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Dynamical Symmetry Breaking of Infinite-Dimensional Stochastic System

Weipeng Hu [1,2,*], Tao Liu [3,*] and Zhengqi Han [1]

[1] School of Civil Engineering and Architecture, Xi'an University of Technology, Xi'an 710048, China
[2] State Key Laboratory of Eco-Hydraulics in Northwest Arid Region of China, Xi'an University of Technology, Xi'an 710048, China
[3] Yulin City Investment Construction Development Co., Ltd., Yulin 719000, China
* Correspondence: wphu@nwpu.edu.cn (W.H.); l13109695016@163.com (T.L.)

Abstract: The mapping relationship between the symmetry and the conserved quantity inspired researchers to seek the conserved quantity from the viewpoint of the symmetry for the dynamic systems. However, the symmetry breaking in the dynamic systems is more common than the symmetry in the engineering. Thus, as the supplement of our previous work on the symmetry breaking of infinite-dimensional deterministic dynamic systems, the dynamical symmetry breaking of infinite-dimensional stochastic systems is discussed in this paper. Following a brief review of the stochastic (multi-)symplectic for the dynamic system excited by stochastic white noise, two types of stochastic symmetry breaking factors, including the general stochastic excitation and the general stochastic parameters of the infinite-dimensional dynamic systems, are investigated in detail. We find that both the general stochastic excitation and the general stochastic parameters will not break the local multi-symplectic structure of the dynamic systems. However, the local energy conservation law will be broken by the general stochastic excitation, as well as by the stochastic parameters, which are given by the local energy dissipation in this paper. To illustrate the validity of the analytical results, the stochastic vibration of a clamped single-walled carbon nanotube is investigated and the critical condition of the appearance of chaos is obtained. The theoretical results obtained can be used to guide us to construct the structure-preserving method for the stochastic dynamic systems.

Keywords: dynamic symmetry breaking; stochastic system; local multi-symplectic structure; Hamiltonian

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1. Introduction

In 1918, Noether [1] proposed the mapping relationship between the mathematical symmetry and the physical conservative quality for the dynamic systems, which permitted the researchers to investigate the physical properties of the dynamic systems by using some mathematical approaches. This mapping relationship guided the theoretical research of the dynamic systems until the presentation of the parity non-conservation [2], which is one of the outstanding contributions of Lee and Yang because it opened another research field of the physical mechanical system. Subsequently, the mathematical theory and the physical essence were revealed by Goldstone et al. [3], Higgs [4], Kibble [5], Anderson [6], Bernstein [7], Adler [8] and Thooft [9].

Non-conservation exists in the practical physical mechanical systems widely, which breaks the mathematical symmetry and determines many nonlinear characteristics of the physical mechanical systems. To perform the above task, many symmetry breaking phenomena were investigated widely, including the chiral symmetry breaking [10–18], the spontaneous symmetry breaking [19–22], the time-reversal symmetry breaking [23–26] and so on. Among these symmetry breaking phenomena, the symmetry breaking related to the time elapse is more complex, which was named as dynamic symmetry breaking [27]. The physical implications of the dynamic symmetry breaking were presented by Weinberg [28].

The bifurcations of the wave propagation in the fluid were investigated from the viewpoint of the dynamic symmetry breaking idea [29,30], which presented a precedent for the application of the dynamic symmetry breaking on the nonlinear dynamics. Then, the dynamical symmetry breaking in 4-fermion models in 2 + 1 dimensions was quantitatively studied [31].

Inspired by the mapping relationship between the symmetry and the conservation property, the symplectic geometry theory was employed in the construction of the numerical scheme for the finite-dimensional Hamiltonian systems by Feng [32], which was named as the symplectic method. To describe the local symmetry of the infinite-dimensional dynamic system, the multi-symplectic structure, as well as the multi-symplectic integrator, was proposed by Bridges [33–35]. Both the symplectic method and the multi-symplectic method were derived from the symmetry of the mechanics [36]. A natural question for the symmetry of the mechanics is how to formulate the dynamic system with the symmetry breaking in the approximate (multi-)symplectic form and construct the structure-preserving scheme. For this topic, the generalized multi-symplectic method was proposed in our previous work [37] for the infinite-dimensional damping system. Recently, the applications of the generalized multi-symplectic method on the complex dynamic problems [38–48] illustrated the structure-preserving characteristics of the generalized multi-symplectic method derived from the tiny residual of the multi-symplectic structure associated with the existence of the dynamic symmetry breaking.

