 \middle| \int_\Omega \left|\frac{u}{\lambda}\right|^p \mathrm{d}x \leq 1 \right\}.$$

In the case of $p = 2, |u|_\infty = \sup\limits_{x\in\Omega} |u(x)|$.

$W^{1,p}(\Omega)$ is the space of all measurable functions, which, together with their first order derivatives, belongs to $L^p(\Omega)$ that is

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) | |\nabla u| \in L^p(\Omega)\},$$

with norm

$$|u|_{W^{1,p}(\Omega)} = |u|_p + |\nabla u|_p, \forall u \in W^{1,p}(\Omega).$$

Let $p \geq 2$. $L^\infty(0,T;W^{1,p}(\Omega))$ be defined as the space of all measurable functions u on Ω_T and for almost all $t \in (0,T)$, $u(\cdot,t) \in W^{1,p}(\Omega)$ and $|u(\cdot,t)|_{W^{1,p}(\Omega)} \in L^\infty(0,T)$. The space $L^\infty(0,T;L^p(\Omega))$ is defined in an obvious way.

If $p = 2$, the space $W^{1,p}(\Omega)$ and $L^\infty(0,T;W^{1,p}(\Omega))$ can be denoted by $H^1(\Omega)$ and $L^\infty(0,T;H^1(\Omega))$, respectively.

In the spirit of [2,3], we introduce the following maximal monotone graph

$$G(x) = \begin{cases} 0, & x > 0, \\ \theta, & x = 0. \end{cases} \tag{3}$$

where $\theta > 0$ and depends only on $|u_0|_\infty$.

The purpose of the paper is to obtain the existence and uniqueness of weak solutions of (1). Let $B = L_2(0, T; H^1(\Omega))$, and the weak solution is defined as follows.

Definition 1. *A pair is called a weak solution of problem (1), if*
(a) $\partial_t u \in L_2(0, T;\ L_2(\Omega))$, *(b)* $u(x,t) \geq u_0(x)$, *(c)* $u(x,0) = u_0(x)$, *(d)* $\xi \in G(u - u_0)$,
(e) for every test-function $\phi \in H_0^1(\Omega)$ *and every* $t \in (0, T)$, *the following identity holds:*

$$\int_\Omega u_t \cdot \phi \mathrm{d}x - \int_\Omega a(u)\nabla u \cdot \nabla\phi \mathrm{d}x = \int_\Omega f\phi \mathrm{d}x + \int_\Omega \xi\phi \mathrm{d}x. \tag{4}$$

It is worth noting that, if $u(x,t) > u_0(x)$, then $\xi = 0$,

$$\int_\Omega u_t \cdot \phi \mathrm{d}x - \int_\Omega a(u)\nabla u \cdot \nabla\phi \mathrm{d}x = \int_\Omega f\phi \mathrm{d}x,$$

if $u(x,t) = u_0(x)$ and $\phi > 0$, then $\xi > 0$, so

$$\int_\Omega u_t \cdot \phi \mathrm{d}x - \int_\Omega a(u)\nabla u \cdot \nabla\phi \mathrm{d}x \geq \int_\Omega f\phi \mathrm{d}x.$$

Hence, ξ plays the same role with $\min\{Lu, u - u_0\} = 0$ in (1). Our main result is the following theorem.

Theorem 1. *Let* $f \in L_2(0, T; H_0^1(\Omega))$. *Under assumption (2), variational inequality problem (1) admits a unique weak solution in the sense of Definition 1.*

We will prove Theorem 1 in Section 4 by means of a parabolic penalty method and end this section by showing the following preliminary result that will be used several times henceforth.

Lemma 1 ([13])**.** *Assume* $p \geq 2$ *and let* $M(s) = |s|^{p(x,t)-2}s$, *then* $\forall \xi, \eta \in \mathrm{R}^N$,

$$(M(\xi) - M(\eta)) \cdot (\xi - \eta) \geq C(p) \cdot |\xi - \eta|^p$$

3. Penalty Problems

Since the problem is degenerate, let us consider the auxiliary penalty problem following the similar method of [1–3],

$$\begin{cases} L_\varepsilon u_\varepsilon + \beta_\varepsilon(u_\varepsilon - u_{0,\varepsilon}) = 0, & (x,t) \in \Omega \times (0,\ T] \\ u_\varepsilon(x,t) = \varepsilon, & (x,t) \in \partial\Omega \times (0,\ T] \\ u_\varepsilon(x,0) = u_{0,\varepsilon}(x) = u_0(x) + \varepsilon, & x \in \Omega \end{cases} \tag{5}$$

where

$$L_\varepsilon u_\varepsilon = -u_\varepsilon \mathrm{div}(a_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u)_\varepsilon - \gamma|\nabla u_\varepsilon|^{p(x,t)} - f(x,t), a_\varepsilon(u_\varepsilon) = \left(\min\{a(u_\varepsilon), K^2\} + \varepsilon\right)^\gamma.$$

with K being a finite parameter to be chosen later. From $\gamma > 0$, it can be easy to see that

$$0 < \varepsilon^\gamma \leq a_\varepsilon(u_\varepsilon) \leq (K^2 + 1)^\gamma < \infty. \tag{6}$$

