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The kth Local Exponent of Doubly Symmetric Primitive Digraphs with d Loops

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Abstract: Let D be a primitive digraph of order n. The exponent of a vertex x in $V(D)$ is denoted $\gamma_D(x)$, which is the smallest integer q such that for any vertex y, there is a walk of length q from x to y. Let $V(D) = \{v_1, v_2, \cdots, v_n\}$. We order the vertices of $V(D)$ so that $\gamma_D(v_1) \le \gamma_D(v_2) \le \cdots \le \gamma_D(v_n)$ is satisfied. Then, for any integer k satisfying $1 \le k \le n$, $\gamma_D(v_k)$ is called the kth local exponent of D and is denoted by $exp_D(k)$. Let $DS_n(d)$ represent the set of all doubly symmetric primitive digraphs with n vertices and d loops, where d is an integer such that $1 \le d \le n$. In this paper, we determine the upper bound for the kth local exponent of $DS_n(d)$, where $1 \le k \le n$. In addition, we find that the upper bound for the kth local exponent of $DS_n(d)$ can be reached, where $1 \le k \le n$.

Keywords: exponent; symmetric digraph; generalized competition index; competition index; scrambling index

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1. Introduction

Let $D = (V, E)$ denote a digraph (directed graph) with n vertices, where the vertex set $V = V(D)$ and the arc set $E = E(D)$. Loops are permitted, but multiple arcs are not. A walk from x to y in D, we mean a sequence of vertices $x, v_1, \cdots, v_t, y$ where each vertex in the sequence of vertices belongs to V, and a sequence of arcs $(x, v_1), (v_1, v_2), \cdots, (v_t, y)$ where each arc in the sequence of arcs belongs to E, and the vertices and arcs are not necessarily distinct. The number of arcs in W is the length of the walk W. The notation $x \xrightarrow{k} y$ means that there exists a walk of length k from x to y. The distance from vertex x to vertex y in D is written as $d_D(x, y)$(for short, $d(x, y)$), which refers to the length of the shortest walk from x to y. If $x = y$, then a walk from x to y is a closed walk. A cycle is a closed walk from x to y with distinct vertices except for $x = y$.

Let x, y be any pair of vertices in a digraph D. The digraph D is called primitive, if there exists a positive integer k such that there is a walk of length k from x to y. This smallest such k is denoted by $exp(D)$, which is called the exponent of D. The greatest common divisor of the lengths of all the cycles in D is recorded as $l(D)$. It is well known (see [1]) that D is primitive if and only if D is strongly connected and $l(D) = 1$.

Brualdi and Liu [2] generalized the concept of exponent for a primitive digraph (primitive matrix). Let D be a primitive digraph with n vertices. The exponent of D can be broken down into more local exponents [3]. For any pair of vertices $x, z \in V(D)$, let $\gamma_D(x, z)$ denote the smallest integer p such that there is a walk of length t from x to z, for each integer $t \ge p$. Since D is a primitive digraph, then $\gamma_D(x, z)$ is a finite number. For any vertex $x \in V(D)$, the exponent of vertex x is written as $\gamma_D(x)$, which is the smallest integer q so that for any vertex $y \in V(D)$, there exists a walk of length q from x to y. Moreover, for any vertex $z \in V(D)$ and any integer $t \ge \gamma_D(x, z)$, there is a walk of length t from x to z. So, we have $q = \max\{\gamma_D(x, z) : z \in V(D)\}$. Then, for any vertex $y \in V(D)$, there is a walk of length t from x to y for each integer $t \ge q$. Therefore, we have

$$\gamma_D(x) = \max\{\gamma_D(x, z) : z \in V(D)\}.$$

Let the vertices of D be ordered as $v_1, v_2, \cdots, v_n$ such that

$$\gamma_D(v_1) \leq \gamma_D(v_2) \leq \cdots \leq \gamma_D(v_n).$$

$\gamma_D(v_k)$ is called the kth local exponent (generalized exponent) of D, and it is denoted by $exp_D(k)$, where $1 \leq k \leq n$. Then,

$$exp_D(1) \leq exp_D(2) \leq \cdots \leq exp_D(n).$$

Furthermore, we have $\gamma(D) = \max\{\gamma_D(x) : x \in V(D)\} = \max\{\gamma_D(x,y) : x,y \in V(D)\}$. Obviously, the exponent of D equals $exp_D(n)$. That is, $\gamma(D) = exp(D) = exp_D(n)$. So, for a primitive digraph D, the local exponents of D are generalizations of the exponent of D.

Brualdi and Liu [2] proposed a memoryless communication system. In the memoryless communication system represented by a primitive digraph D of order n, the kth local exponent is the smallest time for each vertex to simultaneously hold all k bits of the information. For more details, please refer to [2,3].

For any vertices x and y of a digraph D, $(x,y) \in E(D)$ is an arc if and only if $(y,x) \in E(D)$ is an arc, which is represented by $x \leftrightarrow y$, then such a digraph D is called a symmetric digraph. An undirected graph (possibly with loops) can be viewed as a symmetric digraph. For some research on undirected graphs, please see [4–6]. When D is symmetric, the notation $x \xrightarrow{k} y$ indicates that there is a walk of length k from x to y.

Let $D = (V,E)$ be a symmetric digraph, we can regard D as an undirected graph. For convenience, undirected graph terms such as edges, edge set, etc., are used directly to describe a symmetric digraph. Then, let $E(D)$ denote the set of undirected edges (edges) in D. Moreover, we assume that the notation $[x,y] \in E(D)$ represents that there is an edge in D with x,y as end vertices.

Let $D = (V,E)$ be a symmetric digraph, where $V = \{v_1, v_2, \cdots, v_n\}$. If for any vertices v_i and v_j, $[v_i,v_j] \in E(D)$ if and only if $[v_{n+1-i}, v_{n+1-j}] \in E(D)$, then such a symmetric digraph D is called a doubly symmetric digraph. Moreover, $[v_i,v_j]$ and $[v_{n+1-i}, v_{n+1-j}]$ are called a pair of symmetrical edges, or $[v_i,v_j]$ is a symmetrical edge of $[v_{n+1-i}, v_{n+1-j}]$, where $1 \leq i \leq n$ and $1 \leq j \leq n$. The vertices v_{n+1-i}, v_i are called a pair of symmetric vertices, or v_i is a symmetric vertex of v_{n+1-i}, where $1 \leq i \leq n$. According to this definition, when n is odd, $v_{\frac{n+1}{2}}$ is symmetric to itself. If v_i is a loop vertex, then $[v_i,v_i] \in E(D)$ and $[v_{n+1-i}, v_{n+1-i}] \in E(D)$. Therefore, for $i \neq n+1-i$, if $[v_i,v_i]$ is a loop, then $[v_{n+1-i}, v_{n+1-i}]$ is also a loop, the loops appear in pairs. A doubly symmetric digraph D is called a doubly symmetric primitive digraph provided D is primitive.

If a doubly symmetric primitive digraph D contains exactly d loops, then we call D a doubly symmetric primitive digraph with d loops. Let DS_n denote the set of all doubly symmetric primitive digraphs of order n. Let $DS_n(d)$ denote the set of all doubly symmetric primitive digraphs of order n with d loops, where d is an integer such that $1 \leq d \leq n$. Obviously, we have $DS_n(d) \subseteq DS_n$.

Let $D \in DS_n(d)$. After deleting any pair of symmetrical edges $[v_i,v_j]$ and $[v_{n+1-i}, v_{n+1-j}]$ of D, the obtained digraph D' is not a doubly symmetric primitive digraph (that is, D' is not connected), then we call $D \in DS'_n(d)$, where $1 \leq i < j \leq n$. Obviously, we have $DS'_n(d) \subseteq DS_n(d)$.

For example, we consider the kth local exponent of the graph G. Let $V(G) = \{v_1, v_2, \cdots, v_7\}$. Let $E(G) = \{[v_i, v_{i+1}] | 1 \leq i \leq 6\} \cup \{[v_4, v_4]\}$. G is shown in Figure 1.

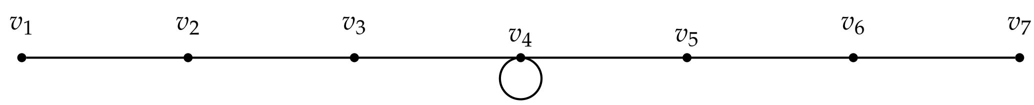

Figure 1. *G*.