In our previous work [49] on the dynamic symmetry breaking of the infinite-dimensional systems, the chronology of current works on the symmetry breaking were reviewed and the effects of two typical dynamic symmetry breaking factors on the local energy dissipation of the infinite-dimensional systems were discussed in detail. These two discussed factors are definitive, while the stochastic factors resulting in the dynamic symmetry breaking were neglected. However, indeterminacy exists in the dynamic systems commonly and it may affect the symmetry of the dynamic systems remarkably, which inspired us to investigate the effects of the indeterminacy on the dynamic symmetry breaking of the infinite-dimensional dynamic systems in this paper. The results of the effects of the indeterminacy on the dynamic symmetry breaking of the infinite-dimensional dynamic systems are the mathematical base of the structure-preserving approaches for the stochastic systems, which can be used to study the local energy dissipation of the stochastic systems.

2. Review of Stochastic (Multi-)Symplectic Method

As the ground-breaking work on the symmetry of the stochastic systems, Zhu [50–52] proposed the stochastic averaging method of the quasi-nonintegrable Hamiltonian systems, in which the external excitation was assumed as the random white noise.

For the following Cauchy problem for the stochastic differential equations in the sense of Stratonovich:

$$\begin{aligned} \mathrm{d}\mathbf{P} &= \mathbf{f}(t,\mathbf{P},\mathbf{Q})\mathrm{d}t + \sum_{r=1}^{m} \boldsymbol{\sigma}_r(t,\mathbf{P},\mathbf{Q}) \circ \mathrm{d}w_r(t),\ \mathbf{P}(t_0) = \mathbf{p} \\ \mathrm{d}\mathbf{Q} &= \mathbf{g}(t,\mathbf{P},\mathbf{Q})\mathrm{d}t + \sum_{r=1}^{m} \boldsymbol{\gamma}_r(t,\mathbf{P},\mathbf{Q}) \circ \mathrm{d}w_r(t),\ \mathbf{Q}(t_0) = \mathbf{q} \end{aligned} \tag{1}$$

Milstein et al. [53] presented the following symplectic structure with several assumptions:

$$\mathrm{d}\mathbf{P} \wedge \mathrm{d}\mathbf{Q} = \mathrm{d}\mathbf{p} \wedge \mathrm{d}\mathbf{q} \tag{2}$$

which implies that the sum of the oriented areas of projections onto the co-ordinate planes is an integral invariant. Where $\mathbf{P}, \mathbf{Q}, \mathbf{f}, \mathbf{g}, \boldsymbol{\sigma}_r, \boldsymbol{\gamma}_r, \mathbf{p}, \mathbf{q}$ are n-dimensional vectors, $w_r(t)$ is an independent Wiener process and '$\circ$' is the Stratonovich product. This work [53] is a supplement for their previous work [54] on the symplectic integration of Hamiltonian systems with the additive noise.

To describe the local dynamic behaviors of the stochastic infinite-dimensional Hamiltonian systems, Jiang et al. [55] and Hong et al. [56] formulated the stochastic nonlinear

Schrödinger equation and the stochastic Maxwell equation in the following stochastic multi-symplectic form uniformly:

$$\mathbf{M}\partial_t\mathbf{z} + \mathbf{K}\partial_x\mathbf{z} = \nabla_{\mathbf{z}}S_1(\mathbf{z}) + \nabla_{\mathbf{z}}S_2(\mathbf{z}) \circ \dot{\chi} \tag{3}$$

where $\mathbf{M}$ and $\mathbf{K}$ are skew-symmetric matrices, $\mathbf{z} \in \mathbf{R}^d$ is a state vector, $S_1(\mathbf{z})$ and $S_2(\mathbf{z})$ are the Hamiltonian function and the additional Hamiltonian function, respectively, and $\dot{\chi}$ is defined as a real-valued white noise, which is delta correlated in time and either smooth or delta correlated in space.

According to the precise mathematical definition of the white noise, the stochastic multi-symplectic Hamiltonian system given by Equation (3) can be rewritten as:

$$\mathbf{M}\mathrm{d}_t\mathbf{z} + \mathbf{K}\partial_x\mathbf{z}\mathrm{d}t = \nabla_{\mathbf{z}}S_1(\mathbf{z})\mathrm{d}t + \nabla_{\mathbf{z}}S_2(\mathbf{z}) \circ \mathrm{d}W \tag{4}$$

where W defines a cylindrical Wiener process.

