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Sampling Techniques and Error Estimation for Linear Canonical S Transform Using MRA Approach

Mohammad Younus Bhat [1,*], Badr Alnssyan [2], Aamir H. Dar [1] and Javid G. Dar [1]

[1] Department of Mathematical Sciences, Islamic University of Science and Technology, Kashmir 192122, India; aamir.dar@islamicuniversity.edu.in or ahdkul740@gmail.com (A.H.D.); javinfo.stat@yahoo.co.in (J.G.D.)

[2] Department of Administrative Sciences and Humanities, Applied College, Qassim University, Buraydah 51452, Saudi Arabia; b.alnssyan@qu.edu.sa

* Correspondence: younis.bhat@islamicuniversity.edu.in

Abstract: A linear canonical S transform (LCST) is considered a generalization of the Stockwell transform (ST). It analyzes signals and has multi-angle, multi-scale, multiresolution, and temporal localization abilities. The LCST is mostly suitable to deal with chirp-like signals. It aims to possess the characteristics lacking in a classical transform. Our aim in this paper was to derive the sampling theorem for the LCST with the help of a multiresolution analysis (MRA) approach. Moreover, we discuss the truncation and aliasing errors for the proposed sampling theory. These types of sampling results, as well as methodologies for solving them, have applications in a wide range of fields where symmetry is crucial.

Keywords: linear canonical S transform (LCST); multiresolution analysis (MRA); sampling theorem; error estimation

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1. Introduction

R. G. Stockwell et al. in [1] studied the generalized version of the integral transform called the Stockwell transform or S transform (ST).

$$S[f](\xi,\varepsilon) = \int_{-\infty}^{\infty} f(t)\frac{|\varepsilon|}{\sqrt{2\pi}}e^{-\frac{(t-\varepsilon)^2\xi^2}{2}}e^{-i2\pi\xi t}dt.$$

The uniqueness of the ST lies in the fact that, on the one hand, it provides a frequency-dependent resolution, while on the other hand, it maintains a direct relationship with the classical Fourier spectrum. Hence, it is used in wider applications in electrocardiograms, seismograms, power-quality, and sound analyses for detecting and interpreting events in time series [2–5]. The LCST was developed by Zhang et al. [6] in 2011 to provide a time linear canonical domain representation. The LCST aims to possess the characteristics lacking in a classical transform. Various types of transformations have been constructed using the LCT in recent times. For a further look at these constructions, we refer the readers to [7–14] and the references therein.

We introduce the sampling theorem with error estimations in this work (based on the LCST). The sampling theorem defines the rate at which a constant time signal is sampled so that the data from the end signal is utilized. In other words, data loss during this process should be zero [15–21]. It helps to recover the continuous-time signals from those of the sampled signals (due to the availability of natural values of time signals at a predetermined rate in reconstruction). However, there is a drawback, i.e., an infinite number of samples is needed for the perfect reconstruction process. A nice and useful feature of sampling is that it measures sampling frequencies at an accurate rate. To reduce the complexity of computing the information, it is necessary to limit the sampling frequencies, which are required to diminish the measured information. However, there is the threat of data loss in the signal whenever we choose a low sampling frequency. Hence, the need of the hour pertains to the trade-off between these two limits. The highlights of our contribution are:

- To develop a sampling theorem in the linear canonical S transform domain via the MRA approach.
- To introduce truncation and aliasing errors for sampling.

The findings of our work can be best utilized in symmetry. The remainder of the paper is organized as follows. In Section 2, we discuss some preliminaries that are required in subsequent sections. In Section 3, we introduce the sampling theorem for LCST associated with the multiresolution analysis (MRA). In Section 4, we discuss the truncation and aliasing errors for sampling.

2. Preliminaries

2.1. Linear Canonical Transform (LCT)

For a uni-modular matrix $M = (A, B, C, D)$, the linear canonical transform of any function $f \in L^2[\mathbb{R}]$ is stated as

$$L^M[f](w) = \begin{cases} \displaystyle\int_{\mathbb{R}} f(t)\, K_M(t,w)\, dt, & B \neq 0 \\ \sqrt{D}\, \exp\left\{\dfrac{iCDw^2}{2}\right\} f(Dw), & B = 0, \end{cases} \tag{1}$$

where $K_M(t,w)$ is called the kernel of the LCT, which is shaped as

$$K_M(t,w) = \frac{1}{\sqrt{2i\pi B}} \exp\left\{\frac{i(At^2 - 2tw + Dw^2)}{2B}\right\}, \quad B \neq 0.$$

note that for $B = 0$, the LCT defined by Equation (1) reduces to the chirp multiplication operator and is of no use to us. Hence, for the sake of brevity, we set $B \neq 0$ in this paper, unless stated otherwise.

The inversion formula for the LCT is given by

$$f(t) = \int_{\mathbb{R}} L^M[f](w)\overline{K_M(t,w)}dw, \quad B \neq 0.$$

2.2. Linear Canonical S Transform (LCST)

The linear canonical S transform (LCST) is a hybrid of the S transform (ST), which is an extension of the LCT and the ST, given by

$$\begin{aligned} \mathcal{S}^M[f](\xi,\varepsilon) &= \int_{-\infty}^{\infty} f(t) g(\xi - t, \varepsilon) K_M(t,\varepsilon) dt \\ &= \frac{1}{\sqrt{2i\pi B}} \int_{-\infty}^{\infty} f(t) g(\xi - t, \varepsilon) e^{\dfrac{i(At^2 - 2t\varepsilon + D\varepsilon^2)}{2B}} dt \end{aligned} \tag{2}$$

where $g(\xi - t, \varepsilon)$ is a Gaussian window scalable function of frequency ξ and time t, and is given by

$$g(\xi - t, \varepsilon) = \frac{|\xi|}{2k\pi\sqrt{2\pi}} e^{-\frac{(\varepsilon - t)^2 \xi^2}{8\pi^2 k^2}}.$$

If we take $M = (0, 1, -1, 0)$, the LCST given by Equation (2) reduces to the novel S transform.

Let us assume that $h(t,\xi,\varepsilon) = g(t-\varepsilon,\xi)K_M(\xi,t)$, then we can introduce a new function $H_M(\varepsilon,\xi,\xi')$, given by

$$H_M(\varepsilon,\xi,\xi') = \int_{\mathbb{R}} h(t,\xi,\varepsilon) K_M(\xi',t) dt.$$

and, thus, with the help of this new function, the LCST defined in (2) can be written as

$$\mathcal{S}^M[f](\xi,\varepsilon) = \mathcal{S}_f^M(\xi,\varepsilon) = \int_{\mathbb{R}} L_f^M(\xi) H_M(\varepsilon,\xi,\xi') d\xi,$$

where $\mathcal{S}_f^M(\xi,\varepsilon)$ and $L_f^M(\xi)$ represent the LCST and LCT of the signal $f(t)$. Now, we shall introduce some notations and definitions that will be used in derivations conducted in later sections.

2.3. Notations and Definitions

The notations used in this paper are presented here for a better understanding of the proposed technique.

- $L^1[0,2\pi]$: denotes the space of absolutely integrable functions on $[0,2\pi]$.
- $L^2[\mathbb{R}]$: denotes the space of all square integral functions on $\mathbb{R}$.
- $\ell^2[\mathbb{Z}]$: denotes the space of all square-summable sequences on $\mathbb{Z}$.
- $a[n]$: represents the discrete signal.
- $\mathcal{H}$: represents the finite-dimensional Hilbert space, whose every basis is a Riesz basis.
- $\chi_E(x)$: denotes the characteristic function of a subset $E \subset \mathbb{R}$.

In the next section, the multiresolution analysis (MRA) associated with LCST is discussed, which will give the time and frequency information simultaneously.

2.4. Multiresolution Analysis

A technique for the L^2-estimation of the function with arbitrary accuracy is known as MRA.

Definition 1. *A multiresolution analysis (MRA) associated with LCST, as defined in [22], is a sequence of closed subspaces $\left\{V_j^M : j \in \mathbb{Z}\right\}$ of $L^2[\mathbb{R}]$, with the following properties:*

(a) $V_j^M \subset V_{j+1}^M$ *for all* $j \in \mathbb{Z}$;
(b) $\bigcup_{j\in\mathbb{Z}} V_j^M$ *is dense in* $L^2[\mathbb{R}]$
(c) $\bigcap_{j\in\mathbb{Z}} V_j^M = \{0\}$, *where* 0 *is the zero element of* $L^2(\mathbb{R})$;
(d) $f(t) \in V_j^M$ *if and only if* $f(2t)e^{i\pi A((2t)^2-t^2)\frac{1}{B}} \in V_{j+1}^M$ *for all* $j \in \mathbb{Z}$;
(e) *There exists a function* φ *in* $L^2[\mathbb{R}]$, *such that* $\phi^M(t) = \varphi(t)e^{\frac{i\pi At^2}{B}}$ *belongs to* V_0^M.

