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Analysis of Solutions to a Free Boundary Problem with a Nonlinear Gradient Absorption

Haihua Lu, Shu Xie and Yujuan Chen *

School of Sciences, Nantong University, Nantong 226019, China
* Correspondence: nttccyj@ntu.edu.cn

Abstract: In this paper, we investigate the blow-up rate and global existence of solutions to a parabolic system with absorption and the free boundary. By using the comparison principle and super-sub solution method, we obtain some sufficient conditions on the global existence, blow-up in finite time of solutions, and blow-up sets when blow-up phenomenon occurs. Furthermore, the global solution is bounded and uniformly tends to zero, and it is either a global fast solution or a global slow solution. Finally, we obtain a trichotomy conclusion by considering the size of parameter σ.

Keywords: global existence; blow-up; blow-up set; absorption

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1. Introduction

Recently, a lot of works have been devoted to studying the solutions of parabolic equations with free boundary conditions. Ghidouche, Souplet, and Tarzia [1] considered the Stefan problem

$$\begin{cases} u_t - u_{xx} = u^p, & t > 0,\ 0 < x < h(t), \\ u(t, h(t)) = u_x(t, 0) = 0, & t > 0, \\ h'(t) = -\mu u_x(t, h(t)), & t > 0, \\ u(0, x) = u_0(x) \geq 0, & 0 \leq x \leq h_0. \end{cases} \tag{1}$$

They gave an energy condition with an initial value. Under this energy condition, the solution blows up in the L^∞ norm sense. At the same time, they obtained that all global solutions are bounded and uniformly tend to zero, and there are only two possibilities:

(i) Global fast solutions: $h_\infty < \infty$ and there exist constants C and α, such that $\|u\|_\infty \leq Ce^{-\alpha t}, t \geq 0$;

(ii) Global slow solutions: $h_\infty = \infty$ and $\lim_{t\to\infty} \|u\|_\infty = 0, \liminf_{t\to\infty} t^{4/(3(p-1))}\|u\|_\infty > 0$.

Fila and Souplet [2] also studied such Equation (1), and they proved the decay of the slow solution and the boundlessness of the free boundary. In other words, they proved the above conclusion of (ii).

In [3], Sun studied the blow-up solution of the following reaction-diffusion equation and the asymptotic behavior of the global solution

$$\begin{cases} u_t = u_{xx} + au + u^p, & t > 0,\ g(t) < x < h(t), \\ u(t, g(t)) = 0, g'(t) = -\mu u(t, g(t)), & t > 0, \\ u(t, h(t)) = 0, h'(t) = -\mu u(t, h(t)), & t > 0, \\ -g(0) = h(0) = h_0,\ \ u(0, x) = u_0(x) \geq 0, & -h_0 \leq x \leq h_0, \end{cases} \tag{2}$$

where $a \in R$ and $p > 1$. He gave the critical value $\sigma^* \geq 0$, and the blow-up set of the solution is compact when the blow up occurs.

Recently, Dancer, Wang, and Zhang considered the problems derived from the Bose–Einstein condensation model and the famous Gross–Pitaevskii equation. In [4], they discussed the case when $\kappa \to \infty$,

$$\frac{\partial u_i}{\partial t} - d_i \triangle u_i = f_i(u_i) - \kappa u_i \Sigma_{j \neq i} b_{ij} u_j^2, \quad B_1(0) \times (-1,0), \quad i,j = 1,2,\cdots,M. \tag{3}$$

where f_i satisfies $f_i(u) = a_i u - u^p$, $a_i > 0, p > 1$. When isolated populations occurred, they obtained the limit form of the Gross–Pitaevskii equation, and the limit is the solution of the following equation

$$\frac{\partial u_i}{\partial t} - d_i \triangle u_i = f_i(u_i) - \kappa u_i \Sigma_{j \neq i} b_{ij} u_j, \quad B_1(0) \times (-1,0), \quad i,j = 1,2,\cdots,M. \tag{4}$$

The model is derived from the competition model of population dynamics.

Zhou, Bao, and Lin [5] considered the heat equation model with localized source term and double free boundaries

$$\begin{cases} u_t - du_{xx} = u^p(t,0), & 0 < t < T,\ g(t) < x < h(t), \\ u(t,g(t)) = 0,\ g'(t) = -\mu u_x(t,g(t)), & t > 0, \\ u(t,h(t)) = 0,\ h'(t) = -\mu u_x(t,h(t)), & t > 0, \\ g(0) = -h_0,\ h(0) = h_0,\ u(0,x) = u_0(x), & -h_0 \le x \le h_0, \end{cases} \tag{5}$$

where $p > 1$, $d > 0$ and $\mu > 0$. They obtained the conditions for the solution blow up at finite time, and also gave the conditions for the global existence of the Equation (5).

In 2018, Lu and Wei [6] considered the following problems with integral source terms

$$\begin{cases} u_t - du_{xx} = au^p \int_{g(t)}^{h(t)} u^q(t,x)\mathrm{d}x, & t > 0,\ g(t) < x < h(t), \\ u(t,g(t)) = 0,\ g'(t) = -\mu u_x(t,g(t)), & t > 0, \\ u(t,h(t)) = 0,\ h'(t) = -\mu u_x(t,h(t)), & t > 0, \\ g(0) = -h_0,\ h(0) = h_0,\ u(0,x) = u_0(x), & -h_0 < x < h_0, \end{cases} \tag{6}$$

they obtained the existence and uniqueness of the solution by using the contraction mapping theorem. At the same time, they discussed conditions for the finite time blow up, global fast solutions, and global slow solutions, separately. Finally, a trichotomy conclusion by considering the size of parameter σ is obtained, where σ satisfies $u_0(x) = \sigma\varphi(x)$ and also can be seen in Section 5.

In 2019, Zhang and Zhang [7] discussed a class of free boundary problems with non-linear gradient absorption terms

$$\begin{cases} u_t - u_{xx} = u^p - \lambda|u_x|^q, & t > 0,\ g(t) < x < h(t), \\ u(t,g(t)) = 0,\ g'(t) = -\mu u_x(t,g(t)), & t > 0, \\ u(t,h(t)) = 0,\ h'(t) = -\mu u_x(t,h(t)), & t > 0, \\ g(0) = -h_0,\ h(0) = h_0,\ u(0,x) = u_0(x), & -h_0 < x < h_0. \end{cases} \tag{7}$$

For $p,q > 1$, the finite time blow-up and global solution are given by constructing super-sub solutions. The similar techniques in dealing with other problems, one can see [8–11].

