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Non-Zero Order of an Extended Temme Integral

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Abstract: A new three-dimensional integral containing $f(x,y,z)I_v(x\alpha)$ is derived where $I_v(x\alpha)$ is the Modified Bessel Function of the first kind and the integral is taken over the infinite cubic space $0 < x < \infty, 0 < y < \infty, 0 < z < \infty$. The integral is not easily evaluated for complex ranges of the parameters. A representation in terms of the Hurwitz–Lerch zeta function, polylogarithm function and Riemann zeta functions are evaluated. This representation yields triple integral representations in terms of fundamental constants that can be derived. Almost all Lerch functions have an asymmetrical zero distribution.

Keywords: modified bessel function; triple integral; Apéry's constant

MSC: Primary 30E20; 33-01; 33-03; 33-04

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1. Introduction

In the work by Temme [1], a double integral containing the modified Bessel function has been computed and asymptotics derived. In this present paper, the authors extend the work previously published by Temme and derive a triple Integral containing the Modified Bessel function of the first kind $I_v(y\alpha)$ in terms of the Hurwitz–Lerch zeta function. The integral formula is then used to derive special cases in terms of other special functions and fundamental constants. In this paper, we will derive the triple definite integral given by:

$$\begin{aligned}\int_{\mathbb{R}^3_+} x^{m-v-1}z^{2v-m}y^{-((m-1)v)-1}e^{-bz-\alpha x}I_v(x\alpha)(p+qy^v)^{m-v-\frac{1}{2}} \\ \log^k\left(\frac{axy^{-v}(p+qy^v)}{z}\right)dxdydz\end{aligned} \tag{1}$$

where the parameters $k,a,p,q,v,m \in \mathbb{C}$ are general complex numbers, such that $0 < Re(m) < 1, -\pi < Im(m) < \pi, Re(\alpha) > 0$ and $Re(v) > 1/2$. This definite integral will be used to derive special cases in terms of special functions and fundamental constants. The derivations follow the method used by us in [2]. This method involves using a form of the generalized Cauchy's integral formula given by:

$$\frac{y^k}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_C \frac{e^{wy}}{w^{k+1}}dw. \tag{2}$$

where C is, in general, an open contour in the complex plane where the bilinear concomitant has the same value at the end points of the contour. We then multiply both sides by a function of x, y, and z, then take a definite triple integral of both sides. This yields a definite integral in terms of a contour integral. Then, we multiply both sides of Equation (2) by another function of y and take the infinite sum of both sides such that the contour integral of both equations are the same.

2. Definite Integral of the Contour Integral

Here, we use the method in [2]. The variable of integration in the contour integral is $r = w + m$. The cut and contour are in the first quadrant of the complex r-plane. The cut approaches the origin from the interior of the first quadrant and the contour goes round the origin with zero radius and is on opposite sides of the cut. Using a generalization of Cauchy's integral formula we form the triple integral by replacing y by:

$$\log\left(\frac{axy^{-\nu}(p+qy^{\nu})}{z}\right)$$

and multiplying by

$$x^{m-v-1}z^{2v-m}y^{-((m-1)\nu)-1}e^{-bz-\alpha x}I_v(x\alpha)(p+qy^{\nu})^{m-v-\frac{1}{2}}$$

followed by taking the definite integral with respect to $x \in [0,\infty)$, $y \in [0,\infty)$ and $z \in [0,\infty)$ to obtain:

$$\begin{aligned}
&\tfrac{1}{\Gamma(k+1)}\int_{\mathbb{R}^3_+} x^{m-v-1}z^{2v-m}y^{-((m-1)\nu)-1}e^{-bz-\alpha x}I_v(x\alpha)(p+qy^{\nu})^{m-v-\frac{1}{2}}\\
&\log^k\left(\tfrac{axy^{-\nu}(p+qy^{\nu})}{z}\right)dxdydz\\
&=\tfrac{1}{2\pi i}\int_{\mathbb{R}^3_+}\int_C a^w w^{-k-1}e^{-bz-\alpha x}x^{m-v+w-1}z^{-m+2v-w}y^{-(\nu(m+w-1))-1}I_v(x\alpha)\\
&(p+qy^{\nu})^{m-v+w-\frac{1}{2}}dwdxdydz\\
&=\tfrac{1}{2\pi i}\int_C\int_{\mathbb{R}^3_+} a^w w^{-k-1}e^{-bz-\alpha x}x^{m-v+w-1}z^{-m+2v-w}y^{-(\nu(m+w-1))-1}I_v(x\alpha)\\
&(p+qy^{\nu})^{m-v+w-\frac{1}{2}}dxdydzdw\\
&=\tfrac{1}{2\pi i}\int_C\tfrac{\sqrt{\pi}a^w w^{-k-1}\Gamma\left(v-\frac{1}{2}\right)2^{-m+v-w}\csc(\pi(m+w))b^{m-2v+w-1}}{\nu}\\
&\left(\tfrac{p}{q}\right)^{-m-w+1}p^{m-v+w-\frac{1}{2}}\alpha^{-m+v-w}dw
\end{aligned} \tag{3}$$

from Equation (3.13.2.2) in [3] and Equations (3.241.4) and (3.326.2) in [4] where $-Re(v) < Re(w+m) < 1/2, Re(\alpha) > 0$ and using the reflection Formula (8.334.3) in [4] for the Gamma function. We are able to switch the order of integration over x, y, and z using Fubini's theorem for multiple integrals page 178 in [5], since the integrand is of bounded measure over the space $\mathbb{C}\times[0,\infty)\times[0,\infty)\times[0,\infty)$.

3. The Hurwitz–Lerch Zeta Function and Infinite Sum of the Contour Integral

In this section we use Equation (2) to derive the contour integral representations for the Hurwitz–Lerch zeta function.

3.1. The Hurwitz–Lerch Zeta Function

The Hurwitz–Lerch zeta function (25.14) in [6,7] has a series representation given by:

$$\Phi(z,s,v) = \sum_{n=0}^{\infty}(v+n)^{-s}z^n \tag{4}$$

where $|z| < 1, v \neq 0, -1, -2, \ldots$ and is continued analytically by its integral representation given by:

$$\Phi(z,s,v) = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-vt}}{1-ze^{-t}}dt = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-(v-1)t}}{e^t-z}dt \tag{5}$$

where $Re(v) > 0$, and either $|z| \leq 1, z \neq 1, Re(s) > 0$, or $z = 1, Re(s) > 1$.

3.2. Infinite Sum of the Contour Integral

Using Equation (2) and replacing y by

$$\log(a) - \log(\alpha) + \log(b) - \log\left(\frac{p}{q}\right) + \log(p) + i\pi(2y+1) - \log(2)$$

then multiplying both sides by

$$-\frac{i\sqrt{\pi}2^{-m+v+1}e^{i\pi m(2y+1)}\Gamma\left(v-\frac{1}{2}\right)b^{m-2v-1}\left(\frac{p}{q}\right)^{1-m}p^{m-v-\frac{1}{2}}\alpha^{v-m}}{\nu}$$

taking the infinite sum over $y \in [0, \infty)$ and simplifying in terms of the Hurwitz–Lerch zeta function we obtain

$$\begin{aligned}
&-\frac{i\pi^{k+\frac{1}{2}}e^{\frac{1}{2}i\pi(k+2m)}\Gamma\left(v-\frac{1}{2}\right)b^{m-2v-1}2^{k-m+v+1}\left(\frac{p}{q}\right)^{1-m}p^{m-v-\frac{1}{2}}\alpha^{v-m}}{\nu\Gamma(k+1)} \\
&\Phi\left(e^{2im\pi}, -k, \frac{-i\log(a)-i\log(b)-i\log(p)+i\log\left(\frac{2p}{q}\right)+i\log(\alpha)+\pi}{2\pi}\right) \\
&= -\frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_C \frac{i\sqrt{\pi}a^w w^{-k-1}\Gamma\left(v-\frac{1}{2}\right)2^{-m+v-w+1}e^{i\pi(2y+1)(m+w)}b^{m-2v+w-1}}{\nu} \\
&\left(\frac{p}{q}\right)^{-m-w+1}p^{m-v+w-\frac{1}{2}}\alpha^{-m+v-w}dw \\
&= -\frac{1}{2\pi i}\int_C \frac{i\sqrt{\pi}a^w w^{-k-1}\Gamma\left(v-\frac{1}{2}\right)2^{-m+v-w+1}b^{m-2v+w-1}}{\nu} \\
&\left(\frac{p}{q}\right)^{-m-w+1}p^{m-v+w-\frac{1}{2}}\alpha^{-m+v-w} \\
&\cdot\left[\sum_{y=0}^{\infty}e^{i\pi(2y+1)(m+w)}\right]dw \\
&= \frac{1}{2\pi i}\int_C \frac{\sqrt{\pi}a^w w^{-k-1}\Gamma\left(v-\frac{1}{2}\right)2^{-m+v-w}\csc(\pi(m+w))b^{m-2v+w-1}}{\nu} \\
&\left(\frac{p}{q}\right)^{-m-w+1}p^{m-v+w-\frac{1}{2}}\alpha^{-m+v-w}dw
\end{aligned} \tag{6}$$