The function in (d) is known as a scaling factor of the given MRA $\left\{V_j^M : j \in \mathbb{Z}\right\}$, assuming that the set of functions is a Riesz basis of the subspace V_0^M. In (e), $\varphi \in L^2[\mathbb{R}]$ is such that $\left\{\varphi_{0,\lambda}^M = \varphi(t-\lambda)e^{-i\pi A(t^2-\lambda^2)\frac{1}{B}}\right\}_{\lambda\in\mathbb{Z}}$, obtained by modulation of $\varphi(t)$, forms an orthonormal basis for subspace V_0^M. The modulated function mentioned in (e) is known as the scaling function of the MRA subspace $\left\{V_j^M : j \in Z\right\}$. Let us accept that sequence $\left\{\varphi_{0,\lambda}^M : \lambda \in \mathbb{Z}\right\}$ is a Riesz basis:

$$V_0^M = \left\{\sum_{\lambda\in\mathbb{Z}} a[n]\varphi_{0,\lambda}^M(t) : a[n] \in \ell^2[\mathbb{Z}]\right\}.$$

Theorem 1. *Let $\varphi(t) \in L^2[\mathbb{R}]$ and $V_0^M = span\left\{\varphi_{0,\lambda}^M = \varphi(t-\lambda)e^{-i\pi A(t^2-\lambda^2)\frac{1}{B}}\right\}_{\lambda\in\mathbb{Z}}$, then $\left\{\varphi_{0,\lambda}^M\right\}_{\lambda\in\mathbb{Z}}$ is a Riesz basis of the subspace V_0^M of $L^2[\mathbb{R}]$, if there exist constants $0 < \mathcal{A} \le \mathcal{B} < +\infty$, such that*

$$\mathcal{A} \le \mathcal{H}_{\varphi,M}^2(\xi,\varepsilon) \le \mathcal{B}, \qquad \forall \xi,\varepsilon \in [0,2\pi B],$$

where

$$\mathcal{H}_{\varphi,M}(\xi,\varepsilon) = \left(\sum_{-\infty}^{\infty}\left|\Phi\left(\frac{\xi}{B}+2\pi k, \varepsilon B+2\pi k\right)\right|^2\right)^{1/2} \tag{3}$$

satisfying the condition

$$0 \leq \|\mathcal{H}_{\varphi,M}(\xi,\varepsilon)\|_k \leq \|\mathcal{H}_{\varphi,M}(\xi,\varepsilon)\|_{k+1} < \infty. \tag{4}$$

where $\Phi\left(\frac{\xi}{B},\varepsilon B\right)$ *denotes the ST of the canonical scaling function* $\varphi(t)$*, where the argument is scaled by* $(\frac{1}{B})$ *and B with respect to frequency and time axis.*

Proof. The proof of the theorem is present in [22]. □

It is clear that $\varphi(t)$ is orthonormal if $\mathcal{A} = \mathcal{B} = 1$, i.e., $\mathcal{H}_{\Phi,M}(\xi,\varepsilon) = 1$, where $\xi \in \mathbb{R}$ and each V_k^M is called a multiscale subspace of LCST. Let $\{\varphi_{0,\lambda}^M : \lambda \in \mathbb{Z}\}$ be the orthonormal basis of V_0^M then $\varphi(t)$ is orthonormal. Assume that the LCT scaling function $\varphi(t)$ is an MRA of $\{V_k^M : k \in \mathbb{Z}\}$, then $\{\varphi_{0,\lambda}^M : \lambda \in \mathbb{Z}\}$ is the Riesz basis of V_1^M [22]. As $\varphi_{0,\lambda}^M \subset V_0^M \subset V_1^M$ is true for all $\lambda \in \mathbb{Z}$, there is a sequence $g[\lambda] \in \ell^2[\mathbb{Z}]$, satisfying

$$\varphi_{0,\lambda}^M(t) = \sum_{\lambda=-\infty}^{\infty} g[\lambda]\varphi_{1,\lambda}^M(t).$$

for simplicity,

$$\varphi_{k,\lambda}^M(t) = 2^{\frac{k}{2}}\phi(2^k t - \lambda)e^{\frac{i\pi A[t^2-(\frac{\lambda}{2^k})^2]}{B}}.$$

Suppose $p[n] \in \ell^2[\mathbb{Z}]$; if $\{\phi_{0,\lambda}^M(t) : \lambda \in \mathbb{Z}\}$ is the Riesz basis of S_0^M, $\phi(t)$ is the canonical function of MRA $\{V_k^M : k \in \mathbb{Z}\}$. Hence, for any $f(t) \in V_{k+1}^M = S_k^M \oplus V_k^M$, then there exits $b[\lambda]$ and $c[\lambda]$ in $\ell^2[\mathbb{Z}]$, such that

$$f(t) = \sum_{\lambda\in\mathbb{Z}} b[\lambda]\Phi_{k,\lambda}^M(t) + \sum_{\lambda\in\mathbb{Z}} c[\lambda]\Phi_{k,\lambda}^M(t),$$

where $\{c[\lambda] : \lambda \in \mathbb{Z}\}$ are the LCST coefficients of $f(t)$ in S_k^M. Subsequently,

$$\Phi\left(\frac{\xi}{B},\varepsilon B\right) = \Lambda\left(\frac{\xi}{2B}\right)\Phi\left(\frac{\xi}{2B},\frac{\varepsilon B}{2}\right) \tag{5}$$

and

$$\Psi\left(\frac{\xi}{B},\varepsilon B\right) = \Gamma\left(\frac{\xi}{2B}\right)\Phi\left(\frac{\xi}{2B},\frac{\varepsilon B}{2}\right). \tag{6}$$

Here, $\Phi\left(\frac{\xi}{B},\varepsilon B\right)$ and $\Psi\left(\frac{\xi}{B},\varepsilon B\right)$ are the STs of $\varphi(t)$ and $\phi(t)$, respectively, with the arguments scaled by $\frac{1}{B}$ and B for the frequency and time axis, respectively, where

$$\Lambda\left(\frac{\xi}{2B}\right) = \frac{1}{\sqrt{2}}\sum_{\lambda\in\mathbb{Z}} g[\lambda]e^{i2\pi\lambda\frac{\xi}{B}}e^{i\pi\frac{\lambda^2}{2B}}$$

and

$$\Gamma\left(\frac{\xi}{2B}\right) = \frac{1}{\sqrt{2}}\sum_{\lambda\in\mathbb{Z}} p[\lambda]e^{i2\pi\lambda\frac{\xi}{B}}e^{i\pi\frac{\lambda^2}{2B}}$$

are defined in $L^\infty[I]$. For any $\tau \in \mathbb{Z}^+ \cup \{0\}$, iterating (5), we have

$$\Phi\left(\frac{2^\tau\xi}{B},2^\tau\varepsilon B\right) = \Upsilon_\tau(\xi)\Phi\left(\frac{\xi}{B},\varepsilon B\right), \tag{7}$$

where $\Upsilon_0(\xi) = 1$ and $\Upsilon_\tau(\xi) = \Pi_{\tau=0}^{\tau-1}\Lambda\left(2^\tau\frac{\xi}{B}\right)$ $\tau \geq 1$. It can be verified that

$$\Upsilon_\tau(\xi) = \Upsilon_\tau(\xi + 2\pi B) \in L^\infty[I].$$

now let us define $D_\tau = \sup \Upsilon_\tau(\xi)$

$$D_\tau = \bigcap_{\tau=0}^{\tau-1} \sup \Lambda\left(\frac{2^\tau \xi}{B}\right); \quad \tau \geq 1. \tag{8}$$

since, $\sup \Lambda\left(\frac{2\xi}{B}\right) = \frac{1}{2} \sup \Lambda\left(\frac{\xi}{B}\right)$,; therefore, from (8), we have

$$D_\tau = \bigcap_{\tau=0}^{\tau-1} \frac{\tau}{2} \sup \Lambda\left(\frac{\xi}{B}\right)$$

and can be defined as

$$\tilde{\Phi}_\tau\left(\frac{2^\tau \xi}{B}, 2^\tau \varepsilon B\right) = \sum_{k\in\mathbb{Z}} \Phi\left(\frac{2^\tau \xi}{B} + 2^{\tau+1}k\pi, 2^\tau \varepsilon B + 2^{\tau+1}k\pi\right). \tag{9}$$

adding (5) and Poisson's summation formula of the ST results, we have

$$\tilde{\Phi}_\tau\left(\frac{2^\tau \xi}{B}, 2^\tau \varepsilon B\right) = \Upsilon_\tau(\xi) \sum_{k\in\mathbb{Z}} \Phi\left(\frac{\xi}{B} + 2^{\tau+1}k\pi, \varepsilon B + 2^{\tau+1}k\pi\right)$$

$$\tilde{\Phi}_\tau\left(\frac{2^\tau \xi}{B}, 2^\tau \varepsilon B\right) = \Upsilon_\tau(\xi) \tilde{\Phi}\left(\frac{\xi}{B}, \varepsilon B\right), \tag{10}$$

where $\tilde{\Phi}\left(\frac{\xi}{B}, \varepsilon B\right)$ represents the discrete-time ST of $\varphi[\lambda]$, being the sampled form of $\varphi(t)$ with the argument scaled by $\frac{1}{B}$ and B for the frequency and time axis.
As the sampling interval (or sampling theorem) plays an important role in MRA, the sampling theorem of LCST (based on the approach of the canonical scaling function of MRA) is discussed in the next section.