Motivated by such interesting models, in this paper we will investigate following free boundary problem with a non-linear gradient absorption

$$\begin{cases} u_t - u_{xx} = au^p \int_0^{h(t)} u^q(t,x)\mathrm{d}x - \lambda|u_x|^r, & t>0,\ 0<x<h(t), \\ u_x(t,0) = u(t,h(t)) = 0 & t>0, \\ h'(t) = -\mu u_x(t,h(t)), & t>0, \\ h(0)=h_0,\ u(0,x)=u_0(x), & 0<x<h_0, \end{cases} \tag{8}$$

where $x = h(t)$ is the free boundary to be determined, $h_0 > 0$, d, a, λ are positive constants. $p \geq 1, q > 0$, $r > 1$, $au^p(t,x)\displaystyle\int_0^{h(t)} u^q(t,x)\mathrm{d}x$ is the integral source term, $\lambda|u_x|^r$ is the absorption term with gradient, which can mean the density function of species, cells, etc. In [12,13], the free boundary problems involving gradients terms are applied for modeling the protein crystal growth. Moreover, it is shown in [12] that blow-up in a finite time may occur. The initial condition $u_0(x)$ indicates the density of the new or invasive species at the beginning domain is $[0,h_0]$. We assume that the species only invades into the new environment from the right side of the initial domain, and the spreading speed of the free boundary is proportional to the density gradient of the population. This condition is a special case of the famous Stefan condition. For classical one-phase Stefan problems, for example, we refer to the melting of ice in contact with water, the population models of a predator-prey system, the diffusive West Nile virus model. Such a condition is also used by many researchers, for example, Kaneko and Yamada [14], and Wang [15]. For its biological background, please refer to [7,16–24]. We also set the boundary condition at the fixed boundary $x = 0$ as a homogeneous Neumann boundary condition, this means that no population passes through the boundary $x = 0$ and the species lives in a self-contained environment.

In this paper, we always assume the initial condition $u_0(x)$ satisfies

$$\begin{cases} u_0 \in C^2([0,h_0]), \quad u_0(x) > 0,\ \ x \in (0,h_0), \\ u_0'(0) = u_0(h_0) = 0. \end{cases} \tag{9}$$

Definition 1. *We say that (u,h) exists globally on $[0,\mathrm{T}_{\max}) \times [0,h(t))$, means that $\mathrm{T}_{\max} = \infty$ and for any $T_1 < \mathrm{T}_{\max}$, $u(t,x)$ is bounded on the domain $[0,T_1] \times [0,h(T_1)]$.*

We say that (u,h) blow-up at finite time $\mathrm{T}_{\max}$ on $[0,\mathrm{T}_{\max}) \times [0,h(t))$, means that $\mathrm{T}_{\max} < \infty$, and

$$\lim_{t\to \mathrm{T}_{\max}} \|u(t,x)\|_{L^\infty([0,t]\times[0,h(t)])} = \infty. \tag{10}$$

Definition 2. *We say that (u,h) is a global fast solution on $[0,\mathrm{T}_{\max}) \times [0,h(t))$, means that $T_{max} = \infty$, and $\lim\limits_{t\to\infty} h(t) := h_\infty < \infty$.*

Definition 3. *We say that (u,h) is a global slow solution on $[0,\mathrm{T}_{\max}) \times [0,h(t))$, means that $T_{max} = \infty$, and $h_\infty = \infty$.*

The organization of this paper is as follows. To study the long time behavior of solutions to the Equation (8), the (local) existence and uniqueness will be discussed in Section 2. In Section 3, we study conditions of the blow-up solution and blow-up sets when the blow-up phenomenon occurs. In Section 4, the results of global fast solutions and global slow solutions are obtained. Finally, we will consider the parameterized initial functions and obtain a trichotomy conclusion.

2. Existence and Uniqueness

In this section, we firstly prove the following local existence and uniqueness result by contraction mapping theorem, and, then, show the monotonicity of the free boundary fronts by Hopf lemma [25–27].

Theorem 1. *For any given $u_0(x)$ satisfies the condtion (9) , and $\alpha \in (0,1)$, then there exists a $T > 0$ such that Equation (8) admits a unique solution*

$$(u,h) \in C^{\frac{1+\alpha}{2},1+\alpha}(\bar{D}_T) \times C^{1+\frac{\alpha}{2}}([0,T]).$$

Furthermore,

$$\|u\|_{C^{\frac{1+\alpha}{2},1+\alpha}(\bar{D}_T)} + \|h\|_{C^{1+\frac{\alpha}{2}}([0,T])} \le C, \tag{11}$$

where

$$D_T = \{(t,x) \in R^2 : \ t \in (0,T], \ x \in [0,h(t)]\},$$

positive constants C and T only depend on h_0, d, a, λ and $\|u_0\|_{C^2([0,h_0])}$.

Proof. The proof is similar to that of [28–30], so we omit it. □

Let (u,h) be the unique positive solution of Equation (8), and $T_{\max} \in (0,\infty]$ is the maximum existence time. Next, the conclusion of the monotonicity of free boundary $h(t)$ will be given.

Theorem 2. *Assume (u,h) defined on $t \in (0,T_0)$ is the positive solution of Equation (8), where $T_0 \in (0,T_{\max})$, and there exists positive constant M_1, such that $u \le M_1$, then there exists positive constant C independent on T_0 and the following inequality*

$$0 < h'(t) \le C, \quad 0 < t < T_0 \tag{12}$$

holds .

Proof. Using Hopf lemma for the Equation (8), it yields

$$u_x(t,h(t)) < 0, \qquad 0 < t < T_0,$$

and thus $h'(t) > 0$ in $(0,T_0)$.

Define

$$\Omega = \Omega_M = \{(t,x) : 0 < t < T_0, \ h(t) - 1/M < x < h(t)\}$$

Constructing the auxiliary function (see [29,31])

$$w(t,x) = M_1[2M(h(t)-x) - M^2(h(t)-x)^2].$$

We will choose M such that $w(t,x) \ge u(t,x)$ on Ω.

Firstly, by direct operation, for any $(t,x) \in \Omega$,

$$\begin{aligned} &w_t = 2M_1Mh'(t)[1 - M(h(t)-x)] \ge 0, \\ &-w_{xx} = 2M_1M^2, \\ &au^p \int_0^{h(t)} u^q dx - \lambda|u_x|^r \le au^p \int_0^{h(t)} u^q dx \le ah(T)M_1^{p+q}, \end{aligned}$$

if $M^2 \geq \frac{ah(T)}{2d} M_1^{p+q-1}$, then

$$w_t - dw_{xx} \geq 2dM_1 M^2 \geq u_t - du_{xx}, \text{ in } \Omega.$$

On the other side, for any $t \in (0, T_0)$

$$w(t, h(t) - M^{-1}) = M_1 \geq u(t, h(t) - M^{-1}),$$

$$w(t, h(t)) = u(t, h(t)) = 0.$$

To take advantage of the comparison principle on Ω, we just need to find a proper M independent of T_0, and satisfy

$$u_0(x) \leq w(0, x), \quad x \in [h_0 - M^{-1}, h_0]. \tag{13}$$

Thus, for all $(t, x) \in \Omega$, we have $u \leq w$, and

$$u_x(t, h(t)) \geq w_x(t, h(t)) = -2M_1 M, \quad t \in (0, T). \tag{14}$$

Next, we will find a proper M independent on T_0, such that (13) holds.
By direct calculation,

$$w_x(0, x) = -2M_1 M[1 - M(h_0 - x)] \leq -M_1 M, \quad x \in [h_0 - (2M)^{-1}, h_0].$$

Thus, we can choose

$$M = \max\left\{\frac{4\|u_0\|_{C^1([0,h_0])}}{3M_1}, \left(\frac{(ah(T))M_1^{p+q-1}}{2d}\right)^{1/2}\right\},$$

thus

$$w_x(0, x) \leq -\frac{4\|u_0\|_{C^1([0,h_0])}}{3M} \leq u_0'(x), \text{ on } [h_0 - (2K)^{-1}, h_0].$$

Integrating the above inequalities on $[x, h_0]$ and by $w(0, h_0) = 0 = u_0(h_0)$, we obtain

$$w(0, x) \geq u_0(x), \quad x \in [h_0 - (2K)^{-1}, h_0]. \tag{15}$$

Meanwhile, utilizing the concavity of $w(0, x)$ and $w_x(0, h_0 - K^{-1}) = 0$, we can see that when $h_0 - K^{-1} \leq x \leq h_0 - (2K)^{-1}$,

$$w(0, x) \geq w(0, h_0 - (2M)^{-1}) = \frac{3M_1}{4} \geq \|u_0\|_{C^1([0,h_0])} \geq u_0(x).$$

Thanks to (15), it is easy to see Equation (13) holds. □

Similar to the method of Theorem 2.2 in [7], we can also prove the continuous dependence on the initial value of the solution of the following problem, and we will omit the specific proof process.