It has been proved that the stochastic multi-symplectic form (4) preserves the following stochastic multi-symplectic conservation law locally [55,56]:

$$\mathrm{d}_t\omega(t,x) + \partial_x\kappa(t,x)\mathrm{d}t = 0 \tag{5}$$

i.e.,

$$\int_{x_0}^{x_1}\omega(t_1,x)\mathrm{d}x + \int_{t_0}^{t_1}\kappa(t,x_1)\mathrm{d}t = \int_{x_0}^{x_1}\omega(t_0,x)\mathrm{d}x + \int_{t_0}^{t_1}\kappa(t,x_0)\mathrm{d}t \tag{6}$$

where $\omega(t,x) = \frac{1}{2}\mathrm{d}\mathbf{z} \wedge \mathbf{M}\mathrm{d}\mathbf{z}$ and $\kappa(t,x) = \frac{1}{2}\mathrm{d}\mathbf{z} \wedge \mathbf{K}\mathrm{d}\mathbf{z}$ are the differential two forms associated with the skew-symmetric matrices $\mathbf{M}$ and $\mathbf{K}$, respectively, which were also defined in the same way in the multi-symplectic theory [33], and $(t_0,t_1) \times (x_0,x_1)$ is the local definition domain of $\mathbf{z}(t,x)$.

It is necessary to clarify that the above symplectic structure and the local multi-symplectic structure for the stochastic Hamiltonian systems only exist when the stochastic excitation is assumed as the white noise process. This assumption implies that, for any time period t, $E(\varepsilon_t) = 0$, $E(\varepsilon_t^2) = \sigma^2$, $E(\varepsilon_t\varepsilon_{t-s}) = E(\varepsilon_{t-j}\varepsilon_{t-j-s}) = 0$ for all j and all $s \neq 0$. In a word, the power spectral density of the white noise process is time-independent, which is the existence prerequisite of the (multi-)symplectic for the stochastic dynamic systems considered in the current documents. Can this conclusion be generalized to the dynamic system with the stochastic parameters or subjected to other common stochastic excitations?

3. Dynamical Symmetry Breaking of Stochastic System

It has been mentioned that the stochastic excitation in the white noise form does not break the local symmetry of the dynamic system in Section 2. However, the white noise process cannot be found in engineering problems. In addition, the stochastic parameter is very common in the physical mechanical systems. Thus, in this section, two dynamical symmetry breaking factors of the stochastic systems will be discussed in detail.

3.1. Case of General Stochastic Excitation

Considering the general stochastic excitation acting on the infinite-dimensional systems without any deterministic dynamical symmetry breaking factors presented in [49], the first-order approximate symmetric form of the systems can be formally written as:

$$\mathbf{M}\partial_t\mathbf{z} + \sum_{i=1}^{n}\mathbf{K}_i\partial_{x_i}\mathbf{z} = \nabla_{\mathbf{z}}[S(\mathbf{z}) + \widetilde{S}(\mathbf{F},\ \mathbf{z},\ \partial_t\mathbf{z},\ \partial_{tt}\mathbf{z})],\ \mathbf{z} \in \mathbf{R}^d \tag{7}$$

where $\mathbf{M}$ and $\mathbf{K}_i$ $(i = 1,2,\cdots,n)$ are the skew-symmetric matrices, $\mathbf{z} \in \mathbf{R}^d$ is a state vector, $S(\mathbf{z})$ and $\widetilde{S}(\mathbf{F},\ \mathbf{z},\ \partial_t\mathbf{z},\ \partial_{tt}\mathbf{z})$ are the Hamiltonian function and the additional Hamiltonian function associated with the stochastic excitation, respectively, and $\mathbf{F}$ is the generalized force vector.

Not limited to the white noise process, $\widetilde{S}(\mathbf{F},\ \mathbf{z},\ \partial_t\mathbf{z},\ \partial_{tt}\mathbf{z})$ describes the general stochastic excitation depending on the stochastic generalized force, as well as the zero/first/second-order derivatives of the stochastic generalized displacement, which implies that there are two types of stochastic excitation (one is the stochastic generalized force acting on the structure; another is the stochastic generalized displacement occurring on the boundary of the structure). According to the multi-symplectic theory [33–35], the local multi-symplectic structure will not be broken because the additional Hamiltonian function $\widetilde{S}(\mathbf{F},\ \mathbf{z},\ \partial_t\mathbf{z},\ \partial_{tt}\mathbf{z})$ is independent of t and x_i explicitly, that is:

$$\begin{aligned}&\sum_{i=1}^{n}\int_{x_i^0}^{x_i^1}\omega(t_1,x_1,x_2,\cdots,x_n)\mathrm{d}x_i+\int_{t_0}^{t_1}\sum_{i=1}^{n}\kappa_i(t,x_1,\cdots x_i^1,\cdots,x_n)\mathrm{d}t\\&=\sum_{i=1}^{n}\int_{x_i^0}^{x_i^1}\omega(t_0,x_1,x_2,\cdots,x_n)\mathrm{d}x_i+\int_{t_0}^{t_1}\sum_{i=1}^{n}\kappa_i(t,x_1,\cdots x_i^0,\cdots,x_n)\mathrm{d}t\end{aligned} \tag{8}$$

However, from the energy viewpoint, the local energy of the system with the additional Hamiltonian function $\widetilde{S}(\mathbf{F},\ \mathbf{z},\ \partial_t\mathbf{z},\ \partial_{tt}\mathbf{z})$ is not always conservative.

For the stochastic generalized force acting on the structure, the local $((t_0,t_1)\times(x_1^0,x_1^1)\times(x_2^0,x_2^1)\times\cdots\times(x_n^0,x_n^1))$ energy conservation law will not be affected by the stochastic generalized force if, and only if:

$$\int_{t_0}^{t_1}\int_{\mathbf{x}^0}^{\mathbf{x}^1}E(\mathbf{F})\mathrm{d}\mathbf{x}\mathrm{d}t=0 \tag{9}$$

where $\mathbf{x}=(x_1,x_2,\cdots,x_n)^{\mathrm{T}}$, $\mathbf{x}^0=(x_1^0,x_2^0,\cdots,x_n^0)^{\mathrm{T}}$, $\mathbf{x}^1=(x_1^1,x_2^1,\cdots,x_n^1)^{\mathrm{T}}$.

If the stochastic generalized force is a white noise, the mathematical expectation of the stochastic force is zero, i.e., $E(\mathbf{F})=0$, which implies that the integration formulated by Equation (9) is zero in any time interval and the local energy conservation law will not be broken by the white noise. Otherwise, the local energy dissipation will occur if $\int_{t_0}^{t_1}\int_{\mathbf{x}^0}^{\mathbf{x}^1}E(\mathbf{F})\mathrm{d}\mathbf{x}\mathrm{d}t\neq 0$. The local energy dissipation in the domain $(t_0,t_1)\times(x_1^0,x_1^1)\times(x_2^0,x_2^1)\times\cdots\times(x_n^0,x_n^1)$ can be formulated formally as:

$$\int_{t_0}^{t_1}\int_{\mathbf{x}^0}^{\mathbf{x}^1}E(\mathbf{F})\mathrm{d}\mathbf{x}\mathrm{d}t=0\ \widetilde{\Delta}_e=\int_{t_0}^{t_1}\int_{\mathbf{x}^0}^{\mathbf{x}^1}E(\mathbf{F})\mathrm{d}\mathbf{x}\mathrm{d}t \tag{10}$$

The effects of the local energy of the structure subjected to the stochastic generalized displacement occurring on the boundary of the structure are determined by the property of the boundary of the structure. If the boundary does not permit the displacement along the direction of the stochastic displacement, the change in the local energy of the structure is zero. Otherwise, the change in the local energy of the structure is not zero. Because of the universality of the stochastic excitation, the change in the local energy cannot be formulated explicitly.

3.2. Case of General Stochastic Parameters

In addition to the stochastic force acting on the structure, another stochastic factor affecting the symmetry of the dynamic systems is the stochastic parameters of the structure. For example, the mass distribution, the stiffness distribution and the damping distribution of the structure are stochastic because of the stochastic manufacturing errors.

Without any deterministic dynamical symmetry breaking factors presented in [49], the first-order approximate symmetric form of the systems with the stochastic parameters can be formally written as:

$$\widetilde{\mathbf{M}}\partial_t\mathbf{z}+\sum_{i=1}^{n}\widetilde{\mathbf{K}}_i\partial_{x_i}\mathbf{z}=\nabla_{\mathbf{z}}S(\mathbf{z}),\ \mathbf{z}\in\mathbf{R}^d \tag{11}$$