Here, $\beta_\varepsilon(\cdot)$ is the penalty function satisfying

$$\begin{aligned} &\varepsilon \in (0,1), \beta_\varepsilon(\cdot) \in C^2(\mathrm{R}), \beta_\varepsilon(x) \le 0, {\beta'}_\varepsilon(x) \ge 0, {\beta''}_\varepsilon(x) \le 0, \\ &\beta_\varepsilon(x) = \begin{cases} 0 & x \ge \varepsilon, \\ -1 & x = 0, \end{cases} \quad \lim_{\varepsilon \to 0+} \beta(x) = \begin{cases} 0, & x > 0, \\ -1, & x = 0. \end{cases} \end{aligned} \tag{7}$$

It is noteworthy that, if $u(x,t) > u_0(x)$ for any $(x,t) \in \Omega_T$, $Lu = 0$ in Ω_T, and, if $u(x,t) = u_0(x)$ for any $(x,t) \in \Omega_T$, one obtains $Lu \ge 0$ in Ω_T, so that $\beta_\varepsilon(u_\varepsilon - u_0)$ plays a similar role in (5). If $u_\varepsilon > u_0 + \varepsilon$,

$$L_\varepsilon u_\varepsilon = -\beta_\varepsilon(u_\varepsilon - u_0) = 0,$$

and, if $u_0 \le u_\varepsilon \le u_0 + \varepsilon$, we have

$$L_\varepsilon u_\varepsilon = -\beta_\varepsilon(u_\varepsilon - u_0) \ge 0.$$

With a similar method as in [8], we can prove that a regularized problem has a unique weak solution

$$u_\varepsilon(x,t) \in L_2(0,T;H^1(\Omega)), \partial_t u_\varepsilon(x,t) \in L_2(0,T;L_2(\Omega))$$

satisfying the following integral identities

$$\int_\Omega \frac{\partial u_\varepsilon}{\partial t} \cdot \phi \mathrm{d}x - \int_\Omega a_\varepsilon(u_\varepsilon) \nabla u_\varepsilon \cdot \nabla \phi \mathrm{d}x = \int_\Omega f \phi \mathrm{d}x - \int_\Omega \beta_\varepsilon(u_\varepsilon - u_0) \phi \mathrm{d}x \tag{8}$$

with $\phi \in H_0^1(\Omega)$ and $t \in (0,T)$.

We start with the following preliminary result that will be used several times henceforth.

Lemma 2 (Comparison principle)**.** *Assume u and v are in $L_2(0,T;H^1(\Omega))$. If $L_\varepsilon u \ge L_\varepsilon v$ in Q_T and $u(x,t) \le v(x,t)$ on ∂Q_T, then $u(x,t) \le v(x,t)$ in Q_T.*

Proof. Argue by contradiction and suppose $u(x,t)$ and $v(x,t)$ satisfies $L_\varepsilon u \ge L_\varepsilon v$ in Q_T, and there is a $\delta > 0$ such that for some $0 < \tau \le T$, $w = u - v > \delta$ on the set

$$\Omega_\delta = \Omega \cap \{x : w(x,t) > \delta\} \tag{9}$$

and $|\Omega_\delta| > 0$. Multiplying $L_\varepsilon u \ge L_\varepsilon v$ by w and integrating in $Q_\delta = \Omega_\delta \times (0,T)$, then

$$J_1 + J_2 \le 0, \tag{10}$$

where

$$J_1 = \int\int_{Q_\delta} \frac{\partial}{\partial t} w \cdot F_\varepsilon(w) \mathrm{d}x\mathrm{d}t, \; J_2 = \int\int_{Q_\delta} [a_\varepsilon(u)\nabla u - a_\varepsilon(v)\nabla v] \nabla w \mathrm{d}x\mathrm{d}t.$$

By virtue of the first inequality of Lemma 2, one gets

$$J_2 \ge c(p) \int\int_{Q_\delta} |w|^p \mathrm{d}x\mathrm{d}t \ge 0. \tag{11}$$

Dropping the nonnegative terms J_2 in (10) obtains

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{Q_\delta} w^2 \mathrm{d}x \le 0 \tag{12}$$