Theorem 3. *Assume that $(u_i(t, x), h_i(t), T_i)$ are the solutions of the problems*

$$\begin{cases} u_{i,t} - du_{i,xx} = au_i^p \int_0^{h(t)} u_i(t, x)^q \mathrm{d}x - \lambda |u_{i,x}|^r, & 0 < t < T_i, \ 0 < x < h_i(t), \\ u_{i,x}(t, 0) = u_i(t, h_i(t)) = 0, & 0 < t < T_i, \\ h_i'(t) = -\mu u_{i,x}(t, h_i(t)), & 0 < t < T_i, \\ h_i(0) = h_{0i}, \ \ u_i(0, x) = u_{0i}(x), & 0 \leq x \leq h_{0i}, \end{cases}$$

where h_{0i}, $u_{0i}(x)$ $(i = 1, 2)$ are the initial value conditions, T_i are the maximum existence time of related problems. Let $T = \min\{T_1,\ T_2\} \in (0, \infty]$, then for all $0 < T' < T$, there exists a constant $K > 0$, which depends only on $\|u_{0i}\|_{C^2([0,h_0])}$, $\|u_i\|_{D_3}$ and μ, such that

$$\begin{aligned} & \sup{}_{t\in[0,T']}\|u_1(t,\cdot) - u_2(t,\cdot)\|_{C([0,h(t)])} + \|h_1(t) - h_2(t)\|_{C^1([0,T'])} \\ \leq\ & K(\|u_{01} - u_{02}\|_{C^2([0,h_0])} + |h_{01} - h_{02}|), \end{aligned}$$

where $h(t) := \min\{h_1(t), h_2(t)\}$, $h_0 := \min\{h_{01}, h_{02}\}$ and $D_3 = [0, T] \times [0, h(t)]$.

Theorem 4. *Assume that $\mathrm{T}_{\max}$ is the maximum existence time and (u, h) is the unique positive solution of Equation (8), then for all $t \in [0, \mathrm{T}_{\max})$, or $\mathrm{T}_{\max} = \infty$, or $\mathrm{T}_{\max} < \infty$ and (10) holds.*

Proof. By unique existence and Zorn lemma [25], we can see that there exists $\mathrm{T}_{\max}$, such that $[0, \mathrm{T}_{\max})$ is the maximum existence interval of solutions. To complete this proof, we will prove Equation (10) holds when the case $\mathrm{T}_{\max} < \infty$ occurs.

Assume that u is bounded for $\mathrm{T}_{\max} < \infty$ and $x \in [0, h(t)]$, i.e., there exists a positive constant M, such that

$$u(t, x) \leq M, \qquad (t, x) \in [0, \mathrm{T}_{\max}) \times [0, h(t)]. \tag{16}$$

By the method of the proof in Theorem 2, there exists a positive C independent on $\mathrm{T}_{\max}$, such that

$$0 \leq h'(t) \leq C,\ \ h_0 \leq h(t) \leq h_0 + Ct \leq h_0 + C\mathrm{T}_{\max},\ \ \forall \mathrm{t} \in [0, \mathrm{T}_{\max}). \tag{17}$$

We will prove that for any $\tau > 0$, (u, h) can also be extended to $[0, \mathrm{T}_{\max} + \tau/2]$, and then we can obtain a conclusion that is contrary to the definition of $\mathrm{T}_{\max}$. To achieve this aim, we restrict $t \in [\varepsilon, \mathrm{T}_{\max})$, where $0 < \varepsilon < \mathrm{T}_{\max}$.

Similar to the proof of Theorem 1, using standard parabolic theory, we can obtain $w, w_y \in C^{\frac{\alpha}{2},\alpha}(D_1)$, where w satisfies the Equation (8) with the free boundary straightened, and $D_1 := [0, \mathrm{T}_{\max}) \times [0, h_0]$, then there exists a positive constant C_7, such that

$$\|w\|_{C^{\frac{\alpha}{2},\alpha}(D_1)} + \|w_y\|_{C^{\frac{\alpha}{2},\alpha}(D_1)} \leq C_7.$$

and

$$\|u\|_{C^{\frac{\alpha}{2},\alpha}(D_2)} + \|u_x\|_{C^{\frac{\alpha}{2},\alpha}(D_2)} \leq C_8,$$

where $D_2 := \{(t, x) \in R^2 :\ t \in [0, \mathrm{T}_{\max}),\ x \in [0, h(t))\}$, and C_8 depends on C_7. Hence, by the equation of $h'(t)$ in problem (8),

$$\|h'(t)\|_{C^{\frac{\alpha}{2}}([0,\mathrm{T}_{\max}))} \leq C_9,$$

where C_9 depends on C_8, μ, ρ. Fixing $0 < \varepsilon < \mathrm{T}_{\max}$, we can use the Schauder theory [25] to w in the domain $D_3 := [\varepsilon, \mathrm{T}_{\max}] \times [0, h_0]$,

$$\|w\|_{C^{1+\frac{\alpha}{2},2+\alpha}(D_3)} \leq C_9.$$

This means

$$\|u\|_{C^{1+\frac{\alpha}{2},2+\alpha}(D_4)} \leq C_{10},$$

where $D_4 := [\varepsilon, \mathrm{T}_{\max}) \times [0, h(t)]$. In the above conclusions, C_7, C_8, C_9, C_{10} are positive constants, and only depend on $M, C, \|u_0\|_{C^2([0,h_0])}$ and $\mathrm{T}_{\max}$.

Repeating the above process, by the proof of Theorem 1, there exists $\tau > 0$ independent of $\mathrm{T}_{\max}$ satisfies that the solution of Equation (8) with the initial time $\mathrm{T}_{\max} - \tau/2$ can be extended to $\mathrm{T}_{\max} - \tau/2 + \tau$. This contradicts the definition of $\mathrm{T}_{\max}$. $\square$

According to Theorem 4, it is easy to draw the following conclusion.

Corollary 1. *Assume that (u, h), defined on the maximum existence interval $(0, \mathrm{T}_{\max})$ with $\mathrm{T}_{\max} < \infty$, is the solution of Equation (8), then (u, h) blows up.*

Lemma 1. *Let M_1 be a positive constant, and for all solutions of Equation (8) defined on the domain $\Omega = \{(t, x) \in R^2 : 0 \le t < \mathrm{T}_{\max} < \infty, 0 < x < h(t)\}$ satisfy $u(t, x) \le M_1$, then there exists a positive constant K satisfies,*

$$-K \le u_x(t, x) < 0, \qquad 0 < t < \mathrm{T}_{\max}, \ h_0 \le x \le h(t). \tag{18}$$

Proof. Let $v = u_x(t, x)$, and v satisfies

$$\begin{cases} v_t - dv_{xx} = au^{p-1}v \displaystyle\int_0^{h(t)} u^q(t, x)\mathrm{d}x - \lambda r|u_x|^{r-2}vv_x, & 0 < t < T_0, \ 0 < x < h(t), \\ v(t, 0) = 0, \ \ v(t, h(t)) < 0, & 0 < t < T_0, \\ h'(t) = -\mu v(t, h(t)), & 0 < t < T_0, \\ h(0) = h_0, \ v(0, x) = u_0'(x), & 0 < x < h_0, \end{cases}$$

Additionally, let $v = e^{bt}w$ with $b > pM_1^{p-1}$, then $w(t, x) \le 0$ by maximum principle, this means that $v \le 0$ when $0 < x < h(t)$. and the maximum value of $|w|$ can only be obtained on $\Gamma := (\{0\} \times [0, h_0]) \cup ((0, T_0) \times \{h(t)\})$, so is $|v|$.

