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Sufficiency for Weak Minima in Optimal Control Subject to Mixed Constraints

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Abstract: For optimal control problems of Bolza involving time-state-control mixed constraints, containing inequalities and equalities, fixed initial end-point, variable final end-point, and nonlinear dynamics, sufficient conditions for weak minima are derived. The proposed algorithm allows us to avoid hypotheses such as the continuity of the second derivatives of the functions delimiting the problems, the continuity of the optimal controls or the parametrization of the final variable end-point. We also present a relaxation relative to some similar works, in the sense that we arrive essentially to the same conclusions but making weaker assumptions.

Keywords: optimal control; mixed constraints; free final end-point; sufficiency; weak minima

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1. Introduction

In this paper, we study sufficiency conditions for a weak minimum in two constrained parametric and nonparametric optimal control problems having nonlinear dynamics, a left fixed end-point, a right variable end-point and mixed time-state-control restrictions involving inequalities and equalities. In the parametric problem, we show how the deviation between admissible costs and optimal costs is derived by some functions playing the role of the square of some norms; in particular, the involvement of a functional whose structure is very similar to the square of the classical norm of the Lebesgue measurable functions is a fundamental component. See [1–4], where the authors study sufficient conditions for optimality, and they obtain a similar behaviour with respect to the corresponding deviations between optimal and feasible costs. In the parametric problem, the variable end-point is subject to a parametrization involving a twice continuously differentiable manifold, and, in the nonparametric problem, we make a relaxation of that concept because of the fact that the final end-point is not only variable but also completely free, in the sense that the final end-point may belong to any set which only must be contained in a surface having continuous second derivatives of the independent variable. Another important relaxation of this paper is that we avoid the imposition of two functional restrictions involving the maximum of some crucial integrals, one of them concerning derivatives of admissible and optimal dynamics and the other concerning the admissible and optimal controls, see [5,6]. In contrast, we show how, by fixing the left end-point, we are able to eliminate the integral depending on the admissible dynamics of the problem and only make a weaker hypothesis only involving the integral of admissible and the optimal controls. It is worth emphasizing that the conclusions are very similar and the hypotheses are weaker.

On the other hand, the sufficiency technique employed to prove the main theorem of the paper is self-contained because it is independent of classical approaches used to obtain sufficiency in optimal control such as the Hamilton–Jacobi theory, the incorporation of symmetric solutions of some matrix-valued Riccati equations or the use of fundamental concepts appealing to Jacobi's theory in terms of conjugate points, see [7–9], respectively.

In contrast, our approach is direct in nature since it strongly depends upon three fundamental concepts; the first one concerns a similar version of the Legendre–Clebsch necessary condition; the second one is related with the positivity of the second variation over the cone of critical directions, and the third one involves a crucial integral inequality involving a Weierstrass verification excess function and the integral of a mapping whose behavior is very similar to the quadratic function around zero and very analogous to the absolute value function around infinity and minus infinity. As the right end-point is variable in the parametric optimal control problem as well as in the nonparametric optimal control problem, our hypotheses also impose a transversality condition and the properties of the proof of the theorem of the article find out the fulfillment of a second order inequality to be crucial. This second order inequality has its origin in a symmetric inequality presented in hypothesis (ii) of Theorem 1 and Corollary 1 of [5,6]. The absence of the continuity of the proposed optimal controls in the content of this paper is also one of the essential components of this work. See [7–21], where that assumption of continuity in the sufficiency approaches containing a degree of generality very similar to that obtained in this article, is a uniform unfortunate assumption since the admissible controls must only lie in the family of measurable functions. To be more precise, it is an unfortunate issue that, in the works mentioned above, their optimal controls need to be confined to the space of continuous functions; meanwhile, all the feasible controls must only be measurable, see [5,6,22], where we show that this assumption of continuity on the optimal controls is very strong.

The paper is organized as follows: In Section 2, we state the parametric optimal control problem that we shall study, some basic definitions, and we enunciate the main theorem of the article. In Section 3, we pose the nonparametric optimal control problem we are going to study together with a fundamental lemma and a corollary which turns out to be the principal result of the paper. In the same section, we illustrate with two examples how even the non-expert can apply the main corollary of the article. In Section 4, we establish three supplementary lemmas whose proofs can be found in [23] and on which the proof of the theorem is strongly based. In Section 5, we make the proof of the theorem of the paper by means of two lemmas. In Section 6, we present a discussion concerning the relations between necessary and sufficient conditions, we add some comments about an experimental economic model, and we exhibit some relevant references containing the fundamental subject of mixed constraints. Finally, in Section 7, we provide the main conclusions of the article.

2. An Auxiliary Theorem

Suppose that we are given an interval $\mathcal{T} := [t_1, t_2]$ in $\mathbf{R}$, a fixed point $\xi_1 \in \mathbf{R}^n$ and C any nonempty subset of $\mathbf{R}^s$, called the set of *parameters*, that we have functions $\gamma\colon \mathbf{R}^n \to \mathbf{R}$, $\Psi\colon \mathbf{R}^n \to \mathbf{R}^n$, $\Gamma(t,x,u)\colon \mathcal{T} \times \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$, $f(t,x,u)\colon \mathcal{T} \times \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ and $\varphi(t,x,u)\colon \mathcal{T} \times \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^q$. Set

$$R := \{(t,x,u) \in \mathcal{T} \times \mathbf{R}^n \times \mathbf{R}^m \mid \varphi_\sigma(t,x,u) \le 0 \ (\sigma \in P),\ \varphi_\varsigma(t,x,u) = 0 \ (\varsigma \in Q)\}$$

where $P := \{1,\ldots,p\}$ and $Q := \{p+1,\ldots,q\}$ $(p = 0,1,\ldots,q)$. If $p = 0$, then P is empty, and we disregard statements about φ_σ. If $p = q$, then Q is empty, and we disregard statements about φ_ς.

Throughout the paper, we suppose that Γ, f and $\varphi = (\varphi_1,\ldots,\varphi_q)$ have first and second derivatives with respect to x and u. Additionally, if we denote by $G(t,x,u)$ either $\Gamma(t,x,u)$, $f(t,x,u)$, $\varphi(t,x,u)$ or any of their partial derivatives of order ≤ 2 with respect to x and u, we are going to assume that, if $\mathcal{G}$ is any bounded subset of $\mathcal{T} \times \mathbf{R}^n \times \mathbf{R}^m$, then $|G(\mathcal{G})|$ is a bounded subset of $\mathbf{R}$. In addition, we suppose that, if $((h_q, l_q))$ is any sequence in $AC(\mathcal{T};\mathbf{R}^n) \times L^\infty(\mathcal{T};\mathbf{R}^m)$ such that for some $(h,l) \in AC(\mathcal{T};\mathbf{R}^n) \times L^\infty(\mathcal{T};\mathbf{R}^m)$, $(h_q(\cdot), l_q(\cdot)) \xrightarrow{L^\infty} (h(\cdot), l(\cdot))$ on $\mathcal{T}$, then, for all $q \in \mathbf{N}$, $G(\cdot, h_q(\cdot), l_q(\cdot))$ is measurable on $\mathcal{T}$ and

$$G(\cdot, h_q(\cdot), l_q(\cdot)) \xrightarrow{L^\infty} G(\cdot, h(\cdot), l(\cdot)) \text{ on } \mathcal{T}.$$

It is worth observing that conditions given above are satisfied if the functions Γ, f, φ and their first and second derivatives relative to x and u are continuous on $\mathcal{T} \times \mathbf{R}^n \times \mathbf{R}^m$. We are going to suppose that the functions γ and Ψ are of class C^2 on $\mathbf{R}^n$.

Designate by $X := \{x \colon \mathcal{T} \to \mathbf{R}^n \mid x \text{ is absolutely continuous}\}$ and for any positive integer s, set $U_s := L^\infty(\mathcal{T}; \mathbf{R}^s)$. Define $A := X \times U_m \times \mathbf{R}^s$. The notation $z_a := (z, a) = (x, u, a)$ denotes any element $z_a \in A$.

We are going to study a parametric optimal control problem, denoted by $P(\gamma, \Gamma, C, f, \xi_1, \Psi, R, s)$, consisting of minimizing a functional of the form

$$I(z_a) := \gamma(a) + \int_{t_1}^{t_2} \Gamma(t, x(t), u(t))dt$$

over all z_a in A satisfying the constraints

$$\begin{cases} a \in C. \\ \dot{x}(t) = f(t, x(t), u(t)) \text{ (a.e. in } \mathcal{T}). \\ x(t_1) = \xi_1,\ x(t_2) = \Psi(a). \\ (t, x(t), u(t)) \in R \ (t \in \mathcal{T}). \end{cases}$$

Elements $a = (a_1, \ldots, a_s)^*$ in $\mathbf{R}^s$ ($*$ denotes transpose) will be called *parameters*, members z_a in A will be called *processes*, and a process is *admissible* if it verifies the constraints.

- A process $\hat{z}_{\hat{a}}$ solves $P(\gamma, \Gamma, C, f, \xi_1, \Psi, R, s)$ if it is admissible and $I(\hat{z}_{\hat{a}}) \leq I(z_a)$ for all admissible processes z_a. An admissible process $\hat{z}_{\hat{a}}$ is a *weak minimum* of $P(\gamma, \Gamma, C, f, \xi_1, \Psi, R, s)$ if it is a minimum of I relative to the norm

$$\|z_a\| := |a| + \|(x, u)\|_\infty,$$

  that is, if, for some $\epsilon > 0$, $I(\hat{z}_{\hat{a}}) \leq I(z_a)$ for all admissible processes z_a verifying $\|z_a - \hat{z}_{\hat{a}}\| < \epsilon$.
- For all $(t, x, u, \omega, \nu) \in \mathcal{T} \times \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^n \times \mathbf{R}^q$, define the augmented Hamiltonian by

$$\mathcal{H}(t, x, u, \omega, \nu) := \omega^* f(t, x, u) - \Gamma(t, x, u) - \nu^* \varphi(t, x, u).$$

  If $\omega \in X$ and $\nu \in U_q$ are given, set, for all $(t, x, u) \in \mathcal{T} \times \mathbf{R}^n \times \mathbf{R}^m$,

$$\mathcal{F}(t, x, u) := -\mathcal{H}(t, x, u, \omega(t), \nu(t)) - \dot{\omega}(t)x$$

  and let

$$J(z_a) := \omega^*(t_2)x(t_2) - \omega^*(t_1)x(t_1) + \gamma(a) + \int_{t_1}^{t_2} \mathcal{F}(t, x(t), u(t))dt.$$

- The *second variation* of J with respect to z_a in the direction w_α, is given by

$$J''(z_a; w_\alpha) := \alpha^* \gamma''(a)\alpha + \int_{t_1}^{t_2} 2\Omega(t, x(t), u(t); y(t), v(t))dt,$$

  where, for all $(t, y, v) \in \mathcal{T} \times \mathbf{R}^n \times \mathbf{R}^m$,

$$2\Omega(t, x(t), u(t); y, v) := y^* \mathcal{F}_{xx}(t, x(t), u(t))y + 2y^* \mathcal{F}_{xu}(t, x(t), u(t))v + v^* \mathcal{F}_{uu}(t, x(t), u(t))v,$$

  and the notation w_α means any element $(y, v, \alpha) \in X \times L^2(\mathcal{T}; \mathbf{R}^m) \times \mathbf{R}^s$. In addition, $\gamma''(a)$ is the second derivative of γ evaluated at a.
- Let

$$E(t, x, u, v) := \mathcal{F}(t, x, v) - \mathcal{F}(t, x, u) - \mathcal{F}_u(t, x, u)(v - u).$$