3. Sampling Theorem of LCST

In this section, the sampling procedure in the sequence of subspace V_0^M for the stable generator function $\varphi(t)$ is a set of $L^2[\mathbb{R}]$ and a Gaussian function $g(t,\xi)$,; its function space is defined as

$$f(t) = \sum_{\lambda\in\mathbb{Z}} f[\lambda]\varphi(t-\lambda)g(t,\xi)e^{-i\pi A(t^2-(\lambda)^2)\frac{1}{B}},$$

where $f[\lambda] \in \ell^2[\mathbb{Z}]$ and $g(t,\xi) \leq 1, \in \mathbb{R}$. We consider $f(t)$ as a pointwise convergent because

$$\left|\sum_{\lambda\in\mathbb{Z}} f[\lambda]\varphi(t-\lambda)g(t,\xi)e^{-i\pi A(t^2-(\lambda)^2)\frac{1}{B}}\right|^2 \leq \left(\sum_{\lambda\in\mathbb{Z}} |f[\lambda]|^2 \sum_{\lambda\in\mathbb{Z}} |\varphi(t-\lambda)|^2 \sum_{\lambda\in\mathbb{Z}} |g(t,\xi)|^2\right).$$

hence, without loss of generality, any continuous function $f(t) \in V_0^M$ can be considered for the sampling. Now, let us begin by presenting the sampling theorem for the LCST.

Theorem 2. *Suppose generator functions $\varphi(t)$ and $g(t,\xi)$ belonging to $L^2[\mathbb{R}]$ are the canonical scaling signals of MRA $\{V_k^M : k \in \mathbb{Z}\}$ related to the LCST and its sampling sequence $\varphi(\lambda)$, which is an integer of $\varphi(t)$ belonging to $\ell^2[\mathbb{Z}]$. Then a continuous function $s(t)$, which is a set of $L^2[\mathbb{R}]$, can be defined with $s(t)e^{-i\frac{A}{2B}t^2} \in V_0^M$, such that*

$$f(t) = \sum_{\eta\in\mathbb{Z}} f[\eta/2^\tau]s(2^\tau t - \eta)g(t,\xi)e^{\frac{-i\pi A}{B}(t^2-(\eta/2^\tau)^2)}, \tag{11}$$

where $\tau \in \mathbb{Z}^+ \cup \{0\}$, *and for all* $f(t), g(t,\xi)$, *are sets of* V_0^M. *Equation (11) holds if*

$$\frac{1}{\sqrt{2\pi}\tilde{\Phi}\left(\frac{\xi}{B},\varepsilon B\right)}\chi_{D_\tau}(\xi) \in L^2(I). \tag{12}$$

moreover, the function $s(t)$ *in (11) satisfies*

$$S\left(\frac{\xi}{B},\varepsilon B\right) = \frac{\Phi\left(\frac{\xi}{B},\varepsilon B\right)}{\sqrt{2\pi}\tilde{\Phi}\left(\frac{\xi}{B},\varepsilon B\right)}, \quad \xi,\varepsilon \in D_\tau, \tag{13}$$

where $S\left(\frac{\xi}{B},\varepsilon B\right)$ *and* $\Phi\left(\frac{\xi}{B},\varepsilon B\right)$ *represent the ST of* $s(t)$*and* $\varphi(t)$ *with the argument scaled by* $\frac{1}{B}$ *and B for the frequency and time axis, respectively.*

Proof. Let us assume that (12) is true; hence, $\tilde{\Phi}\left(\frac{\xi}{B},\varepsilon B\right) \neq 0$, holds for a.e ξ,ε in D_τ, by [22]; we have a sequence $a[n] \in \ell^2[\mathbb{Z}]$, such that

$$\frac{1}{\sqrt{2\pi}\tilde{\Phi}\left(\frac{\xi}{B},\varepsilon B\right)}\chi_{D_\tau}(\xi,\varepsilon) = \sum_{\lambda\in\mathbb{Z}}\sum_{\lambda\in\mathbb{Z}} a[\lambda]e^{i\pi A\frac{\lambda^2}{B}}e^{-i2\pi\xi'\frac{\lambda}{B}}H_M^*(\mu,\xi,\xi'), \tag{14}$$

holds in the $L^2[I]$ sense. As $\tilde{\Phi}\left(\frac{\xi}{B},\varepsilon B\right)$ is periodic with period $2\pi B$, (14) can be written as

$$\int_{\mathbb{R}}\left|\frac{\Phi\left(\frac{\xi}{B},\varepsilon B\right)}{\sqrt{2\pi}\tilde{\Phi}\left(\frac{\xi}{B},\varepsilon B\right)}\chi_{D_\tau}(\xi,\varepsilon)\right|^2 d\xi = \sum_{k\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\int_I\left|\frac{\Phi\left(\frac{\xi}{B}+2k\pi,\varepsilon B+2k\pi\right)}{\sqrt{2\pi}\tilde{\Phi}\left(\frac{\xi}{B},\varepsilon B\right)}\right|^2\chi_{D_\tau}(\xi,\varepsilon)d\xi.$$

now applying (3), the above equation becomes

$$\int_{\mathbb{R}}\left|\frac{\Phi\left(\frac{\xi}{B},\varepsilon B\right)}{\sqrt{2\pi}\tilde{\Phi}\left(\frac{\xi}{B},\varepsilon B\right)}\chi_{D_\tau}(\xi,\varepsilon)\right|^2 d\xi = \int_I \frac{\mathcal{H}^2_{\varphi,M}(\xi,\varepsilon)}{|\sqrt{2\pi}\tilde{\Phi}\left(\frac{\xi}{B},\varepsilon B\right)|^2}\chi_{D_\tau}(\xi,\varepsilon)d\xi. \tag{15}$$

from (4) and (15), we can establish

$$\int_{\mathbb{R}}\left|\frac{\Phi\left(\frac{\xi}{B},\varepsilon B\right)}{\sqrt{2\pi}\tilde{\Phi}\left(\frac{\xi}{B},\varepsilon B\right)}\chi_{D_\tau}(\xi,\varepsilon)\right|^2 d\xi = \int_I \frac{1}{|\sqrt{2\pi}\tilde{\Phi}\left(\frac{\xi}{B},\varepsilon B\right)|^2}\chi_{D_\tau}(\xi,\varepsilon)d\xi\|\mathcal{H}^2_{\varphi,M}(\xi,\varepsilon)\|^2,$$

which implies that $\frac{\Phi\left(\frac{\xi}{B},\varepsilon B\right)}{\sqrt{2\pi}\tilde{\Phi}\left(\frac{\xi}{B},\varepsilon B\right)}\chi_{D_\tau}(\xi,\varepsilon) \in L^2[\mathbb{R}]$. Thus, we can obtain

$$\begin{aligned} S\left(\frac{\xi}{B},\varepsilon B\right) &= \mathcal{S}\{s(t)\}\left(\frac{\xi}{B},\varepsilon B\right) \\ &= \chi_{D_\tau}(\xi,\varepsilon)\frac{\Phi\left(\frac{\xi}{B},\varepsilon B\right)}{\sqrt{2\pi}\tilde{\Phi}\left(\frac{\xi}{B},\varepsilon B\right)}\chi_{D_\tau}(\xi,\varepsilon), \end{aligned} \tag{16}$$

where $\mathcal{S}$ is the ST operator. Further simplifying, we have

$$\Phi\left(\frac{\xi}{B},\varepsilon B\right)\chi_{D_\tau}(\xi,\varepsilon) = \sqrt{2\pi}S\left(\frac{\xi}{B},\varepsilon B\right)\tilde{\Phi}\left(\frac{\xi}{B},\varepsilon B\right). \tag{17}$$

now inserting (14) into (17),we obtain

$$S\left(\frac{\xi}{B},\varepsilon B\right)=\Phi\left(\frac{\xi}{B},\varepsilon B\right)\sum_{\lambda\in\mathbb{Z}}\sum_{\lambda\in\mathbb{Z}}f[\lambda]e^{i\pi A\frac{\lambda^2}{B}}e^{-i2\pi\xi'\frac{\lambda}{B}}H_M^*(\mu,\xi,\xi'). \tag{18}$$

next, by the relation between LCST and ST, we have

$$\mathcal{S}^M\{s(t)e^{-i\pi A\frac{t^2}{B}}\}(\xi,\varepsilon)=\sqrt{2\pi}\mathcal{A}_M e^{i\pi A\frac{\lambda^2}{B}}\mathcal{S}\{s(t)\}\left(\frac{\xi}{B},\varepsilon B\right). \tag{19}$$