Noting that $u(x,t) \leq v(x,t)$ on ∂Q_T, one gets

$$\int_{\Omega_\delta} w^2 \mathrm{d}x \leq \int_{Q_\delta} |u(x,0) - v(x,0)|^2 \mathrm{d}x = 0$$

This leads to $|\Omega_\delta| = 0$, and a contradiction is obtained. □

Lemma 3. *Let there be weak solutions of (5). Then,*

$$u_{0\varepsilon} \leq u_\varepsilon \leq |u_0|_\infty + \varepsilon, \tag{13}$$

$$u_{\varepsilon_1} \leq u_{\varepsilon_2} \text{ for } \varepsilon_1 \leq \varepsilon_2, \tag{14}$$

where $|u_0|_\infty = \sup\limits_{x\in\Omega} |u_0(x)|$.

Proof. First, prove $u_\varepsilon \geq u_{0\varepsilon}$ by contradiction. Assume $u_\varepsilon \leq u_{0\varepsilon}$ in Q_T^0 , $Q_T^0 \subset Q_T$. Noting $u_\varepsilon \geq u_{0\varepsilon}$ on ∂Q_T, we assume that $u_\varepsilon = u_{0\varepsilon}$ on $\partial \mathrm{Q}_T^0$. With (5) and letting $t = 0$, it is easy to see that

$$L_\varepsilon u_{0,\varepsilon} = -\beta_\varepsilon(u_{0,\varepsilon} - u_{0,\varepsilon}) = 1, \tag{15}$$

$$L_\varepsilon u_\varepsilon = -\beta_\varepsilon(u_\varepsilon - u_{0,\varepsilon}) \leq 1. \tag{16}$$

From Lemma 2, it holds that

$$u_\varepsilon(x,t) \geq u_{0,\varepsilon}(x) \text{ for any } (x,t) \in Q_T. \tag{17}$$

Therefore, we obtain a contradiction.

Second, pay attention to

$$u_\varepsilon(t,x) \leq |u_0|_\infty + \varepsilon.$$

Applying the definition of $\beta_\varepsilon(\cdot)$ yields

$$L_\varepsilon(|u_0|_\infty + \varepsilon) = 0,\ L_\varepsilon u_\varepsilon = -\beta_\varepsilon(u_\varepsilon - u_{0,\varepsilon}) \geq 0 \tag{18}$$

From (18), applying Lemma 2 obtains

$$u_\varepsilon(t,x) \leq |u_0|_\infty + \varepsilon \text{ on } \partial \mathrm{Q_T} \tag{19}$$

and $u_\varepsilon(t,x) \leq |u_0|_\infty + \varepsilon$ in Ω. Thus, combining (18) and (19) and using Lemma 2, one obtains

$$u_\varepsilon(t,x) \leq |u_0|_\infty + \varepsilon \text{ in } Q_T. \tag{20}$$

Third, aim to prove (14). From (5), it yields

$$L_{\varepsilon_1} u_{\varepsilon_1} = \beta_{\varepsilon_1}(u_{\varepsilon_1} - u_{0,\varepsilon_1}), \tag{21}$$

$$L_{\varepsilon_2} u_{\varepsilon_2} = \beta_{\varepsilon_2}(u_{\varepsilon_2} - u_{0,\varepsilon_2}). \tag{22}$$

It follows by $\varepsilon_1 \leq \varepsilon_2$ and the definition of $\beta_\varepsilon(\cdot)$ that

$$\begin{aligned} &L_{\varepsilon_2} u_{\varepsilon_2} - \beta_{\varepsilon_1}(u_{\varepsilon_2} - u_{0,\varepsilon_1}) \\ &= \beta_{\varepsilon_2}(u_{\varepsilon_2} - u_{0,\varepsilon_2}) - \beta_{\varepsilon_1}(u_{\varepsilon_1} - u_{0,\varepsilon_1}) \geq \beta_{\varepsilon_2}(u_{\varepsilon_2} - u_{0,\varepsilon_2}) - \beta_{\varepsilon_1}(u_{\varepsilon_2} - u_{0,\varepsilon_2}) \geq 0. \end{aligned} \tag{23}$$

Thus, combining initial and boundary conditions in (5), (14) can be proved by Lemma 1. □

Lemma 4. *Let u_ε be a weak solution of Problem (5). If $u_0 \in L_{2k}(\Omega)$ and $f \in L_1(0,T;L_{2k}(\Omega))$, for any $k \in \mathrm{N}$, then*

$$\|u_\varepsilon\|_{L_{2k}(\Omega)} \leq \|u_0\|_{L_{2k}(\Omega)} + \int_0^T \|f\|_{L_{2k}(\Omega)} \mathrm{d}t + M \cdot T \cdot |\Omega| \leq C, \tag{24},$$

where C does not depend on ε.

