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An Asymptotic Expansion for the Generalized Gamma Function

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Abstract: The symmetric patterns that inequalities contain are reflected in researchers' studies in many mathematical sciences. In this paper, we prove an asymptotic expansion for the generalized gamma function $\Gamma_\mu(v)$ and study the completely monotonic (CM) property of a function involving $\Gamma_\mu(v)$ and the generalized digamma function $\psi_\mu(v)$. As a consequence, we establish some bounds for $\Gamma_\mu(v)$, $\psi_\mu(v)$ and polygamma functions $\psi_\mu^{(r)}(v)$, $r \geq 1$.

Keywords: gamma function; digamma function; polygamma functions; asymptotic expansion; CM function; inequalities

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1. Introduction

Stirling's formula is given by

$$\Gamma(r+1) \sim \sqrt{2\pi r}\left(\frac{r}{e}\right)^r, \quad r \longrightarrow \infty \tag{1}$$

where Γ is the classical Gamma function [1]. An elementary and complete proof of this formula is available at [2]. Moreover, many mathematicians have used the logarithm of gamma function to deduce several useful properties of the gamma function, and their powerful tool for such investigations was the digamma function

$$\psi(v) = \frac{d}{dv}\ln\Gamma(v) = -\sum_{r=0}^{\infty}\left[\frac{1}{v+r} - \frac{1}{r+1}\right] - \gamma, \quad v > 0$$

where $\gamma = \lim_{r\to\infty}\left(\sum_{k=1}^{r}\frac{1}{k} - \log r\right) \approx 0.5772156649$ is Euler–Mascheroni's constant. For more details on bounds of the functions $\Gamma(v)$ and $\frac{d^r}{dv^r}\psi(v)$, please refer to [3–7] and the references therein. Many of such bounds deduced from the monotonicity properties of some functions involving Γ or ψ. An infinitely differentiable real valued function M defined on $v > 0$ is said to be CM if $(-1)^r M^{(r)}(v) \geq 0$ for all $r \geq 0$ on $v > 0$. For more details about CM functions and their applications, we refer to [8–11]. According to Bernstein theorem [12], function M is CM if and only if $M(v) = \int_0^\infty e^{-vu}d\nu(u)$, where $\nu(u)$ is a non-negative measure on $u \geq 0$ such that the integral converges for $v > 0$.

In 2007, Alzer and Batir [13] studied the completely monotonicity of the function

$$S_\rho(v) = \ln\Gamma(v) + \frac{1}{2}\psi(v+\rho) + v - \frac{1}{2}\ln(2\pi) - v\ln v, \quad \rho \geq 0; \quad v > 0 \tag{2}$$

and deduced the following double inequality:

$$\sqrt{2\pi}\ \exp\left[-\frac{1}{2}\psi(a+v)-v\right] < v^{-v}\Gamma(v) < \sqrt{2\pi}\exp\left[-\frac{1}{2}\psi(b+v)-v\right], \quad v>0 \tag{3}$$

with the constants $a=\frac{1}{3}$ and $b=0$ being the best possible constants. In 2008, Batir [14] modified $S_\rho(v)$ and deduced some bounds for $\Gamma(v)$ in terms of digamma and polygamma functions.

Euler [15] originally defined gamma function as $\Gamma(v)=\lim_{\mu\to\infty}\Gamma_\mu(v)$, where

$$\Gamma_\mu(v)=\frac{\mu^v\ \mu!}{(\mu+v)\cdots(v+2)(v+1)v}, \quad v>0,\ \mu=1,2,\cdots$$

which satisfies the following recurrence relation.

$$\Gamma_\mu(r+v)=\frac{v\ \mu^r\ \Gamma_\mu(v)}{v+n+\mu}\prod_{s=1}^{r-1}\left[\frac{s+v}{s+\mu+v}\right], \quad v>0,\ r\in\mathbb{N}. \tag{4}$$

In 2010, Krasniqi and Shabani [16] presented the strictly CM property of function ψ'_μ on $(0,\infty)$, where the following is the case.

$$\psi_\mu(v)=\ln\mu-\sum_{i=0}^{\mu}\frac{1}{v+i}. \tag{5}$$

Krasniqi and Merovci [17] introduced the following integral representations for ψ_μ and its derivatives:

$$\psi_\mu(v)=\ln\mu+\int_0^\infty\left(\frac{e^{-(\mu+1)u}-1}{1-e^{-u}}\right)e^{-vu}du, \quad v>0, \quad \mu\in\mathbb{N} \tag{6}$$

and

$$\psi_\mu^{(r)}(v)=(-1)^r\int_0^\infty\left(\frac{e^{-(\mu+1)u}-1}{1-e^{-u}}\right)u^re^{-vu}du, \quad v>0, \quad \mu,r\in\mathbb{N}. \tag{7}$$