- Define

$$\mathcal{D}(u) := \int_{t_1}^{t_2} L(u(t))dt \quad \text{where} \quad L(c) := (1+|c|^2)^{1/2} - 1 \ (c \in \mathbf{R}^m).$$

Finally, if $(t,x,u) \in \mathcal{T} \times \mathbf{R}^n \times \mathbf{R}^m$ is given, denote by

$$i(t,x,u) := \{\sigma \in P \mid \varphi_\sigma(t,x,u) = 0\},$$

the set of active indices of (t,x,u) relative to the inequality constraints. For all $z_a \in A$, let $Y(z_a)$ be the cone of all $w_\alpha \in X \times L^2(T;\mathbf{R}^m) \times \mathbf{R}^s$ satisfying

$$\begin{cases} \dot{y}(t) = f_x(t,x(t),u(t))y(t) + f_u(t,x(t),u(t))v(t) \ \text{(a.e. in } \mathcal{T}). \\ y(t_1) = 0,\ y(t_2) = \Psi'(a)\alpha. \\ \varphi_{\sigma x}(t,x(t),u(t))y(t) + \varphi_{\sigma u}(t,x(t),u(t))v(t) \leq 0 \ \text{(a.e. in } \mathcal{T}, \sigma \in i(t,x(t),u(t))). \\ \varphi_{\varsigma x}(t,x(t),u(t))y(t) + \varphi_{\varsigma u}(t,x(t),u(t))v(t) = 0 \ \text{(a.e. in } \mathcal{T}, \varsigma \in Q). \end{cases}$$

The set $Y(z_a)$ is the *cone of critical directions* with respect to z_a.

Theorem 1. *Let $\hat{z}_{\hat{a}}$ be an admissible process. Assume that $i(\cdot,\hat{x}(\cdot),\hat{u}(\cdot))$ is piecewise constant on $\mathcal{T}$ that there exist $\omega \in X$, $\nu \in U_q$ with $\nu_\sigma(t) \geq 0$, $\nu_\sigma(t)\varphi_\sigma(t,\hat{x}(t),\hat{u}(t)) = 0$ $(\sigma \in P, t \in \mathcal{T})$ and $\delta, \epsilon > 0$, such that*

$$\dot{\omega}(t) = -\mathcal{H}_x^*(t,\hat{x}(t),\hat{u}(t),\omega(t),\nu(t)) \ \textit{(a.e. in } \mathcal{T}),$$

$$\mathcal{H}_u^*(t,\hat{x}(t),\hat{u}(t),\omega(t),\nu(t)) = 0 \ (t \in \mathcal{T}),$$

and the following is satisfied:

(i) $\gamma'^*(\hat{a}) + \Psi'^*(\hat{a})\omega(t_2) = 0.$
(ii) $\omega^*(t_2)\Psi''(\hat{a};h) \geq 0$ *for all* $h \in \mathbf{R}^s$.
(iii) $\mathcal{H}_{uu}(t,\hat{x}(t),\hat{u}(t),\omega(t),\nu(t)) \leq 0$ *(a.e. in* $\mathcal{T}$*).*
(iv) $J''(\hat{z}_{\hat{a}};w_\alpha) > 0$ *for all* $w_\alpha \in Y(\hat{z}_{\hat{a}})$, $w_\alpha \not\equiv (0,0,0)$.
(v) z_a *admissible with* $\|(x,u) - (\hat{x},\hat{u})\|_\infty < \epsilon$ *implies that* $\int_{t_1}^{t_2} E(t,x(t),\hat{u}(t),u(t)) \geq \delta\mathcal{D}(u - \hat{u})$.

Then, for some $\rho_1, \rho_2 > 0$ and all admissible processes z_a satisfying $\|z_a - \hat{z}_{\hat{a}}\| < \rho_1$,

$$I(z_a) \geq I(\hat{z}_{\hat{a}}) + \rho_2 \min\{|a-\hat{a}|^2, \mathcal{D}(u-\hat{u})\}.$$

In particular, $\hat{z}_{\hat{a}}$ is a weak minimum of $P(\gamma,\Gamma,C,f,\xi_1,\Psi,R,s)$.

3. The Principal Result

Suppose that an interval $\mathcal{T} := [t_1,t_2]$ in $\mathbf{R}$ is given, a fixed point $\Upsilon_1 \in \mathbf{R}^n$, a set $B \subset \mathbf{R}^n$ and functions $\ell\colon \mathbf{R}^n \to \mathbf{R}$, $\mathcal{L}(t,x,u)\colon \mathcal{T} \times \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$, $g(t,x,u)\colon \mathcal{T} \times \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ and $\phi(t,x,u)\colon \mathcal{T} \times \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^q$. Set

$$\mathcal{R} := \{(t,x,u) \in \mathcal{T} \times \mathbf{R}^n \times \mathbf{R}^m \mid \phi_\sigma(t,x,u) \leq 0 \ (\sigma \in P),\ \phi_\varsigma(t,x,u) = 0 \ (\varsigma \in Q)\}$$

where $P := \{1,\ldots,p\}$ and $Q := \{p+1,\ldots,q\}$ $(p = 0,1,\ldots,q)$. If $p = 0$, then P is empty, and we disregard statements about ϕ_σ. If $p = q$, then Q is empty, and we disregard statements about ϕ_ς.

In this section, we shall assume that $\mathcal{L}$, g and $\phi = (\phi_1,\ldots,\phi_q)$ satisfy the regularity hypotheses mentioned in Section 2. In particular, if $\mathcal{L}$, g, and ϕ have first and second continuous partial derivatives with respect to x and u on $\mathcal{T} \times \mathbf{R}^n \times \mathbf{R}^m$, then they verify the previously mentioned regularity hypotheses. Moreover, we shall be assuming that the function ℓ is of class C^2 on $\mathbf{R}^n$.

Set $\mathcal{A} := X \times U_m$, where usually X is the space of absolutely continuous functions mapping $\mathcal{T}$ to $\mathbf{R}^n$, and U_m is the space of all essentially bounded measurable functions mapping $\mathcal{T}$ to $\mathbf{R}^m$.

In this section, we are going to study the non-parametric optimal control problem $\mathcal{P}(\ell, \mathcal{L}, g, \Upsilon_1, B, \mathcal{R}, n)$ of finding a minimum value to the functional

$$\mathcal{J}(x,u) := \ell(x(t_2)) + \int_{t_1}^{t_2} \mathcal{L}(t, x(t), u(t))dt$$

over all pairs (x, u) in $\mathcal{A}$ verifying the constraints

$$\begin{cases} \dot{x}(t) = g(t, x(t), u(t)) \text{ (a.e. in } \mathcal{T}). \\ x(t_1) = \Upsilon_1, \ x(t_2) \in B. \\ (t, x(t), u(t)) \in \mathcal{R} \ (t \in \mathcal{T}). \end{cases}$$

The elements (x, u) in $\mathcal{A}$ will be called *processes*. A process is admissible if it satisfies the restrictions.

A process $(\hat{x}, \hat{u})$ is a global solution of $\mathcal{P}(\ell, \mathcal{L}, g, \Upsilon_1, B, \mathcal{R}, n)$ if it is admissible and $\mathcal{J}(\hat{x}, \hat{u}) \leq \mathcal{J}(x, u)$ for all (x, u) admissible. An admissible process $(\hat{x}, \hat{u})$ is a *weak minimum* of $\mathcal{P}(\ell, \mathcal{L}, g, \Upsilon_1, B, \mathcal{R}, n)$ if it is a minimum of $\mathcal{J}$ with respect to the essential supremum norm, that is, $\mathcal{J}(\hat{x}, \hat{u}) \leq \mathcal{J}(x, u)$ for all admissible processes verifying $\|(x, u) - (\hat{x}, \hat{u})\|_\infty < \epsilon$, for some $\epsilon > 0$.

Let $\Psi\colon \mathbf{R}^n \to \mathbf{R}^n$ be any twice continuously differentiable function such that $B \subset \Psi(\mathbf{R}^n)$. Connect the nonparametric optimal control problem $\mathcal{P}(\ell, \mathcal{L}, g, \Upsilon_1, B, \mathcal{R}, n)$ with the parametric optimal control problem stated in Section 2, denoted by $P(\gamma, \Gamma, C, f, \xi_1, \Psi, R, s)$, that is, $P(\gamma, \Gamma, C, f, \xi_1, \Psi, R, s)$ is the parametric problem stated in Section 2, with the next data; $\gamma = \ell \circ \Psi$, $\Gamma = \mathcal{L}$, $C = \Psi^{-1}(B)$, $f = g$, $\xi_1 = \Upsilon_1$, Ψ the function given above, $R = \mathcal{R}$ and $s = n$.

Lemma 1. *The following conditions are satisfied:*

(i) z_a *is an admissible process of* $P(\gamma, \Gamma, C, f, \xi_1, \Psi, R, n)$ *if and only if* (x, u) *is a feasible process of* $\mathcal{P}(\ell, \mathcal{L}, g, \Upsilon_1, B, \mathcal{R}, n)$ *and* $a \in \Psi^{-1}(x(t_2))$.

(ii) If z_a *is an admissible process of* $P(\gamma, \Gamma, C, f, \xi_1, \Psi, R, n)$, *then*

$$\mathcal{J}(x, u) = I(z_a).$$

(iii) If $\hat{z}_{\hat{a}}$ *solves* $P(\gamma, \Gamma, C, f, \xi_1, \Psi, R, n)$, *then* $(\hat{x}, \hat{u})$ *solves* $\mathcal{P}(\ell, \mathcal{L}, g, \Upsilon_1, B, \mathcal{R}, n)$.

Proof. Index (i) follows from the definition of the problems. In order to prove (ii), note that, if z_a is an admissible process of $P(\gamma, \Gamma, C, f, \xi_1, \Psi, R, n)$, then, by (i), (x, u) is an admissible process of $\mathcal{P}(\ell, \mathcal{L}, g, \Upsilon_1, B, \mathcal{R}, n)$ and $x(t_2) = \Psi(a)$. Then,

$$\begin{aligned} \mathcal{J}(x,u) &= \ell(x(t_2)) + \int_{t_1}^{t_2} \mathcal{L}(t, x(t), u(t))dt \\ &= \ell(\Psi(a)) + \int_{t_1}^{t_2} \Gamma(t, x(t), u(t))dt \\ &= \gamma(a) + \int_{t_1}^{t_2} \Gamma(t, x(t), u(t))dt = I(z_a). \end{aligned}$$

Finally, in order to prove (iii), let z_a be an admissible process of $P(\gamma, \Gamma, C, f, \xi_1, \Psi, R, n)$. By (i), $(\hat{x}, \hat{u})$ and (x, u) are admissible of $\mathcal{P}(\ell, \mathcal{L}, g, \Upsilon_1, B, \mathcal{R}, n)$. Then, by (ii) and (iii),

$$\mathcal{J}(\hat{x}, \hat{u}) = I(\hat{z}_{\hat{a}}) \leq I(z_a) = \mathcal{J}(x, u).$$

□

Corollary 1 below is a straightforward implication of Theorem 1 and Lemma 1. It provides sufficient conditions for weak minima of the nonparametric problem $\mathcal{P}(\ell, \mathcal{L}, g, \Upsilon_1, B, \mathcal{R}, n)$. It is worth observing that the proposed optimal control is not necessarily continuous but only measurable as was the case of Theorem 1.