substituting, (16) and (18) into (19) the LCST of the modulated signal can be written as

$$\begin{aligned}\mathcal{S}^M\{s(t)e^{-i\pi A\frac{t^2}{B}}\}(\xi,\varepsilon) &= \sqrt{2\pi}\Phi\left(\frac{\xi}{B},\varepsilon B\right)\sum_{\lambda\in\mathbb{Z}}\sum_{\lambda\in\mathbb{Z}}a[\lambda]K_M(\xi,\lambda)\\ &= \sqrt{2\pi}\tilde{A}_M(\xi,\varepsilon)\Phi\left(\frac{\xi}{B},\varepsilon B\right),\end{aligned}$$

where $\tilde{A}_M(\xi,\varepsilon)$ represents the DTLCST of $a[\lambda]$. Implementing the semi-discrete canonical convolution theorem [23], we have

$$s(t)e^{-i\pi A\frac{t^2}{B}}=\sum_{\lambda\in\mathbb{Z}}\varphi(t-\lambda)a[\lambda]e^{-iA\frac{\pi}{B}(t^2-\lambda^2)},$$

where $s(t)e^{-i\pi A\frac{t^2}{B}}\in V_0^M$ as $\varphi(t-\lambda)e^{-iA\frac{\pi}{B}(t^2-\lambda^2)}$ is the Riesz basis of V_0^M. Now, adding (17) and (8) results in

$$\Upsilon_\tau(\xi)\Phi\left(\frac{\xi}{B},\varepsilon B\right)\chi_{D\tau}(\xi,\varepsilon)=\Upsilon_\tau(\xi)\sqrt{2\pi}S\left(\frac{\xi}{B},\varepsilon B\right)\tilde{\Phi}\left(\frac{\xi}{B},\varepsilon B\right). \tag{20}$$

implementing (7) and (9) in (20) and scaling by 2^τ, we have

$$\Phi\left(\frac{\xi}{B},\varepsilon B\right)=\sqrt{2\pi}S\left(\frac{\xi}{2^\tau B},\frac{\varepsilon B}{2^\tau}\right)\tilde{\Phi}_\tau\left(\frac{\xi}{B},\varepsilon B\right). \tag{21}$$

making use of Poisson's summation formula of the ST from [16] in (21), we have

$$\tilde{\Phi}_\tau\left(\frac{\xi}{B},\varepsilon B\right)=\frac{1}{2^\tau}\sum_{\lambda\in\mathbb{Z}}\varphi\left[\frac{\lambda}{2^\tau}\right]e^{-i2\pi\xi\frac{\eta}{2^\tau B}}. \tag{22}$$

applying IST on both sides of (22)

$$\varphi(t)=\sum_{\lambda\in\mathbb{Z}}\varphi\left[\frac{\lambda}{2^\tau}\right]S(2^\tau t-\lambda). \tag{23}$$

now for any function $f(t)\in V_0^M$, there is a sequence $d[k]\in\ell^2[\mathbb{Z}]$, such that

$$f(t)=\sum_{k\in\mathbb{Z}}d[k]\varphi(t-k)e^{-i\pi A(t^2-k^2)\frac{1}{B}}. \tag{24}$$

from (24) and (23), we have

$$f(t)=\sum_{k\in\mathbb{Z}}d[k]\sum_{\lambda\in\mathbb{Z}}\varphi\left[\frac{\lambda}{2^\tau}\right]S\{2^\tau(t-k)-\lambda\}e^{-i\pi A(t^2-k^2)\frac{1}{B}}. \tag{25}$$

setting $\lambda=2^\tau k+\eta$ in (25), we have

$$f(t)=\sum_{\lambda\in\mathbb{Z}}S(2^\tau t+\eta)\sum_{k\in\mathbb{Z}}d[k]\varphi\left[\frac{\lambda}{2^\tau}-k\right]e^{-i\pi A(t^2-k^2)\frac{1}{B}}. \tag{26}$$

using (24) above can be rewritten as

$$f(t) = \sum_{k \in \mathbb{Z}} d[k]\varphi(\lambda - k)e^{-i\pi A(t^2-k^2)\frac{1}{B}}. \tag{27}$$

therefore, $\{f[\lambda] : \lambda \in \mathbb{Z}\}$ is well defined as $d[\lambda], \varphi[\lambda] \in \ell^2\mathbb{Z}$. This satisfies the condition of convergence, i.e.,

$$f[n] \to 0 as[n] \to \infty.$$

Let $\tilde{\mathcal{A}}_M(\xi, \varepsilon)$ represent the discrete-time LCST of $d[k]$. $\tilde{\mathcal{A}}_M(\xi, \varepsilon)$ and $\tilde{\Phi}\left(\frac{\xi}{B}, \varepsilon B\right)$ belong to $L^1[I]$, thus,

$$\tilde{\mathcal{A}}_M(\xi, \varepsilon)\tilde{\Phi}\left(\frac{\xi}{B}, \varepsilon B\right)\frac{2\pi e^{-i\pi A\frac{\xi^2}{B}}}{\sqrt{1 - \frac{iA}{B}}} \in L^1[I]. \tag{28}$$

evaluating the Fourier coefficients in (28), we have

$$\frac{1}{2\pi B}\int_I \tilde{\mathcal{A}}_M(\xi, \varepsilon)\tilde{\Phi}\left(\frac{\xi}{B}, \varepsilon B\right)\frac{2\pi e^{-i\pi A\frac{\xi^2}{B}}}{\sqrt{1 - \frac{iA}{B}}}e^{i\pi\frac{\xi}{B}}d\xi.$$

substituting the expression of $\tilde{\mathcal{A}}_M(\xi, \varepsilon)$ in terms of $d[k]$ in the above expression results in

$$\frac{1}{B\sqrt{1 - \frac{iA}{B}}}\int_I \sum_{k \in \mathbb{Z}} d[k]\tilde{\Phi}\left(\frac{\xi}{B}, \varepsilon B\right)K_M(\xi, k)e^{-i\pi A\frac{\xi^2}{B}}e^{i2\pi\frac{\xi}{B}}. \tag{29}$$

substituting the expression of $K_M(\xi, k)$ in (29), it can be solved as

$$\sum_{k \in \mathbb{Z}} d[k]\varphi[\lambda - k]e^{i\pi A\frac{k^2}{B}} = f[n]e^{i\pi A\frac{\lambda^2}{B}}.$$

as $\lambda \to \frac{\lambda}{2^\tau}$ in (27) gives

$$f\left[\frac{\lambda}{2^\tau}\right] = \sum_{k \in \mathbb{Z}} d[k]\varphi\left(\frac{\lambda}{2^\tau} - k\right)e^{-i\pi\left((\frac{\lambda}{2^\tau})^2 - k^2\right)}. \tag{30}$$

however, if (30) is substituted in (26), we have (11). This actually proves the proposed sampling theorem presented in (11). Let us suppose that $s(t) \in L^2[\mathbb{R}]$ with $s(t)e^{-i\pi A\frac{t^2}{B}} \in V_0^M$, such that (11) holds in $L^2[\mathbb{R}]$. It is clear that $\varphi(t)e^{-i\pi A\frac{t^2}{B}} \in V_0^M$; therefore, taking $\varphi(t)e^{-i\pi A\frac{t^2}{B}}$ for $f(t)$ in (11) gives

$$\varphi(t)e^{-i\pi A\frac{t^2}{B}} = \sum_{\lambda \in \mathbb{Z}} \varphi\left[\frac{\lambda}{2^\tau}\right]S(2^\tau t + m)e^{-i\pi A\frac{t^2}{B}}. \tag{31}$$

with the help of LCST, (31) becomes

$$\Phi\left(\frac{\xi}{B}, \varepsilon B\right) = \sqrt{2\pi}\tilde{\Phi}\left(\frac{\xi}{B}, \varepsilon B\right)S\left(\frac{\xi}{2^\tau B}, \frac{\varepsilon B}{2^\tau}\right). \tag{32}$$

modifying by using the scaling operation, we have

$$\Phi\left(\frac{2^\tau\xi}{B}, \varepsilon 2^\tau B\right) = \sqrt{2\pi}\tilde{\Phi}\left(\frac{2^\tau\xi}{B}, \varepsilon 2^\tau B\right)S\left(\frac{\xi}{B}, \varepsilon B\right). \tag{33}$$

applying (7) and (10) into LHS and RHS of (33), we obtain

$$\Upsilon_\tau(\xi)\Phi\left(\frac{\xi}{B},\varepsilon B\right)=\sqrt{2\pi}\Upsilon_\tau(\xi)\tilde{\Phi_\tau}\left(\frac{\xi}{B},\varepsilon B\right)S\left(\frac{\xi}{B},\varepsilon B\right).$$

solving the above equation yields

$$S\left(\frac{\xi}{B},\varepsilon B\right)=\frac{\Phi\left(\frac{\xi}{B},\varepsilon B\right)}{\sqrt{2\pi}\tilde{\Phi}\left(\frac{\xi}{B},\varepsilon B\right)},\tag{34}$$