Proof. Multiplying the first equation of Problem (5) by u_ε^{2k-1} and integrating in Ω, for any $t \in (0,\ T]$,

$$\frac{1}{2k}\frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon\|_{L_{2k}(\Omega)}^{2k} + (2k-1)a_\varepsilon(u_\varepsilon)\int_\Omega u_\varepsilon^{2k-2}|\nabla u_\varepsilon|^2\mathrm{d}x = \int_\Omega f u_\varepsilon^{2k-1}\mathrm{d}x - \int_\Omega \beta_\varepsilon(u_\varepsilon - u_0)u_\varepsilon^{2k-1}\mathrm{d}x. \tag{25}$$

Applying the Hölder inequality, we have

$$\int_\Omega f u_\varepsilon^{2k-1}\mathrm{d}x \le \|f\|_{L_{2k}(\Omega)} \cdot \|u_\varepsilon\|_{L_{2k}(\Omega)}^{2k-1}, \tag{26}$$

$$\int_\Omega \beta_\varepsilon(u_\varepsilon - u_0)u_\varepsilon^{2k-1}\mathrm{d}x \le M \cdot \int_\Omega |u_\varepsilon|^{2k-1}\mathrm{d}x \le M \cdot \left(\int_\Omega |u_\varepsilon|^{2k}\mathrm{d}x\right)^{\frac{2k-1}{2k}} \cdot |\Omega|^{\frac{1}{2k}} \tag{27}$$

Substituting (26) and (27) into (28) and dropping the non-negative term $a(u_\varepsilon)\int_\Omega u_\varepsilon^{2k-2}|\nabla u_\varepsilon|^2\mathrm{d}x$,

$$\frac{1}{2k}\frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon\|_{L_{2k}(\Omega)}^{2k} \le \|f\|_{L_{2k}(\Omega)} \cdot \|u_\varepsilon\|_{L_{2k}(\Omega)}^{2k-1} + M \cdot |\Omega|^{\frac{1}{2k}} \cdot \|u_\varepsilon\|_{L_{2k}(\Omega)}^{2k-1}.$$

Simplifying the factor $\|u_\varepsilon\|_{L_{2k}(\Omega)}^{2k-1}$ and integrating in t, (24) follows. □

Lemma 5. *If $u_0 \in H_0^1(\Omega)$, $f \in L_2(0,T;H_0^1(\Omega))$, and $\gamma \ge 0$, then*

$$\int_0^T \int_\Omega |\frac{\partial u_\varepsilon}{\partial t}|^2\mathrm{d}x\mathrm{d}t \le C\left(\int_\Omega |u_0|^2\mathrm{d}x + \int_0^T \int_\Omega |\nabla f|^2 + f^2\mathrm{d}x\mathrm{d}t + 2MT|\Omega|\right), \tag{28}$$

where C does not depend on ε.

Proof. Multiplying the first equation of (5) by $\frac{\partial u_\varepsilon}{\partial t}$ and integrating in $\Omega \times [0,T]$,

$$\begin{aligned} &\int_0^T \int_\Omega |\tfrac{\partial u_\varepsilon}{\partial t}|^2\mathrm{d}x\mathrm{d}t + \int_0^T a(u_\varepsilon)\int_\Omega \nabla u_\varepsilon \tfrac{\partial}{\partial t}\nabla u_\varepsilon\mathrm{d}x\mathrm{d}t \\ &= \int_0^T \int_\Omega f\tfrac{\partial u_\varepsilon}{\partial t}\mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \beta_\varepsilon(u_\varepsilon - u_0)\tfrac{\partial u_\varepsilon}{\partial t}\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{29}$$

First, estimate $\int_0^T \int_\Omega f\frac{\partial u_\varepsilon}{\partial t}\mathrm{d}x\mathrm{d}t$ and use Holder and Young inequalities to arrive at

$$\begin{aligned} &\int_0^T \int_\Omega f\tfrac{\partial u_\varepsilon}{\partial t}\mathrm{d}x\mathrm{d}t \\ &= \int_0^T \int_\Omega (2f)\cdot\left(\tfrac{1}{2}\tfrac{\partial u_\varepsilon}{\partial t}\right)\mathrm{d}x\mathrm{d}t \le 2\int_0^T \int_\Omega f^2\mathrm{d}x\mathrm{d}t + \tfrac{1}{8}\int_0^T \int_\Omega \left|\tfrac{\partial u_\varepsilon}{\partial t}\right|^2\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{30}$$