Corollary 1. *Let $\Psi\colon \mathbf{R}^n \to \mathbf{R}^n$ be any twice continuously differentiable function such that $B \subset \Psi(\mathbf{R}^n)$ and let $P(\gamma, \Gamma, C, f, \xi_1, \Psi, R, n)$ be the parametric optimal control problem before pronouncing Lemma 1. Let $\hat{z}_{\hat{a}}$ be an admissible process of $P(\gamma, \Gamma, C, f, \xi_1, \Psi, R, n)$. Suppose that $i(\cdot, \hat{x}(\cdot), \hat{u}(\cdot))$ is piecewise constant on $\mathcal{T}$, there exist $\omega \in X$, $\nu \in U_q$ satisfying $\nu_\sigma(t) \geq 0$ and $\nu_\sigma(t)\varphi_\sigma(t, \hat{x}(t), \hat{u}(t)) = 0$ ($\sigma \in P, t \in \mathcal{T}$), two positive numbers δ, ϵ such that*

$$\dot{\omega}(t) = -\mathcal{H}_x^*(t, \hat{x}(t), \hat{u}(t), \omega(t), \nu(t)) \text{ (a.e. in } \mathcal{T}),$$

$$\mathcal{H}_u^*(t, \hat{x}(t), \hat{u}(t), \omega(t), \nu(t)) = 0 \ (t \in \mathcal{T}),$$

and the following conditions are satisfied:

(i) $\gamma'^*(\hat{a}) + \Psi'^*(\hat{a})\omega(t_2) = 0$.
(ii) $\omega^*(t_2)\Psi''(\hat{a}; h) \geq 0$ *for all* $h \in \mathbf{R}^n$.
(iii) $\mathcal{H}_{uu}(t, \hat{x}(t), \hat{u}(t), \omega(t), \nu(t)) \leq 0$ (*a.e. in* $\mathcal{T}$).
(iv) $J''(\hat{z}_{\hat{a}}; w_\alpha) > 0$ *for all* $w_\alpha \in Y(\hat{z}_{\hat{a}})$, $w_\alpha \not\equiv (0, 0, 0)$.
(v) z_a *admissible with* $\|(x, u) - (\hat{x}, \hat{u})\|_\infty < \epsilon$ *implies that* $\int_{t_1}^{t_2} E(t, x(t), \hat{u}(t), u(t)) \geq \delta\mathcal{D}(u - \hat{u})$.

Then, $(\hat{x}, \hat{u})$ *is a weak minimum of* $\mathcal{P}(\ell, \mathcal{L}, g, \Upsilon_1, B, \mathcal{R}, n)$.

Examples 1 and 2 below show how even a non-expert can apply Corollary 1. Examples 1 and 2 are concerned with an inequality-equality restrained optimal control problem in which one has to verify that an element $(\hat{x}, \hat{u}, \omega, \nu)$ satisfies the sufficient conditions

$$\dot{\omega}(t) = -\mathcal{H}_x^*(t, \hat{x}(t), \hat{u}(t), \omega(t), \nu(t)) \text{ (a.e. in } \mathcal{T}), \quad \mathcal{H}_u^*(t, \hat{x}(t), \hat{u}(t), \omega(t), \nu(t)) = 0 \ (t \in \mathcal{T}),$$

and that the former also satisfies conditions (i), (ii), (iii), (iv), and (v) of Corollary 1, implying that it is a weak minimum of $\mathcal{P}(\ell, \mathcal{L}, g, \Upsilon_1, B, \mathcal{R}, n)$.

Example 1. *Consider the nonparametric optimal control problem $\mathcal{P}(\ell, \mathcal{L}, g, \Upsilon_1, B, \mathcal{R}, n)$ of finding a minimum value to the functional*

$$\mathcal{J}(x, u) = x^2(1) - x(1) + \int_0^1 \{\exp(tu(t)) + \sinh x(t)\} dt$$

over all (x, u) in $\mathcal{A}$ verifying the constraints

$$\begin{cases} \dot{x}(t) = u(t) \text{ almost everywhere in } [0, 1]. \\ x(0) = 0, \ x(1) \in (-\infty, 0]. \\ (t, x(t), u(t)) \in \mathcal{R} \ (t \in [0, 1]) \end{cases}$$

where

$$\mathcal{R} := \{(t, x, u) \in [0, 1] \times \mathbf{R} \times \mathbf{R} \mid (3/2)u^2 - x^2 - \exp(-x) - x + 1 \leq 0\},$$

$$\mathcal{A} := X \times U_1,$$

$$X := \{x\colon [0, 1] \to \mathbf{R} \mid x \text{ is absolutely continuous on } [0, 1]\},$$

$$U_1 := \{u\colon [0, 1] \to \mathbf{R} \mid u \text{ is essentially bounded on } [0, 1]\}.$$

For this event, the data of the proposed nonparametric problem are given by $\mathcal{T} = [0, 1]$, $m = 1$, $p = 1$, $q = 1$, $\ell(\cdot) = x^2(\cdot) - x(\cdot)$, $\mathcal{L}(t, x, u) = \exp(tu) + \sinh x$, $g(t, x, u) = u$,

$\Upsilon_1 = 0$, $B = (-\infty, 0]$, $\mathcal{R} = \{(t, x, u) \in \mathcal{T} \times \mathbf{R} \times \mathbf{R} \mid (3/2)u^2 - x^2 - \exp(-x) - x + 1 \leq 0\}$ and $n = 1$. Observe that

$$\phi_1(t, x, u) = (3/2)u^2 - x^2 - \exp(-x) - x + 1.$$

We have that the functions $\mathcal{L}$, g, $\phi = \phi_1$, and their first and second derivatives relative to x and u are continuous on $\mathcal{T} \times \mathbf{R} \times \mathbf{R}$. Additionally, the function ℓ is C^2 in $\mathbf{R}$.

Moreover, one can verify that the process $(\hat{x}, \hat{u}) \equiv (0, 0)$ is admissible of $\mathcal{P}(\ell, \mathcal{L}, g, \Upsilon_1, B, \mathcal{R}, n)$. Let $\Psi \colon \mathbf{R} \to \mathbf{R}$ be given by $\Psi(b) := b$. Clearly, Ψ is C^2 in $\mathbf{R}$ and $B \subset \Psi(\mathbf{R})$. The connected parametric problem designated by $P(\gamma, \Gamma, C, f, \xi_1, \Psi, R, s)$ has the next data; $\gamma = \ell \circ \Psi$, $\Gamma = \mathcal{L}$, $C = \Psi^{-1}(B)$, $f = g$, $\xi_1 = \Upsilon_1$, Ψ the function given above, $R = \mathcal{R}$ and $s = n$.

Observe that, if we set $\hat{a} := 0$, then $\hat{z}_{\hat{a}} = (\hat{x}, \hat{u}, \hat{a}) \equiv (0, 0, 0)$ is admissible of $P(\gamma, \Gamma, C, f, \xi_1, \Psi, R, n)$. Moreover, $i(\cdot, \hat{x}(\cdot), \hat{u}(\cdot)) \equiv \{1\}$ is constant on $\mathcal{T}$. Let $\omega \equiv t$, $\nu_1 \equiv 1$ and observe that $(\omega, \nu) \in X \times U_1$, $\nu_\sigma \geq 0$ and $\nu_\sigma(t)\varphi_\sigma(t, \hat{x}(t), \hat{u}(t)) = 0$ $(t \in \mathcal{T}, \sigma = 1)$. Recall that $\varphi = \phi$.

Now,

$$\mathcal{H}(t, x, u, \omega, \nu) = \omega u - \exp(tu) - \sinh x - \nu_1[(3/2)u^2 - x^2 - \exp(-x) - x + 1],$$

and observe that

$$\mathcal{H}_x(t, x, u, \omega, \nu) = -\cosh x - \nu_1[-2x + \exp(-x) - 1],$$

$$\mathcal{H}_u(t, x, u, \omega, \nu) = \omega - t\exp(tu) - 3\nu_1 u.$$

Then,

$$\dot{\omega}(t) = -\mathcal{H}_x(t, \hat{x}(t), \hat{u}(t), \omega(t), \nu(t)) \text{ (a.e. in } \mathcal{T}) \quad \text{and} \quad \mathcal{H}_u(t, \hat{x}(t), \hat{u}(t), \omega(t), \nu(t)) = 0 \ (t \in \mathcal{T})$$

and hence $(\hat{x}, \hat{u}, \omega, \nu)$ verifies the first order sufficiency conditions of Corollary 1. Since $\Psi(b) = b$ $(b \in \mathbf{R})$, we have that $\gamma(b) = b^2 - b$ $(b \in \mathbf{R})$. Then,

$$\gamma'(\hat{a}) + \Psi'(\hat{a})\omega(1) = 0$$

and hence condition (i) of Corollary 1 is verified. Moreover, one can verify that

$$\omega(1)\Psi''(\hat{a}; h) = 0 \text{ for all } h \in \mathbf{R}$$

and then condition (ii) of Corollary 1 is verified.

Now, for all $(t, x, u) \in \mathcal{T} \times \mathbf{R} \times \mathbf{R}$,

$$\mathcal{H}(t, x, u, \omega(t), \nu(t)) = tu - \exp(tu) - \sinh x - [(3/2)u^2 - x^2 - \exp(-x) - x + 1]$$

and hence, for all $t \in \mathcal{T}$,

$$\mathcal{H}_{uu}(t, \hat{x}(t), \hat{u}(t), \omega(t), \nu(t)) = -t^2 - 3 \leq 0$$

implying that $(\hat{x}, \hat{u}, \omega, \nu)$ satisfies condition (iii) of Corollary 1.

Additionally, note that, for all $t \in \mathcal{T}$,

$$f_x(t, \hat{x}(t), \hat{u}(t)) = 0 \quad \text{and} \quad f_u(t, \hat{x}(t), \hat{u}(t)) = 1,$$

$$\varphi_x(t, \hat{x}(t), \hat{u}(t)) = 0 \quad \text{and} \quad \varphi_u(t, \hat{x}(t), \hat{u}(t)) = 0.$$

Consequently, $Y(\hat{z}_{\hat{a}})$ is given by all $w_\alpha \in X \times L^2(\mathcal{T}; \mathbf{R}) \times \mathbf{R}$ verifying

$$\begin{cases} \dot{y}(t) = v(t) \text{ (a.e. in } \mathcal{T}). \\ y(0) = 0,\ y(1) = \alpha. \end{cases}$$

In addition, observe that, for all $(t, x, u) \in \mathcal{T} \times \mathbf{R} \times \mathbf{R}$,

$$\mathcal{F}(t, x, u) = -tu + \exp(tu) + (3/2)u^2 + \sinh x - x^2 - \exp(-x) - 2x + 1$$

and, for all $t \in \mathcal{T}$,

$$\mathcal{F}_{xx}(t, \hat{x}(t), \hat{u}(t)) = -3, \quad \mathcal{F}_{xu}(t, \hat{x}(t), \hat{u}(t)) = 0, \quad \mathcal{F}_{uu}(t, \hat{x}(t), \hat{u}(t)) = t^2 + 3.$$

Thus, for all $w_\alpha \in Y(\hat{z}_{\hat{a}})$,

$$J''(\hat{z}_{\hat{a}}; w_\alpha) = 2\alpha^2 + \int_0^1 3\{v^2(t) - y^2(t)\}dt + \int_0^1 3t^2 v^2(t)dt \geq 2\alpha^2 + \int_0^1 3\{\dot{y}^2(t) - y^2(t)\}dt.$$

Hence,

$$J''(\hat{z}_{\hat{a}}; w_\alpha) > 0$$

for all $w_\alpha \in Y(\hat{z}_{\hat{a}})$, $w_\alpha \not\equiv (0, 0, 0)$, and hence condition (iv) of Corollary 1 is fulfilled.