where $\widetilde{\mathbf{M}}$ and $\widetilde{\mathbf{K}}_i$ $(i = 1, 2, \cdots, n)$ are stochastic skew-symmetric matrices. Generally, the matrices $\widetilde{\mathbf{M}}$ and $\widetilde{\mathbf{K}}_i$ $(i = 1, 2, \cdots, n)$ are the probability density functions respective to $\mathbf{x} = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}$, i.e., Equation (11) can be rewritten as:

$$\widetilde{\mathbf{M}}(x_1, x_2, \cdots, x_n)\partial_t \mathbf{z} + \sum_{i=1}^{n} \widetilde{\mathbf{K}}_i(x_1, x_2, \cdots, x_n)\partial_{x_i}\mathbf{z} = \nabla_{\mathbf{z}} S(\mathbf{z}),\ \mathbf{z} \in \mathbf{R}^d \tag{12}$$

According to the multi-symplectic theory [33] for the deterministic infinite-dimensional systems, the local multi-symplectic structure (8) will not be broken by the stochastic parameters.

Following the outline of our previous work [49], to formulate the local energy dissipation derived from the stochastic parameters, we define $\widetilde{e} = S(\mathbf{z}) - \frac{1}{2}\sum\limits_{i=1}^{n}\left\langle \widetilde{\mathbf{K}}_i(x_1, x_2, \cdots, x_n)\partial_{x_i}\mathbf{z},\ \mathbf{z}\right\rangle$ as the modified energy density and $\widetilde{f}_i = \frac{1}{2}\left\langle \widetilde{\mathbf{K}}_i(x_1, x_2, \cdots, x_n)\partial_t \mathbf{z},\ \mathbf{z}\right\rangle$ $(i = 1, 2, \cdots, n)$ as the modified energy fluxes for system (12). The partial derivative of $\widetilde{f}_i$ with respect to x_i $(i = 1, 2, \cdots, n)$ is:

$$\begin{aligned} \partial_{x_i}\widetilde{f}_i &= \tfrac{1}{2}\left\langle [\partial_{x_i}\widetilde{\mathbf{K}}_i(x_1, x_2, \cdots, x_n)]\partial_t \mathbf{z},\ \mathbf{z}\right\rangle + \tfrac{1}{2}\left\langle \widetilde{\mathbf{K}}_i(x_1, x_2, \cdots, x_n)\partial_{tx_i}\mathbf{z},\ \mathbf{z}\right\rangle + \tfrac{1}{2}\left\langle \widetilde{\mathbf{K}}_i(x_1, x_2, \cdots, x_n)\partial_t \mathbf{z},\ \partial_{x_i}\mathbf{z}\right\rangle \\ &= \tfrac{1}{2}\left\langle \partial_{x_i}[\widetilde{\mathbf{K}}_i(t, x_1, x_2, \cdots, x_n)]\partial_t \mathbf{z},\ \mathbf{z}\right\rangle - \mathrm{d}S(\mathbf{z})/\mathrm{d}t \end{aligned} \tag{13}$$

Then, the local energy dissipation resulting from the stochastic parameters can be formulated as:

$$\widetilde{\Delta}_e = \partial_t \widetilde{e} + \sum_{i=1}^{n}\partial_{x_i}\widetilde{f}_i = \frac{1}{2}\sum_{i=1}^{n}\left\langle \partial_{x_i}[\widetilde{\mathbf{K}}_i(x_1, x_2, \cdots, x_n)]\partial_t \mathbf{z},\ \mathbf{z}\right\rangle \tag{14}$$

A brief summary for the dynamical symmetry breaking of the infinite-dimensional stochastic system follows: the local multi-symplectic structure will not be broken by the above two stochastic factors, while the local energy dissipation will depend on the above stochastic factors, which is formulated by Equations (10) and (14), respectively.

4. Symmetry Breaking of the Dynamic Model Controlling the Stochastic Vibration of SWCNT

Because of the tiny size and the high sensitivity of the carbon nanotube devices [57], both the geometric parameters and the tiny external excitation can be considered as a series of stochastic variables. Thus, the dynamic model controlling the stochastic vibration of the embedded single-walled carbon nanotube can be formulated referring to our previous work [58] when the damping of the nanotube is neglected:

$$E_0\widetilde{I}\partial_{xxxx}u + \rho\widetilde{A}\partial_{tt}u + ku - \widetilde{F}(t, x) - \frac{E\widetilde{A}}{2\widetilde{L}}\partial_{xx}u\int_0^{\widetilde{L}}(\partial_x u)^2\,\mathrm{d}x = 0 \tag{15}$$

where the geometric parameters $(\widetilde{A}, \widetilde{I}, \widetilde{L})$ and the external excitation $(\widetilde{F}(t, x))$ are stochastic, and the elastic constant k of the elastic medium is a constant. In this section, we only highlight the effects of the stochastic parameters on the symmetry of the dynamic system. Thus, we assume the external excitation $\widetilde{F}(t, x)$ as a stochastic white noise.