Second, focus on $\int_0^T \int_\Omega \beta_\varepsilon(u_\varepsilon - u_0)\frac{\partial u_\varepsilon}{\partial t}\mathrm{d}x\mathrm{d}t$. It follows by the definition of $\beta_\varepsilon(\cdot)$ that

$$\int_0^T \int_\Omega \beta_\varepsilon(u_\varepsilon - u_0)\frac{\partial u_\varepsilon}{\partial t}\mathrm{d}x\mathrm{d}t \le M\int_0^T \int_\Omega \left|\frac{\partial u_\varepsilon}{\partial t}\right|\mathrm{d}x\mathrm{d}t. \tag{31}$$

Using Holder and Young inequalities [12], then

$$\int_0^T \int_\Omega \beta_\varepsilon(u_\varepsilon - u_0)\frac{\partial u_\varepsilon}{\partial t}\mathrm{d}x\mathrm{d}t \le 2M^2T\cdot|\Omega| + \frac{1}{8}\int_0^T \int_\Omega \left|\frac{\partial u_\varepsilon}{\partial t}\right|^2\mathrm{d}x\mathrm{d}t. \tag{32}$$

Third, pay attention to $\int_0^T a(u_\varepsilon)\int_\Omega \nabla u_\varepsilon \frac{\partial}{\partial t}\nabla u_\varepsilon \mathrm{d}x\mathrm{d}t$. Applying (6) gives

$$\begin{aligned}
&\left|\int_0^T a(u_\varepsilon)\int_\Omega \nabla u_\varepsilon \tfrac{\partial}{\partial t}\nabla u_\varepsilon \mathrm{d}x\mathrm{d}t\right| \\
=&\left|\tfrac{1}{2}\int_0^T a(u_\varepsilon)\int_\Omega \tfrac{\partial}{\partial t}(\nabla u_\varepsilon)^2\mathrm{d}x\mathrm{d}t\right| \leq \left|\tfrac{1}{2}(K^2+1)^\gamma \int_0^T\int_\Omega \tfrac{\partial}{\partial t}(\nabla u_\varepsilon)^2\mathrm{d}x\mathrm{d}t\right| \\
=&\left|\tfrac{1}{2}(K^2+1)^\gamma \int_\Omega (\nabla u_{0\varepsilon})^2\mathrm{d}x - \int_\Omega \nabla u_\varepsilon(\cdot,T)^2\mathrm{d}x\right|.
\end{aligned} \tag{33}$$

Since $0 \leq u_0 \in H_0^1(\Omega)$, it is easy to see that

$$\left|\int_0^T a(u_\varepsilon)\int_\Omega \nabla u_\varepsilon \frac{\partial}{\partial t}\nabla u_\varepsilon \mathrm{d}x\mathrm{d}t\right| \leq \frac{1}{2}(K^2+1)^\gamma ||\nabla u_{0\varepsilon}||_{L_2}^2. \tag{34}$$

Combining (29), (30), (32), and (34), then

$$\int_0^T\int_\Omega |\frac{\partial u_\varepsilon}{\partial t}|^2\mathrm{d}x\mathrm{d}t \leq C\int_\Omega |\Delta u_0|^2\mathrm{d}x + C\int_0^T\int_\Omega |\nabla f|^2\mathrm{d}x\mathrm{d}t + 2\int_0^T\int_\Omega f^2\mathrm{d}x\mathrm{d}t + 2M^2T\cdot|\Omega| \tag{35}$$

and the result follows. □

4. Proof of Theorem 1

From Lemmas 3–5, we see that u_ε is bounded and increasing in ε, which implies the existence of a function u and subsequences such that

$$u_\varepsilon \to u \text{ a.e. in } Q_T, \tag{36}$$

$$\nabla u_\varepsilon \to \nabla u \text{ weakly in } L_{2k}(Q_T), \tag{37}$$

$$\frac{\partial}{\partial t}u_\varepsilon \to \frac{\partial}{\partial t}u \text{ weakly in } L_2(Q_T), \tag{38}$$

Since $a(\cdot)$ is continuous, we have that

$$a_\varepsilon(u_\varepsilon) \to a(u) \text{ a.e. in } L_2(\Omega\times(0,T]). \tag{39}$$

Next, we pay attention to the limitation of $\beta_\varepsilon(u_\varepsilon - u_0)$.