We easily get $\gamma_G(v_4) = 3$, $\gamma_G(v_3) = \gamma_G(v_5) = 4$, $\gamma_G(v_2) = \gamma_G(v_6) = 5$, $\gamma_G(v_1) = \gamma_G(v_7) = 6$. Then, we have $exp_G(1) = 3, exp_G(2) = exp_G(3) = 4, exp_G(4) = exp_G(5) = 5, exp_G(6) = exp_G(7) = 6$. Moreover, we have $\gamma(G) = exp(G) = exp_G(7) = 6$.

Some studies [7–12] have investigated exponents and their generalization. Chen and Liu [11] studied the kth local exponent of doubly symmetric primitive matrices (primitive digraphs). Chen and Liu [12] characterized the doubly symmetric primitive digraphs with the kth local exponent reaching the maximum value. A doubly symmetric primitive digraph with d loops is a special doubly symmetric primitive digraph. It is important to mention that the kth local exponent of such a class of digraphs has not been studied before. Using graph theory methods, we obtain the upper bound of the kth local exponent of digraphs in $DS_n(d)$, where $1 \leq k \leq n$. Some studies have investigated the scrambling index [13–16] and generalized competition index [17–23]. Several studies explored the generalized μ-scrambling indices, please refer to [24–26].

Let $D \in DS_n(d)$. Let $V(L(D))$ represent the set of d loop vertices in D. Let $E(L(D))$ denote the set of d loops in D. Let v_i, v_j be any pair of vertices of the digraph D. If the walk from v_i to v_j in D is denoted as $W_D(v_i, v_j)$ (for short, $W(v_i, v_j)$), then $|W(v_i, v_j)|$ is used to denote the length of the walk $W(v_i, v_j)$, and $V(W(v_i, v_j))$ is used to denote the set of all vertices in this walk $W(v_i, v_j)$. If there is a unique path from v_i to v_j in D, then let $P_D(v_i, v_j)$ (for short, $P(v_i, v_j)$) denote the unique path, and let $V(P_D(v_i, v_j))$ (for short, $V(P(v_i, v_j))$) denote the set of all vertices on the path. If $v_i = v_j$, then $V(P(v_i, v_j)) = \{v_i\} = \{v_j\}$. If a walk $W(v_i, v_j)$ from v_i to v_j in D does not pass through a loop vertex, then let $V(W(v_i, v_j)) \cap V(L(D)) = \emptyset$, otherwise $V(W(v_i, v_j)) \cap V(L(D)) \neq \emptyset$. Similarly, if the unique path from v_i to v_j passes through a loop vertex, that is $V(P(v_i, v_j)) \cap V(L(D)) \neq \emptyset$, otherwise $V(P(v_i, v_j)) \cap V(L(D)) = \emptyset$.

For a vertex $v \in V(D)$ and a set $X \subseteq V(D)$, let $d(v, X) = \min\{d(v, v_i) : v_i \in X\}$. If $v \in X$, let $d(v, X) = 0$. For any vertex $u \in V(D)$ and $v \in V(D)$, if $u = v$, let $d(u, v) = 0$. If T is a set, the notation $|T|$ is used to denote the number of all elements in T. The notation $\lfloor a \rfloor$ is used to denote the largest integer not greater than a, and the notation $\lceil b \rceil$ is used to denote the smallest integer not less than b.

In this paper, let n, d and k be integers with $n \geq 5, 1 \leq k \leq n, 1 \leq d \leq n$. We give the upper bound of the kth local exponent of digraphs in $DS_n(d)$, where $1 \leq k \leq n$.

2. The Upper Bound for the kth Local Exponent of $DS_n(d)$

In this section, let $D = (V, E)$, where $V = \{v_1, v_2, \cdots, v_n\}$.

In the case of $D \in DS_n(d)$, we observe the exponent of any vertex in D, it is easy to get the following Proposition 1, let us omit the proof.

Proposition 1. *Let $D \in DS_n(d)$ and let v_i be any vertex of D, then $\gamma_D(v_i) = \gamma_D(v_{n+1-i})$, where $1 \leq i \leq n$.*

Lemma 1 (Lemma 3.3 [2])**.** *Let D be a primitive digraph with n vertices. Then, $exp_D(k+1) \leq exp_D(k) + 1$, where $1 \leq k \leq n-1$.*

Remark 1. *Lemma 1 is actually very useful. Next, we repeat the proof of Brualdi and Liu (see [2]). Since D is strongly connected, for any integer k such that $1 \leq k \leq n-1$, there is a vertex x that is joined by an arc to one of the vertices with the k smallest exponents. Therefore, $exp_D(k+1) \leq exp_D(k) + 1$, where $1 \leq k \leq n-1$.*

Lemma 2. *Let $D \in DS_n(d)$ and let v_i, v_j be any pair of vertices of D. Then, $\gamma_D(v_j) \leq \gamma_D(v_i) + d(v_i, v_j)$.*

Proof. For any vertex $x \in V(D)$, there is a walk of length t from v_i to x, which is $v_i \xrightarrow{t} x$, for each integer $t \geq \gamma_D(v_i)$. So, there is a walk of length s from v_j to x, which is $v_j \xrightarrow{d(v_j,v_i)} v_i \xrightarrow{t} x$, for each integer $s \geq \gamma_D(v_i) + d(v_i, v_j)$. Therefore, $\gamma_D(v_j) \leq \gamma_D(v_i) + d(v_i, v_j)$. □

Lemma 3. *Let $D \in DS_n(d)$ and let x, y be any pair of vertices of D. If there exists a walk $W(x,y)$ from x to y such that $V(W(x,y)) \cap V(L(D)) \neq \emptyset$, then $\gamma_D(x,y) \leq |W(x,y)|$.*

Proof. Let $h = |W(x,y)|$. We consider the following.

Case 1 $x \in V(D) \setminus V(L(D))$ and $y \in V(D) \setminus V(L(D))$.

Suppose a walk from x to y through a loop vertex is denoted by $x \xrightarrow{a} v_i \xrightarrow{h-a} y$, where v_i is a loop vertex and a is an integer such that $1 \leq a \leq h-1$. Then, the length of the walk $x \xrightarrow{a} v_i \xrightarrow{1} v_i \xrightarrow{h-a} y$ is $h+1$. The length of the walk $x \xrightarrow{a} v_i \xrightarrow{1} v_i \xrightarrow{1} v_i \xrightarrow{h-a} y$ is $h+2$. Similarly, we can easily conclude that there is a walk of length s from x to y, for each integer $s \geq h$. So, $\gamma_D(x,y) \leq |W(x,y)|$.

Case 2 $x \in V(L(D))$ or $y \in V(L(D))$.

Similar to Case 1, it is easy to get that there is a walk of length s from x to y, for each integer $s \geq h$. So, $\gamma_D(x,y) \leq |W(x,y)|$.

Therefore, the lemma holds. $\square$

Lemma 4 (Lemma 1 [23])**.** *Let $D \in DS_n$. If n is odd and x is any vertex of D, then $d(x, v_{\frac{n+1}{2}}) \leq \frac{n-1}{2}$.*

Theorem 1. *Let $D \in DS_n(d)$. If n is odd and d is odd, then $exp_D(k) \leq \frac{n-1}{2} + \left\lfloor \frac{k}{2} \right\rfloor$, where $1 \leq k \leq n$.*