For any given $l \in [h_0, h^*)$, where $h^* := \lim_{t \to \mathrm{T}_{\max}} h(t)$, we can find a unique $T \ge 0$, such that $h(T) = l$. Additionally, for any $\delta : 0 < \delta \ll \mathrm{T}_{\max} - \mathrm{T}$, we define $\Omega_l := \{(t, x) : T < t < \mathrm{T}_{\max} - \delta, \ l < x < h(t)\}$, and

$$w(t, x) = u(t, x) - u(t, 2l - x), \quad (t, x) \in \Omega_l.$$

Then,

$$\begin{cases} w_t = dw_{xx} + c_1(t, x)w, & (t, x) \in \Omega_l, \\ w(t, h(t)) < 0, & T < t < \mathrm{T}_{\max} - \delta, \\ w(t, l) = 0, & T < t < \mathrm{T}_{\max} - \delta, \end{cases}$$

with $c_1(t, x)$ is a bounded function. Therefore, by using the strong maximum principle and the arbitrariness of δ,

$$w(t, x) < 0, \ \ \{(t, x) : T < t < \mathrm{T}_{\max}, \ l < x < h(t)\}$$

and by Hopf lemma, we can obtain

$$w_x(t, l) < 0, \qquad T < t < \mathrm{T}_{\max}.$$

Since

$$w_x(t, l) = 2u_x(t, l),$$

thus

$$u_x(t, h(T)) < 0, \qquad T < t < \mathrm{T}_{\max}.$$

Then, for any $(t, x) \in \{(t, x) : 0 < t < \mathrm{T}_{\max}, \ h_0 \le x < h(t)\}$, we can find a unique $T \in (0, t)$, such that $x = h(T)$. Thus, $u_x(t, x) < 0$. This inequality holds for the case $x = h(t)$, since this conclusion is directly obtained by using Hopf lemma for the Equation (8). □

Remark 1. *From the discussion of the global existence of the following solutions in this paper, if* $\mathrm{T}_{\max} = \infty$, *then the global solution* u *is bounded, thus* c_1 *in Lemma 1 is still a bounded function. Therefore, when* $\{(t,x) : t > 0,\ h_0 \le x \le h(t)\}$, *then* $w(t,x) < 0$. *We also can prove Equation (18) holds. Thus, the assumption* $\mathrm{T}_{\max} < \infty$ *in Lemma 1 can be deleted.*

Next, we will give the comparison principle that plays an important role in studying the positive solution of the Equation (8).

Lemma 2. *(Comparison principle). Let* $T \in (0,\infty), \bar{h} \in C^1([0,T]),\ \bar{u},\ \bar{v} \in C(\bar{D}_T^*) \cap C^{1,2}(D_T^*)$ *where*

$$D_T^* = \big\{(t,x) \in R^2 : 0 < t \le T,\ 0 < x < \bar{h}(t)\big\}.$$

If $(\bar{u},\bar{h})$ *satisfies*

$$\begin{cases} \bar{u}_t - d\bar{u}_{xx} = a\bar{u}^p \displaystyle\int_0^{\bar{h}(t)} \bar{u}^q(t,x)\mathrm{d}x - \lambda|\bar{u}_x|^r, & t > 0,\ 0 < x < \bar{h}(t), \\ \bar{u}_x(t,0) = \bar{u}(t,\bar{h}(t)) = 0, & t > 0, \\ \bar{h}'(t) = -\mu\bar{u}_x(t,\bar{h}(t)), & t > 0, \\ \bar{h}(0) = \bar{h}_0,\ \ \bar{u}(0,x) = \bar{u}_0(x), & 0 < x < \bar{h}_0, \end{cases}$$

with

$$\bar{h}(0) \ge h_0, \ \ and\ \ \bar{u}(0,x) \ge u_0(x), \quad x \in [0,h_0],$$

then the solution (u,h) *of Equation (8) satisfies*

$$h(t) \le \bar{h}(t), \quad t \in (0,T]; \qquad u(t,x) \le \bar{u}(t,x), \quad (t,x) \in (0,T] \times (0,h(t)).$$

Proof. Motivated by the Lemma 3.5 in [29] (also see Lemma 2.1 in [32] and Lemma 2.1 in [7]), we give the following proof. For appropriately small $\varepsilon > 0$, let $(u_\varepsilon(t,x), h_\varepsilon(t))$ is the solution of the problem

$$\begin{cases} u_{\varepsilon,t} - du_{\varepsilon,xx} = au_\varepsilon^p \displaystyle\int_0^{h(t)} u_\varepsilon(t,x)^q\mathrm{d}x - \lambda|u_{\varepsilon,x}|^r, & t > 0,\ 0 < x < h_\varepsilon(t), \\ u_{\varepsilon,x}(t,0) = u_\varepsilon(t,h(t)) = 0, & t > 0, \\ h_\varepsilon'(t) = -\mu u_{\varepsilon,x}(t,h_\varepsilon(t)), & t > 0, \\ h_\varepsilon(0) = h_0^\varepsilon,\ \ u_\varepsilon(0,x) = u_0^\varepsilon(x), & 0 < x < h_0^\varepsilon, \end{cases}$$

where $h_0^\varepsilon := h_0(1-\varepsilon)$, u_0^ε is a function defined on $C^2([0,h_0^\varepsilon])$, and satisfies

$$0 < u_0^\varepsilon(x) \le u_0(x), \quad x \in [0,h_0^\varepsilon), \quad u_0^\varepsilon(h_0^\varepsilon) = 0,$$

and when $\varepsilon \to 0$,

$$u_0^\varepsilon\Big(\frac{h_0}{h_0^\varepsilon}x\Big) \to u_0(x)$$

in the sense of $C^2([0,h_0])$ norm. Let $u_\varepsilon, h_\varepsilon$ be the unique solution of Equation (8) with u_0 and h_0 replaced by u_0^ε and h_0^ε, respectively.

We first prove the following assertion:

$$h_\varepsilon(t) < \bar{h}(t), \quad \text{for all } t \in (0,T].$$

Obviously, using the continuity, the above conclusion holds for small $t > 0$. Otherwise, we can find the first $t^* \leq T$, such that

$$h_\varepsilon(t) < \bar{h}(t), \quad t \in (0, t^*), \quad \text{and } \; h_\varepsilon(t^*) = \bar{h}(t^*).$$

Therefore,

$$h'_\varepsilon(t^*) \geq \bar{h}'(t^*). \tag{19}$$

Next, we compare u_ε with $\bar{u}$ on the domain

$$\Omega_{t^*} := \{(t, x) \in R : \; 0 < t \leq t^*, \; 0 \leq x < h_\varepsilon(t)\}.$$

By the maximum principle [25], we can see that $u_\varepsilon(t, x) < \bar{u}(t, x)$ in Ω_{t^*}. Thus, $w(t, x) := \bar{u}(t, x) - u_\varepsilon(t, x) > 0$ in Ω_{t^*} , $w(t^*, h_\varepsilon(t^*)) = 0$, and $w_x(t^*, h_\varepsilon(t^*)) < 0$. Therefore, by $u_{\varepsilon,x}(t^*, h(t^*)) < 0$, we can see $h'_\varepsilon(t^*) < \bar{h}'(t^*)$. This contradicts with inequality (19).