Now, note that, if z_a is admissible, for all $t \in \mathcal{T}$,

$$E(t, x(t), \hat{u}(t), u(t)) = -tu(t) + \exp(tu(t)) + (3/2)u^2(t) - 1.$$

Thus, if z_a is admissible,

$$\begin{aligned} \int_0^1 E(t, x(t), \hat{u}(t), u(t))dt &= \int_0^1 \{-tu(t) + \exp(tu(t)) + (3/2)u^2(t) - 1\}dt \geq \int_0^1 (1/2)u^2(t)dt \\ &\geq \int_0^1 L(u(t) - \hat{u}(t))dt = \mathcal{D}(u - \hat{u}). \end{aligned}$$

Therefore, condition (v) of Corollary 1 is satisfied for any $\epsilon > 0$ and $\delta = 1$. By Corollary 1, $(\hat{x}, \hat{u})$ is a weak minimum of $\mathcal{P}(\ell, \mathcal{L}, g, \Upsilon_1, B, \mathcal{R}, n)$.

Example 2. *Let us study the nonparametric optimal control problem $\mathcal{P}(\ell, \mathcal{L}, g, \Upsilon_1, B, \mathcal{R}, n)$ of minimizing the functional*

$$\mathcal{J}(x, u) = x^2(1) + \int_0^1 \{\tfrac{1}{2}(u_1(t) + u_2(t))^2 + u_1(t)\}dt$$

over all (x, u) in $\mathcal{A}$ satisfying the constraints

$$\begin{cases} \dot{x}(t) = u_1(t) + u_2(t) + x^3(t) \textit{ almost everywhere in } [0, 1]. \\ x(0) = 0, \ x(1) \in \mathbf{R}. \\ (t, x(t), u(t)) \in \mathcal{R} \ (t \in [0, 1]) \end{cases}$$

where

$$\mathcal{R} := \{(t, x, u) \in [0, 1] \times \mathbf{R} \times \mathbf{R}^2 \mid -\tfrac{1}{2}x^2 - u_1 \leq 0, \ \sin u_2 = 0\},$$

$$\mathcal{A} := X \times U_2,$$

$$X := \{x \colon [0, 1] \to \mathbf{R} \mid x \textit{ is absolutely continuous on } [0, 1]\},$$

$$U_2 := \{u \colon [0, 1] \to \mathbf{R}^2 \mid u \textit{ is essentially bounded on } [0, 1]\}.$$

For this event, the data of the nonparametric problem are given by $\mathcal{T} = [0, 1]$, $m = 2$, $p = 1$, $q = 2$, $\ell(\cdot) = x^2(\cdot)$, $\mathcal{L}(t, x, u) = \frac{1}{2}(u_1 + u_2)^2 + u_1$, $g(t, x, u) = u_1 + u_2 + x^3$, $\Upsilon_1 = 0$, $B = \mathbf{R}$, $\mathcal{R} = \{(t, x, u) \in \mathcal{T} \times \mathbf{R} \times \mathbf{R}^2 \mid -\frac{1}{2}x^2 - u_1 \leq 0, \ \sin u_2 = 0\}$ and $n = 1$. Observe that

$$\phi_1(t, x, u) = -\tfrac{1}{2}x^2 - u_1 \quad \text{and} \quad \phi_2(t, x, u) = \sin u_2.$$

We have that the functions $\mathcal{L}$, g, $\phi = (\phi_1, \phi_2)$ and their first and second derivatives with respect to x and u are continuous on $\mathcal{T} \times \mathbf{R} \times \mathbf{R}^2$. Additionally, the function ℓ is C^2 in $\mathbf{R}$.

Moreover, as one readily verifies, the process $(\hat{x}, \hat{u}) \equiv (0,0,0)$ is admissible of $\mathcal{P}(\ell, \mathcal{L}, g, \Upsilon_1, B, \mathcal{R}, n)$. Let $\Psi\colon \mathbf{R} \to \mathbf{R}$ be defined by $\Psi(b) := b$. Clearly, Ψ is C^2 in $\mathbf{R}$ and $B \subset \Psi(\mathbf{R})$. The connected parametric problem designated by $P(\gamma, \Gamma, C, f, \xi_1, \Psi, R, s)$ has the next data; $\gamma = \ell \circ \Psi$, $\Gamma = \mathcal{L}$, $C = \Psi^{-1}(B)$, $f = g$, $\xi_1 = \Upsilon_1$, Ψ the function given above, $R = \mathcal{R}$ and $s = n$.

Observe that, if we set $\hat{a} := 0$, then $\hat{z}_{\hat{a}} = (\hat{x}, \hat{u}, \hat{a}) \equiv (0,0,0,0)$ is admissible of $P(\gamma, \Gamma, C, f, \xi_1, \Psi, R, n)$. Moreover, $i(\cdot, \hat{x}(\cdot), \hat{u}(\cdot)) \equiv \{1\}$ is constant on $\mathcal{T}$. Let $\omega \equiv 0$, $\nu_1 \equiv 1$, $\nu_2 \equiv 0$ and observe that $(\omega, \nu) \in X \times U_2$, $\nu_\sigma \geq 0$ and $\nu_\sigma(t)\varphi_\sigma(t, \hat{x}(t), \hat{u}(t)) = 0$ $(t \in \mathcal{T}, \sigma = 1)$. Recall that $\varphi = \phi$.

Now,

$$\mathcal{H}(t,x,u,\omega,\nu) = \omega u_1 + \omega u_2 + \omega x^3 - \tfrac{1}{2}(u_1+u_2)^2 - u_1 + \tfrac{1}{2}\nu_1 x^2 + \nu_1 u_1 - \nu_2 \sin u_2,$$

and observe that

$$\mathcal{H}_x(t,x,u,\omega,\nu) = 3\omega x^2 + \nu_1 x,$$
$$\mathcal{H}_u(t,x,u,\omega,\nu) = (\omega - u_1 - u_2 - 1 + \nu_1, \omega - u_1 - u_2 - \nu_2 \cos u_2).$$

Consequently,

$$\dot{\omega}(t) = -\mathcal{H}_x(t, \hat{x}(t), \hat{u}(t), \omega(t), \nu(t)) \text{ (a.e. in } \mathcal{T}) \quad \text{and} \quad \mathcal{H}_u(t, \hat{x}(t), \hat{u}(t), \omega(t), \nu(t)) = (0,0) \ (t \in \mathcal{T})$$

and hence $(\hat{x}, \hat{u}, \omega, \nu)$ satisfies the first order sufficiency conditions of Corollary 1. Since $\Psi(b) = b$ $(b \in \mathbf{R})$, we have that $\gamma(b) = b^2$ $(b \in \mathbf{R})$. Then,

$$\gamma'(\hat{a}) + \Psi'(\hat{a})\omega(1) = 0$$

and then condition (i) of Corollary 1 is satisfied. Moreover, one can verify that

$$\omega(1)\Psi''(\hat{a}; h) = 0 \text{ for all } h \in \mathbf{R}$$

and hence condition (ii) of Corollary 1 is fulfilled.

Now, for all $(t, x, u) \in \mathcal{T} \times \mathbf{R} \times \mathbf{R}^2$,

$$\mathcal{H}(t,x,u,\omega(t),\nu(t)) = -\tfrac{1}{2}(u_1+u_2)^2 + \tfrac{1}{2}x^2$$

and hence, for all $t \in \mathcal{T}$,

$$\mathcal{H}_{uu}(t, \hat{x}(t), \hat{u}(t), \omega(t), \nu(t)) = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \leq 0$$

implying that $(\hat{x}, \hat{u}, \omega, \nu)$ verifies condition (iii) of Corollary 1.

Additionally, note that, for all $t \in \mathcal{T}$,

$$f_x(t, \hat{x}(t), \hat{u}(t)) = 0 \quad \text{and} \quad f_u(t, \hat{x}(t), \hat{u}(t)) = (1,1),$$

$$\varphi_x(t, \hat{x}(t), \hat{u}(t)) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \varphi_u(t, \hat{x}(t), \hat{u}(t)) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, $Y(\hat{z}_{\hat{a}})$ is given by all $w_\alpha \in X \times L^2(\mathcal{T}; \mathbf{R}^2) \times \mathbf{R}$ verifying

$$\begin{cases} \dot{y}(t) = v_1(t) + v_2(t) \text{ (a.e. in } \mathcal{T}). \\ y(0) = 0,\ y(1) = \alpha. \\ -v_1(t) \leq 0,\ v_2(t) = 0 \text{ (a.e. in } \mathcal{T}). \end{cases}$$

In addition, observe that, for all $(t,x,u) \in \mathcal{T} \times \mathbf{R} \times \mathbf{R}^2$,

$$\mathcal{F}(t,x,u) = \tfrac{1}{2}(u_1+u_2)^2 - \tfrac{1}{2}x^2$$

and, for all $t \in \mathcal{T}$,

$$\mathcal{F}_{xx}(t,\hat{x}(t),\hat{u}(t)) = -1, \quad \mathcal{F}_{xu}(t,\hat{x}(t),\hat{u}(t)) = (0,0), \quad \mathcal{F}_{uu}(t,\hat{x}(t),\hat{u}(t)) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Thus, for all $w_\alpha \in Y(\hat{z}_{\hat{a}})$,

$$J''(\hat{z}_{\hat{a}}; w_\alpha) = 2\alpha^2 + \int_0^1 \{(v_1(t)+v_2(t))^2 - y^2(t)\}dt = 2\alpha^2 + \int_0^1 \{\dot{y}^2(t) - y^2(t)\}dt.$$

Hence,

$$J''(\hat{z}_{\hat{a}}; w_\alpha) > 0$$

for all $w_\alpha \in Y(\hat{z}_{\hat{a}})$, $w_\alpha \not\equiv (0,0,0,0)$, and then condition (iv) of Corollary 1 is verified.

Now, note that, if z_a is admissible, for all $t \in \mathcal{T}$,

$$E(t,x(t),\hat{u}(t),u(t)) = \tfrac{1}{2}(u_1(t)+u_2(t))^2.$$

Therefore, if z_a is admissible,

$$\int_0^1 E(t,x(t),\hat{u}(t),u(t))dt = \int_0^1 \tfrac{1}{2}(u_1(t)+u_2(t))^2 dt \geq \int_0^1 L(u(t)-\hat{u}(t))dt = \mathcal{D}(u-\hat{u}).$$

Thus, condition (v) of Corollary 1 is verified for any $\epsilon > 0$ and $\delta = 1$. By Corollary 1, $(\hat{x},\hat{u})$ is a weak minimum of $\mathcal{P}(\ell,\mathcal{L},g,\Upsilon_1,B,\mathcal{R},n)$.