Proof. If d is odd, then $v_{\frac{n+1}{2}}$ is a loop vertex. Let x be any vertex. Then, a shortest path from x to $v_{\frac{n+1}{2}}$ goes through the loop vertex $v_{\frac{n+1}{2}}$. According to Lemma 4, we have $d(x, v_{\frac{n+1}{2}}) \leq \frac{n-1}{2}$. Furthermore, according to Lemma 3, we have $\gamma_D(v_{\frac{n+1}{2}}, x) \leq d(x, v_{\frac{n+1}{2}})$. Further, we have $\gamma_D(v_{\frac{n+1}{2}}, x) = d(x, v_{\frac{n+1}{2}})$. So, $\gamma_D(v_{\frac{n+1}{2}}) = \max\{d(v_{\frac{n+1}{2}}, x) : x \in V(D)\} \leq \frac{n-1}{2}$. Then, we have $exp_D(1) \leq \gamma_D(v_{\frac{n+1}{2}}) \leq \frac{n-1}{2}$. Further, according to Proposition 1, we have $\gamma_D(v_1) = \gamma_D(v_n), \gamma_D(v_2) = \gamma_D(v_{n-1}), \cdots, \gamma_D(v_{\frac{n-1}{2}}) = \gamma_D(v_{\frac{n+3}{2}})$. So, according to Lemma 1, we conclude $exp_D(2) = exp_D(3) \leq exp_D(1) + 1 \leq \frac{n-1}{2} + 1$, $exp_D(4) = exp_D(5) \leq exp_D(1) + 2 \leq \frac{n-1}{2} + 2, \cdots, exp_D(n-1) = exp_D(n) \leq exp_D(1) + \frac{n-1}{2} \leq \frac{n-1}{2} + \frac{n-1}{2}$. Therefore, we have $exp_D(k) \leq exp_D(1) + \left\lfloor \frac{k}{2} \right\rfloor \leq \frac{n-1}{2} + \left\lfloor \frac{k}{2} \right\rfloor$, where $1 \leq k \leq n$. $\square$

Let $D' \in DS_n(d)$ and $D \in DS_n(d)$. If D is a subgraph of D' such that $V(D) = V(D')$ and $E(D) \subseteq E(D')$, then $exp_{D'}(k) \leq exp_D(k)$, where $1 \leq k \leq n$. So, if we investigate the upper bound of the kth local exponent of digraphs in $DS_n(d)$, we only need to investigate the digraphs in $DS'_n(d)$.

Referring to Definition 3 in [23], we give the following Definition 1.

Definition 1. *Let $D \in DS'_n(d)$, where n is odd, d is even such that $d \geq 2$. There exist two connected subgraphs $D_* = (V_*, E_*)$ and $D_{**} = (V_{**}, E_{**})$ of D, and D_*, D_{**} satisfy $V(D) = V_* \cup V_{**}$ and $E(D) = E_* \cup E_{**} \cup E(L(D))$. Where $V_* = \{v_{n+1-i} : v_i \in V_{**}\}$ and $E_* = \{[v_{n+1-i}, v_{n+1-j}] : [v_i, v_j] \in E_{**}\}$. Moreover, $|V_*| = |V_{**}| = \frac{n+1}{2}, |E_*| = |E_{**}| = \frac{n-1}{2}$.*

Remark 2. *Suppose n is odd and d is even that satisfies $d \geq 2$. If $D \in DS'_n(d)$, then there are two connected subgraphs D_* and D_{**} of D. In addition, there is a unique path for any two different vertices in D_* and D_{**}, respectively. Moreover, there is a unique path for any two different vertices in D. Let x, y be any pair of vertices of D such that $x \in V_*$ and $y \in V_{**}$, then $V(P(x, v_{\frac{n+1}{2}})) \cap V(P(v_{\frac{n+1}{2}}, y)) = \{v_{\frac{n+1}{2}}\}$ (see [23]). After removing d loops from D, the obtained graph is a tree. Therefore, D is a special tree with loops that satisfies $[v_i, v_j] \in E(D)$ if and only if $[v_{n+1-i}, v_{n+1-j}] \in E(D)$, where $1 \leq i < j \leq n$.*

Lemma 5 (Lemma 3 [23]). *Let $D \in DS'_n(d)$, where n is odd, d is even such that $d \geq 2$. Let x, y be any pair of vertices of D such that $V(P(x, v_{\frac{n+1}{2}})) \cap V(L(D)) = \emptyset$ and $V(P(y, v_{\frac{n+1}{2}})) \cap V(L(D)) = \emptyset$. Then, there is a walk $W(x, y)$ from x to y such that $V(W(x, y)) \cap V(L(D)) \neq \emptyset$, and $|W(x, y)| \leq n - d + 1$.*

Corollary 1. *Let $D \in DS'_n(d)$, where n is odd, d is even and $d \geq 2$. Let x, y be any pair of vertices of D satisfying $V(P(x, v_{\frac{n+1}{2}})) \cap V(L(D)) = \emptyset$ and $V(P(y, v_{\frac{n+1}{2}})) \cap V(L(D)) = \emptyset$. Then, $\gamma_D(x, y) \leq n - d + 1$.*

Proof. According to Lemma 5, there is a walk $W(x, y)$ from x to y passing through a loop vertex, and $|W(x, y)| \leq n - d + 1$. Moreover, according to Lemma 3, we have $\gamma_D(x, y) \leq |W(x, y)| \leq n - d + 1$. $\square$

In Corollary 1, if $x = v_{\frac{n+1}{2}}$ and x isn't a loop vertex, then $V(P(x, v_{\frac{n+1}{2}})) = \{v_{\frac{n+1}{2}}\}$, we have $V(P(x, v_{\frac{n+1}{2}})) \cap V(L(D)) = \emptyset$.

Theorem 2. *Let $D \in DS_n(d)$. If n is odd, d is even and $d \geq 2$, then*

(1) If $n \leq 2d - 3$, then $exp_D(k) \leq \frac{n-1}{2} + \left\lfloor \frac{k}{2} \right\rfloor$, where $1 \leq k \leq n$.

(2) If $n \geq 2d - 1$, then

$$exp_D(k) \leq \begin{cases} n - d + 1, & \text{where } 1 \leq k \leq n - 2d + 4, \\ n - d + 1 + \left\lceil \frac{k-(n-2d+4)}{2} \right\rceil, & \text{where } n - 2d + 4 \leq k \leq n. \end{cases}$$

Proof. We only need to consider $D \in DS'_n(d)$. Since d is even, then $v_{\frac{n+1}{2}}$ isn't a loop vertex. Let x be any vertex. By Lemma 4, we have $d(x, v_{\frac{n+1}{2}}) \leq \frac{n-1}{2}$. If the path from $v_{\frac{n+1}{2}}$ to x passes through a loop vertex, that is $V(P(v_{\frac{n+1}{2}}, x)) \cap V(L(D)) \neq \emptyset$, then $\gamma_D(v_{\frac{n+1}{2}}, x) = d(x, v_{\frac{n+1}{2}}) \leq \frac{n-1}{2}$. If vertex x satisfies $V(P(v_{\frac{n+1}{2}}, x)) \cap V(L(D)) = \emptyset$, according to Corollary 1, we have $\gamma_D(v_{\frac{n+1}{2}}, x) \leq n - d + 1$. Therefore, we have $\gamma_D(v_{\frac{n+1}{2}}) \leq \max\{\frac{n-1}{2}, n - d + 1\}$.

(1) If $n \leq 2d - 3$, then $n - d + 1 \leq \frac{n-1}{2}$. Then, we have $exp_D(1) \leq \gamma_D(v_{\frac{n+1}{2}}) \leq \frac{n-1}{2}$. According to Proposition 1, we have $\gamma_D(v_1) = \gamma_D(v_n), \gamma_D(v_2) = \gamma_D(v_{n-1}), \cdots, \gamma_D(v_{\frac{n-1}{2}}) = \gamma_D(v_{\frac{n+3}{2}})$. Then, according to Lemma 1, we can conclude that $exp_D(2) = exp_D(3) \leq exp_D(1) + 1 \leq \frac{n-1}{2} + 1, exp_D(4) = exp_D(5) \leq exp_D(1) + 2 \leq \frac{n-1}{2} + 2, \cdots$. Therefore, we have $exp_D(k) \leq exp_D(1) + \left\lfloor \frac{k}{2} \right\rfloor \leq \frac{n-1}{2} + \left\lfloor \frac{k}{2} \right\rfloor$, where $1 \leq k \leq n$.

(2) If $n \geq 2d - 1$, then $n - d + 1 \geq \frac{n+1}{2}$. We have $exp_D(1) \leq \gamma_D(v_{\frac{n+1}{2}}) \leq n - d + 1$.

Next, we construct a set $V(M)$ such that $|V(M)| \geq n - 2d + 4$, and for any vertex $v_i \in V(M)$, $\gamma_D(v_i) \leq n - d + 1$ holds. Suppose $V(M) = \{v : d(v, v_{\frac{n+1}{2}}) \leq \frac{n-2d+3}{2}\}$. Then, $v_{\frac{n+1}{2}} \in V(M)$.