Next, by common comparison principles on domain Ω_T (see [33]), we can see that $u_\varepsilon < \bar{u}$ on Ω_T. Additionally, by the continuous dependence of the unique solution on parameters of the Equation (8), $(u_\varepsilon, h_\varepsilon)$ converges to the unique solution (u, h) of (8) when $\varepsilon \to 0$. Thus, let $\varepsilon \to 0$, then $u_\varepsilon < \bar{u}$, $h_\varepsilon < \bar{h}$. □

Remark 2. *$(\bar{u}, \bar{h})$ in Lemma 2 is called super-solution of Equation (8), meanwhile, we can define the sub-solution of the Equation (8) by changing the above inequalities direction. Furthermore, the existence of the sub-solution can be proved by using similar methods.*

3. Blow-Up Solutions and Blow-Up Sets

In this section, we will give the blow-up results of the solution in the sense of L^∞ norm under the condition of large initial value when $p + q > r$. At the same time, we also obtain the result of blow-up sets.

Theorem 5. *Let (u, h) is the solution of the Equation (8), $p + q > r$, $u_0(x) = \sigma\varphi(x)$, where $\varphi(x)$ satisfies the condition (9). If σ is large enough, then the solution (u, h) blows up in finite time.*

Proof. We will construct a self-similar sub-solution and prove it by using the comparison principle. Let

$$\begin{aligned} &D = [t_0, \frac{1}{\varepsilon}) \times [0, \rho(t)), \\ &V(t, x) = \frac{1}{(1 - \varepsilon t)^k} W(\frac{x}{(1 - \varepsilon t)^m}), \quad (t, x) \in D, \\ &\rho(t) = \delta M (1 - \varepsilon t)^m, \quad t_0 \leq t < \frac{1}{\varepsilon}, \end{aligned}$$

where $W(y) = \delta^2 + \delta^2 A/2 - y^2/(2A), 0 \leq y \leq \delta M$, $M = \sqrt{A(2 + A)}$ and $t_0, \varepsilon, k, m, \delta, A$ are positive constants to be determined.

By simple calculations,

$$\delta^2 \leq W(y) \leq \delta^2(1 + \frac{A}{2}), \quad -\delta \leq W'(y) \leq 0, \quad 0 \leq y \leq \delta A, \tag{20}$$

$$0 \leq W(y) \leq \delta^2, \; -\frac{\delta M}{A} \leq W'(y) \leq -\delta, \quad \delta A \leq y \leq \delta M, \tag{21}$$

$$W''(y) = -\frac{1}{A}, \quad 0 < y < \delta M, \tag{22}$$

and

$$\begin{aligned} LV: \quad &= V_t - dV_{xx} - aV^p \int_0^{h(t)} V(t,x)^q \mathrm{d}x + \lambda |V_x|^r \\ &= \frac{k\varepsilon}{(1-\varepsilon t)^{k+1}} W(y) + \frac{dmy\varepsilon}{(1-\varepsilon t)^{k+1}} W'(y) - \frac{d}{(1-\varepsilon t)^{k+2m}} W''(y) \\ &\quad - \frac{a}{(1-\varepsilon t)^{k(p+q)}} W^p(y) \int_0^{h(t)} W(y)^q \mathrm{d}x + \frac{\lambda}{(1-\varepsilon t)^{(k+m)r}} |W'(y)|^r. \end{aligned} \tag{23}$$

We can choose

$$k = \frac{1}{p+q-1}, \quad 0 < m < \min\left\{\frac{1}{2}, \frac{p+q-r}{r(p+q-1)}\right\},$$

then $k+1 = k(p+q) > k+2m$, and $k+1 > (k+m)r$.

Next, we also can choose

$$A > \frac{k}{m}, \quad \delta \le \frac{h_0}{M}, \quad \varepsilon < \frac{\delta^{2(p+q-1)}}{k(1+\frac{A}{2})}.$$

Case 1. $0 \le y \le \delta A$.

By Equations (20), (22), and (23), and selecting t_0 that is sufficiently close to $\frac{1}{\varepsilon}$, we can see

$$\begin{aligned} LV &\le \frac{\delta^2 k\varepsilon(1+\frac{A}{2}) - \delta^{2p}}{(1-\varepsilon t)^{k+1}} + \frac{\frac{1}{A}}{(1-\varepsilon t)^{k+2m}} + \frac{\lambda\delta^r}{(1-\varepsilon t)^{(k+m)r}} \\ &\le (1-\varepsilon t)^{-(k+1)}\left[\delta^2 k\varepsilon(1+\frac{A}{2}) - \delta^{2(p+q)} + \frac{1}{A}(1-\varepsilon t_0)^{1-2m} + \lambda\delta^r(1-\varepsilon t_0)^{k+1-(k+m)r}\right] \\ &\le 0. \end{aligned}$$

Case 2. $\delta A \le y \le \delta M$.

By Equations (21)–(23), and choosing t_0 that is also sufficiently close to $\frac{1}{\varepsilon}$, we can obtain

$$\begin{aligned} LV &\le \frac{\delta^2 k\varepsilon - m\varepsilon A\delta^2}{(1-\varepsilon t)^{k+1}} + \frac{\frac{1}{A}}{(1-\varepsilon t)^{k+2m}} + \frac{\lambda(\frac{\delta M}{A})^r}{(1-\varepsilon t)^{(k+m)r}} \\ &\le (1-\varepsilon t)^{-(k+1)}\left[\delta^2 k\varepsilon(k-mA) - \delta^{2(p+q)} + \frac{1}{A}(1-\varepsilon t_0)^{1-2m} + \lambda\left(\frac{\delta M}{A}\right)^r (1-\varepsilon t_0)^{k+1-(k+m)r}\right] \\ &\le 0. \end{aligned}$$

Additionally, we find that

$$\begin{aligned} V_x(t,0) = V(t,\rho(t)) = 0, &\quad t_0 \le t \le \tfrac{1}{\varepsilon}, \\ \rho'(t) < \mu V_x(t,\rho(t)), &\quad t_0 \le t \le \tfrac{1}{\varepsilon}, \end{aligned}$$

and by the selection of δ we can know that $\rho(0) \le h_0$.

On the other hand, for any t_0 that is sufficiently close to $1/\varepsilon$ and sufficiently large σ, $u_0(x) \ge V(t_0,x)$.