Generally, both the cross-section area and the length of the SWCNT obey normal distributions. Thus, in a random small interval $(t, t + \Delta t)$, we can assume the density functions of $\widetilde{A}$ and $\widetilde{L}$ as:

$$\widetilde{A}(x) \sim N(A_0, \sigma_1^2),\ \widetilde{L}(x) \sim N(L_0, \sigma_2^2) \tag{16}$$

where $A_0 = E(\widetilde{A})$, $\sigma_1^2 = D(\widetilde{A})$, $L_0 = E(\widetilde{L})$, $\sigma_2^2 = D(\widetilde{L})$. Then, $\widehat{I} = \frac{\widetilde{I} - A_0^2/4}{\sigma_1^2/16} \sim \chi^2$ (χ^2 distribution) approximately.

With the intermediate variables defined by $\partial_t u = v$, $\partial_x z = w$, $\partial_x w = \psi$, $\partial_x \psi = q$, Equation (15) can be rewritten as the following first-order form:

$$\widetilde{\mathbf{M}}\mathrm{d}_t\mathbf{z} + \widetilde{\mathbf{K}}\partial_x\mathbf{z}\mathrm{d}t = \nabla_{\mathbf{z}}S_1(\mathbf{z})\mathrm{d}t + \nabla_{\mathbf{z}}S_2(\mathbf{z}) \circ \mathrm{d}W \tag{17}$$

where $S_1(\mathbf{z}) = ku^2 - \frac{1}{2}\rho\widetilde{A}v^2 + \frac{1}{2}E\widetilde{I}\psi^2 - E\widetilde{I}wq$, $S_2(\mathbf{z}) = \widetilde{F}(t,x) + \frac{E\widetilde{A}}{2\widetilde{L}}\psi\int_0^{\widetilde{L}} w^2\mathrm{d}x$, $\mathbf{z} = (u, v, w, \psi, q)^{\mathrm{T}}$, and the skew-symmetric matrices are:

$$\widetilde{\mathbf{M}} = \begin{bmatrix} 0 & \rho\widetilde{A} & 0 & 0 & 0 \\ -\rho\widetilde{A} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \widetilde{\mathbf{K}} = \begin{bmatrix} 0 & 0 & 0 & 0 & E\widetilde{I} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -E\widetilde{I} & 0 \\ 0 & 0 & E\widetilde{I} & 0 & 0 \\ -E\widetilde{I} & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then, in a local domain $(t_0, t_1) \times (x_0, x_1)$, a stochastic multi-symplectic structure of Equation (17) can be formulated as Equation (6), in which $\omega(t,x) = \frac{1}{2}\mathrm{d}\mathbf{z} \wedge \widetilde{\mathbf{M}}\mathrm{d}\mathbf{z}$, $\kappa(t,x) = \frac{1}{2}\mathrm{d}\mathbf{z} \wedge \widetilde{\mathbf{K}}\mathrm{d}\mathbf{z}$. The local energy dissipation can be calculated by Equation (14).

For the existence of the stochastic parameters and the external stochastic excitation, the analytical condition of the existence of the chaos in the embedded SWCNT clamped at both ends cannot be obtained following our previous work [58]. To illustrate the above mapping relationship between the symmetry breaking and the local energy dissipation, the structure-preserving numerical scheme presented in [58] is employed with zero damping and stochastic parameters, as well as the stochastic white noise excitation to reproduce the local energy dissipation of the dynamic system given by Equation (15).

Some parameters of the carbon nanotube are assumed as: $A_0 = 2.3562 \times 10^{-18}$ m^2, $L_0 = 100$ nm, $\sigma_1^2 = 0.01 \times 10^{-36}$ m^4, $\sigma_2^2 = 5$ nm^2, and $E(F)$ increases from 5×10^{-11} N to 10×10^{-11} N. The stochastic vibration of the carbon nanotube is simulated by using the structure-preserving numerical scheme and the stochastic chaos is reproduced with the following boundary conditions and the initial conditions.

$$\begin{gathered} u(0,t) = \frac{\partial u}{\partial x}(0,t) = u(\widetilde{L},t) = \frac{\partial u}{\partial x}(\widetilde{L},t) = 0 \\ u(x,0) = 0 \end{gathered} \tag{18}$$

Figures 1 and 2 show the Poincaré section when $E(F) = 9.25 \times 10^{-11}$ N and $E(F) = 9.3 \times 10^{-11}$ N, respectively. Figures 3 and 4 show the maximum Lyapunov exponents (denoted by MLEs in the figures) when $E(F) = 9.25 \times 10^{-11}$ N and $E(F) = 9.3 \times 10^{-11}$ N, respectively.