which proves the expression of the interpolation function, defined in (13), which is true $\forall\xi,\varepsilon\in D_J$, Therefore, (34) can be written as

$$S\left(\frac{\xi}{B},\varepsilon B\right)=\frac{\Phi\left(\frac{\xi}{B},\varepsilon B\right)}{\sqrt{2\pi}\tilde{\Phi}\left(\frac{\xi}{B},\varepsilon B\right)}\chi_{D_J}(\xi,\varepsilon).\tag{35}$$

as $S\left(\frac{\xi}{B},\varepsilon B\right)\in L^2[\mathbb{R}]$, then by using (4), the bounds of the square summable function of (35) can be defined as

$$0\le\|H_{\varphi,M}(\xi,\varepsilon)\|_k\le\|H_{\varphi,M}(\xi,\varepsilon)\|_{k+1}<\infty.$$

hence,

$$\begin{aligned}\int_{\mathbb{R}}\left|S\left(\frac{\xi}{B},\varepsilon B\right)\right|^2 d\xi &= \sum_{k\in\mathbb{Z}}\int_I\left|\frac{\Phi\left(\frac{\xi}{B},\varepsilon B\right)}{\sqrt{2\pi}\tilde{\Phi}\left(\frac{\xi}{B},\varepsilon B\right)}\right|^2\\ &= \int_I\frac{H^2_{\varphi,M}(\xi,\varepsilon)}{|\sqrt{2\pi}\tilde{\Phi}\left(\frac{\xi}{B},\varepsilon B\right)|^2}\chi_{D_J}(\xi,\varepsilon)<\infty.\end{aligned}$$

therefore,

$$0\le\|H_{\varphi,M}(\xi,\varepsilon)\|_0^2\int_I\frac{1}{|\sqrt{2\pi}\tilde{\Phi}\left(\frac{\xi}{B},\varepsilon B\right)|^2}\chi_{D_J}(\xi,\varepsilon)<\infty.\tag{36}$$

Expression (36) asserts that $\dfrac{1}{|\sqrt{2\pi}\tilde{\Phi}\left(\frac{\xi}{B},\varepsilon B\right)|^2}\chi_{D_J}(\xi,\varepsilon)$ represents a square integral function. Thus, $\dfrac{1}{\sqrt{2\pi}\tilde{\Phi}\left(\frac{\xi}{B},\varepsilon B\right)}\chi_{D_J}(\xi,\varepsilon)\in L^2[I]$, as $L^2[I]$ represents the space of all square integral functions on I. Thus, the condition for the existence of the sampling theorem for LCST, defined in (12), is proved. □

4. Error Estimation

Error estimation is important to study. So, we devote this section to the study of truncation and aliasing errors.

4.1. Truncation Error

The truncation error can be expressed as

$$\epsilon(t)=\sum_{|\lambda|\ge N}f\left[\frac{\lambda}{2^\tau}\right]s(2^\tau t-\lambda)g(t,\xi)e^{-iA\pi\left(t^2-(\frac{\lambda}{2^\tau})^2\right)\frac{1}{B}},\tag{37}$$

where $f(t)$ and $g(t,\xi)$ are a set of V_0^M.

Theorem 3. *Let $\varphi(t) \in L^2[\mathbb{Z}]$ be a continuous scaling function of MRA $\{V_k^M : k \in \mathbb{Z}\}$ alongside the LCST, then the sampling sequence $\{\varphi(\lambda) : \lambda \in \mathbb{Z}\} \in \ell^2[\mathbb{Z}]$ and $\dfrac{1}{\sqrt{2\pi\tilde{\Phi}\left(\frac{\xi}{B}, \varepsilon B\right)}}\chi_{D_\tau} \in L^\infty[I]$. Then, the truncation error is bounded by*

$$\|\epsilon(t)\|_{L^2} \le 2^{-\tau/2}\sqrt{\sum_{|\lambda|\ge N}\left|f\left[\frac{\lambda}{2^\tau}\right]g\left(\frac{\lambda}{2^\tau}, \xi\right)\right|^2}\left\|\frac{H_{\varphi,M}(\xi,\varepsilon)}{\Phi\left(\frac{\xi}{B}, \varepsilon B\right)}\chi_{D_\tau}\right\|_\infty. \tag{38}$$

Proof. By taking the LCST on both sides of (37), we have

$$\begin{aligned} E_M(\xi,\varepsilon) &= \mathcal{S}^M\{e(t)\}(\xi,\varepsilon) \\ &= \sqrt{2\pi}2^{-\tau}\sum_{|\lambda|\ge N} f\left[\frac{\lambda}{2^\tau}\right]g\left(\frac{\lambda}{2^\tau}, \xi\right)K_M(2^{-\tau}\lambda, \xi)S\left(\frac{\xi}{2^\tau B}\frac{\varepsilon B}{2^\tau}\right). \end{aligned}$$

using Parseval's Theorem [24], we have

$$\|e(t)\|_{L^2}^2 = \frac{1}{2^{2\tau}B}\int_{\mathbb{R}}\left|\sum_{|\lambda|\ge N} f\left[\frac{\lambda}{2^\tau}\right]g\left(\frac{\lambda}{2^\tau}, \xi\right)e^{i\pi A(\frac{\lambda}{2^\tau})^2\frac{1}{B}}e^{i2\pi\xi\frac{\lambda}{2^\tau B}}S\left(\frac{\xi}{2^\tau B}, \frac{\varepsilon B}{2^\tau}\right)\right|^2 d\xi. \tag{39}$$

setting $\tilde{f}[\lambda] = f[\lambda]e^{i\pi B\frac{\lambda^2}{B}}$, $\tilde{g}(\lambda,\xi) = g(\lambda,\xi)$ and taking $\xi' = \frac{\xi}{2^\tau}$ in (39). Moreover, since $e^{i2\pi\xi\frac{\lambda}{2^\tau B}}$ is $2\pi B$ periodic, we obtain

$$\begin{aligned} &\|e(t)\|_{L^2}^2 \\ &= \frac{1}{2^\tau B}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\left|\sum_{|\lambda|\ge N}\tilde{f}\left[\frac{\lambda}{2^\tau}\right]g\left(\frac{\lambda}{2^\tau}, \xi'\right)e^{-i2\pi\xi'(\frac{\lambda}{2^\tau})^2\frac{1}{B}}\right|^2\left|S\left(\frac{\xi'}{B}, \varepsilon' B\right)\right|^2 d\xi' \\ &= \frac{1}{2^\tau B}\sum_{k\in\mathbb{Z}}\int_I\left|\sum_{|\lambda|\ge N}\tilde{f}\left[\frac{\lambda}{2^\tau}\right]\tilde{g}\left(\frac{\lambda}{2^\tau}, \xi'\right)e^{-i2\pi\xi'(\frac{\lambda}{2^\tau})^2\frac{1}{B}}\right|^2\left|S\left(\frac{\xi'}{B}+2k\pi, \varepsilon' B+2k\pi\right)\right|^2 d\xi' \\ &= \frac{1}{2^\tau B}\int_I\left|\sum_{|\lambda|\ge N}\tilde{f}\left[\frac{\lambda}{2^\tau}\right]\tilde{g}\left(\frac{\lambda}{2^\tau}, \xi'\right)e^{-i2\pi\xi'(\frac{\lambda}{2^\tau})^2\frac{1}{B}}\right|^2\sum_{k\in\mathbb{Z}}\left|S\left(\frac{\xi'}{B}+2k\pi, \varepsilon' B+2k\pi\right)\right|^2 d\xi. \end{aligned} \tag{40}$$

making use of (4), (32), and by Parseval's theorem of the DTST, (40) can be rewritten as

$$\begin{aligned} &\|e(t)\|_{L^2}^2 \\ &= 2^{-\tau}\int_I\left|\frac{1}{\sqrt{2\pi}}\sum_{|\lambda|\ge N}\tilde{f}\left[\frac{\lambda}{2^\tau}\right]\tilde{g}\left(\frac{\lambda}{2^\tau}, \xi'\right)e^{-i2\pi\xi'(\frac{\lambda}{2^\tau})^2\frac{1}{B}}\right|^2\frac{H^2_{\varphi,M}(\xi',u)}{B|\tilde{\Phi}\left(\frac{\xi'}{B}, \varepsilon' B\right)|^2}\chi_{D_\tau}(\xi',\varepsilon')d\xi' \\ &\le \int_I\left|\frac{1}{\sqrt{2\pi}}\sum_{|\lambda|\ge N}\tilde{f}\left[\frac{\lambda}{2^\tau}\right]\tilde{g}\left(\frac{\lambda}{2^\tau}, \xi'\right)e^{-i2\pi\xi'(\frac{\lambda}{2^\tau})^2\frac{1}{B}}\right|^2 d\xi'\frac{1}{2^\tau B}\left\|\frac{H_{\varphi,M}(\xi',u)}{\tilde{\Phi}\left(\frac{\xi'}{B}, \varepsilon'\right)}\chi_{D_\tau}(\xi',\varepsilon')\right\|_\infty^2 \\ &= 2^{-\tau}\sum_{|\lambda|\ge N}\left|\tilde{f}\left[\frac{\lambda}{2^\tau}\right]\tilde{g}\left(\frac{\lambda}{2^\tau}, \xi'\right)\right|^2\left\|\frac{H_{\varphi,M}(\xi',u)}{\tilde{\Phi}\left(\frac{\xi'}{B}, \varepsilon'\right)}\chi_{D_\tau}(\xi',\varepsilon')\right\|_\infty^2, \end{aligned}$$

which validates (38). □

4.2. Aliasing Error

The aliasing error for any signal $f(t)$ is a set of V_1^M defined by

$$\|e_a(t)\|_L^2 = f(t) - \sum_{\eta\in\mathbb{Z}} f\left[\frac{m}{2^\tau}\right] s(2^\tau t - \eta) g(t,\xi) e^{-i\pi A\left(t^2 - (\frac{\eta}{2^\tau})^2\right)\frac{1}{B}}. \tag{41}$$