Nantomah, Prempeh and Twum [18] presented the following generalization of the functions Γ and ψ:

$$\Gamma_{\mu,k}(v) \quad = \quad \frac{k^{\mu+1}\ (1+\mu)!\ (\mu\ k)^{\frac{v}{k}-1}}{v(v+k)(v+2k)\cdots(v+\mu\ k)}=\int_0^\mu\frac{u^v}{u}\left(1-\frac{u^k}{\mu\ k}\right)^\mu du, \quad v,k>0, \qquad \mu\in\mathbb{N}$$

and

$$\psi_{\mu,k}(v) \quad = \quad \frac{1}{k}\ln(k\ \mu)-\sum_{i=0}^{\mu}\frac{1}{v+ik}=\frac{1}{k}\ln(k\ \mu)+\int_0^\infty\left(\frac{e^{-k(\mu+1)u}-1}{1-e^{-ku}}\right)e^{-vu}du.$$

Nantomah, Merovci and Nasiru [19] presented bounds

$$\frac{1}{(k+v+\mu\ k)}-v^{-1}\le\psi_{\mu,k}(v)-k^{-1}\ln\left(\frac{v\ k\ \mu}{v+k+\mu\ k}\right)\le 0, \quad \mu\in\mathbb{N}, \quad v,k>0 \tag{8}$$

and

$$\frac{1}{(k+\mu\ k+v)^2}-\frac{1}{v^2}\le-\psi'_{\mu,k}(v)+k^{-1}\left[v^{-1}-\frac{1}{(v+k+\mu\ k)}\right]\le 0, \quad \mu\in\mathbb{N}, \quad v,k>0. \tag{9}$$

Recently, several inequalities involving the generalized gamma function have been presented [17–19].

Research involving Gamma function is conducted by many authors currently. Results concerning extensions of Gamma function involving Mittag–Leffler function are presented in [20]. Other extensions of Gamma function are investigated in [21] using the two-parameter Mittag–Leffler matrix function and some important properties of these extended matrix functions are proved. A new series representation of the extended $k-$gamma function is provided in [22] and particular cases involving the original gamma function are discussed as corollaries.

In the following, we will present the asymptotic expansion $\Gamma_\mu(r+1) \sim \frac{\mu^{r+1}\, r^{r+\frac{1}{2}}\, e^{\mu+1}\, \mu!}{(r+\mu+1)^{r+\mu+\frac{3}{2}}}$ for large values of r and we will discuss the property of completely monotonicity of the function

$$S_{\alpha,\mu}(v) = \ln\Gamma_\mu(v) + 1/2\ \psi_\mu(\alpha+v) - v\ln\left(\frac{\mu\, v}{v+\mu+1}\right) + (\mu+1)\ln(v+\mu+1) - \ln\left(\mu!\ \sqrt{\mu}\ e^{\mu+1}\right),$$

for $\mu \in \mathbb{N}$, $v > 0$ and different values of $\alpha \geq 0$. As a consequence, we establish some bounds for $\Gamma_\mu(v)$, $\psi_\mu(v)$ and $\psi_\mu^{(r)}(v)$, $r \geq 1$.

2. An Asymptotic Expansion for $\Gamma_\mu(v)$

Theorem 1. *For all $\mu \in \mathbb{N}$,*

$$\Gamma_\mu(r+1) \sim \frac{\mu^{r+1}\, r^{r+\frac{1}{2}}\, e^{1+\mu}\, \mu!}{(r+\mu+1)^{\frac{3}{2}+\mu+r}}, \quad r \longrightarrow \infty. \tag{10}$$

Proof. For $\mu = 1$, we have

$$\lim_{r\to\infty} \frac{\Gamma_1(1+r)}{\frac{r^{r+\frac{1}{2}}e^2}{(r+2)^{r+\frac{5}{2}}}} = \frac{1}{e^2}\lim_{r\to\infty}\left[\frac{r!}{(r+2)!}\,(r+2)^2\left(1+2r^{-1}\right)^{\frac{1}{2}+r}\right] = 1.$$

Similarly, for $\mu = 2$, we have $\lim_{r\to\infty} \frac{\Gamma_2(r+1)}{\frac{2^{r+2}\, r^{r+\frac{1}{2}}\, e^3}{(r+3)^{r+\frac{7}{2}}}} = 1$. Now, for $\mu \geq 3$, taking logarithm for both sides of (4) at $v = 1$ yields the following.

$$\ln\Gamma_\mu(r+1) = \ln\left(\frac{\mu^2}{(1+\mu)(2+\mu)}\right) + \sum_{s=2}^{r}\ln\left(\frac{\mu\, s}{\mu+1+s}\right). \tag{11}$$