Lemma 6. *For any $(x,t)\in\Omega_T$, let u_ε be the solution of (5). Then,*

$$\beta_\varepsilon(u_\varepsilon - u_0) \to \xi \in G(u-u_0) \text{as} \varepsilon \to 0. \tag{40}$$

Proof. Using (14) and the definition of β_ε, one has

$$\beta_\varepsilon(u_\varepsilon - u_0) \to \xi \text{ as } \varepsilon \to 0. \tag{41}$$

Now, consider $\xi \in G(u-u_0)$. According to the definition of $G(\cdot)$, we only need to prove that, if $u(x_0,t_0) > u_0(x_0)$,

$$\xi(x_0,t_0) = 0.$$

In fact, if $u(x_0,t_0) > u_0(x_0)$, there exist a constant $\lambda > 0$ and a δ-neighborhood $B_\delta(x_0,t_0)$ such that, if ε is small enough,

$$u_\varepsilon(x,t) \geq u_0(x) + \lambda, \forall (x,t) \in B_\delta(x_0,t_0).$$

Thus, if ε is small enough, such that

$$0 \geq \beta_\varepsilon(u_\varepsilon - u_0) \geq \beta_\varepsilon(\lambda) = 0, \forall (x,t) \in B_\delta(x_0,t_0).$$

Furthermore, it follows by $\varepsilon \to 0$ that

$$\xi(x,t) = 0, \forall (x,t) \in B_\delta(x_0,t_0).$$

Hence, (41) holds, and the proof of Lemma 6 completes. □

On the one hand, when $u \geq u_0$, $Lu = 0$, and when $u = u_0$, we have $Lu \geq 0$ in (1). On the other hand, when $u_\varepsilon \geq u_{0\varepsilon}$, $L_\varepsilon u_\varepsilon = -\beta(u_\varepsilon - u_{0\varepsilon}) = 0$, and, when $u_\varepsilon = u_{0\varepsilon}$, we have $L_\varepsilon u_\varepsilon = -\beta(u_\varepsilon - u_{0\varepsilon}) \geq 0$ in (5). When $\beta(u_\varepsilon - u_{0\varepsilon})$ converges to ξ, ξ plays the same role in weak solution.

Now, we prove the existence of the weak solutions in the sense of Definition 1.

Proof of Existence of Theorem 1. Combining (36)–(40) and Lemma 6, passing to the limit in

$$\int_\Omega \frac{\partial u_\varepsilon}{\partial t} \cdot \phi \mathrm{d}x - \int_\Omega a_\varepsilon(u_\varepsilon)\nabla u_\varepsilon \cdot \nabla\phi \mathrm{d}x = \int_\Omega f\phi \mathrm{d}x - \int_\Omega \beta_\varepsilon(u_\varepsilon - u_0)\phi \mathrm{d}x,$$

we arrive at

$$\int_\Omega \frac{\partial u}{\partial t} \cdot \phi \mathrm{d}x - \int_\Omega a(u)\nabla u \cdot \nabla\phi \mathrm{d}x = \int_\Omega f\phi \mathrm{d}x + \int_\Omega \xi\phi \mathrm{d}x. \tag{42}$$

Applying (36), (46), and Lemma 6, it is clear that

$$u(x,t) \leq u_0(x) \text{ in } \Omega_T, \;\; u(x,0) = u_0(x) \text{ in } \Omega, \; \xi \in G(u - u_0),$$

thus (a), (b), and (c) hold. Hence, u is a weak solution of Problem (3) in the sense of Definition 1. □

Proof of Uniqueness of Theorem 1. Finally, we study the uniqueness of the weak solutions to Problem (1). Argue by contradiction and suppose (u,ξ_1) and (v,ξ_2) are two nonnegative weak solutions of Problem (1).

Define $w = u - v$,

$$F(w) = \begin{cases} -\frac{1}{\alpha-1}w^{1-\alpha}, & \text{if } w > 0, \\ 0, \quad \text{if } w \leq 0, \end{cases} \tag{43}$$

and let $F(w) \in H_0^1(\Omega)$ be a test-function in (42),

$$0 \geq \int\int_{\Omega_T} w_t F(w) + [a(u)\nabla u - a(v)\nabla v]\nabla F(w)\mathrm{d}x\mathrm{d}t - \int\int_{Q_T} (\xi_1 - \xi_2)F(w)\mathrm{d}x\mathrm{d}t. \tag{44}$$

Now, analyze $\int_\Omega (\xi_1 - \xi_2)F(w)\mathrm{d}x\mathrm{d}t$. On one hand, if $u_1(x,t) > u_2(x,t)$, then using (13) yields

$$u_1(x,t) > u_2(x,t) \geq u_0(x). \tag{45}$$

From (3) and (45), it is easy to see

$$\xi_1 = 0 < \xi_2. \tag{46}$$

Combining (45) and (46) and the fact that $\alpha = \frac{1}{2}\sigma > 1$,

$$\int_\Omega (\xi_1 - \xi_2)F(w)\mathrm{d}x\mathrm{d}t \leq 0. \tag{47}$$

On the other hand, if $u_1(x,t) < u_2(x,t)$, it is easy to have that $F(w) = 0$. In this case, (47) still holds.