Suppose $v_h \in V(M) \cap V_*$ and $v_h \neq v_{\frac{n+1}{2}}$. If the vertex sequence of the unique path in D_* from v_h to $v_{\frac{n+1}{2}}$ is $v_h, \cdots, v_t, \cdots, v_{\frac{n+1}{2}}$, then the vertex sequence of the unique path in D_{**} from v_{n+1-h} to $v_{\frac{n+1}{2}}$ is $v_{n+1-h}, \cdots, v_{n+1-t}, \cdots, v_{\frac{n+1}{2}}$. Moreover, $V(P(v_h, v_{\frac{n+1}{2}})) \cap V(P(v_{n+1-h}, v_{\frac{n+1}{2}})) = \{v_{\frac{n+1}{2}}\}$. So, $d(v_h, v_{\frac{n+1}{2}}) = d(v_{n+1-h}, v_{\frac{n+1}{2}}) \leq \frac{n-2d+3}{2}$. Therefore, we have $v_{n+1-h} \in V(M)$. Next, we prove that $|V(M)| \geq n - 2d + 4$. For any vertex $v_s \in V(D)$, if $v_s \in V(M)$, then $|V(M)| = n$. If there is a vertex v_l satisfying $v_l \in V(D) \setminus V(M)$, then $d(v_l, v_{\frac{n+1}{2}}) \geq \frac{n-2d+3}{2} + 1$. So, $V(P(v_l, v_{\frac{n+1}{2}})) \cap V(M) = \frac{n-2d+3}{2} + 1$. Moreover, $V(P(v_{n+1-l}, v_{\frac{n+1}{2}})) \cap V(M) = \frac{n-2d+3}{2} + 1$ and $V(P(v_l, v_{\frac{n+1}{2}})) \cap V(P(v_{n+1-l}, v_{\frac{n+1}{2}})) = \{v_{\frac{n+1}{2}}\}$. Then, we have $\left|\left(V(P(v_l, v_{\frac{n+1}{2}})) \cup V(P(v_{n+1-l}, v_{\frac{n+1}{2}}))\right) \cap V(M)\right| = n - 2d + 4$. Therefore, we have $|V(M)| \geq n - 2d + 4$. For any vertex $v_i \in V(M)$, next we consider $\gamma_D(v_i)$.

For any vertex $v_i \in V(M)$, let the walk $V(W(v_i, x))$ from v_i to x be $v_i \xrightarrow{d(v_i, v_{\frac{n+1}{2}})} v_{\frac{n+1}{2}} \xrightarrow{d(v_{\frac{n+1}{2}}, x)} x$. If the walk $V(W(v_i, x))$ passes through a loop vertex, we have $\gamma_D(v_i, x) \leq d(v_i, v_{\frac{n+1}{2}}) + d(v_{\frac{n+1}{2}}, x) \leq \frac{n-2d+3}{2} + \frac{n-1}{2} = n - d + 1$. If the walk $V(W(v_i, x))$ doesn't pass through a loop vertex, that is, $V(W(v_i, x)) \cap V(L(D)) = \emptyset$. Then, we have $V(P(v_i, v_{\frac{n+1}{2}})) \cap V(L(D)) = \emptyset$ and $V(P(x, v_{\frac{n+1}{2}})) \cap V(L(D)) = \emptyset$. So, if $V(W(v_i, x)) \cap V(L(D)) = \emptyset$, according to Corollary 1, we have $\gamma_D(v_i, x) \leq n - d + 1$. Therefore, we have $\gamma_D(v_i) \leq n - d + 1$. According to Proposition 1, we have $\gamma_D(v_{n+1-i}) = \gamma_D(v_i) \leq n - d + 1$. Therefore, we have $exp_D(k) \leq n - d + 1$, where $1 \leq k \leq |V(M)|$.

For any vertex v_j such that $v_j \in V(D) \setminus V(M)$, then $v_{n+1-j} \in V(D) \setminus V(M)$. There is a unique path for any pair of vertices in D. So for any vertex $v_i \in V(M)$, we have $d(v_j, v_i) = d(v_{n+1-j}, v_{n+1-i})$. Suppose $d(v_j, V(M)) = d(v_j, v_l)$, where $v_l \in V(M)$. We have $d(v_{n+1-j}, v_{n+1-l}) = d(v_j, v_l) \leq d(v_j, v_i) = d(v_{n+1-j}, v_{n+1-i})$. So, we have $d(v_j, V(M)) = d(v_{n+1-j}, V(M))$. Furthermore, according to Lemma 2, we can easily conclude that $\gamma_D(v_j) = \gamma_D(v_{n+1-j}) \leq n - d + 1 + d(v_j, V(M))$. Since $|V(M)| \geq n - 2d + 4$, the conclusion is clearly established.

Therefore, the theorem holds. □

Referring to Definition 4 in [23], we give the following Definition 2.

Definition 2. *Let $D \in DS'_n(d)$, where n is even, d is even such that $d \geq 2$.*

(1) *There exist two connected subgraphs $D_1 = (V_1, E_1)$ and $D_2 = (V_2, E_2)$ of D, and D_1, D_2 satisfy $V(D) = V_1 \cup V_2$ and $E(D) = E_1 \cup E_2 \cup \{[v_f, v_{n+1-g}]\} \cup \{[v_{n+1-f}, v_g]\} \cup E(L(D))$. Where $V_1 = \{v_{n+1-i} : v_i \in V_2\}$ and $E_1 = \{[v_{n+1-i}, v_{n+1-j}] : [v_i, v_j] \in E_2\}$, $v_f \in V_1$ and $v_g \in V_1$. Moreover, $|V_1| = |V_2| = \frac{n}{2}$, $|E_1| = |E_2| = \frac{n}{2} - 1$.*
(2) *If $\{v_f, v_{n+1-f}\} \cap V(L(D)) = \emptyset$. Let $V(H_1) = \{v : V(P_{D_1}(v_f, v)) \cap V(L(D)) = \emptyset$, where $v \in V_1\}$ and $V(H_2) = \{v : V(P_{D_2}(v_{n+1-f}, v)) \cap V(L(D)) = \emptyset$, where $v \in V_2\}$. Suppose $V(H) = V(H_1) \cup V(H_2)$.*

Definition 3. *Let $D \in DS'_n(d)$, where n is even, d is even such that $d \geq 2$. In Definition 2(1), we give the following definition:*

(1) *Let $W(v_f, v_f)$ be $v_f \xrightarrow{d_{D_1}(v_f, v_g)} v_g \xrightarrow{d_D(v_g, v_{n+1-f})} v_{n+1-f} \xrightarrow{d_{D_2}(v_{n+1-f}, v_{n+1-g})} v_{n+1-g} \xrightarrow{d_D(v_{n+1-g}, v_f)} v_f$, then $W(v_f, v_f)$ is a closed walk from v_f to v_f. Let us write $V(W(v_f, v_f)) = V(R)$.*
(2) *If $f \neq g$, let $D \in DS'_{n,1}(d)$. If $f = g$, let $D \in DS'_{n,2}(d)$.*

Remark 3. *Suppose n is even and d is even that satisfies $d \geq 2$. If $D \in DS'_n(d)$, then there are two connected subgraphs D_1 and D_2 of D. Moreover, there is a unique path for any two different vertices in D_1 and D_2, respectively. Since D is connected, then there are edges $[v_f, v_{n+1-g}] \in E(D)$ and $[v_{n+1-f}, v_g] \in E(D)$. Then, $d_D(v_g, v_{n+1-f}) = d_D(v_{n+1-g}, v_f) = 1$. If v_f and v_{n+1-f} are not loop vertices, then $v_f \in V(H)$ and $v_{n+1-f} \in V(H)$. If $D \in DS'_{n,1}(d)$, then $f \neq g$, and $|V(R)|$ is even such that $|V(R)| \geq 4$. Furthermore, if $D \in DS'_{n,1}(d)$, after removing d loops from D, the obtained graph D^* is not a tree. According to Definitions 2 and 3, it is not difficult to see that $|V(D^*)| = n$ and $|E(D^*)| = n$. If $D \in DS'_{n,2}(d)$, then $f = g$, $V(R) = \{v_f, v_{n+1-f}\}$ and $|V(R)| = 2$. If $D \in DS'_{n,2}(d)$, then D is a special tree with loops that satisfies $[v_i, v_j] \in E(D)$ if and only if $[v_{n+1-i}, v_{n+1-j}] \in E(D)$, where $1 \leq i < j \leq n$. In fact, $D \in DS'_{n,2}(d)$ can be regarded as a special case of $f = g$ in $D \in DS'_{n,1}(d)$.*

In Lemma 2 in [23], let $D \in DS'_n(d)$, where n is even and d be even such that $d \geq 2$. We can directly get the following Lemma 6.