Theorem 4. *If $\varphi(t)$ is the canonical scaling function of an MRA $\{V_k^M : k \in \mathbb{Z}\}$, with the sampling sequence $\varphi[n] \in \ell^2[\mathbb{Z}]$ and $\frac{1}{\sqrt{2\pi}\tilde{\Phi}\left(\frac{\xi}{B},\varepsilon B\right)}\chi_{D_\tau}(\xi,\varepsilon) \in L^2[I]$ for some $\tau \in \mathbb{Z}\cup\{0\}$, then the aliasing error is bounded by*

$$\begin{aligned}
&\|e_a(t)\|_L^2 \\
&\leq \sqrt{2\pi}2^{(\tau+\delta_\tau)\frac{1}{2}} \sqrt{\sum_{\eta\in\mathbb{Z}} |d[n]|^2} \left\| \left\{ \frac{\tilde{\Phi}\left(\frac{\xi}{B}+\pi, \varepsilon B+\pi\right)}{\tilde{\Phi}\left(\frac{2\xi}{B}+\pi, 2\varepsilon B+\pi\right)} \Delta\left\{W\left(\frac{\xi}{B}\right)\right\} \right\}^{\delta_J} \left\{\Gamma\left(\frac{\xi}{B}\right)\right\}^{1-\delta_J} \right. \\
&\left. \times \mathcal{H}_{\varphi,M}\left(\frac{\xi}{2^{\tau+\delta_\tau-1}}, \frac{\varepsilon}{2^{\tau+\delta_\tau-1}}\right) \left\{\Pi_{\tau=1}^{\tau-1}\Lambda\left(\frac{\xi}{2^\tau B}\right)\right\}^{1-\delta_\tau-\delta_{\tau-1}} \right\|_\infty,
\end{aligned} \tag{42}$$

where LCST coefficients of $f(t)$ in W_0^M are denoted by $\{d[\lambda] : \lambda \in \mathbb{Z}\}$ and $W\left(\frac{\xi}{B}\right)$ is defined in ([22], Theorem 3) as

$$W\left(\frac{\xi}{B}\right) = \begin{pmatrix} \Lambda\left(\frac{\xi}{B}\right) & \Lambda\left(\frac{\xi}{B}+\pi\right) \\ \Gamma\left(\frac{\xi}{B}\right) & \Gamma\left(\frac{\xi}{B}+\pi\right). \end{pmatrix}$$

Proof. Let us suppose that $W_0^M = V_0^M \oplus V_1^M$ is the direct complement of V_0^M and V_1^M, from (11), it will be required to show that (42) satisfies for any $f(t) \in W_0^M$. Let $\varphi(t) \in L^2[\mathbb{R}]$, $\varphi(t)$ be set to W_0^M begin the LCST coefficients of MRA $\{V_k\}_{k\in\mathbb{Z}}$, as $\varphi_{0,\lambda}(t) = \varphi(t-\lambda)e^{-i\pi A(t^2-\lambda^2)\frac{1}{B}}$ from the Riesz basis of W_0^M, and $d[\lambda] \in \ell^2[\mathbb{Z}]$, such that,

$$f(t)g(t,\xi) = \sum_{\lambda\in\mathbb{Z}} d[\lambda]\varphi(t-\lambda)e^{-i\pi A(t^2-\lambda^2)\frac{1}{B}}.$$

let $\mathcal{S}_f^M(\xi,\varepsilon)$ denote the LCST OF $f(t)$, $\tilde{D}_M(\xi,\varepsilon)$ denote the discrete-time LCST of the product $d[\lambda]$, and $G_M(\varepsilon,\xi)$ be the LCST coefficients of $g(t,\xi)$.

Taking LCST on both sides of (41) and using (6), we have

$$\begin{aligned}
\mathcal{S}_f^M(\xi,\varepsilon)G_M(\varepsilon,\xi) &= \sqrt{2\pi}\tilde{D}_M(\xi,\varepsilon)\Psi(\xi,\varepsilon) \\
&= \sqrt{2\pi}\tilde{D}_M(\xi,\varepsilon)\Phi\left(\frac{\xi}{2B},\frac{\varepsilon B}{2}\right)\Gamma\left(\frac{\xi}{2B}\right).
\end{aligned} \tag{43}$$

taking the LCST on both (41) and (32) gives

$$\begin{aligned}
\mathcal{E}_a^M(\xi,\varepsilon) &= \mathcal{S}_f^M(\xi,\varepsilon)G_M(\varepsilon,\xi) - \sqrt{2\pi}2^{-\tau}\sum_{\lambda\in\mathbb{Z}} f\left[\frac{\lambda}{2^\tau}\right] g\left(\frac{\lambda}{2^\tau},\xi\right) K_M\left(\xi,\frac{\lambda}{2^\tau}\right) S\left(\frac{\xi}{2^\tau B},\frac{\varepsilon B}{2^\tau}\right) \\
&= \mathcal{S}_f^M(\xi,\varepsilon)G_M(\varepsilon,\xi) - \sqrt{2\pi}2^{-\tau}\sum_{\lambda\in\mathbb{Z}} f\left[\frac{\lambda}{2^\tau}\right] g\left(\frac{\lambda}{2^\tau},\xi\right) K_M\left(\xi,\frac{\lambda}{2^\tau}\right) \\
&\times \frac{\Phi\left(\frac{\xi}{2^\tau B},\frac{\varepsilon B}{2^\tau}\right)}{\sqrt{2\pi}\tilde{\Phi}_{\frac{\xi}{2^\tau B},\frac{\varepsilon B}{2^\tau}}} \aleph_{D_\tau}\left(\frac{\xi}{2^\tau},\frac{\varepsilon}{2^\tau}\right),
\end{aligned} \tag{44}$$

where $\mathcal{E}_a^M(\xi,\varepsilon)$ denotes the LCST of $e_a(t)$. Now, by the Poisson summation formula [25],

$$\begin{aligned} & 2^{-\tau}\sum_{\lambda\in\mathbb{Z}} f\left[\frac{\lambda}{2^\tau}\right] g\left(\frac{\lambda}{2^\tau},\xi\right) K_M\left(\xi,\frac{\lambda}{2^\tau}\right) \\ & = e^{i\pi A\frac{\xi^2}{B}} \sum_{\lambda\in\mathbb{Z}} S_f^M(\xi+2^{\tau+1}\lambda\pi B,\varepsilon+2^{\tau+1}\lambda\pi B) \\ & \times e^{-i\pi A(\xi+2^{\tau+1}\lambda\pi B)^2\frac{1}{B}}. \end{aligned} \tag{45}$$

by inserting (43) into (45), we obtain

$$\begin{aligned} & 2^{-\tau}\sum_{\lambda\in\mathbb{Z}} f\left[\frac{\lambda}{2^\tau}\right] g\left(\frac{\lambda}{2^\tau},\xi\right) K_M\left(\xi,\frac{\lambda}{2^\tau}\right) \\ & = \sqrt{2\pi} e^{i\pi A\frac{\xi^2}{B}} \sum_{\lambda\in\mathbb{Z}} \tilde{D}_M(\xi+2^{\tau+1}\lambda\pi B,\varepsilon+2^{\tau+1}\lambda\pi B) \\ & \times e^{-i\pi A(\xi+2^{\tau+1}\lambda\pi B)^2\frac{1}{B}} \Psi(\xi+2^{\tau+1}\lambda\pi B,\varepsilon+2^{\tau+1}\lambda\pi B). \end{aligned}$$