Therefore, by comparison principle,

$$u(t-t_0,x) \ge V(t,x), \quad (t,x) \in D.$$

Indeed, $V(t_0,x) \to \infty$ when $t_0 \to 1/\varepsilon$, then we can obtain $\mathrm{T}_{\max}(u_0) \le 1/\varepsilon - t_0 < \infty$, and

$$\lim_{t\to \mathrm{T}_{\max}} \|u(t,x)\|_{L^\infty([0,h(t)])} = \infty.$$

□

Remark 3. *Assume $V(t,x)$ is the solution of corresponding fixed boundary Equation (8), then it is easy to prove that the solution of such problem will blows up in finite time under the condition of large initial value $u_0(x)$, and $V(t,x)$ is a sub-solution of Equation (8). The conclusion of Theorem 5 can also be obtained by using comparison principle.*

Next, we will consider the blow-up sets of Equation (8). Define

$$B(u_0) = \{x \in [0,h^*] : \text{there exist } x_n \to x,\ t_n \to \mathrm{T}_{\max} \text{ such that } u(t_n,x_n) \to \infty\},$$

where u_0 denotes the initial condition, $h^* = \lim_{t\to\mathrm{T}_{\max}} h(t)$. Such h^* is well defined, since for all $0 < t < \mathrm{T}_{\max}$ satisfies $h'(t) > 0$.

Theorem 6. *Assume u_0 satisfies the condition (9) and the solution (u,h) of Equation (8) will blow-up in finite time $\mathrm{T}_{\max} < \infty$, then:*
(i) The blow-up set $B(u_0)$ is a compact set of $[0,h_0]$;
(ii) There exists a constant $C > 0$, such that $h^* = \lim_{t\to\mathrm{T}_{\max}} h(t) \le C$.

Proof. (i) We declare that (h_0,h^*) is not included in $B(u_0)$.

We will prove such a declaration by contradictions. Assume there exists $h_1 \in (h_0,h^*)$ satisfies $h_1 \in B(u_0)$, then by Lemma 1, $u_x < 0$ on the domain $(0,\mathrm{T}_{\max}) \times [h_0,h(t)]$. That is, u is monotonic decreasing in x on the domain $(0,\mathrm{T}_{\max}) \times [h_0,h(t)]$, then $[h_0,h_1] \subseteq B(u_0)$. Since $h'(t) > 0$ when $0 < t < \mathrm{T}_{\max}$, and $h_1 < h^*$, then there exists a unique $\tau \in (0,\mathrm{T}_{\max})$ satisfies $h(\tau) = h_1$.

The auxiliary function is constructed as below, for any $0 < \delta \ll 1$ and $(t,x) \in [\tau,\mathrm{T}_{\max}-\delta) \times [h_0,h_1]$,

$$P(t,x) = u_x(t,x) + q(x)u^2(t,x),$$

where $q(x) := \varepsilon \sin\frac{\pi(x-h_0)}{h_1-h_0}$, $\varepsilon > 0$ is sufficiently small. Obviously, for all $\tau < t < \mathrm{T}_{\max}-\delta$,

$$P(t,h_0) = u_x(t,h_0) < 0, \quad P(t,h_1) = u_x(t,h_1) < 0,$$

and by direct calculations,

$$\begin{aligned} P_t &= u_{xt} + 2quu_t, \\ P_x &= u_{xx} + q'u^2 + 2quu_x, \\ P_{xx} &= u_{xxx} + q''u^2 + 4q'uu_x + 2q(u_x)^2 + 2quu_{xx}. \end{aligned}$$

Then, by the definition of P, we can see

$$\begin{aligned} & P_t - dP_{xx} \\ \le\ & (u_t - du_{xx})_x + 2qu(u_t - du_{xx}) - dq''u^2 - 4dq'uu_x \\ =\ & (apu^{p-1}\int_0^{h(t)} u^q \mathrm{d}x - \lambda r|u_x|^{r-2}u_{xx})u_x + 2qu(au^p\int_0^{h(t)} u^q \mathrm{d}x - \lambda|u_x|^r) - dq''u^2 - 4dq'uu_x \\ =\ & (apu^{p-1}\int_0^{h(t)} u^q \mathrm{d}x - \lambda r|u_x|^{r-2}u_{xx} - 4dq'u)(P - qu^2) \\ & +2qu(au^p\int_0^{h(t)} u^q \mathrm{d}x - \lambda|u_x|^r) - dq''u^2. \end{aligned}$$

Thus,

$$\begin{aligned} & P_t - dP_{xx} - b_1P \\ \le\ & -q(apu^{p+1}\int_0^{h(t)} u^q \mathrm{d}x - \lambda r|u_x|^{r-2}u^2u_{xx} - 4dq'u^3) + 2qu(au^p\int_0^{h(t)} u^q \mathrm{d}x - \lambda|u_x|^r) - dq''u^2 \\ =\ & -q\big(apu^{p+1}\int_0^{h(t)} u^q \mathrm{d}x - \lambda r|u_x|^{r-2}u^2u_{xx} - 4dq'u^3 - 2au^{p+1}\int_0^{h(t)} u^q \mathrm{d}x + 2\lambda u|u_x|^r + \frac{dq''}{q}u^2\big), \end{aligned}$$

where the bounded function $b_1 = apu^{p-1}\int_0^{h(t)} u^q \mathrm{d}x - \lambda r|u_x|^{r-2}u_{xx} - 4dq'u$. Using $[h_0, h_1] \subseteq B(u_0)$ and the boundedness of q' and $\dfrac{q''}{q}$, we can find a $T_1 \in (\tau, \mathrm{T}_{\max} - \delta)$, thus

$$P_t - dP_{xx} - b_1 P \leq 0, \ \text{ in } [T_1, \mathrm{T}_{\max} - \delta) \times [h_0, h_1].$$

At the same time, since $T_1 < \mathrm{T}_{\max} - \delta$, $u_x(T_1, x) < 0$ in $[h_0, h_1]$, and $u(T_1, x) < \infty$ in $[h_0, h_1]$, then we can choose small ε, such that $P(T_1, x) \leq 0$ in $[h_0, h_1]$. Hence, we can apply the comparison principle and arbitrariness of δ to deduce that, for $(t, x) \in [T_1, \mathrm{T}_{\max}) \times [h_0, h_1]$,

$$-u_x \geq \varepsilon u^2 \sin \frac{\pi(x - h_0)}{h_1 - h_0}. \tag{24}$$

For any $h_0 < y \leq h_1$, integrating inequality (24) with respect to x from h_0 to y, we have

$$G(u(t, y)) - G(u(t, h_0)) \geq \varepsilon \int_{h_0}^{y} \sin \frac{\pi(x - h_0)}{h_1 - h_0} \mathrm{d}x, \tag{25}$$

where $G(u) := \displaystyle\int_u^\infty \frac{\mathrm{d}s}{s^2}$. Let $t \uparrow \mathrm{T}_{\max}$, then the left-hand side of inequality (25) tends to 0, since $G(\infty) = 0$. Meanwhile, the right-hand side of inequality (25) is positive. This is a contraction.

Obviously, $[h^*, \infty)$ is not included in $B(u_0)$. Hence, $B(u_0)$ is a compact subset of the initial domain $[0, h_0]$, since $B(u_0)$ is a closed set.

(ii) Let $\tilde{t} = \mathrm{T}_{\max}/4$. Then, $h(\tilde{t}) > h_0$, since $h'(t) > 0$ and $h(0) = h_0$. Therefore, by $B(u_0) \subseteq [0, h_0]$, we have

$$\widetilde{M} := \limsup_{t \to \mathrm{T}_{\max}} u(t, h(\tilde{t})) < \infty.$$

The rest proof is similar to the proof of Theorem 4.1 in [3], we omit it. □

4. Global Fast Solutions and Global Slow Solutions

In this section, we will prove that the global solution (u, h) of Equation (8) is bounded and uniformly tends to 0, and (u, h) is either a global fast solution, or a global slow solution.