Lemma 6. *Let $D \in DS'_n(d)$. Let n be even and d be even such that $d \geq 2$. Let x be any vertex of D. Then, for any vertex $v_s \in V(R)$, we have $d(x, v_s) \leq \frac{n}{2}$.*

Lemma 7 (Lemma 5 [23])**.** *Let $D \in DS'_n(d)$, where n is even, d is even and $d \geq 2$. Let x, y be any pair of vertices of D such that $x, y \in V(H)$. If $V(R) \cap V(L(D)) = \emptyset$, then there exists a walk $W(x, y)$ from x to y such that $V(W(x, y)) \cap V(L(D)) \neq \emptyset$, and $|W(x, y)| \leq n - d + 1$.*

Corollary 2. *Let $D \in DS'_n(d)$, where n is even, d is even such that $d \geq 2$. Let x, y be any pair of vertices of D satisfying $x, y \in V(H)$. If $V(R) \cap V(L(D)) = \emptyset$, then $\gamma_D(x, y) \leq n - d + 1$.*

Proof. According to Lemma 7, there is a walk $W(x, y)$ from x to y passing through a loop vertex, and $|W(x, y)| \leq n - d + 1$. Furthermore, according to Lemma 3, we have $\gamma_D(x, y) \leq |W(x, y)| \leq n - d + 1$. □

Theorem 3. *Let $D \in DS_n(d)$. If n is even, d is even and $d \geq 2$, then*

(1) If $n \leq 2d - 2$, then $exp_D(k) \leq \frac{n}{2} - 1 + \left\lceil \frac{k}{2} \right\rceil$, where $1 \leq k \leq n$.

(2) If $n \geq 2d$, then

$$exp_D(k) \leq \begin{cases} n - d + 1, & \text{where } 1 \leq k \leq n - 2d + 4, \\ n - d + 1 + \left\lceil \frac{k-(n-2d+4)}{2} \right\rceil, & \text{where } n - 2d + 4 \leq k \leq n. \end{cases}$$

Proof. We only need to consider $D \in DS'_n(d)$. Let x be any vertex. Let us consider the following two cases.

Case 1 $D \in DS'_{n,1}(d)$.

Since $v_f \in V_1$ and $v_g \in V_1$, then $v_{n+1-f} \in V_2$ and $v_{n+1-g} \in V_2$.

Case 1.1 $V(R) \cap V(L(D)) \neq \emptyset$.

Suppose $\{v_m, v_{n+1-m}\} \subseteq V(R) \cap V(L(D))$. Then, v_m and v_{n+1-m} are loop vertices. According to Lemma 6, we have $d(x, v_m) \leq \frac{n}{2}$. Further, $\gamma_D(v_m) = \gamma_D(v_{n+1-m}) \leq \frac{n}{2}$. Then, $exp_D(1) = exp_D(2) \leq \gamma_D(v_m) \leq \frac{n}{2}$. Therefore, according to Proposition 1 and Lemma 1, we have $exp_D(k) \leq \frac{n}{2} - 1 + \left\lceil \frac{k}{2} \right\rceil$, where $1 \leq k \leq n$.

(1) If $n \leq 2d - 2$, then the conclusion is clearly established.

(2) If $n \geq 2d$, for $1 \leq k \leq n - 2d + 4$, then $exp_D(k) \leq \frac{n}{2} - 1 + \left\lceil \frac{k}{2} \right\rceil \leq \frac{n}{2} - 1 + \frac{n}{2} - d + 2 = n - d + 1$. For $n - 2d + 4 \leq k \leq n$, we have $exp_D(k) \leq \frac{n}{2} - 1 + \left\lceil \frac{k}{2} \right\rceil = \frac{n}{2} - 1 + \left\lceil \frac{k-(n-2d+4)}{2} \right\rceil + \frac{n}{2} - d + 2 = n - d + 1 + \left\lceil \frac{k-(n-2d+4)}{2} \right\rceil$.

Case 1.2 $V(R) \cap V(L(D)) = \emptyset$.

Then, $v_f \in V(H) \cap V(R), v_{n+1-f} \in V(H) \cap V(R)$.

For $x \in V(H)$, according to Corollary 2, we have $\gamma_D(v_f, x) \leq n - d + 1$.

For any vertex $x \in V_1 \setminus V(H)$, then $V(P_{D_1}(v_f, x)) \cap V(L(D)) \neq \emptyset$. We have $\gamma_D(v_f, x) \leq d_{D_1}(v_f, x) \leq \frac{n}{2}$.

For any vertex $x \in V_2 \setminus V(H)$, then $V(P_{D_2}(v_{n+1-f}, x)) \cap V(L(D)) \neq \emptyset$. Let the walk $W_D(v_f, x)$ be $v_f \xrightarrow{d_D(v_f, v_{n+1-g})} v_{n+1-g} \xrightarrow{d_{D_2}(v_{n+1-g}, x)} x$. Since $V(R) \cap V(L(D)) = \emptyset$, then $V(P_{D_2}(v_{n+1-f}, v_{n+1-g})) \cap V(L(D)) = \emptyset$. In addition, $V(P_{D_2}(v_{n+1-f}, x)) \cap V(L(D)) \neq \emptyset$. We have $V(P_{D_2}(v_{n+1-g}, x)) \cap V(L(D)) \neq \emptyset$. So $V(W_D(v_f, x)) \cap V(L(D)) \neq \emptyset$. Moreover, $|W_D(v_f, x)| = 1 + d_{D_2}(v_{n+1-g}, x) \leq \frac{n}{2}$. We have $\gamma_D(v_f, x) \leq |W_D(v_f, x)| \leq \frac{n}{2}$.

Therefore, we have $\gamma_D(v_f) = \gamma_D(v_{n+1-f}) \leq \max\{\frac{n}{2}, n - d + 1\}$.

(1) If $n \leq 2d - 2$, then $n - d + 1 \leq \frac{n}{2}$. Then, we have $exp_D(1) = exp_D(2) \leq \gamma_D(v_f) \leq \frac{n}{2}$. Therefore, according to Proposition 1 and Lemma 1, $exp_D(k) \leq \frac{n}{2} - 1 + \left\lceil \frac{k}{2} \right\rceil$, where $1 \leq k \leq n$.

(2) If $n \geq 2d$, then $n - d + 1 \geq \frac{n}{2} + 1$. We have $exp_D(1) = exp_D(2) \leq \gamma_D(v_f) \leq n - d + 1$. Next, we construct a set $V(L_1)$ such that $|V(L_1)| \geq n - 2d + 4$, and for any vertex $v_i \in V(L_1)$, $\gamma_D(v_i) \leq n - d + 1$ holds. Let $V(L_{11}) = \{v : d_{D_1}(v, v_f) \leq \frac{n-2d+2}{2}$, where $v \in V_1\}$. Let $V(L_{12}) = \{v : d_{D_2}(v, v_{n+1-f}) \leq \frac{n-2d+2}{2}$, where $v \in V_2\}$. Suppose $V(L_1) = V(L_{11}) \cup V(L_{12})$. Then, $v_f \in V(L_1)$ and $v_{n+1-f} \in V(L_1)$. Suppose $v_h \in V(L_{11})$ and $v_h \neq v_f$. If the vertex sequence of the unique path in D_1 from v_h to v_f is $v_h, \cdots, v_t, \cdots, v_f$, then the vertex sequence of the unique path in D_2 from v_{n+1-h} to v_{n+1-f} is $v_{n+1-h}, \cdots, v_{n+1-t}, \cdots, v_{n+1-f}$. So, $d_{D_1}(v_h, v_f) = d_{D_2}(v_{n+1-h}, v_{n+1-f}) \leq \frac{n-2d+2}{2}$. Therefore, we have $v_{n+1-h} \in V(L_1)$. Next, we prove that $|V(L_1)| \geq n - 2d + 4$. For any vertex $v_s \in V(D)$, if $v_s \in V(L_1)$, then $|V(L_1)| = n$. If there is a vertex v_l satisfying $v_l \in V(D) \setminus V(L_1)$, we might as well assume $v_l \in V_1$. Then $d_{D_1}(v_l, v_f) \geq \frac{n-2d+2}{2} + 1$. So, $V(P_{D_1}(v_l, v_f)) \cap V(L_{11}) = \frac{n-2d+2}{2} + 1$. Moreover, $V(P_{D_2}(v_{n+1-l}, v_{n+1-f})) \cap V(L_{12}) = \frac{n-2d+2}{2} + 1$. Then, $\big|(V(P_{D_1}(v_l, v_f)) \cup V(P_{D_2}(v_{n+1-l}, v_{n+1-f}))) \cap V(L_1)\big| = n - 2d + 4$. Therefore, we have $|V(L_1)| \geq n - 2d + 4$. Let us assume $v_i \in V(L_{11})$. Next, we consider $\gamma_D(v_i)$.