upon further simplification, we have

$$\begin{aligned} & 2^{-\tau}\sum_{\lambda\in\mathbb{Z}} f\left[\frac{\lambda}{2^\tau}\right] g\left(\frac{\lambda}{2^\tau},\xi\right) K_M\left(\xi,\frac{\lambda}{2^\tau}\right) \\ & = \sqrt{2\pi} e^{i\pi A\frac{\xi^2}{B}} \sum_{\lambda\in\mathbb{Z}} \tilde{D}_M(\xi,\varepsilon) \sum_{\lambda\in\mathbb{Z}} \\ & \times \Psi\left(\frac{\xi}{B}+2^{\tau+1}\lambda\pi,\varepsilon B+2^{\tau+1}\lambda\pi\right). \end{aligned} \tag{46}$$

inserting (43) and (46) into (44) yields

$$\begin{aligned} & \mathcal{E}_a^M(\xi,\varepsilon) \\ & = \sqrt{2\pi}\tilde{D}_M(\xi,\varepsilon)\Gamma\left(\frac{\xi}{2B}\right)\Phi\left(\frac{\xi}{2B},\frac{\varepsilon B}{2}\right) \\ & -\sqrt{2\pi}\tilde{D}_M(\xi,\varepsilon)\sum_{\lambda\in\mathbb{Z}}\Psi\left(\frac{\xi}{B}+2^{\tau+1}\lambda\pi,\varepsilon B+2^{\tau+1}\lambda\pi\right)\frac{\Phi\left(\frac{\xi}{2^\tau B},\frac{\varepsilon B}{2^\tau}\right)}{\sqrt{2\pi}\tilde{\Phi}_{\frac{\xi}{2^\tau B},\frac{\varepsilon B}{2^\tau}}}\aleph_{D_\tau}\left(\frac{\xi}{2^\tau}\right). \end{aligned} \tag{47}$$

Case I. When $\tau=0$. Adding (47) and Parseval's theorem of the LCT results in

$$\begin{aligned} \|e_a(t)\|_{L^2}^2 &= 2\pi\Bigg\|\tilde{D}_M(\xi,\varepsilon)\Gamma\left(\frac{\xi}{2B}\right)\Phi\left(\frac{\xi}{2B},\frac{\varepsilon B}{2}\right) \\ & -\tilde{D}_M(\xi)\sum_{\lambda\in\mathbb{Z}}\Psi\left(\frac{\xi}{B}+2\lambda\pi,\varepsilon B+2\lambda\pi\right)\frac{\Phi\left(\frac{\xi}{B},\varepsilon B\right)}{\sqrt{2\pi}\tilde{\Phi}\left(\frac{\xi}{B},\varepsilon B\right)}\Bigg\|_{L^2}^2. \end{aligned} \tag{48}$$

using (5) and (48), it can be written as

$$\begin{aligned} \|e_a(t)\|_{L^2}^2 &= 2\pi\Bigg\|\tilde{D}_M(\xi,\varepsilon)\Gamma\left(\frac{\xi}{2B}\right)\Phi\left(\frac{\xi}{2B},\frac{\varepsilon B}{2}\right) \\ & -\frac{\sum_{\lambda\in\mathbb{Z}}\Psi\left(\frac{\xi}{B}+2\lambda\pi,\varepsilon B+2\lambda\pi\right)}{\tilde{\Phi}\left(\frac{\xi}{B},\varepsilon B\right)}\Phi\left(\frac{\xi}{2B},\frac{\varepsilon B}{2}\right)\Lambda\left(\frac{\xi}{2B}\right)\Bigg\|_{L^2}^2. \end{aligned}$$

assuming that $I_\eta = [0, 2^{\eta+1}\pi B]$, then

$$\begin{aligned} \|e_a(t)\|^2_{L^2} &= 2\pi \sum_{k\in\mathbb{Z}} \int_I \left|\Phi\left(\frac{\xi}{2B} + 2\pi k, \frac{\varepsilon B}{2} + 2\pi k\right)\right|^2 \left|\Gamma\left(\frac{\xi}{2B}\right)\right| |\tilde{D}_M(\xi,\varepsilon)|^2 \\ &\quad - \frac{\sum_{\lambda\in\mathbb{Z}} \Psi\left(\frac{\xi}{B} + 2\lambda\pi, \varepsilon B + 2\lambda\pi\right)}{\tilde{\Phi}\left(\frac{\xi}{B}, \varepsilon B\right)} \Lambda\left(\frac{\xi}{2B}\right)\Bigg|^2 d\xi \end{aligned}$$

$$\begin{aligned} \|e_a(t)\|^2_{L^2} &= 2\pi \int_{I_1} \mathcal{H}^2_{\varphi,M}\left(\frac{\xi}{2}, \frac{\varepsilon}{2}\right) |\tilde{D}_M(\xi,\varepsilon)|^2 \Bigg|\Gamma\left(\frac{\xi}{2B}\right) \\ &\quad - \frac{\sum_{\lambda\in\mathbb{Z}} \Psi\left(\frac{\xi}{B} + 2\lambda\pi, \varepsilon B + 2\lambda\pi\right)}{\tilde{\Phi}\left(\frac{\xi}{B}, \varepsilon B\right)} \Lambda\left(\frac{\xi}{2B}\right)\Bigg|^2 d\xi. \end{aligned} \tag{49}$$

then (5) and (6) yield

$$\begin{aligned} &\sum_{\lambda\in\mathbb{Z}} \Psi\left(\frac{\xi}{B} + 2\lambda\pi, \varepsilon B + 2\lambda\pi\right) \\ &= \sum_{\lambda\in\mathbb{Z}} \Gamma\left(\frac{\xi}{2B} + 2\pi\lambda\right) \Phi\left(\frac{\xi}{2B} + \pi\lambda, \frac{\varepsilon B}{2} + 2\pi\lambda\right) \\ &= \sum_{k\in\mathbb{Z}} \Gamma\left(\frac{\xi}{2B} + \pi\lambda\right) \Phi\left(\frac{\xi}{2B} + \pi\lambda, \frac{\varepsilon B}{2} + \pi\lambda\right) \\ &+ \sum_{k\in\mathbb{Z}} \Gamma\left(\frac{\xi}{2B} + \pi(2k+1)\right) \Phi\left(\frac{\xi}{2B} + \pi(2k+1), \frac{\varepsilon B}{2} + \pi(2k+1)\right) \\ &= \Gamma\left(\frac{\xi}{2B}\right) \tilde{\Phi}\left(\frac{\xi}{2B}, \frac{\varepsilon B}{2}\right) + \tilde{\Phi}\left(\frac{\xi}{2B} + \pi, \frac{\varepsilon B}{2} + \pi\right) \Gamma\left(\frac{\xi}{2B} + \pi\right). \end{aligned} \tag{50}$$

and

$$\begin{aligned} \tilde{\Phi}\left(\frac{\xi}{B}, \varepsilon B\right) &= \sum_{\lambda\in\mathbb{Z}} \Phi\left(\frac{\xi}{B} + 2\lambda\pi, \varepsilon B + 2\lambda\pi\right) \\ &= \tilde{\Phi}\left(\frac{\xi}{2B}, \frac{\varepsilon B}{2}\right) \Lambda\left(\frac{\xi}{2B}\right) + \tilde{\Phi}\left(\frac{\xi}{2B} + \pi, \frac{\varepsilon B}{2} + \pi\right) \Lambda\left(\frac{\xi}{2B} + \pi\right). \end{aligned} \tag{51}$$

upon substituting (50) and (51) in (49), we have

$$\begin{aligned} \|e_a(t)\|^2_{L^2} &= 2\pi \int_{I_1} \mathcal{H}^2_{\varphi,M}\left(\frac{\xi}{2}, \frac{\varepsilon}{2}\right) |\tilde{D}_M(\xi,\varepsilon)|^2 \Bigg| \frac{\tilde{\Phi}\left(\frac{\xi}{2B} + \pi, \frac{\varepsilon B}{2} + \pi\right)}{\tilde{\Phi}\left(\frac{\xi}{2B}, \varepsilon B\right)} \\ &\quad \times \left\{\Gamma\left(\frac{\xi}{2B}\right) \Lambda\left(\frac{\xi}{2B} + \pi\right) - \Gamma\left(\frac{\xi}{2B} + \pi\right) \Lambda\left(\frac{\xi}{2B}\right)\right\}\Bigg|^2 d\xi \\ &\quad \le 2\pi \left\| \mathcal{H}^2_{\varphi,M}\left(\frac{\xi}{2}, \frac{\varepsilon}{2}\right) \frac{\tilde{\Phi}\left(\frac{\xi}{2B} + \pi, \frac{\varepsilon B}{2} + \pi\right)}{\tilde{\Phi}\left(\frac{\xi}{B}, \varepsilon B\right)} \Delta\{W(\frac{\xi}{2B})\} \right\|^2 \int_{I_1} |\tilde{D}_M(\xi,\varepsilon)|^2 d\xi \\ &= 4\pi \left\| \mathcal{H}^2_{\varphi,M}\left(\frac{\xi}{2}, \frac{\varepsilon}{2}\right) \frac{\tilde{\Phi}\left(\frac{\xi}{2B} + \pi, \frac{\varepsilon B}{2} + \pi\right)}{\tilde{\Phi}\left(\frac{2\xi}{B}, 2\varepsilon B\right)} \Delta\{W(\frac{\xi}{2B})\} \right\|^2 \sum_{\lambda\in\mathbb{Z}} |d[\lambda]|^2. \end{aligned}$$