4. Supplementary Lemmas

Now, we enunciate three supplementary lemmas which are going to be fundamental in proving Theorem 1. These lemmas are direct consequences of Lemmas 3.1–3.3 of [23].

If (Σ_n) is a sequence of measurable functions and Σ is a measurable function, we shall designate uniform convergence of (Σ_n) to Σ by $\Sigma_n \xrightarrow{\mathrm{u}} \Sigma$. Similarly, strong convergence in L^p by $\Sigma_n \xrightarrow{L^p} \Sigma$ and weak convergence by $\Sigma_n \overset{L^p}{\rightharpoonup} \Sigma$.

In the next three lemmas, we suppose that $\hat{u} \in L^1(\mathcal{T};\mathbf{R}^m)$ is given and a sequence (u_q) in $L^1(T;\mathbf{R}^m)$ such that

$$\lim_{q\to\infty} \mathcal{D}(u_q - \hat{u}) = 0 \quad \text{and} \quad d_q := [2\mathcal{D}(u_q-\hat{u})]^{1/2} > 0 \quad (q \in \mathbf{N}).$$

For all $q \in \mathbf{N}$, define

$$v_q := \frac{u_q - \hat{u}}{d_q}.$$

Lemma 2. *For some $\hat{v} \in L^2(\mathcal{T};\mathbf{R}^m)$ and some subsequence of (u_q) (without relabeling), $v_q \overset{L^1}{\rightharpoonup} \hat{v}$ on $\mathcal{T}$.*

Lemma 3. *Let $A_q \in L^\infty(\mathcal{T};\mathbf{R}^{n\times n})$ and $B_q \in L^\infty(\mathcal{T};\mathbf{R}^{n\times m})$ be matrix-valued functions for which we have the existence of some constants $m_0, m_1 > 0$ such that $\|A_q\|_\infty \leq m_0$, $\|B_q\|_\infty \leq m_1$ $(q \in \mathbf{N})$, and for all $q \in \mathbf{N}$ indicate by y_q the solution of the initial value problem*

$$\dot{y}(t) = A_q(t)y(t) + B_q(t)v_q(t) \ (\text{a.e. in } \mathcal{T}), \quad y(t_1) = 0.$$

Then, there exist $\zeta \in L^2(\mathcal{T};\mathbf{R}^n)$ and a subsequence (without relabeling), such that $\dot{y}_q \overset{L^1}{\rightharpoonup} \zeta$ on $\mathcal{T}$, and hence, if $\hat{y}(t) := \int_{t_1}^t \zeta(\tau)d\tau$ $(t \in \mathcal{T})$, then $y_q \xrightarrow{u} \hat{y}$ on $\mathcal{T}$.

Lemma 4. *Suppose $u_q \xrightarrow{L^\infty} \hat{u}$ on $\mathcal{T}$, let $\Phi_q, \Phi \in L^\infty(\mathcal{T}; \mathbf{R}^{m\times m})$; suppose that $\Phi_q \xrightarrow{L^\infty} \Phi$ on $\mathcal{T}$, $\Phi(t) \geq 0$ (a.e. in $\mathcal{T}$) and let $\hat{v}$ be the function given in Lemma 2. Then,*

$$\liminf_{q\to\infty} \int_{t_1}^{t_2} v_q^*(t)\Phi_q(t)v_q(t)dt \geq \int_{t_1}^{t_2} \hat{v}^*(t)\Phi(t)\hat{v}(t)dt.$$

5. Proof of Theorem 1

The proof of Theorem 1 will be divided into two Lemmas. In Lemmas 5 and 6 below, we shall suppose that all the hypotheses of Theorem 1 are verified. Before stating the lemmas, let us present some definitions.

Note first that, given $x = (x_1, \ldots, x_n)^*$ in $\mathbf{R}^n$ and $a = (a_1, \ldots, a_s)^*$ in $\mathbf{R}^s$, if we set $x\mathbf{i}$, $a\mathbf{j}$ in $\mathbf{R}^{n+s}$ by $x\mathbf{i} := (x_1, \ldots, x_n, 0, \ldots, 0)^*$ and $a\mathbf{j} := (0, \ldots, 0, a_1, \ldots, a_s)^*$, then

$$x\mathbf{i} + a\mathbf{j} = (x_1, \ldots, x_n, a_1, \ldots, a_s)^* = \begin{pmatrix} x \\ a \end{pmatrix} \in \mathbf{R}^{n+s}.$$

Define $\tilde{\mathcal{F}}\colon \mathcal{T} \times \mathbf{R}^{n+s} \times \mathbf{R}^m \to \mathbf{R}$ by

$$\tilde{\mathcal{F}}(t, \xi, u) := \frac{\gamma(\xi_{n+1}, \ldots, \xi_{n+s})}{t_2 - t_1} + \mathcal{F}(t, \xi_1, \ldots, \xi_n, u).$$

Observe that the Weierstrass function $\tilde{E}\colon \mathcal{T} \times \mathbf{R}^{n+s} \times \mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}$ of $\tilde{\mathcal{F}}$ is given by

$$\tilde{E}(t, \xi, u, v) := \tilde{\mathcal{F}}(t, \xi, v) - \tilde{\mathcal{F}}(t, \xi, u) - \tilde{\mathcal{F}}_u(t, \xi, u)(v - u).$$

It is not difficult to see that, for all $(t, x, u, v) \in \mathcal{T} \times \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^m$ and all a in $\mathbf{R}^s$,

$$\tilde{E}(t, x\mathbf{i} + a\mathbf{j}, u, v) = E(t, x, u, v).$$

Set

$$\tilde{J}(z_a) := \omega^*(t_2)x(t_2) - \omega^*(t_1)x(t_1) + \int_{t_1}^{t_2} \tilde{\mathcal{F}}(t, x(t)\mathbf{i} + a\mathbf{j}, u(t))dt.$$

As one readily verifies, $J(z_a) = \tilde{J}(z_a)$ for all z_a in A, and

$$\tilde{J}(z_a) = \tilde{J}(\hat{z}_{\hat{a}}) + \tilde{J}'(\hat{z}_{\hat{a}}; z_a - \hat{z}_{\hat{a}}) + \tilde{\mathcal{K}}(\hat{z}_{\hat{a}}; z_a) + \tilde{\mathcal{E}}(\hat{z}_{\hat{a}}; z_a) \tag{1}$$

where

$$\tilde{\mathcal{E}}(\hat{z}_{\hat{a}}; z_a) := \int_{t_1}^{t_2} \tilde{E}(t, x(t)\mathbf{i} + a\mathbf{j}, \hat{u}(t), u(t))dt,$$

$$\tilde{\mathcal{K}}(\hat{z}_{\hat{a}}; z_a) := \int_{t_1}^{t_2} \{\tilde{\mathcal{M}}(t, x(t)\mathbf{i} + a\mathbf{j}) + [u^*(t) - \hat{u}^*(t)]\tilde{\mathcal{N}}(t, x(t)\mathbf{i} + a\mathbf{j})\}dt,$$

$$\begin{aligned} \tilde{J}'(\hat{z}_{\hat{a}}; z_a - \hat{z}_{\hat{a}}) \quad := \quad & \omega^*(t_2)[x(t_2) - \hat{x}(t_2)] - \omega^*(t_1)[x(t_1) - \hat{x}(t_1)] \\ & + \int_{t_1}^{t_2} \{\tilde{\mathcal{F}}_\xi(t, \hat{x}(t)\mathbf{i} + \hat{a}\mathbf{j}, \hat{u}(t))([x(t) - \hat{x}(t)]\mathbf{i} + [a - \hat{a}]\mathbf{j}) \\ & \qquad + \tilde{\mathcal{F}}_u(t, \hat{x}(t)\mathbf{i} + \hat{a}\mathbf{j}, \hat{u}(t))(u(t) - \hat{u}(t))\}dt, \end{aligned}$$

and $\tilde{\mathcal{M}}, \tilde{\mathcal{N}}$ are defined by

$$\begin{aligned} \tilde{\mathcal{M}}(t, x\mathbf{i} + a\mathbf{j}) \quad := \quad & \tilde{\mathcal{F}}(t, x\mathbf{i} + a\mathbf{j}, \hat{u}(t)) - \tilde{\mathcal{F}}(t, \hat{x}(t)\mathbf{i} + \hat{a}\mathbf{j}, \hat{u}(t)) \\ & - \tilde{\mathcal{F}}_\xi(t, \hat{x}(t)\mathbf{i} + \hat{a}\mathbf{j}, \hat{u}(t))([x - \hat{x}(t)]\mathbf{i} + [a - \hat{a}]\mathbf{j}), \end{aligned}$$

$$\tilde{\mathcal{N}}(t, x\mathbf{i} + a\mathbf{j}) := \tilde{\mathcal{F}}_u^*(t, x\mathbf{i} + a\mathbf{j}, \hat{u}(t)) - \tilde{\mathcal{F}}_u^*(t, \hat{x}(t)\mathbf{i} + \hat{a}\mathbf{j}, \hat{u}(t)).$$

By Taylor's theorem,

$$\tilde{\mathcal{M}}(t, x\mathbf{i} + a\mathbf{j}) = \tfrac{1}{2}([x^* - \hat{x}^*(t)]\mathbf{i} + [a^* - \hat{a}^*]\mathbf{j})\tilde{\mathcal{P}}(t, x\mathbf{i} + a\mathbf{j})([x - \hat{x}(t)]\mathbf{i} + [a - \hat{a}]\mathbf{j}), \quad (2a)$$

$$\tilde{\mathcal{N}}(t, x\mathbf{i} + a\mathbf{j}) = \tilde{\mathcal{Q}}(t, x\mathbf{i} + a\mathbf{j})([x - \hat{x}(t)]\mathbf{i} + [a - \hat{a}]\mathbf{j}), \quad (2b),$$

where

$$\tilde{\mathcal{P}}(t, x\mathbf{i} + a\mathbf{j}) := 2\int_0^1 (1-\theta)\tilde{\mathcal{F}}_{\xi\xi}(t, [\hat{x}(t) + \theta(x - \hat{x}(t))]\mathbf{i} + [\hat{a} + \theta(a - \hat{a})]\mathbf{j}, \hat{u}(t))d\theta,$$

$$\tilde{\mathcal{Q}}(t, x\mathbf{i} + a\mathbf{j}) := \int_0^1 \tilde{\mathcal{F}}_{u\xi}(t, [\hat{x}(t) + \theta(x - \hat{x}(t))]\mathbf{i} + [\hat{a} + \theta(a - \hat{a})]\mathbf{j}, \hat{u}(t))d\theta.$$

Lemma 5. *If the deduction of Theorem 1 is false, then we have the existence of a subsequence* $(z^q_{a_q})$ *of admissible processes such that*

$$\lim_{q\to\infty} \mathcal{D}(u_q - \hat{u}) = 0 \quad \text{and} \quad d_q := [2\mathcal{D}(u_q - \hat{u})]^{1/2} > 0 \quad (q \in \mathbf{N}).$$

Proof. If the deduction of Theorem 1 is false, then, for all $\rho_1, \rho_2 > 0$, there exists an admissible process z_a such that

$$\|z_a - \hat{z}_{\hat{a}}\| < \rho_1 \quad \text{and} \quad I(z_a) < I(\hat{z}_{\hat{a}}) + \rho_2 \min\{|a - \hat{a}|^2, \mathcal{D}(u - \hat{u})\}. \quad (3)$$