Case 1.2.1 $x \in V_1$.

Let $W_{D_1}(v_i, x)$ be the walk from v_i to x, which is $v_i \xrightarrow{d_{D_1}(v_i, v_f)} v_f \xrightarrow{d_{D_1}(v_f, x)} x$. If the walk $W_{D_1}(v_i, x)$ passes through a loop vertex, we have $|W_{D_1}(v_i, x)| = d_{D_1}(v_i, v_f) + d_{D_1}(v_f, x) \leq \frac{n-2d+2}{2} + \frac{n}{2} = n - d + 1$. If the walk $W_{D_1}(v_i, x)$ does not pass through a loop vertex, we have $x \in V(H)$ and $v_i \in V(H)$. According to Corollary 2, then $\gamma_D(v_i, x) \leq n - d + 1$.

Case 1.2.2 $x \in V_2$.

Let the walk $W_D(v_i, x)$ be $v_i \xrightarrow{d_{D_1}(v_i, v_f)} v_f \xrightarrow{d_D(v_f, v_{n+1-g})} v_{n+1-g} \xrightarrow{d_{D_2}(v_{n+1-g}, x)} x$. If the walk $W_D(v_i, x)$ passes through a loop vertex, we have $|W_D(v_i, x)| = d_{D_1}(v_i, v_f) + 1 + d_{D_2}(v_{n+1-g}, x) \leq \frac{n-2d+2}{2} + \frac{n}{2} = n - d + 1$. If the walk $W_D(v_i, x)$ does not pass through a loop vertex. Then, $v_i \in V(H)$. Since $V(R) \cap V(L(D)) = \emptyset$, then $V(P_{D_2}(v_{n+1-f}, v_{n+1-g})) \cap V(L(D)) = \emptyset$. Moreover, $V(P_{D_2}(v_{n+1-g}, x)) \cap V(L(D)) = \emptyset$, we have $V(P_{D_2}(v_{n+1-f}, x)) \cap V(L(D)) = \emptyset$. So, we have $x \in V(H)$. Since $x \in V(H)$ and $v_i \in V(H)$, according to Corollary 2, then $\gamma_D(v_i, x) \leq n - d + 1$.

Therefore, whether $x \in V_1$ or $x \in V_2$, we have $\gamma_D(v_i, x) \leq n - d + 1$. So $\gamma_D(v_i) \leq n - d + 1$. According to Proposition 1, we have $\gamma_D(v_{n+1-i}) = \gamma_D(v_i) \leq n - d + 1$. Therefore, we have $exp_D(k) \leq n - d + 1$, where $1 \leq k \leq |V(L_1)|$. For any vertex v_j such that $v_j \in V_1 \setminus V(L_{11})$, then $v_{n+1-j} \in V_2 \setminus V(L_{12})$. We have $d_{D_1}(v_j, V(L_{11})) = d_{D_2}(v_{n+1-j}, V(L_{12}))$. Furthermore, according to Lemma 2, we can conclude that $\gamma_D(v_j) = \gamma_D(v_{n+1-j}) \leq n - d + 1 + d_{D_1}(v_j, V(L_{11}))$. Since $|V(L_1)| \geq n - 2d + 4$, the conclusion is clearly established.

Case 2 $D \in DS'_{n,2}(d)$.

For $D \in DS'_{n,2}(d)$, it is equivalent to $f = g$ in $D \in DS'_{n,1}(d)$, let us omit it. $\square$

3. The kth Local Exponents of H^* and H^{**}

In this section, we study the kth local exponents of the special graphs H^* and H^{**}.

Definition 4. *Suppose* $1 \leq d \leq n$. *Let* $V(H^*) = \{v_1, v_2, \cdots, v_n\}$ *and* $E(H^*) = \{[v_i, v_{i+1}] | 1 \leq i \leq n-1\} \cup E(L(H^*))$, *where* $E(L(H^*))$ *are* d *loops arranged arbitrarily such that* $H^* \in DS'_n(d)$.

Definition 5. *Suppose* d *is even and* $d \geq 2$. *Let* $V(H^{**}) = \{v_1, v_2, \cdots, v_n\}$ *and let* $E(H^{**}) = \{[v_i, v_{i+1}] | 1 \leq i \leq n-1\} \cup \{[v_i, v_i] :$ *where* $1 \leq i \leq \frac{d}{2}$ *and* $n - \frac{d}{2} + 1 \leq i \leq n\}$.

From the definition of H^* and H^{**}, we know that $H^* \in DS'_n(d)$ and $H^{**} \in DS'_n(d)$. Furthermore, there is a unique path for any two different vertices in H^* and H^{**}, respectively.

Remark 4. *In Figure 2, n can be either odd or even. In Figure 3, since n is odd and $n \geq 2d-1$, then $v_{\frac{n+1}{2}}$ is not a loop vertex. In Figure 4, since n is even and $n \geq 2d$, then $v_{\frac{n}{2}}$ and $v_{\frac{n}{2}+1}$ are not loop vertices.*

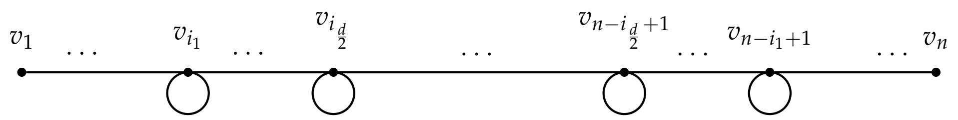

Figure 2. The H^* for d is even and $d \geq 2$.

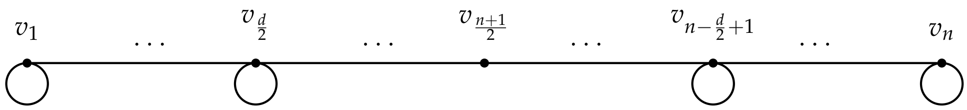

Figure 3. The H^{**} for n is odd, d is even such that $d \geq 2$, and $n \geq 2d-1$.

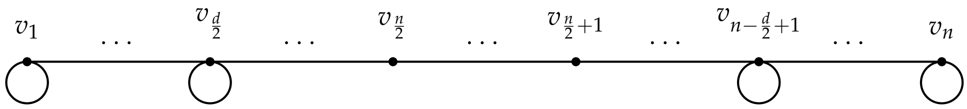

Figure 4. The H^{**} for n is even, d is even such that $d \geq 2$, and $n \geq 2d$.