Case II. When $\tau = 1$. Adding (47) and Parseval's theorem of the LCST gives

$$\begin{aligned}\|e_a(t)\|_{L^2}^2 &= 2\pi\bigg\|\tilde{D}_M(\xi,\varepsilon)\Gamma\left(\frac{\xi}{2B}\right)\Phi\left(\frac{\xi}{2B},\frac{\varepsilon B}{2}\right)\\ &\quad -\tilde{D}_M(\xi,\varepsilon)\sum_{\lambda\in\mathbb{Z}}\Psi\left(\frac{\xi}{B}+4\lambda\pi\right)\frac{\Phi\left(\frac{\xi}{2B},\frac{\varepsilon B}{2}\right)}{\tilde{\Phi}\left(\frac{\xi}{2B},\frac{\varepsilon B}{2}\right)}\aleph_{2supp\Lambda(\frac{\xi}{B})}(\xi,\varepsilon)\bigg\|_{L^2}^2.\end{aligned}$$

now by (6), it gives

$$\begin{aligned}\|e_a(t)\|_{L^2}^2 &= 2\pi\bigg\|\tilde{D}_M(\xi,\varepsilon)\Gamma\left(\frac{\xi}{2B}\right)\Phi\left(\frac{\xi}{2B},\frac{\varepsilon B}{2}\right)-\tilde{D}_M(\xi,\varepsilon)\Gamma\left(\frac{\xi}{2B}\right)\\ &\quad\times\sum_{\lambda\in\mathbb{Z}}\Phi\left(\frac{\xi}{2B}+2\lambda\pi,\frac{\varepsilon B}{2}+2\lambda\pi\right)\frac{\Phi\left(\frac{\xi}{2B},\frac{\varepsilon B}{2}\right)}{\tilde{\Phi}\left(\frac{\xi}{2B},\frac{\varepsilon B}{2}\right)}\aleph_{2supp\Lambda(\frac{\xi}{B})}(\xi,\varepsilon)\bigg\|_{L^2}^2.\end{aligned}$$

using (37) and (51)

$$\begin{aligned}\|e_a(t)\|_{L^2}^2 &= 2\pi\bigg\|\tilde{D}_M(\xi,\varepsilon)\Gamma\left(\frac{\xi}{2B}\right)\Phi\left(\frac{\xi}{2B},\frac{\varepsilon B}{2}\right)-\tilde{D}_M(\xi,\varepsilon)\Gamma\left(\frac{\xi}{2B}\right)\\ &\quad\times\Phi\left(\frac{\xi}{2B},\frac{\varepsilon B}{2}\right)\frac{\Phi\left(\frac{\xi}{2B},\frac{\varepsilon B}{2}\right)}{\tilde{\Phi}\left(\frac{\xi}{2B},\frac{\varepsilon B}{2}\right)}\aleph_{2supp\Lambda(\frac{\xi}{B})}(\xi,\varepsilon)\bigg\|_{L^2}^2\\ &= 2\pi\bigg\|\tilde{D}_M(\xi,\varepsilon)\Gamma\left(\frac{\xi}{2B}\right)\Phi\left(\frac{\xi}{2B},\frac{\varepsilon B}{2}\right)\{1-\aleph_{2supp\Lambda(\frac{\xi}{B})}(\xi,\varepsilon)\}\bigg\|_{L^2}^2\\ &= 2\pi\sum_{k\in\mathbb{Z}}\int_{I_1}|\tilde{D}_M(\xi,\varepsilon)|^2\left|\Gamma\left(\frac{\xi}{2B}\right)\right|^2\left|\Phi\left(\frac{\xi}{2B}+2k\pi,\frac{\varepsilon B}{2}+2k\pi\right)\right|^2\\ &\quad\times|1-\aleph_{2supp\Lambda(\frac{\xi}{B})}(\xi,\varepsilon)|d\xi\\ &\quad\leq 2\pi\int_{I_1}|\tilde{D}_M(\xi,\varepsilon)|^2 d\xi\bigg\|\Gamma\left(\frac{\xi}{2B}\right)\mathcal{H}_{\phi,M}\left(\frac{\xi}{2},\frac{\varepsilon}{2}\right)(1-\aleph_{supp\Lambda(\frac{\xi}{2B})}(\xi,\varepsilon))\bigg\|_{\infty}^2\\ &= 4\pi\sum_{\lambda\in\mathbb{Z}}|d[n]|^2\bigg\|\Gamma\left(\frac{\xi}{B}\right)\mathcal{H}_{\phi,M}(\xi,\varepsilon)\aleph_{\mathbb{R}\oplus supp\Lambda(\frac{\xi}{B})(\xi,\varepsilon)}\bigg\|_{\infty}^2.\end{aligned}$$

thus,

$$\|e_a(t)\|_{L^2}^2 \leq 4\pi\sum_{\lambda\in\mathbb{Z}}|d[n]|^2\bigg\|\Gamma\left(\frac{\xi}{B}\right)\mathcal{H}_{\phi,M}(\xi,\varepsilon)\bigg\|_{\infty}^2.$$

Case III. For $\tau \geq 2$. Using (47) and Parseval's theorem for LCST gives

$$\begin{aligned}&\|e_a(t)\|_{L^2}^2\\ &= 2\pi\bigg\|\tilde{D}_M(\xi,\varepsilon)\Gamma\left(\frac{\xi}{2B}\right)\prod_{J=2}^{j}\Lambda\left(\frac{\xi}{2^J B}\right)\Phi\left(\frac{\xi}{2^J B},\frac{\varepsilon B}{2^J}\right)\\ &-\tilde{D}_M(\xi,\varepsilon)\Gamma\left(\frac{\xi}{2B}\right)\prod_{\tau=2}^{\tau}\Lambda\left(\frac{\xi}{2^\tau B}\right)\tilde{\Phi}\left(\frac{\xi}{2^\tau B},\frac{\varepsilon B}{2^\tau}\right)\frac{\Phi\left(\frac{\xi}{2^\tau B},\frac{\varepsilon B}{2^\tau}\right)}{\tilde{\Phi}\left(\frac{\xi}{2^\tau B},\frac{\varepsilon B}{2^\tau}\right)}\aleph_{\cap_{\tau=1}^{\tau}2^\tau supp\Lambda(\frac{\xi}{B})}(\xi,\varepsilon)\bigg\|_{L^2}^2\\ &= 2\pi\bigg\|\tilde{D}_M(\xi,\varepsilon)\Gamma\left(\frac{\xi}{2B}\right)\prod_{\tau=2}^{\tau}\Lambda\left(\frac{\xi}{2^\tau B}\right)\Phi\left(\frac{\xi}{2^\tau B},\frac{\varepsilon B}{2^\tau}\right)(1-\aleph_{\cap_{\tau=1}^{\tau}2^\tau supp\Lambda(\frac{\xi}{B})}(\xi,\varepsilon))\bigg\|_{L^2}^2.\end{aligned}$$

upon further simplification, we can write it as

$$\|e_a(t)\|^2_{L^2} = 2\pi \int_{I_\tau} \left| \tilde{D}_M(\xi,\varepsilon)\Gamma\left(\frac{\xi}{2B}\right) \prod_{\tau=2}^{\tau} \Lambda\left(\frac{\xi}{2^\tau B}\right) \right|^2 \sum_{k\in\mathbb{Z}} \left| \Phi\left(\frac{\xi}{2^\tau B} + 2k\pi, \frac{\varepsilon B}{2^\tau} + 2k\pi\right) \right|^2 \times |1 - \aleph_{\cap_{\tau=1}^{\tau} 2^\tau supp\Lambda(\frac{\xi}{B})}(\xi,\varepsilon)| d\xi.$$

using the same procedure as used in previous cases, we have

$$\|e_a(t)\|^2_{L^2} \le 2\pi 2^\tau \sum_{\lambda\in\mathbb{Z}} |d[\lambda]|^2 \left\| \Gamma\left(\frac{\xi}{2^\tau B}\right) \mathcal{H}_{\varphi,M}\left(\frac{\xi}{2^{\tau-1}}, \frac{\varepsilon}{2^{\tau-1}}\right) \prod_{\tau=1}^{\tau-1} \Lambda\left(\frac{\xi}{2^\tau B}\right) \right\|^2_\infty .$$

by combining all three cases, (42) is validated. □

5. Conclusions

In this work, a sampling theorem for LCST was proposed with help from the sampling kernel in the multiresolution subspace. Moreover, for the proposed sampling theory, the truncation and aliasing errors were determined with their bounds. In future works, we will extend the current study to quaternion algebra, which will lead the researchers to focus on quaternion-valued signals and their samplings.

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Abbreviations

The following abbreviations are used in this manuscript:

LCST	linear canonical S transform
LCT	linear canonical transform
ST	Stockwell transform

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