Since

$$\nu_\sigma(t) \geq 0 \ (\sigma \in P, \text{ a.e. in } \mathcal{T}),$$

if z_a is admissible, then $I(z_a) \geq J(z_a)$. Additionally, as

$$\nu_\sigma(t)\varphi_\sigma(t, \hat{x}(t), \hat{u}(t)) = 0 \ (\sigma \in P, \text{ a.e. in } \mathcal{T})$$

then $I(\hat{z}_{\hat{a}}) = J(\hat{z}_{\hat{a}})$. Thus, (3) implies that, for $\rho_1, \rho_2 > 0$, we have the existence of z_a admissible such that

$$\|z_a - \hat{z}_{\hat{a}}\| < \rho_1 \quad \text{and} \quad J(z_a) < J(\hat{z}_{\hat{a}}) + \rho_2 \min\{|a - \hat{a}|^2, \mathcal{D}(u - \hat{u})\}.$$

Therefore, if the deduction of Theorem 1 is false, then, for all $q \in \mathbf{N}$, we have the existence of a sequence of admissible processes $(z^q_{a_q})$ such that

$$\|z^q_{a_q} - \hat{z}_{\hat{a}}\| < \min\{\epsilon, 1/q\}, \quad J(z^q_{a_q}) - J(\hat{z}_{\hat{a}}) < \min\left\{\frac{|a_q - \hat{a}|^2}{q}, \frac{\mathcal{D}(u_q - \hat{u})}{q}\right\}. \quad (4)$$

The first relation in (4) assures that

$$\lim_{q\to\infty} \mathcal{D}(u_q - \hat{u}) = 0.$$

Moreover, as $(z^q_{a_q})$ is a sequence of admissible processes, we see that $\mathcal{D}(u_q - \hat{u}) = 0$ if and only if $z^q = \hat{z}$. Hence, the second relation of (4) implies that

$$\mathcal{D}(u_q - \hat{u}) = 0 \Longrightarrow a_q \neq \hat{a}.$$

Assume that $\mathcal{D}(u_q - \hat{u}) = 0$ for infinitely many q's. We have

$$0 = x_q(t_2) - \hat{x}(t_2) = \Psi(a_q) - \Psi(\hat{a}) = \int_0^1 \Psi'(\hat{a} + \theta[a_q - \hat{a}])(a_q - \hat{a})d\theta, \quad (5)$$

$$0 = \Psi(a_q) - \Psi(\hat{a}) = \Psi'(\hat{a})(a_q - \hat{a}) + \int_0^1 (1-\theta)\Psi''(\hat{a} + \theta[a_q - \hat{a}]; a_q - \hat{a})d\theta. \quad (6)$$

If we designate by $(a_q, \hat{a})$ the line segment in $\mathbf{R}^s$ joining the points a_q and $\hat{a}$, by the second relation of (4), by hypothesis (i) of Theorem (1), by (6), and the mean value theorem, we have the existence of $\Theta_q \in (a_q, \hat{a})$ such that

$$\begin{aligned} 0 &> J(\hat{z}_{a_q}) - J(\hat{z}_{\hat{a}}) \\ &= \gamma(a_q) - \gamma(\hat{a}) \\ &= \gamma'(\hat{a})(a_q - \hat{a}) + \tfrac{1}{2}(a_q - \hat{a})^* \gamma''(\Theta_q)(a_q - \hat{a}) \\ &= -\omega^*(t_2)\Psi'(\hat{a})(a_q - \hat{a}) + \tfrac{1}{2}(a_q - \hat{a})^* \gamma''(\Theta_q)(a_q - \hat{a}) \\ &= \int_0^1 (1-\theta)\omega^*(t_2)\Psi''(\hat{a} + \theta[a_q - \hat{a}]; a_q - \hat{a})d\theta + \tfrac{1}{2}(a_q - \hat{a})^* \gamma''(\Theta_q)(a_q - \hat{a}). \end{aligned} \tag{7}$$

Select an adequately subsequence of $((a_q - \hat{a})/|a_q - \hat{a}|)$, such that

$$\lim_{q\to\infty} \frac{a_q - \hat{a}}{|a_q - \hat{a}|} = \hat{\alpha} \tag{8}$$

for some $\hat{\alpha} \in \mathbf{R}^s$ satisfying $|\hat{\alpha}| = 1$. By (5),

$$\Psi'(\hat{a})\hat{\alpha} = 0.$$

By (7) and (8) and hypothesis (ii) of Theorem 1, we see that

$$0 \geq \tfrac{1}{2}\omega^*(t_2)\Psi''(\hat{a}; \hat{\alpha}) + \tfrac{1}{2}\hat{\alpha}^* \gamma''(\hat{a})\hat{\alpha} \geq \tfrac{1}{2}\hat{\alpha}^* \gamma''(\hat{a})\hat{\alpha} = \tfrac{1}{2}J''(\hat{z}_{\hat{a}}; 0_{\hat{\alpha}})$$

contradicting (iv) of Theorem 1. Consequently, we may suppose that, for all $q \in \mathbf{N}$,

$$d_q = [2\mathcal{D}(u_q - \hat{u})]^{1/2} > 0.$$

□

Lemma 6. *If the deduction of Theorem 1 is false, then condition (iv) of Theorem 1 is false.*

Proof. Let $(z^q_{a_q})$ be the sequence of admissible processes provided in Lemma 5. Hence,

$$\lim_{q\to\infty} \mathcal{D}(u_q - \hat{u}) = 0 \quad \text{and} \quad d_q = [2\mathcal{D}(u_q - \hat{u})]^{1/q} > 0 \quad (q \in \mathbf{N}).$$

Case(1): First, assume that the sequence $((a_q - \hat{a})/d_q)$ is bounded in $\mathbf{R}^s$. For all $q \in \mathbf{N}$, set

$$y_q := \frac{x_q - \hat{x}}{d_q}, \quad v_q := \frac{u_q - \hat{u}}{d_q}, \quad \varpi_q := y_q\mathbf{i} + \frac{a_q - \hat{a}}{d_q}\mathbf{j}.$$

By Lemma 2, there exist $\hat{v} \in L^2(\mathcal{T}; \mathbf{R}^m)$ and a subsequence of $(z^q_{a_q})$ (without relabeling) such that $v_q \xrightarrow{L^1} \hat{v}$ on $\mathcal{T}$. We have, for all $q \in \mathbf{N}$, that

$$\dot{y}_q(t) = A_q(t)y_q(t) + B_q(t)v_q(t) \text{ (a.e. in } \mathcal{T}), \quad y_q(t_1) = 0,$$

where

$$A_q(t) := \int_0^1 f_x(t, \hat{x}(t) + \theta[x_q(t) - \hat{x}(t)], \hat{u}(t) + \theta[u_q(t) - \hat{u}(t)])d\theta,$$

$$B_q(t) := \int_0^1 f_u(t, \hat{x}(t) + \theta[x_q(t) - \hat{x}(t)], \hat{u}(t) + \theta[u_q(t) - \hat{u}(t)])d\theta.$$

We obtain the existence of $m_0, m_1 > 0$ such that $\|A_q\|_\infty \leq m_0$, $\|B_q\|_\infty \leq m_1$ $(q \in \mathbf{N})$. By Lemma 3, there exist $\zeta \in L^2(\mathcal{T}; \mathbf{R}^n)$ and some subsequence of $(z^q_{a_q})$ (we do not relabel) such that, if for all $t \in \mathcal{T}$, $\hat{y}(t) := \int_{t_1}^t \zeta(\tau)d\tau$, then

$$y_q \xrightarrow{\mathrm{u}} \hat{y} \text{ on } \mathcal{T}. \tag{9}$$

As the sequence $((a_q - \hat{a})/d_q)$ is bounded in $\mathbf{R}^s$, then we can suppose that there exists some $\hat{\alpha} \in \mathbf{R}^s$ such that

$$\lim_{q\to\infty} \frac{a_q - \hat{a}}{d_q} = \hat{\alpha}. \tag{10}$$

First, we shall show that

$$\hat{y}(t_2) = \Psi'(\hat{a})\hat{\alpha}. \tag{11}$$

Note that, we have, for all $q \in \mathbf{N}$, that

$$y_q(t_2) = \int_0^1 \Psi'(\hat{a} + \theta[a_q - \hat{a}]) \frac{(a_q - \hat{a})}{d_q} d\theta. \tag{12}$$

By (9), (10), and (12), as one readily verifies, (11) holds. Now, we claim that

$$J''(\hat{z}_{\hat{a}}; \hat{w}_{\hat{\alpha}}) \leq 0 \quad \text{and} \quad \hat{w}_{\hat{\alpha}} = (\hat{y}, \hat{v}, \hat{\alpha}) \not\equiv (0, 0, 0). \tag{13}$$

In order to prove it, note that, by (2), (9), and (10),

$$\begin{aligned} \frac{\tilde{\mathcal{M}}(\cdot, x_q(\cdot)\mathbf{i} + a_q\mathbf{j})}{d_q^2} &= \tfrac{1}{2}\varpi_q^*(\cdot)\tilde{\mathcal{P}}(\cdot, x_q(\cdot)\mathbf{i} + a_q\mathbf{j})\varpi_q(\cdot) \xrightarrow{L^\infty} \\ &\quad \tfrac{1}{2}[\hat{y}^*(\cdot)\mathbf{i} + \hat{\alpha}^*\mathbf{j}]\tilde{\mathcal{F}}_{\xi\xi}(\cdot, \hat{x}(\cdot)\mathbf{i} + \hat{a}\mathbf{j}, \hat{u}(\cdot))[\hat{y}(\cdot)\mathbf{i} + \hat{\alpha}\mathbf{j}], \end{aligned}$$

$$\frac{\tilde{\mathcal{N}}(\cdot, x_q(\cdot)\mathbf{i} + a_q\mathbf{j})}{d_q} = \tilde{\mathcal{Q}}(\cdot, x_q(\cdot)\mathbf{i} + a_q\mathbf{j})\varpi_q(\cdot) \xrightarrow{L^\infty} \tilde{\mathcal{F}}_{u\xi}(\cdot, \hat{x}(\cdot)\mathbf{i} + \hat{a}\mathbf{j}, \hat{u}(\cdot))[\hat{y}(\cdot)\mathbf{i} + \hat{\alpha}\mathbf{j}]$$

both on $\mathcal{T}$. This fact together with Lemma 2 implies that

$$\begin{aligned} \lim_{q\to\infty} \frac{\tilde{\mathcal{K}}(\hat{z}_{\hat{a}}; z_{a_q}^q)}{d_q^2} &= \frac{1}{2}\int_{t_1}^{t_2} \{[\hat{y}^*(t)\mathbf{i} + \hat{\alpha}^*\mathbf{j}]\tilde{\mathcal{F}}_{\xi\xi}(t, \hat{x}(t)\mathbf{i} + \hat{\alpha}\mathbf{j}, \hat{u}(t))[\hat{y}(t)\mathbf{i} + \hat{\alpha}\mathbf{j}] \\ &\quad + 2\hat{v}^*(t)\tilde{\mathcal{F}}_{u\xi}(t, \hat{x}(t)\mathbf{i} + \hat{\alpha}\mathbf{j}, \hat{u}(t))[\hat{y}(t)\mathbf{i} + \hat{\alpha}\mathbf{j}]\}dt. \end{aligned} \tag{14}$$