Using (45) in (44) and dropping the nonnegative term, (44) becomes

$$\int\int_{Q_T} w_t F(w) + [a(u)\nabla u - a(v)\nabla v]\nabla F(w)\mathrm{d}x\mathrm{d}t \leq 0.$$

By the above inequality and combining initial and boundary condition in Problem (1), the uniqueness of solution can be proved following the similar proof of Lemma 2. □

5. Numerical Examples

In order to observe the application of parabolic variational inequalities (3), we consider an American call option. An American option is the extension of a European option. An American option is a contract in which the investor has the right to purchase a certain amount of risky assets at a predetermined price K during the duration $[0, T]$. Let S be the risk asset price, then American barrier option C at time t can be written as

$$\begin{cases} \min\{LC, C - \max(e^x - K, 0)\} = 0, & (x,t) \in \mathrm{R} \times [0,T], \\ V(T,x) = \max(e^x - K, 0), & x \in \mathrm{R}, \end{cases} \tag{48}$$

where $x = lnS$,

$$LC = \partial_t V + \frac{1}{2}\sigma^2 \partial_{xx} V + (r - q - \frac{1}{2}\sigma^2)\partial_x V - rV.$$

Here σ is the volatility of risk assets, q is the return rate of risk assets, and r is the yield of risk-free assets.

Compared with American options, European options can only be exercised on the expiration date T. The American barrier option c at time t can be written as

$$\begin{cases} Lc = 0, & (x,t) \in \mathrm{R} \times [0,T], \\ V(T,x) = \max(e^x - K, 0), & x \in \mathrm{R}, \end{cases} \tag{49}$$

Calculate the price of European options and American options written on the stock price $\exp\{x_0\}$ at time 0. Define space step h and time step Δt and denote $x_i = i \times h$ for $i = 0, \pm 1, \pm 2, \cdots$, and $t_k = k \times \Delta t$, for $k = 0, 1, 2, \cdots, N_T$. Similar to the discussion in [1–3], the value of American call options satisfies the explicit difference scheme:

$$\begin{cases} (1 + r\Delta t)V_j^k = \max\{(1-\alpha)V_j^{k+1} + \frac{1}{2}(\alpha+\beta)V_{j+1}^{k+1} + \frac{1}{2}(\alpha-\beta)V_{j-1}^{k+1}, V_j^n\}, \\ V_j^n = \max\{\exp\{j\Delta x\} - K, 0\}, \end{cases} \tag{50}$$

where $\alpha = \sigma^2 \frac{\Delta t}{\Delta x^2}$, $\beta = (r - q - \frac{1}{2}\sigma^2)\frac{\Delta t}{\Delta x}$,

$$\Delta t \cdot L_j^k V_j^k = -(1 + r\Delta t)V_j^k + (1-\alpha)V_j^{k+1} + \frac{1}{2}(\alpha+\beta)V_{j+1}^{k+1} + \frac{1}{2}(\alpha-\beta)V_{j-1}^{k+1}.$$

The value of European call options satisfies the explicit difference scheme

$$\begin{cases} (1 + r\Delta t)V_j^k = (1-\alpha)V_j^{k+1} + \frac{1}{2}(\alpha+\beta)V_{j+1}^{k+1} + \frac{1}{2}(\alpha-\beta)V_{j-1}^{k+1}, \\ V_j^n = \max\{\exp\{j\Delta x\} - K, 0\}. \end{cases} \tag{51}$$

Next, we numerically simulate the difference scheme (50) and (51) to compare the difference between the variational inequality (48) and the corresponding parabolic Equation (49). The parameters' values are chosen as $K = 18, r = 0.1, \sigma = 0.2, T = 1, \Delta t = 0.1, h = 0.1$. The results are shown in Figures 1–3. We found that the value of European and American options is increasing with the increase of stock price. Compared with Figures 1–3, it can be found that American options and European options have the same value when dividends are not paid (q = 0). It is unwise to implement American options in advance in this case. When the rate of return $q > 0$, the American option value obtained by variational inequality (49) is greater than the corresponding European option value. This shows that the early exercise clause of American options brings additional value compared with European Options.