Theorem 4. *If n is odd and d is odd, then $exp_{H^*}(k) = \frac{n-1}{2} + \left\lfloor \frac{k}{2} \right\rfloor$, where $1 \leq k \leq n$.*

Proof. Since d is odd, then $v_{\frac{n+1}{2}}$ is a loop vertex. We have $\gamma_{H^*}(v_{\frac{n+1}{2}}) = \max\{d(v_{\frac{n+1}{2}}, x) : x \in V(H^*)\} = d(v_{\frac{n+1}{2}}, v_n) = \frac{n-1}{2}$. Moreover, according to Lemma 2, we have $\gamma_{H^*}(v_{\frac{n+1}{2}-a}) \leq \gamma_{H^*}(v_{\frac{n+1}{2}}) + d(v_{\frac{n+1}{2}-a}, v_{\frac{n+1}{2}}) = \frac{n-1}{2} + a$, where $1 \leq a \leq \frac{n-1}{2}$. Further, $\gamma_{H^*}(v_{\frac{n+1}{2}-a}) = \gamma_{H^*}(v_{\frac{n+1}{2}+a}) = \gamma_{H^*}(v_{\frac{n+1}{2}-a}, v_n) = d(v_{\frac{n+1}{2}-a}, v_n) = \frac{n-1}{2} + a$, where $1 \leq a \leq \frac{n-1}{2}$. Therefore, the theorem now holds. □

Theorem 5. *If n is odd, d is even and $d \geq 2$, then*

(1) If $n \leq 2d-3$, then $exp_{H^*}(k) = \frac{n-1}{2} + \left\lfloor \frac{k}{2} \right\rfloor$, where $1 \leq k \leq n$.

(2) $n \geq 2d-1$, then

$$exp_{H^{**}}(k) = \begin{cases} n-d+1, & \text{where } 1 \leq k \leq n-2d+4, \\ n-d+1+\left\lceil \frac{k-(n-2d+4)}{2} \right\rceil, & \text{where } n-2d+4 \leq k \leq n. \end{cases}$$

Proof. Since d is even, then $v_{\frac{n+1}{2}}$ is not a loop vertex.

(1) H^* is shown in Figure 2. Let x be any vertex of H^*. If $V(P(v_{\frac{n+1}{2}}, x)) \cap V(L(H^*)) \neq \emptyset$, then $\gamma_{H^*}(v_{\frac{n+1}{2}}, x) = d(v_{\frac{n+1}{2}}, x) \leq \frac{n-1}{2}$. If $V(P(v_{\frac{n+1}{2}}, x)) \cap V(L(H^*)) = \emptyset$, according to Corollary 1, we have $\gamma_{H^*}(v_{\frac{n+1}{2}}, x) \leq n-d+1$. Then, $\gamma_{H^*}(v_{\frac{n+1}{2}}) \leq \max\{\frac{n-1}{2}, n-d+1\}$. If $n \leq 2d-3$, then $n-d+1 \leq \frac{n-1}{2}$. Then, we have $\gamma_{H^*}(v_{\frac{n+1}{2}}) \leq \frac{n-1}{2}$. Further, $\gamma_{H^*}(v_{\frac{n+1}{2}}, v_n) = d(v_{\frac{n+1}{2}}, v_n) = \frac{n-1}{2}$. Therefore, $\gamma_{H^*}(v_{\frac{n+1}{2}}) = \frac{n-1}{2}$. Let a be an integer satisfying $1 \leq a \leq \frac{n-1}{2}$. According to Lemma 2, we have $\gamma_{H^*}(v_{\frac{n+1}{2}-a}) \leq d(v_{\frac{n+1}{2}-a}, v_{\frac{n+1}{2}}) + \gamma_{H^*}(v_{\frac{n+1}{2}}) = a + \frac{n-1}{2}$. Further, we have $\gamma_{H^*}(v_{\frac{n+1}{2}-a}, v_n) = d(v_{\frac{n+1}{2}-a}, v_n) = \frac{n-1}{2} + a$. Then, $\gamma_{H^*}(v_{\frac{n+1}{2}-a}) = \gamma_{H^*}(v_{\frac{n+1}{2}+a}) = \frac{n-1}{2} + a$ by Proposition 1, where $1 \leq a \leq \frac{n-1}{2}$. Therefore, we have $exp_{H^*}(k) = \frac{n-1}{2} + \left\lfloor \frac{k}{2} \right\rfloor$, where $1 \leq k \leq n$.

(2) H^{**} is shown in Figure 3. If $n \geq 2d-1$, then $n-d+1 \geq \frac{n+1}{2}$. Since $n \geq 5$ and $n \geq 2d-1$, then $\frac{d}{2} \leq \frac{n+1}{4} \leq \frac{n-1}{2}$ and $n - \frac{d}{2} + 1 \geq n - \frac{n+1}{4} + 1 \geq \frac{n+3}{2}$. Then, $v_{\frac{n+1}{2}}$ is not a loop vertex.

Suppose any integer a satisfies $d-1 \le a \le \frac{n+1}{2}$. Since $d \ge 2$, then $a \ge d-1 \ge \frac{d}{2}$. Let x be any vertex of H^{**}. Let the walk $W(v_a, x)$ be $v_a \xrightarrow{d(v_a, v_{\frac{n+1}{2}})} v_{\frac{n+1}{2}} \xrightarrow{d(v_{\frac{n+1}{2}}, x)} x$. If $V(W(v_a,x)) \cap V(L(H^{**})) \neq \emptyset$, then $\gamma_{H^{**}}(v_a, x) \le (\frac{n+1}{2} - a) + \frac{n-1}{2} = n-a$. If $V(W(v_a,x)) \cap V(L(H^{**})) = \emptyset$, we have $V(P(x, v_{\frac{n+1}{2}})) \cap V(L(D)) = \emptyset$ and $V(P(v_a, v_{\frac{n+1}{2}}))$ $\cap V(L(D)) = \emptyset$. According to Corollary 1, we have $\gamma_{H^{**}}(v_a, x) \le n-d+1$. Therefore, we have $\gamma_{H^{**}}(v_a) \le \max\{n-a, n-d+1\} = n-d+1$, where $d-1 \le a \le \frac{n+1}{2}$. Since $d \ge 2$ and $d-1 \le a \le \frac{n+1}{2}$, then $\frac{n+1}{2} \le n-a+1 \le n-d+2 \le n-\frac{d}{2}+1$. Further, $\gamma_{H^{**}}(v_a, v_{n-a+1}) = d(v_a, v_{\frac{d}{2}}) + d(v_{\frac{d}{2}}, v_{n-a+1}) = d(v_a, v_{n-\frac{d}{2}+1}) + d(v_{n-\frac{d}{2}+1}, v_{n-a+1}) = n-d+1$. Therefore, $\gamma_{H^{**}}(v_a) = n-d+1$, where $d-1 \le a \le \frac{n+1}{2}$. According to Proposition 1, if $\frac{n+1}{2} \le a \le n-d+2$, we have $\gamma_{H^{**}}(v_a) = \gamma_{H^{**}}(v_{n-a+1}) = n-d+1$. So, for $d-1 \le a \le n-d+2$, we have $\gamma_{H^{**}}(v_a) = n-d+1$. Therefore, we have $exp_{H^{**}}(k) = n-d+1$, where $1 \le k \le n-2d+4$.

Suppose a satisfies $1 \le a \le d-1$. We have $\gamma_{H^{**}}(v_a) \le d(v_a, v_{d-1}) + \gamma_{H^{**}}(v_{d-1}) = n-d+1+(d-1-a) = n-a$. Moreover, $\gamma_{H^{**}}(v_a) = \gamma_{H^{**}}(v_{n-a+1}) = d(v_a, v_n) = n-a$. So, we can get $exp_{H^{**}}(k) = n-d+1+\left\lceil \frac{k-(n-2d+4)}{2} \right\rceil$, where $n-2d+4 \le k \le n$.

Therefore, the theorem now holds. □

Theorem 6. *If n is even, d is even and $d \ge 2$, then*

(1) *If $n \le 2d-2$, then $exp_{H^*}(k) = \frac{n}{2} - 1 + \left\lceil \frac{k}{2} \right\rceil$, where $1 \le k \le n$.*

(2) *$n \ge 2d$, then*

$$exp_{H^{**}}(k) = \begin{cases} n-d+1, & \text{where } 1 \le k \le n-2d+4, \\ n-d+1+\left\lceil \frac{k-(n-2d+4)}{2} \right\rceil, & \text{where } n-2d+4 \le k \le n. \end{cases}$$

Proof. (1) H^* is shown in Figure 2. We have $H^* \in DS'_{n,2}(d)$, $f = g = \frac{n}{2}$ or $f = g = \frac{n}{2}+1$. Let us assume $f = g = \frac{n}{2}$, then $n+1-f = n+1-g = \frac{n}{2}+1$. If $n \le 2d-2$, let us consider the following two cases.