The function $A_\mu(v) = \frac{\mu\, v}{\mu+v+1}$ is strictly increasing on $v > 0$ and, hence, $A_\mu(v) > A_\mu(2) \geq 1$ for all $\mu \geq 3$. Using the relation between the integral and the Riemann sums, we have

$$\int_2^r \ln\left(\frac{v\,\mu}{\mu+v+1}\right)dv < \sum_{s=2}^{r}\ln\left(\frac{s\,\mu}{\mu+s+1}\right) < \int_2^{r+1}\ln\left(\frac{\mu\, v}{\mu+v+1}\right)dv$$

and hence, we have the following:

$$\ln u_\mu(r) < \ln\Gamma_\mu(1+r) < \ln u_\mu(1+r), \tag{12}$$

where

$$u_\mu(r) = \left[\frac{r\,\mu}{1+r+\mu}\right]^r\left[\frac{(3+\mu)^{\mu+3}}{4(1+r+\mu)^{1+\mu}(\mu+1)(\mu+2)}\right].$$

Sequence $u_\mu(r)$ satisfies

$$u_\mu(r+1) = \left[\frac{1}{r+\mu+1}+1\right]^{-(r+2+\mu)}\left[\frac{r\,\mu}{r+\mu+1}\right]\left[r^{-1}+1\right]^r\left[r^{-1}+1\right]\, u_\mu(r)$$

and using inequality $(r^{-1}+1)^r < e < (r^{-1}+1)^{r+1}$ for $r \in \mathbb{N}$, we have the following.

$$u_\mu(r+1) < 2\left[\frac{\mu\, r}{\mu+r+1}\right] u_\mu(r). \tag{13}$$

Substituting (13) into (12) yields

$$\Gamma_\mu(1+r) = u_\mu(r)\theta_\mu(r),$$

where

$$1 < \theta_\mu(r) < 2\left[\frac{\mu\, r}{r+1+\mu}\right].$$

Let

$$a_\mu(r) = \frac{\Gamma_\mu(r+1)}{(r\,\mu)^{\frac{2r+1}{2}}(\mu+r+1)^{-\mu-r-\frac{3}{2}}}, \qquad r,\mu \in \mathbb{N}.$$

Using (4) at $v=1$, we have

$$\ln a_\mu(r) = -(r+1/2)\ln r + \ln\left(\frac{\sqrt{\mu}}{1+\mu}\right) + \sum_{s=1}^{r} \ln\left(\frac{s}{s+1+\mu}\right) + (3/2+\mu+r)\ln(\mu+1+r)$$

and then

$$\ln a_\mu(r) - \ln a_\mu(1+r) = H(r) - H(1+\mu+r), \tag{14}$$

where $H(v) = (v+\frac{1}{2})\ln\left(1+\frac{1}{v}\right)$, $v>0$ is decreasing function on $v>0$ and consequently $\left(\ln a_\mu(r)\right)_{r\in\mathbb{N}}$ is decreasing. Using (14), we obtain the following.

$$\ln a_\mu(1) - \ln a_\mu(r) = \sum_{m=1}^{r-1}[H(m) - H(m+\mu+1)]. \tag{15}$$

Using well-known series $\ln\left(\frac{1+y}{1-y}\right) = 2\left(y + \frac{y^3}{3} + \frac{y^5}{5} + \cdots\right)$, $|y|<1$ and letting $y = \frac{1}{2v+1}$, we obtain the following.

$$H(v) - 1 = \sum_{\tau=1}^{\infty} \frac{1}{1+2\tau}\left(\frac{1}{2v+1}\right)^{2\tau} < \sum_{\tau=1}^{\infty} \frac{1}{3}\left(\frac{1}{2v+1}\right)^{2\tau} = \frac{1}{12v(1+v)}. \tag{16}$$

As $H(v)$ is decreasing, we have the following.

$$H(v) > \lim_{v\to\infty} H(v) = 1. \tag{17}$$

Substituting (16) and (17) into (15) produces

$$\ln a_\mu(1) - \ln a_\mu(r) < \frac{1}{12}\sum_{\tau=1}^{r-1}\left[\frac{1}{\tau} - \frac{1}{(\tau+1)}\right] < \frac{1}{12}$$

and then $(\ln a_\mu(r))_{r\in\mathbb{N}}$ is bounded from below by $(\ln a_\mu(1) - \frac{1}{12})$. Hence, $(\ln a_\mu(r))_{r\in\mathbb{N}}$ is convergent to some constant and depends on μ. Then, the following is obtained.

$$\Gamma_\mu(1+r) \sim \frac{C_\mu (r\,\mu)^{r+\frac{1}{2}}}{(\mu+r+1)^{\mu+r+3/2}}. \tag{18}$$

Using Stirling's Formula (1), we obtain $\lim\limits_{r\to\infty} \frac{2^{4r}(r!)^4}{\pi r[(2r)!]^2} = 1$ and $\lim\limits_{r\to\infty} \frac{\pi(r+\mu)[(2r+2\mu)!]^2}{2^{4(r+\mu)}[(r+\mu)!]^4} = 1$.