from Equation (1.232.2) in [4] where $Im(w+m) > 0$ in order for the sum to converge.

4. Definite Integral in Terms of the Hurwitz–Lerch Zeta Function

Theorem 1. *For all* $0 < Re(m) < 1, -\pi < Im(m) < \pi, 1/2 < Re(\nu)$ *then,*

$$\begin{aligned}
&\int_{\mathbb{R}_+^3} x^{m-v-1}z^{2v-m}y^{-((m-1)\nu)-1}e^{-bz-\alpha x}I_v(x\alpha)(p+qy^\nu)^{m-v-\frac{1}{2}} \\
&\log^k\left(\frac{axy^{-\nu}(p+qy^\nu)}{z}\right)dxdydz \\
&= -\frac{i\pi^{k+\frac{1}{2}}e^{\frac{1}{2}i\pi(k+2m)}\Gamma\left(v-\frac{1}{2}\right)b^{m-2v-1}2^{k-m+v+1}\left(\frac{p}{q}\right)^{1-m}p^{m-v-\frac{1}{2}}\alpha^{v-m}}{\nu} \\
&\Phi\left(e^{2im\pi}, -k, \frac{-i\log(a)-i\log(b)-i\log(p)+i\log\left(\frac{2p}{q}\right)+i\log(\alpha)+\pi}{2\pi}\right)
\end{aligned} \tag{7}$$

Proof. The right-hand sides of relations (3) and (6) are identical; hence, the left-hand sides of the same are identical too. Simplifying with the Gamma function yields the desired conclusion. □

Example 1. *The degenerate case.*

$$\begin{aligned}
&\int_{\mathbb{R}_+^3} x^{m-v-1}z^{2v-m}y^{-m\nu+\nu-1}e^{-bz-\alpha x}I_v(x\alpha)(p+qy^\nu)^{m-v-\frac{1}{2}}dxdydz \\
&= \frac{\sqrt{\pi}2^{v-m}\csc(\pi m)\Gamma\left(v-\frac{1}{2}\right)b^{m-2v-1}\left(\frac{p}{q}\right)^{1-m}p^{m-v-\frac{1}{2}}\alpha^{v-m}}{\nu}
\end{aligned} \tag{8}$$

Proof. Use Equation (7) and set $k = 0$ and simplify using entry (2) in Table below (64:12:7) in [8]. □

Example 2. *The polylogarithm function* $Li_n(z)$.

$$
\begin{aligned}
&\int_{\mathbb{R}^3_+} e^{-x-2z} x^{m-v-1} z^{2v-m} y^{-mv+v-1} I_v(x) \left(y^v + \frac{1}{2}\right)^{m-v-\frac{1}{2}} \\
&\qquad \log^k\left(-\frac{x(y^{-v}+2)}{2z}\right) dxdydz \\
&= -\frac{i 2^{k-\frac{1}{2}} \pi^{k+\frac{1}{2}} e^{\frac{1}{2} i\pi(k-2m)} \Gamma\left(v-\frac{1}{2}\right) Li_{-k}\left(e^{2im\pi}\right)}{v}
\end{aligned} \tag{9}
$$

Proof. Use Equation (7) and set $a = -1, b = 2, p = 1/2, q = 1/2, \alpha = 1$ simplify using Equation (64:12:2) in [8]. □