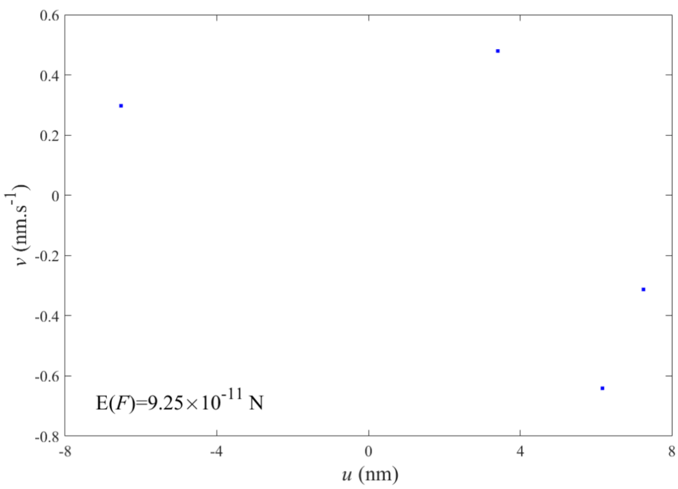

Figure 1. Poincaré section with $E(F) = 9.25 \times 10^{-11}$ N.

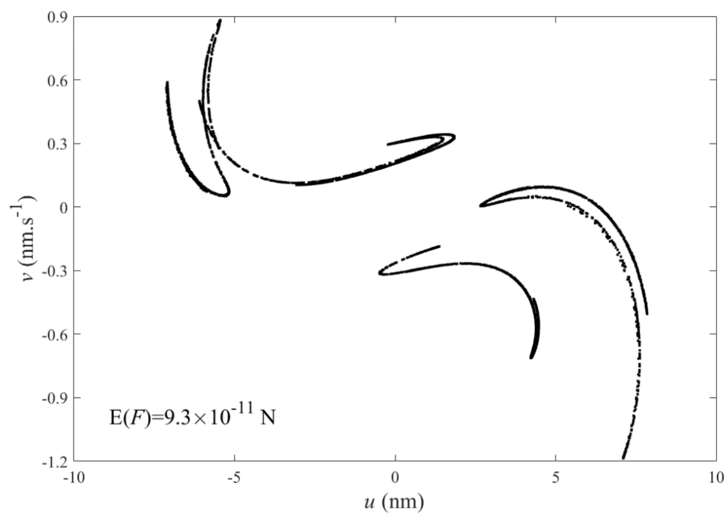

Figure 2. Poincaré section with $E(F) = 9.3 \times 10^{-11}$ N.

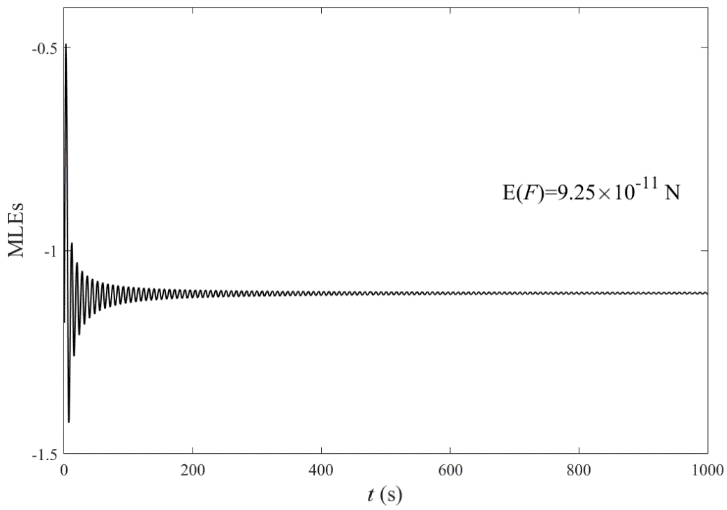

Figure 3. Maximum Lyapunov exponents with $E(F) = 9.25 \times 10^{-11}$ N.