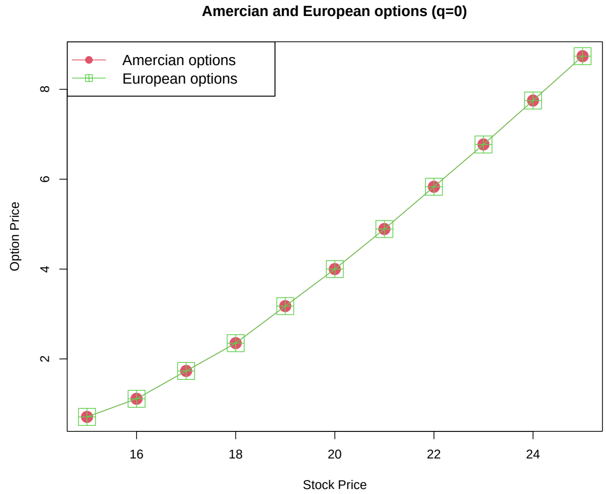

Figure 1. American and European call options with different stock prices ($q = 0$).

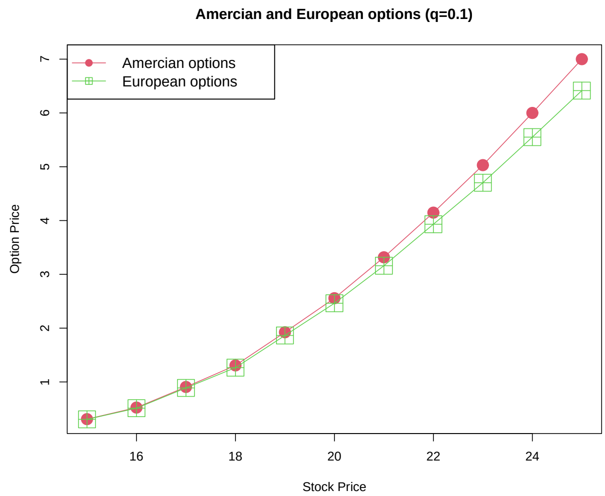

Figure 2. American and European call options with different stock prices ($q = 0.1$).

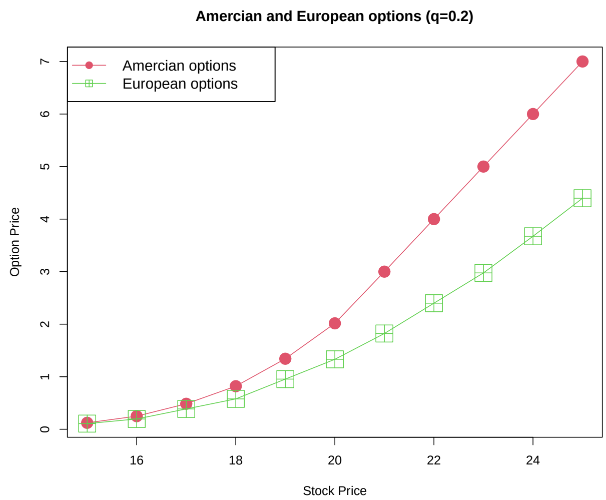

Figure 3. American and European call options with different stock prices ($q = 0.2$).

6. Discussion

In this paper, we study the existence and uniqueness of solutions of variational inequality (1) by penalty function $\beta_\varepsilon(u_\varepsilon - u_0)$. Since the penalty function $\beta_\varepsilon(\cdot)$ is controlled by ε, we integrate it into the degenerate parabolic operator Lu and form a new parabolic operator $L_\varepsilon u_\varepsilon$. More importantly, the weak solution of the penalty problem under this operator exists. After giving some estimates of the penalty problem, the existence of weak solutions is given by the convergence method. The uniqueness of the weak solution is proved by the method of proof and Lemma 2.

Compared with other literature, Ref. [8] analyzes the existence and uniqueness of solutions to variational inequality problems by using quasi-linear parabolic operators. The advantage of [8] is that a nonlinear term related to u is constructed in the Quasilinear Parabolic operator. In this connection, this paper constructs a nonlinear term related to $\triangle u$ of Lu using L^2 norm. Ref. [10] is similar to [8], and studies variational inequalities formed by linear parabolic operators with a nonlinear term related to u .

Refs. [12–16] study the existence, uniqueness, solvability, and stability of solutions of parabolic equations. Since these literature works are not concerned with variational inequalities, it is not necessary to give the comparison principle of parabolic operator and construct penalty functions when analyzing the existence.

7. Conclusions

This paper studies a class of variational inequalities with degenerate parabolic operators

$$\begin{cases} \min\{Lu, u-u_0\} = 0, & (x,t) \in Q_T, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T], \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

with a degenerate parabolic operator, which satisfies

$$Lu = \partial_t u - a(u)\Delta u - f(x,t), \gamma > 0.$$

The existence and uniqueness of the solutions in the weak sense are proved by using the penalty method and the reduction method. However, there are some problems that have not been solved: when $0 < p(x,t) < 2$, we cannot use Lemmas 2 and 3 to prove Lemmas 4–6. We will continue to study this problem in the future.

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