Let $\mathrm{T}_{\max}$ is the maximum existence time, $h_\infty := \lim_{t \to \mathrm{T}_{\max}} h(t)$.

Theorem 7. *(Global fast solutions) Assume that $u(t, x)$ is the solution of Equation (8), initial condition $u_0(x)$ satisfies*

$$\|u_0\|_{L^\infty(0, h_0)} \leq \frac{1}{2} \min\{[d/(128h_0^3)]^{\frac{1}{p+q-1}}, d/(8\mu)\},$$

then $\mathrm{T}_{\max} = \infty$. Furthermore, $h_\infty < \infty$, there exist positive constants C_5, β only depend on u_0, such that

$$\|u(t, \cdot)\|_{L^\infty[g(t), h(t)]} \leq C_5 e^{-\beta t}, \ \ t > 0. \tag{26}$$

Proof. Obviously, it is only necessary to construct an appropriate global super-solution. Inspired by [21], we define

$$\vartheta(t) = 2h_0(2 - e^{-\gamma t}), \ \ t \geq 0, \ \ V(y) = 1 - y^2, \ \ 0 \leq y \leq 1,$$

and

$$v(t, x) = \varepsilon e^{-\beta t} V(x/\vartheta), \ \ 0 \leq x \leq \vartheta(t), \ t \geq 0,$$

where γ, β and $\varepsilon > 0$ are constants to be determined.

Directly calculated, for all $t > 0$ and $0 < x < \vartheta(t)$,

$$\begin{aligned} & v_t - dv_{xx} - av^p \int_0^{\vartheta(t)} v(t,x)^q dx + \lambda |v_x|^r \\ = \; & \varepsilon e^{-\beta t} \Big[-\beta V - x\vartheta'\vartheta^{-2}V' - d\vartheta^{-2}V'' - aV^p \int_0^{\vartheta(t)} \varepsilon^{p+q-1} e^{-\beta(p+q-1)t} V^q dx + \frac{\lambda \varepsilon^{r-1} e^{-\beta t(r-1)}}{\theta^r} (2y)^r \Big] \\ \geq \; & \varepsilon e^{-\beta t} \Big[-\beta + \frac{d}{8h_0} - 8h_0 \varepsilon^{p+q-1} \Big]. \end{aligned}$$

On the other hand, it is easy to see $\vartheta'(t) = 2\gamma h_0 e^{-\gamma t}$ and $-v_x(t, \vartheta(t)) = 2\varepsilon e^{-\beta t}/\vartheta(t)$. Let $\gamma = \beta = \frac{d}{16h_0^2}$, and $\varepsilon \leq \varepsilon_0 = \min\{[d/(128h_0^3)]^{\frac{1}{p+q-1}}, d/(8\mu)\}$, thus

$$\begin{cases} v_t - dv_{xx} - av^p \int_0^{\vartheta(t)} v(t,x)^q dx + \lambda |v_x|^r \geq 0, & t > 0, \; 0 < x < \vartheta(t), \\ \vartheta'(t) > -\mu v_x(t, \vartheta(t)), & t > 0, \\ v_x(t,0) = v(t, \vartheta(t)) = 0, & t > 0, \\ \vartheta(0) = 2h_0 > h_0. \end{cases}$$

Suppose $\|u_0\|_{L^\infty(0,h_0)} \leq \frac{1}{2} \min\{[d/(128h_0^3)]^{\frac{1}{p+q-1}}, d/(8\mu)\}$, we can also obtain

$$u_0(x) \leq v(x,0), \quad x \in [0, h_0].$$

According to the comparison principle,

$$h(t) < \vartheta(t), \text{ for all } t > 0,$$

and

$$u(t,x) \leq v(t,x) \text{ for all } t > 0, \; 0 < x < h(t).$$

Using the above inequality, we can obtain the conclusion immediately. □

Proposition 1. *Assume that (u,h) is the solution of Equation (8), $T_{\max} = \infty$ is the maximum existence time, and $h_\infty < +\infty$. Then, there exists $C > 0$, such that*

$$\sup_{t \geq 0} \|u(t,x)\|_{L^\infty(0,h(t))} \leq C.$$

Proof. The proof of this theorem is basically similar to that of Proposition 2 in [2], so we omit it. □

The above conclusion shows that the global solution is uniformly bounded. In order to further analyze the global solution, we give the following theorem. The theorem shows that the global solution uniformly decays to 0.

Theorem 8. *Under the conditions of Proposition 1, the solution of Equation (8) satisfies*

$$\lim_{t \to +\infty} \|u(t,x)\|_{L^\infty(0,h(t))} = 0.$$

Proof. The proof is similar to Lemma 4.3 in [34], Proposition A in [35] and Lemma 4.2 in [7]. So we also omit it. □

Theorem 9. *(Global slow solutions) Assume that $\varphi(x)$ satisfies the condition (9), $p + q > r$, then there exists $\sigma > 0$, such that the solution of Equation (8) with initial data $u_0 = \sigma\varphi(x)$ is a global slow solution, that is $T^* = \infty$, $h_\infty = \infty$.*

Proof. Motivated by [5,34], we will give the detail proof for this theorem. We denote the solution to (8) by $u(u_0; \cdot)$ to emphasize the dependence of u_0 on the initial data when necessary. So do $h(t), h_\infty$ and the maximal existence time $T = \mathrm{T}_{\max}$.

By [2], we define

$$\Sigma = \{\sigma > 0;\ T(\sigma(\varphi)) = \infty,\ \text{ and }\ h_\infty(\sigma(\varphi)) < \infty,\ \varphi \text{ satisfy } \textit{condition}(9)\}.$$

According to Theorem 7, we know $\sigma \in \Sigma$ if σ is small, so Σ is not empty. Conversely, when σ is large enough, it follows from Theorem 5 that the corresponding solution will blow up, i.e., $T(\sigma(\varphi)) < \infty$, hence Σ is bounded.

Assume

$$\sigma_1^* = \sup \Sigma \in (0, \infty), \quad w = u(\sigma_1^*(\varphi);\ \cdot), \ \zeta = h(\sigma_1^*(\varphi);\ \cdot), \quad \text{and} \quad \tau = T(\sigma_1^*(\varphi);\ \cdot).$$

First of all, we declare $\tau = \infty$. In fact, by continuous dependence (see [22,35]), for any fixed $t \in [0, \tau), u(\sigma(\varphi); t, x)$ converges to $w(t, x)$ in $L^\infty(0, \infty)$ and $h(\sigma(\varphi); t) \to \rho(t)$ as $\sigma \uparrow \sigma^*$. Here, we extend $u(t, x)$ by 0 for $x \in (h(t), \infty)$. It follows from proposition 1 that $\|w(t, \cdot)\|_{L^\infty(0,\infty)} < C$ for all $t \in [0, \tau)$ because $T(\sigma(\varphi)) = \infty$ for all $\sigma \in (0, \sigma^*)$. Thus, we have $\tau = \infty$ since non-global solutions should satisfy $\limsup_{t \to T}(\|u(t, \cdot)\|_{L^\infty(0,h(t))}) = \infty$.