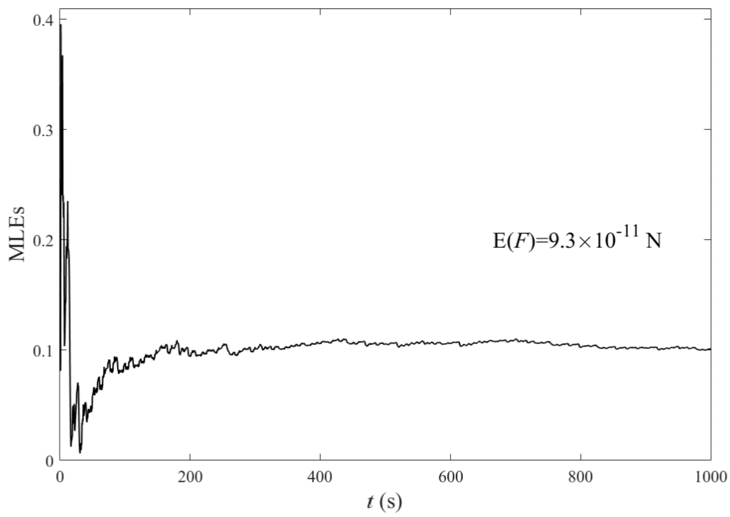

Figure 4. Maximum Lyapunov exponents with $E(F) = 9.3 \times 10^{-11}$ N.

In the Poincaré section presented in Figure 1, the number of the points is finite, while the number of the points in the Poincaré section presented in Figure 2 is infinite. In addition, the maximum Lyapunov exponents with $E(F) = 9.25 \times 10^{-11}$ N shown in Figure 3 are negative, while the maximum Lyapunov exponents with $E(F) = 9.3 \times 10^{-11}$ N shown in Figure 4 are positive. Both the infinite number of the points in the Poincaré section and the positive maximum Lyapunov exponent are the proofs that the stochastic vibration of the SWCNT goes into chaos when $E(F) = 9.3 \times 10^{-11}$ N.

Comparing the Poincaré section and the maximum Lyapunov exponents presented in the above figures, we can find that the critical condition of the appearance of the chaos in the embedded SWCNT clamped at both ends considered in this paper is about $E(F) \geq 9.3 \times 10^{-11}$ N, which is larger than that obtained in our previous work [58]. It can be concluded that the effect of the stochastic parameters, as well as the stochastic excitation on the vibration of the clamped SWCNT is increasing the critical condition of the appearance of the chaos in the SWCNT remarkably. Reproducing the critical condition of the appearance of the chaos during the stochastic vibration accurately is just benefited from the calculation on the local energy dissipation referring to the theoretical results presented in Section 3.

5. Conclusions and Remarks

As the foundation of the structure-preserving method, the symmetry mapping to the conserved quantity and the symmetry breaking resulting in the energy dissipation of the dynamic systems have aroused considerable interests in the last century. To generalize the results of our previous work on the symmetry breaking of the infinite-dimensional deterministic systems, the dynamical symmetry breaking of the infinite-dimensional stochastic systems is investigated in this paper. Referring to the exploratory works on the stochastic symplectic method for the finite-dimensional stochastic systems and the stochastic multi-symplectic method for the infinite-dimensional stochastic systems, two typical dynamical symmetry breaking factors of the infinite-dimensional stochastic systems are discussed. One is the stochastic excitation and another is the stochastic parameters. We find that the local multi-symplectic structure of the system is not broken by the above two factors according to the classic multi-symplectic theory, but the local energy conservation law of the system is broken by the above two factors. The effect of the stochastic excitation with the non-zero mathematical expectation and the effect of the stochastic parameters on the local energy conservation law are formulated by the local energy dissipation, which is the main contribution of this work. The mathematical expression on the local energy dissipation of the stochastic dynamic system permits us to develop the structure-preserving method to study the nonlinear dynamic behaviors of the stochastic dynamic systems in detail, such as chaos and bifurcation phenomena. To help the researchers to perform the stochastic multi-symplectic scheme for the infinite-dimensional dynamic system with the stochastic parameters, the local energy dissipation resulting from the stochastic parameters is proposed. In the example of the stochastic vibration of the clamped SWCNT, the effect of the stochastic parameters, as well as the stochastic excitation, on the vibration of the clamped SWCNT is reproduced and the dynamical symmetry breaking result of the infinite-dimensional stochastic systems is illustrated.

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