Case 1 $\{v_{\frac{n}{2}}, v_{\frac{n}{2}+1}\} \cap V(L(H^*)) \neq \emptyset$.

It is easy to see that $\gamma_{H^*}(v_{\frac{n}{2}}) = \gamma_{H^*}(v_{\frac{n}{2}+1}) = \max\{d(v_{\frac{n}{2}}, x) : x \in V(H^*)\} = d(v_{\frac{n}{2}}, v_n) = \frac{n}{2}$. Suppose a satisfies $1 \le a \le \frac{n}{2}-1$. According to Lemma 2, we can get $\gamma_{H^*}(v_{\frac{n}{2}-a}) \le d(v_{\frac{n}{2}-a}, v_{\frac{n}{2}}) + \gamma_{H^*}(v_{\frac{n}{2}}) = \frac{n}{2}+a$. Further, $\gamma_{H^*}(v_{\frac{n}{2}-a}) = \gamma_{H^*}(v_{\frac{n}{2}-a}, v_n) = d(v_{\frac{n}{2}-a}, v_n) = \frac{n}{2}+a$. According to Proposition 1, we have $\gamma_{H^*}(v_{\frac{n}{2}-a}) = \gamma_{H^*}(v_{\frac{n}{2}+a+1}) = \frac{n}{2}+a$, where $1 \le a \le \frac{n}{2}-1$. Therefore, we have $exp_{H^*}(k) = \frac{n}{2}-1+\left\lceil \frac{k}{2} \right\rceil$, where $1 \le k \le n$.

Case 2 $\{v_{\frac{n}{2}}, v_{\frac{n}{2}+1}\} \cap V(L(H^*)) = \emptyset$.

Let x be any vertex of H^*. If $V(P(v_{\frac{n}{2}}, x)) \cap V(L(H^*)) \neq \emptyset$, then $\gamma_{H^*}(v_{\frac{n}{2}}, x) = d(v_{\frac{n}{2}}, x) \le \frac{n}{2}$. If $V(P(v_{\frac{n}{2}}, x)) \cap V(L(H^*)) = \emptyset$, according to Corollary 2, we have $\gamma_{H^*}(v_{\frac{n}{2}}, x)$ $\le n-d+1$. Then, $\gamma_{H^*}(v_{\frac{n}{2}}) \le \max\{\frac{n}{2}, n-d+1\} = \frac{n}{2}$. Further, we have $\gamma_{H^*}(v_{\frac{n}{2}}) = \gamma_{H^*}(v_{\frac{n}{2}}, v_n) = d(v_{\frac{n}{2}}, v_n) = \frac{n}{2}$. Let a satisfy that $1 \le a \le \frac{n}{2}-1$. Then, $\gamma_{H^*}(v_{\frac{n}{2}-a}) \le d(v_{\frac{n}{2}-a}, v_{\frac{n}{2}}) + \gamma_{H^*}(v_{\frac{n}{2}}) = \frac{n}{2}+a$. Further, we have $\gamma_{H^*}(v_{\frac{n}{2}-a}) = \gamma_{H^*}(v_{\frac{n}{2}-a}, v_n) = d(v_{\frac{n}{2}-a}, v_n)$ $= \frac{n}{2}+a$. According to Proposition 1, $\gamma_{H^*}(v_{\frac{n}{2}+1+a}) = \gamma_{H^*}(v_{\frac{n}{2}-a}) = \frac{n}{2}+a$, where $0 \le a \le \frac{n}{2}-1$. Therefore, we have $exp_{H^*}(k) = \frac{n}{2}-1+\left\lceil \frac{k}{2} \right\rceil$, where $1 \le k \le n$.

(2) H^{**} is shown in Figure 4. We have $H^{**} \in DS'_{n,2}(d)$, $f = g = \frac{n}{2}$ or $f = g = \frac{n}{2}+1$. Let us assume $f = g = \frac{n}{2}$, then $n+1-f = n+1-g = \frac{n}{2}+1$. If $n \ge 2d$, then $n-d+1 \ge \frac{n}{2}+1$. Since $n \ge 6$ and $n \ge 2d$, then $\frac{d}{2} \le \frac{n}{4} \le \frac{n}{2}-1$ and $n-\frac{d}{2}+1 \ge n-\frac{n}{4}+1 \ge \frac{n}{2}+2$. Then $v_{\frac{n}{2}}$ and $v_{\frac{n}{2}+1}$ are not loop vertices.

Suppose any integer a satisfies $d-1 \le a \le \frac{n}{2}$. Since $d \ge 2$, then $a \ge d-1 \ge \frac{d}{2}$. Let x be any vertex of H^{**}. Let the walk $W(v_a, x)$ be $v_a \xrightarrow{d(v_a, v_{\frac{n}{2}})} v_{\frac{n}{2}} \xrightarrow{d(v_{\frac{n}{2}}, x)} x$. If $V(W(v_a,x)) \cap V(L(H^{**})) \neq \emptyset$, then $\gamma_{H^{**}}(v_a, x) \le (\frac{n}{2}-a) + \frac{n}{2} = n-a$. If $V(W(v_a,x)) \cap V(L(H^{**})) = \emptyset$, we have $V(P(v_a, v_{\frac{n}{2}})) \cap V(L(D)) = \emptyset$ and $V(P(x, v_{\frac{n}{2}})) \cap V(L(D)) = \emptyset$. According to Corollary 2, we have $\gamma_{H^{**}}(v_a, x) \leq n - d + 1$. So $\gamma_{H^{**}}(v_a) \leq \max\{n - a, n - d + 1\} = n - d + 1$, where $d - 1 \leq a \leq \frac{n}{2}$. Since $d \geq 2$ and $d - 1 \leq a \leq \frac{n}{2}$, then $\frac{n}{2} + 1 \leq n - a + 1 \leq n - d + 2 \leq n - \frac{d}{2} + 1$. Further, we have $\gamma_{H^{**}}(v_a) = \gamma_{H^{**}}(v_a, v_{n-a+1}) = d(v_a, v_{\frac{d}{2}}) + d(v_{\frac{d}{2}}, v_{n-a+1}) = d(v_a, v_{n-\frac{d}{2}+1}) + d(v_{n-\frac{d}{2}+1}, v_{n-a+1}) = n - d + 1$, where $d - 1 \leq a \leq \frac{n}{2}$. According to Proposition 1, if $\frac{n}{2} + 1 \leq a \leq n - d + 2$, we have $\gamma_{H^{**}}(v_a) = \gamma_{H^{**}}(v_{n-a+1}) = n - d + 1$. So, for $d - 1 \leq a \leq n - d + 2$, we have $\gamma_{H^{**}}(v_a) = n - d + 1$. Therefore, we have $exp_{H^{**}}(k) = n - d + 1$, where $1 \leq k \leq n - 2d + 4$.

Suppose a satisfies $1 \leq a \leq d - 1$. We have $\gamma_{H^{**}}(v_a) \leq d(v_a, v_{d-1}) + \gamma_{H^{**}}(v_{d-1}) = n - d + 1 + (d - 1 - a) = n - a$. Moreover, $\gamma_{H^{**}}(v_a) = \gamma_{H^{**}}(v_{n-a+1}) = d(v_a, v_n) = n - a$. So, we can get $exp_{H^{**}}(k) = n - d + 1 + \left\lceil \frac{k-(n-2d+4)}{2} \right\rceil$, where $n - 2d + 4 \leq k \leq n$.

Therefore, the theorem now holds. □

It can be seen from Theorems 4–6, the upper bound for the kth local exponent of doubly symmetric primitive digraphs of order n with d loops can be reached, where $1 \leq k \leq n$.

4. Conclusions

In this paper, we study the upper bound for the kth local exponent of doubly symmetric primitive digraphs of order n with d loops, where d is an integer such that $1 \leq d \leq n$. For the class of doubly symmetric primitive digraphs, we get the upper bound for the kth local exponent, where $1 \leq k \leq n$. Furthermore, for the class of doubly symmetric primitive digraphs, we find that the upper bound for the kth local exponent can be reached, where $1 \leq k \leq n$. The upper bounds of the generalized μ-scrambling indices for a doubly symmetric primitive digraph are not given. It would be meaningful and interesting to solve the problems in future research.

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