As $(\hat{x}, \hat{u}, \omega, \nu)$ satisfies the first order sufficient conditions

$$\dot{\omega}(t) = -\mathcal{H}_x^*(t, \hat{x}(t), \hat{u}(t), \omega(t), \nu(t)) \text{ (a.e. in } \mathcal{T}), \quad \mathcal{H}_u^*(t, \hat{x}(t), \hat{u}(t), \omega(t), \nu(t)) = 0 \ (t \in \mathcal{T}),$$

and, by condition (i) of Theorem 1, we obtain

$$\begin{aligned} \lim_{q\to\infty} \frac{\tilde{J}'(\hat{z}_{\hat{a}}; z_{a_q}^q - \hat{z}_{\hat{a}})}{d_q^2} &= \lim_{q\to\infty} \frac{1}{d_q^2}[\omega^*(t_2)(x_q(t_2) - \hat{x}(t_2)) + \gamma'(\hat{a})(a_q - \hat{a})] \\ &= \lim_{q\to\infty} \frac{1}{d_q^2}[\omega^*(t_2)(\Psi(a_q) - \Psi(\hat{a})) - \omega^*(t_2)\Psi'(\hat{a})(a_q - \hat{a})] \\ &= \lim_{q\to\infty} \frac{1}{d_q^2}\omega^*(t_2)(\Psi(a_q) - \Psi(\hat{a}) - \Psi'(\hat{a})(a_q - \hat{a})) \\ &= \lim_{q\to\infty} \frac{1}{d_q^2}\int_0^1 \omega^*(t_2)(1-\theta)\Psi''(\hat{a} + \theta[a_q - \hat{a}]; a_q - \hat{a})d\theta \\ &= \tfrac{1}{2}\omega^*(t_2)\Psi''(\hat{a}; \hat{\alpha}). \end{aligned} \tag{15}$$

Then, by (1), the fact that

$$J(z^q_{a_q}) - J(\hat{z}_{\hat{a}}) < \min\left\{ \frac{|a_q - \hat{a}|^2}{q}, \frac{\mathcal{D}(u_q - \hat{u})}{q} \right\},$$

Equation (15) and hypothesis (ii) of Theorem 1,

$$0 \geq \lim_{q\to\infty} \frac{\tilde{\mathcal{K}}(\hat{z}_{\hat{a}}; z^q_{a_q})}{d_q^2} + \liminf_{q\to\infty} \frac{\tilde{\mathcal{E}}(\hat{z}_{\hat{a}}; z^q_{a_q})}{d_q^2}. \tag{16}$$

Now, we have, for all $t \in \mathcal{T}$ and $q \in \mathbf{N}$, that

$$\frac{1}{d_q^2}\tilde{E}(t, x_q(t)\mathbf{i} + a_q\mathbf{j}, \hat{u}(t), u_q(t)) = \tfrac{1}{2} v_q^*(t)\Phi_q(t)v_q(t),$$

where

$$\Phi_q(t) := 2\int_0^1 (1-\theta)\tilde{\mathcal{F}}_{uu}(t, x_q(t)\mathbf{i} + a_q\mathbf{j}, \hat{u}(t) + \theta[u_q(t) - \hat{u}(t)])d\theta.$$

We have

$$\Phi_q(\cdot) \xrightarrow{L^\infty} \Phi(\cdot) := \tilde{\mathcal{F}}_{uu}(\cdot, \hat{x}(\cdot)\mathbf{i} + \hat{a}\mathbf{j}, \hat{u}(\cdot)) \text{ on } \mathcal{T}.$$

By condition (iii) of Theorem 1, we have

$$\tilde{\mathcal{F}}_{uu}(t, \hat{x}(t)\mathbf{i} + \hat{a}\mathbf{j}, \hat{u}(t)) = \Phi(t) \geq 0 \text{ (a.e. in } \mathcal{T}). \tag{17}$$

By the fact that

$$\|z^q_{a_q} - \hat{z}_{\hat{a}}\| < \frac{1}{q},$$

$u_q \xrightarrow{L^\infty} \hat{u}$ on $\mathcal{T}$. Keeping this in mind, by (17) and Lemma 4,

$$\begin{aligned}
\liminf_{q\to\infty} \frac{\tilde{\mathcal{E}}(\hat{z}_{\hat{a}}; z^q_{a_q})}{d_q^2} &= \liminf_{q\to\infty} \frac{1}{d_q^2}\int_{t_1}^{t_2} \tilde{E}(t, x_q(t)\mathbf{i} + a_q\mathbf{j}, \hat{u}(t), u_q(t))dt \\
&= \frac{1}{2}\liminf_{q\to\infty}\int_{t_1}^{t_2} v_q^*(t)\Phi_q(t)v_q(t)dt \geq \frac{1}{2}\int_{t_1}^{t_2} \hat{v}^*(t)\Phi(t)\hat{v}(t)dt. \qquad (18)
\end{aligned}$$

By (16) and (18), we have

$$\begin{aligned}
0 &\geq \int_{t_1}^{t_2} \{\hat{v}^*(t)\tilde{\mathcal{F}}_{uu}(t, \hat{x}(t)\mathbf{i} + \hat{a}\mathbf{j}, \hat{u}(t))\hat{v}(t) + 2\hat{v}^*(t)\tilde{\mathcal{F}}_{u\xi}(t, \hat{x}(t)\mathbf{i} + \hat{a}\mathbf{j}, \hat{u}(t))[\hat{y}(t)\mathbf{i} + \hat{\alpha}\mathbf{j}] \\
&\qquad + [\hat{y}^*(t)\mathbf{i} + \hat{\alpha}^*\mathbf{j}]\tilde{\mathcal{F}}_{\xi\xi}(t, \hat{x}(t)\mathbf{i} + \hat{a}\mathbf{j}, \hat{u}(t))[\hat{y}(t)\mathbf{i} + \hat{\alpha}\mathbf{j}]\}dt \\
&= \hat{\alpha}^*\gamma''(\hat{a})\hat{\alpha} + \int_{t_1}^{t_2} \{\hat{v}^*(t)\mathcal{F}_{uu}(t, \hat{x}(t), \hat{u}(t))\hat{v}(t) + 2\hat{v}^*(t)\mathcal{F}_{ux}(t, \hat{x}(t), \hat{u}(t))\hat{y}(t) \\
&\qquad + \hat{y}^*(t)\mathcal{F}_{xx}(t, \hat{x}(t), \hat{u}(t))\hat{y}(t)\}dt \\
&= \hat{\alpha}^*\gamma''(\hat{a})\hat{\alpha} + \int_{t_1}^{t_2} 2\Omega(t, \hat{x}(t), \hat{u}(t); \hat{y}(t), \hat{v}(t))dt = J''(\hat{z}_{\hat{a}}; \hat{w}_{\hat{\alpha}}).
\end{aligned}$$

Now, let us prove that $\hat{w}_{\hat{\alpha}} \not\equiv (0,0,0)$. By (16) and hypothesis (v) of Theorem 1, we have

$$0 \geq \lim_{q\to\infty} \frac{\tilde{\mathcal{K}}(\hat{z}_{\hat{a}}; z^q_{a_q})}{d_q^2} + \liminf_{q\to\infty} \frac{\delta}{d_q^2}\mathcal{D}(u_q - \hat{u}) = \lim_{q\to\infty} \frac{\tilde{\mathcal{K}}(\hat{z}_{\hat{a}}; z^q_{a_q})}{d_q^2} + \frac{\delta}{2}.$$

Keeping this in mind together with (14), if we assume that $\hat{w}_{\hat{\alpha}} \equiv (0,0,0)$, then δ would be nonpositive, which is a contradiction, and this proves (13). Now, let us show that

$$\frac{d}{dt}\hat{y}(t) = f_x(t, \hat{x}(t), \hat{u}(t))\hat{y}(t) + f_u(t, \hat{x}(t), \hat{u}(t))\hat{v}(t) \text{ (a.e. in } \mathcal{T}). \tag{19}$$

In fact, since

$$A_q(\cdot) \xrightarrow{L^\infty} f_x(\cdot, \hat{x}(\cdot), \hat{u}(\cdot)), \quad B_q(\cdot) \xrightarrow{L^\infty} f_u(\cdot, \hat{x}(\cdot), \hat{u}(\cdot)), \quad y_q \xrightarrow{\mathrm{u}} \hat{y}, \quad v_q \overset{L^1}{\rightharpoonup} \hat{v}$$

all on $\mathcal{T}$, we see that

$$\dot{y}_q(\cdot) \overset{L^1}{\rightharpoonup} f_x(\cdot, \hat{x}(\cdot), \hat{u}(\cdot))\hat{y}(\cdot) + f_u(\cdot, \hat{x}(\cdot), \hat{u}(\cdot))\hat{v}(\cdot) \text{ on } \mathcal{T}.$$

By Lemma 3, $\dot{y}_q \overset{L^1}{\rightharpoonup} \zeta = \frac{d\hat{y}}{dt}$ on $\mathcal{T}$. Consequently, (19) is fulfilled. Additionally, we claim that

i. $\varphi_{\sigma x}(t, \hat{x}(t), \hat{u}(t))\hat{y}(t) + \varphi_{\sigma u}(t, \hat{x}(t), \hat{u}(t))\hat{v}(t) \leq 0$ (a.e. in $\mathcal{T}$, $\sigma \in i(t, \hat{x}(t), \hat{u}(t))$).
ii. $\varphi_{\varsigma x}(t, \hat{x}(t), \hat{u}(t))\hat{y}(t) + \varphi_{\varsigma u}(t, \hat{x}(t), \hat{u}(t))\hat{v}(t) = 0$ (a.e. in $\mathcal{T}$, $\varsigma \in Q$).

As one readily verifies, (i) and (ii) above follows if one copies the proofs from (13) to (15) of [24].

Hence, from (11), (19), (i) and (ii), above, we see that $\hat{w}_{\tilde{\alpha}} \in Y(\hat{z}_{\hat{a}})$. This fact combined with (13) contradict condition (iv) of Theorem 1.