Example 3. *The Riemann zeta function* $\zeta(s)$.

$$
\begin{aligned}
&\int_{\mathbb{R}^3_+} x^{-v-\frac{1}{2}} z^{2v-\frac{1}{2}} e^{-x-2z} y^{\frac{v}{2}-1} I_v(x) \left(y^v + \frac{1}{2}\right)^{-v} \log^k\left(-\frac{x(y^{-v}+2)}{2z}\right) dxdydz \\
&= -\frac{2^{k-\frac{1}{2}} \left(2^{k+1}-1\right) e^{\frac{i\pi k}{2}} \pi^{k+\frac{1}{2}} \zeta(-k) \Gamma\left(v-\frac{1}{2}\right)}{v}
\end{aligned} \tag{10}
$$

Proof. Use Equation (9) and set $m = 1/2$ and simplify using Equation (25.12.10) in [6]. □

Example 4. *The constant* $\log(2)$.

$$
\int_{\mathbb{R}^3_+} \frac{x^{-v-\frac{1}{2}} z^{2v-\frac{1}{2}} e^{-x-2z} y^{\frac{v}{2}-1} I_v(x) \left(y^v + \frac{1}{2}\right)^{-v}}{\log\left(-\frac{x(y^{-v}+2)}{2z}\right)} dxdydz = -\frac{i \log(2) \Gamma\left(v - \frac{1}{2}\right)}{2\sqrt{2\pi} v} \tag{11}
$$

Proof. Use Equation (10) and apply l'Hopital's rule as $k \to -1$ and simplify using Equation (25.4.1) in [6]. □

Example 5. *Apery's constant* $\zeta(3)$.

$$
\int_{\mathbb{R}^3_+} \frac{x^{-v-\frac{1}{2}} z^{2v-\frac{1}{2}} e^{-x-2z} y^{\frac{v}{2}-1} I_v(x) \left(y^v + \frac{1}{2}\right)^{-v}}{\log^3\left(-\frac{x(y^{-v}+2)}{2z}\right)} dxdydz = \frac{3i\zeta(3) \Gamma\left(v - \frac{1}{2}\right)}{32\sqrt{2}\pi^{5/2} v} \tag{12}
$$

Proof. Use Equation (10) and set $k = -3$ and simplify. □

5. Conclusions

In this paper, we have presented a novel method for deriving a new triple integral transform containing the modified Bessel function along with some interesting definite integrals using contour integration. The results presented were numerically verified for both real and imaginary and complex values of the parameters in the integrals using Mathematica by Wolfram.

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References

1. Temme, N.M. A Double Integral Containing the Modified Bessel Function: Asymptotics and Computation. *Math. Comput.* **1986**, *47*, 683–691. [CrossRef]
2. Reynolds, R.; Stauffer, A. A Method for Evaluating Definite Integrals in Terms of Special Functions with Examples. *Int. Math. Forum* **2020**, *15*, 235–244. [CrossRef]
3. Brychkov, Y.A.; Marichev, O.I.; Savischenko, N.V. *Handbook of Mellin Transforms*; Taylor & Francis, Chapman and Hall/CRC: Abingdon, UK, 2019; 607p.
4. Gradshteyn, I.S.; Ryzhik, I.M. *Tables of Integrals, Series and Products*, 6th ed.; Academic Press: Cambridge, MA, USA, 2000.
5. Gelca, R.; Andreescu, T. *Putnam and Beyond*, 1st ed.; Springer: New York, NY, USA, 2007.
6. Olver, F.W.J.; Lozier, D.W.; Boisvert, R.F.; Clark, C.W. (Eds.) *NIST Digital Library of Mathematical Functions*; U.S. Department of Commerce, National Institute of Standards and Technology: Washington, DC, USA; Cambridge University Press: Cambridge, UK, 2010; With 1 CD-ROM (Windows, Macintosh and UNIX). MR 2723248 (2012a:33001).
7. Laurincikas, A.; Garunkstis, R. *The Lerch Zeta-Function*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2013.
8. Oldham, K.B.; Myland, J.C.; Spanier, J. *An Atlas of Functions: With Equator, the Atlas Function Calculator*, 2nd ed.; Springer: New York, NY, USA, 2009.