Next, we claim $\zeta_\infty = \infty$. In what follows, we use the contradiction argument. Without loss of generality, we assume $\zeta_\infty < \infty$. Since $\|w(t, \cdot)\|_{L^\infty(0,\infty)} \to 0$ as $t \to \infty$, by Theorem 8, we can choose t_0 sufficiently large, such that

$$\|w(t_0, \cdot)\|_{L^\infty(0,h(t))} \le \frac{1}{4} \min\{[d/(128h_0^3)]^{\frac{1}{p+q-1}}, d/(8\mu)\}.$$

By continuous dependence, we can deduce that

$$\|u(\sigma(\varphi, \psi); t_0, x)\|_{L^\infty[0,h(t_0)]} \le \frac{1}{2} \min\{[d/(128h_0^3)]^{\frac{1}{p+q-1}}, d/(8\mu)\}$$

for $\sigma > \sigma^*$ sufficiently close to σ^*. However, this implies that $h_\infty(\sigma(\varphi)) < \infty$ by Theorem 7, which is a contradiction to the definition of σ^*. $\square$

5. Parameterized Initial Value and Trichotomy

In this section, we will parameterize the initial value. Let φ satisfy the condition (9) and for any $\sigma > 0$, (u_σ, h_σ) is the unique positive solution of Equation (8) with initial value $u_0 = \sigma\varphi$. For convenience, define the solution (u_σ, h_σ) with the maximum existence time of T_σ. Additionally, $h_{\sigma,\infty} = \lim_{t \to T_\sigma} h_\sigma(t)$. In this section, we always write them as $h_{\sigma,\infty}$ even if $T_\sigma < \infty$.

By the comparison principle, Theorems 5 and 7, we can obtain the following Lemma immediately.

Lemma 3. *(i) If $(u_{\sigma_1}, h_{\sigma_1})$ is the global fast solution, then for all $0 < \sigma \le \sigma_1$, (u_σ, h_σ) is also a global fast solution.*

(ii) If $(u_{\sigma_1}, h_{\sigma_1})$ blows up in finite time, then for all $\sigma \ge \sigma_1$, (u_σ, h_σ) also blows up in finite time.

Theorem 10. *There exist $\sigma_*, \sigma^* \in (0, \infty)$ satisfies $\sigma_* \le \sigma^*$, such that:*

(i) (u_σ, h_σ) is a global fast solution when $\sigma \in (0, \sigma_*)$;

(ii) (u_σ, h_σ) is a global slow solution when $\sigma \in [\sigma_*, \sigma^*]$;

(iii) (u_σ, h_σ) blows up in finite time when $\sigma \in (\sigma^*, \infty)$.

Proof. This proof is similar to that of [6], now we give the details. If $\sigma_* = \infty$, then for all $\sigma > 0$, then the solution of Equation (8) is a global fast solution with initial value $u_0 = \sigma\varphi$. If $\sigma^* = \infty$, then the Equation (8) will not have a solution that blows up at finite time.

Firstly, we denote

$$S_1 = \{\sigma \in (0, +\infty] : T_\sigma = +\infty, h_{\sigma,\infty} < +\infty\},$$

$$S_2 = \{\sigma \in (0, +\infty] : T_\sigma = +\infty, h_{\sigma,\infty} = +\infty\},$$

$$S_3 = \{\sigma \in (0, +\infty] : T_\sigma < +\infty, h_{\sigma,\infty} < +\infty\}.$$

Let

$$\sigma_* = \sup S_1 \text{ and } \sigma^* = \inf S_3.$$

It is easy to see $\sigma_* \leq \sigma*$.

By Theorem 7, we can see $S_1 \neq \emptyset$. By Theorem 9, we can see $S_2 \neq \emptyset$ when $p + q > r$. By Theorem 1, we also can see $S_3 \neq \emptyset$ when $p + q > r$.

Next, we consider the case $\sigma_* \leq \sigma^* < \infty$. Additionally, we divide the proof into three steps.

Step 1. We will prove $\sigma_* \notin S_1$.

Assume this conclusion does not hold, then $\sigma_* \in S_1$. Thus, we have

$$T_{\sigma_*} = +\infty,\ h_{\sigma_*,\infty} < +\infty.$$

By Theorem 8, there exists some large $t_0 > 0$, such that

$$\|u_{\sigma_*}(t_0, \cdot)\|_\infty \leq \frac{1}{4} \min\{d/(128 h_{\sigma_*}(t_0)^3)]^{\frac{1}{p+q-1}},\ d/(8\mu)\}.$$

By the continuity of solutions with respect to σ, we can take a small $\epsilon > 0$ such that the corresponding solution $(u_{\sigma_*+\epsilon}, h_{\sigma_*+\epsilon})$ satisfies

$$\|u_{\sigma_*+\epsilon}(t_0, \cdot)\|_\infty \leq \frac{1}{2} \min\{d/(128 h_{\sigma_*+\epsilon}(t_0)^3)]^{\frac{1}{p+q-1}},\ d/(8\mu)\}.$$

Thus, Theorem 7 indicates $(u_{\sigma_*+\epsilon}, h_{\sigma_*+\epsilon})$ is a global fast solution, which is a contradiction to the definition of σ_*.

Step 2. We will prove $\sigma^* \in S_2$.

This proof is similar to that of Theorem 1.3 in [3] and Theorem 5.2 in [36]. Firstly, we claim that $T_{\sigma^*} = +\infty$. Indeed, by continuous dependence, for any given $t \in [0, T_{\sigma^*})$, u_σ tends to u_{σ^*} in the sense of L^∞ norm when $\sigma \uparrow \sigma^*$. Here, we extend $u(t, x)$ by 0 in $(h(t), +\infty)$. Since for all $\sigma \in (0, \sigma^*)$, $T_\sigma = +\infty$, by Proposition 1,

$$\|u_{\sigma^*}\|_\infty \leq \widetilde{D}, \quad \|v_{\sigma^*}\|_\infty \leq \widetilde{D}, \qquad \text{for all } t \in [0, T_{\sigma^*}),$$

where $\widetilde{D}$ is a positive constant. Therefore, by Corollary 1, we can obtain $T_{\sigma^*} = +\infty$.

On the other hand, it is easy to prove $\sigma^* \geq \sigma_*$. Additionally, $h_{\sigma^*,\infty} = +\infty$ holds. Thus, we can see $\sigma^* \in S_2$.

Step 3. We prove that for any $\sigma \in [\sigma_*, \sigma^*]$, (u_σ, h_σ) is global slow solution. By using the comparison principle and due to Step 1 and Step 2, we have $\sigma \in S_2$.

Finally, by Lemma 3, step 1 to step 3, the conclusions (i), (ii), and (iii) hold. □

6. Conclusions

In this paper, we assume $r > 1$ instead of $r \geq 1$. Indeed, the results of this paper are valid for $r = 1$ in the sense of mathematics. However, from the derivation process of the biological background of the absorption term of the model, we still need to assume that $r > 1$ (see the Formula (1.9) in [7]).

Compare this paper with [6], the model in [6] may be seen as the special case of this paper with zero absorption term. If the source term controls the absorption gradient term, the solutions blows up in finite time with sufficient large initial data. Global fast solutions and global slow solutions are also given in these papers. Because of the appearance of absorption gradient term, the proof process is complicated. Compare this paper with [7], we can see that the non-local term $\int_0^{h(t)} u^q(t,x)\mathrm{d}x$ has the same effect as $u^q(t,x)$.

In this paper, we also obtain a trichotomy conclusion which is not considered in [6], whereas the result that the blow-up can not occur in finite time if absorption gradient term controls the source term is not considered in this paper, which will be studied in our future work.

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