Case (2): Now, suppose that the sequence $((a_q - \hat{a})/d_q)$ is not bounded. Then,

$$\lim_{q\to\infty} \left| \frac{a_q - \hat{a}}{d_q} \right| = +\infty. \tag{20}$$

Select an adequately subsequence of $((a_q - \hat{a})/|a_q - \hat{a}|)$ (without relabeling), and $\tilde{\alpha} \in \mathbf{R}^s$ satisfying $|\tilde{\alpha}| = 1$, such that

$$\lim_{q\to\infty} \frac{a_q - \hat{a}}{|a_q - \hat{a}|} = \tilde{\alpha}. \tag{21}$$

For all $q \in \mathbf{N}$ and $t \in \mathcal{T}$, set

$$\tilde{\omega}(t) := \frac{x_q(t) - \hat{x}(t)}{|a_q - \hat{a}|}\mathbf{i} + \frac{a_q - \hat{a}}{|a_q - \hat{a}|}\mathbf{j}.$$

By Lemma 2 and (20),

$$\frac{x_q(\cdot) - \hat{x}(\cdot)}{|a_q - \hat{a}|} = y_q(\cdot) \cdot \frac{d_q}{|a_q - \hat{a}|} \xrightarrow{\mathrm{u}} \hat{y}(\cdot) \cdot 0 = 0 \text{ on } \mathcal{T}. \tag{22}$$

For all $q \in \mathbf{N}$, we have

$$\frac{x_q(t_2) - \hat{x}(t_2)}{|a_q - \hat{a}|} = \int_0^1 \Psi'(\hat{a} + \theta[a_q - \hat{a}]) \left(\frac{a_q - \hat{a}}{|a_q - \hat{a}|} \right) d\theta. \tag{23}$$

By (21)–(23),

$$\Psi'(\hat{a})\tilde{\alpha} = 0. \tag{24}$$

Now, by (2), (21), and (22),

$$\begin{aligned} \frac{\tilde{\mathcal{M}}(\cdot, x_q(\cdot)\mathbf{i} + a_q\mathbf{j})}{|a_q - \hat{a}|^2} &= \tfrac{1}{2}\tilde{\omega}_q^*(\cdot)\tilde{\mathcal{P}}(\cdot, x_q(\cdot)\mathbf{i} + a_q\mathbf{j})\tilde{\omega}_q(\cdot) \\ &\xrightarrow{L^\infty} \tfrac{1}{2}0_{\tilde{\alpha}}^*\tilde{\mathcal{F}}_{\xi\xi}(\cdot, \hat{x}(\cdot)\mathbf{i} + \hat{a}\mathbf{j}, \hat{u}(\cdot))0_{\tilde{\alpha}} = \frac{\tilde{\alpha}^*\gamma''(\hat{a})\tilde{\alpha}}{2(t_2 - t_1)}, \end{aligned}$$

$$\begin{aligned}\frac{\tilde{\mathcal{N}}(\cdot, x_q(\cdot)\mathbf{i} + a_q\mathbf{j})}{|a_q - \hat{a}|} &= \tilde{\mathcal{Q}}(\cdot, x_q(\cdot)\mathbf{i} + a_q\mathbf{j})\tilde{\omega}_q(\cdot) \\ &\xrightarrow{L^\infty} \tilde{\mathcal{F}}_{u\xi}(\cdot, \hat{x}(\cdot)\mathbf{i} + \hat{a}\mathbf{j}, \hat{u}(\cdot))0_{\tilde{\alpha}} = 0\end{aligned}$$

both on $\mathcal{T}$. Combined this fact with Lemma 2, this implies that

$$\begin{aligned}\lim_{q\to\infty} \frac{\tilde{\mathcal{K}}(\hat{z}_{\hat{a}}; z^q_{a_q})}{|a_q - \hat{a}|^2} &= \tfrac{1}{2}\tilde{\alpha}^*\gamma''(\hat{a})\tilde{\alpha} + \lim_{q\to\infty}\int_{t_1}^{t_2} \frac{d_q}{|a_q - \hat{a}|} \cdot v_q^*(t)\frac{\tilde{\mathcal{N}}(t, x_q(t)\mathbf{i} + a_q\mathbf{j})}{|a_q - \hat{a}|}dt \\ &= \tfrac{1}{2}\tilde{\alpha}^*\gamma''(\hat{a})\tilde{\alpha}.\end{aligned} \tag{25}$$

As in (15), we have

$$\lim_{q\to\infty} \frac{\tilde{J}'(\hat{z}_{\hat{a}}; z^q_{a_q} - \hat{z}_{\hat{a}})}{|a_q - \hat{a}|^2} = \tfrac{1}{2}\omega^*(t_2)\Psi''(\hat{a}; \tilde{\alpha}). \tag{26}$$

In addition, by (1), (4), and (26) and condition (ii) of Theorem 1,

$$0 \geq \lim_{q\to\infty} \frac{\tilde{\mathcal{K}}(\hat{z}_{\hat{a}}; z^q_{a_q})}{|a_q - \hat{a}|^2} + \liminf_{q\to\infty} \frac{\tilde{\mathcal{E}}(\hat{z}_{\hat{a}}; z^q_{a_q})}{|a_q - \hat{a}|^2}. \tag{27}$$

Hence, as $\tilde{\mathcal{E}}(\hat{z}_{\hat{a}}; z^q_{a_q}) \geq 0$ $(q \in \mathbf{N})$, by (25) and (27),

$$0 \geq \tfrac{1}{2}\tilde{\alpha}^*\gamma''(\hat{a})\tilde{\alpha} = \tfrac{1}{2}J''(\hat{z}_{\hat{a}}; 0_{\tilde{\alpha}}). \tag{28}$$

Accordingly, (24) and (28) contradict condition (iv) of Theorem 1. □

6. Discussion Part

Let us point out that our hypotheses try to respect the property that the first and second order sufficient conditions are closely related to the necessary conditions for optimality. For instance, the sufficient conditions

$$\dot{\omega}(t) = -\mathcal{H}_x^*(t, \hat{x}(t), \hat{u}(t), \omega(t), \nu(t)) \text{ (a.e. in } \mathcal{T}), \quad \mathcal{H}_u^*(t, \hat{x}(t), \hat{u}(t), \omega(t), \nu(t)) = 0 \ (t \in \mathcal{T}),$$

are the Pontryagin maximum principle in normal form. On the other hand, a cone of critical directions that we strengthen in the article is the following:

$$\mathcal{Y}(\hat{z}_{\hat{a}}) := \begin{cases} \dot{y}(t) = f_x(t, x(t), u(t))y(t) + f_u(t, x(t), u(t))v(t) \text{ (a.e. in } \mathcal{T}). \\ y(t_1) = 0,\ y(t_2) = \Psi'(a)\alpha. \\ \varphi_{\sigma x}(t, x(t), u(t))y(t) + \varphi_{\sigma u}(t, x(t), u(t))v(t) \leq 0 \text{ a.e. in } \mathcal{T}, \sigma \in i(t, x(t), u(t))) \text{ with } \nu_\sigma(t) = 0. \\ \varphi_{\varsigma x}(t, x(t), u(t))y(t) + \varphi_{\varsigma u}(t, x(t), u(t))v(t) = 0 \text{ a.e. in } \mathcal{T}, \varsigma \in P \text{ with } \nu_\varsigma(t) > 0 \text{ or } \varsigma \in Q. \end{cases}$$

Here, condition (iv) of Theorem 1 and Corollary 1 asks for

$$J''(\hat{z}_{\hat{a}}; w_\alpha) > 0 \text{ for all } w_\alpha \in Y(\hat{z}_{\hat{a}}),\ w_\alpha \not\equiv (0, 0, 0),$$

that is, the positivity of the second variation on $Y(\hat{z}_{\hat{a}})$, which can be considered as a strengthening of the second order necessary condition

$$J''(\hat{z}_{\hat{a}}; w_\alpha) \geq 0 \text{ for all } w_\alpha \in \mathcal{Y}(\hat{z}_{\hat{a}}).$$

Additionally, condition (i),

$$\gamma'^*(\hat{a}) + \Psi'^*(\hat{a})\omega(t_2) = 0,$$

is the classical transversality condition. It is well-known that the transversality condition is a necessary condition for a weak minimum of problem $P(\gamma, \Gamma, C, f, \xi_1, \Psi, R, s)$. As explained in the article, condition (iii),

$$\mathcal{H}_{uu}(t, \hat{x}(t), \hat{u}(t), \omega(t), \nu(t)) \leq 0 \text{ (a.e. in } \mathcal{T}),$$

is a similar version of the Legendre–Clebsch necessary condition. It is not the necessary condition of Legendre–Clebsch because the former is less restrictive, that is,

$$\mathcal{H}_{uu}(t, \hat{x}(t), \hat{u}(t), \omega(t), \nu(t))$$

must be less or equal than zero almost everywhere on $\mathcal{T}$, but only in a subset related with the kernel of the linear transformation $\varphi_u(t, \hat{x}(t), \hat{u}(t))$. In the fixed-endpoints problem of calculus of variations, it is well-known that, if $\hat{x}$ is a smooth nonsingular extremal satisfying Legendre necessary condition, then, for some $\epsilon > 0$,

$$E(t, x, \dot{x}, u) > 0 \text{ for } (t, x, \dot{x}, u) \in T(\hat{x}, \epsilon), u \neq \dot{x},$$

is a sufficient condition for a weak minimum. Here,

$$T(\hat{x}, \epsilon) := \{(t, x, \dot{x}, u) \in \mathcal{T} \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \mid |x - \hat{x}(t)| < \epsilon, \ |\dot{x} - (d/dt)\hat{x}(t)| < \epsilon\}.$$

In fact, as one can be seen in [10], the above condition implies that

$$E(t, x, \dot{x}, u) \geq \delta L(u - \dot{x}) \text{ for } (t, x, \dot{x}, u) \in T(\hat{x}, \epsilon) \tag{29}$$

for some $\delta, \epsilon > 0$. Then, (29) implies that for some $\delta, \epsilon > 0$,

$$\int_{t_1}^{t_2} E(t, x(t), (d/dt)\hat{x}(t), \dot{x}(t))dt \geq \delta \int_{t_1}^{t_2} L(\dot{x}(t) - (d/dt)\hat{x}(t))dt = \delta\mathcal{D}(\dot{x} - (d/dt)\hat{x}), \tag{30}$$

whenever x is such that $\|x - \hat{x}\|_1 < \epsilon$, where

$$\|x\|_1 := \|x\|_\infty + \|\dot{x}\|_\infty.$$

It is worth to say that (30) gave us the inspiration to obtain the sufficient condition (v) of Theorem 1 and Corollary 1. Condition (ii) arises from the properties of the algorithm established to prove Theorem 1. In summary, our goal consists of providing an alternate model of sufficiency. Even though we do not necessarily obtain no gap hypotheses between necessary and sufficient conditions for optimality, we follow a classical way of obtaining sufficient conditions by strengthening the necessary ones. Finally, in [25], one could find an experimental application involving an economic model of population growth. More precisely, in [25], an application concerning a model for a one sector economy taking into consideration population growth is presented. In the proposed economic model, it is shown that the only factor decreasing the capital per worker is the inclusion of additional workers to the economy, and the only factor increasing the economy is the rate of production. The presence of nonlinear time-state-control mixed constraints plays a crucial role in that model, see [25], for details. For comparison reasons, it is worthwhile mentioning some of the bibliography studying necessary and sufficient conditions involving mixed constraints. Some relevant works we found convenient for that issue are the following [26–36].

7. Conclusions

In this article, we derive sufficiency conditions for weak minima in optimal control problems of Bolza in the parametric as well as in the nonparametric forms. These problems include nonlinear dynamics, a fixed initial end-point, a variable final end-point, and nonlinear mixed time-state-control constraints involving inequalities and equalities. In the nonparametric optimal control problem, the final end-point is not only variable, but also completely free, in the sense that it must not be confined to a parametrization, but it only must be contained in the image of a twice continuously differentiable manifold. Due to the fact that the left end-point is fixed, we were able to make a relaxation, in the sense that we arrived essentially to the same conclusions, but we made weaker assumptions. This relaxation is relative to some recently published works whose initial left end-point is not necessarily fixed. The algorithm used to prove the main theorem of the paper is independent of some classical concepts such as the Hamilton–Jacobi theory, the verification of bounded solutions of certain matrix Riccati equations, or extended notions of the conjugate points theory. Finally, in the parametric problem, we were able to present how the deviation between optimal costs and admissible costs is estimated by quadratic functions, in particular, the square of the norm of the classical Banach space of integrable functions in the deviation mentioned above, is a fundamental component.

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