Multiplying these two limits produces the following.

$$\lim_{r\to\infty}\left(\frac{r!}{(\mu+r)!}\right)^4\left(\frac{(2\mu+2r)!}{(2r)!}\right)^2 = 2^{4\mu}. \tag{19}$$

Letting $v = 1$ in (4) and using relation $(2r+2\mu)! = (2r+\mu+1)!\prod_{s=2+\mu}^{2\mu}(2r+s)$, we have

$$\Gamma_\mu(2r+1) = \frac{(2r)!\ \mu^{1+2r}\ \mu!\ \prod_{s=\mu+2}^{2\mu}(s+2r)}{(2r+2\mu)!}. \tag{20}$$

Now, using (4) at $v = 1$ and inserting (20) into (19) yields

$$\lim_{r\to\infty}\frac{\Big(\Gamma_\mu(r+1)\Big)^4}{\Big(\Gamma_\mu(2r+1)\Big)^2}(r+\mu+1)^4\left(\prod_{s=\mu+2}^{2\mu}(2r+s)\right)^2 = 2^{4\mu}\ \mu^2\ (\mu!)^2 \tag{21}$$

and using (18), we obtain the following:

$$C_\mu^2\lim_{r\to\infty}\left(\frac{2r+\mu+1}{2r+2\mu+2}\right)^{4r}\left(\prod_{s=\mu+2}^{2\mu}\frac{s+2r}{r+1+\mu}\right)^2 = (\mu!)^2\ 2^{2\mu-2}\ \mu$$

which leads to

$$C_\mu = \sqrt{\mu}\ \mu!\ e^{\mu+1}, \qquad \mu \geq 3,$$

which completes the proof. □

In the next part, we provide a double inequality involving $\Gamma_\mu(v+1)$.

Some Bounds for the Function $\Gamma_\mu(v)$

Theorem 2. *Assume that $\mu \in \mathbb{N}$ and $v \in (0,\infty)$. Then, we have*

$$\exp\Big[-\lambda(\mu+1+v)+\lambda(v)\Big] < \frac{\Gamma_\mu(1+v)}{\left(\frac{\mu^{1+v}v^{\frac{1}{2}+v}\ e^{\mu+1}\ \mu!}{(\mu+v+1)^{\mu+v+3/2}}\right)} < \exp\Big[-\beta(\mu+v+1)+\beta(v)\Big], \tag{22}$$

where $\beta(v) = \frac{1}{12v}$ and $\lambda(v) = \beta(v) - \frac{1}{360v^3} - \frac{1}{120v^4}$.

Proof. Let

$$W_\mu(v) = -H(v+\mu+1)+H(v)+\Delta\Big[\beta(v+\mu+1)-\beta(v)\Big],$$

where $\Delta g(v) = g(v) - g(v+1)$. Then $W_\mu''(v) = -\frac{(1+\mu)(2+\mu+2v)D_\mu(v)}{6v^3(1+v)^3(1+\mu+v)^3(2+\mu+v)^3} < 0$, where

$$D_\mu(v) = 4+12\mu+13\mu^2+6\mu^3+\mu^4+(14+29\mu+19\mu^2+4\mu^3)v+(19+23\mu+7\mu^2)v^2+(12+6\mu)v^3+3v^4.$$

Hence, $W_\mu'(v)$ is decreasing on $v > 0$ with $\lim_{v\to\infty} W_\mu'(v) = 0$; thus , $W_\mu'(v) > 0$ for all $v > 0$. Hence, $W_\mu(v)$ is increasing on $v > 0$ with $\lim_{v\to\infty} W_\mu(v) = 0$. Then, we have the following.

$$H(v) - H(v+\mu+1) < \Delta\Big[\beta(v)-\beta(v+\mu+1)\Big]. \tag{23}$$

Similarly, we obtain the following.

$$\Delta\Big[\lambda(v) - \lambda(v+\mu+1)\Big] < H(v) - H(v+\mu+1). \tag{24}$$

Combining (23) with (24) provides the following:

$$\Delta\Big[\lambda(v) - \lambda(v+\mu+1)\Big] < H(v) - H(v+\mu+1) < \Delta\Big[\beta(v) - \beta(v+\mu+1)\Big] \tag{25}$$

and using (14), we obtain the following.

$$\exp\left(\Delta\Big[\lambda(v) - \lambda(v+\mu+1)\Big]\right) < \frac{a_\mu(v)}{a_\mu(v+1)} < \exp\left(\Delta\Big[\beta(v) - \beta(v+\mu+1)\Big]\right).$$

It follows that

$$a_\mu(v+1)\exp\Big[\lambda(v+2+\mu) - \lambda(1+v)\Big] < a_\mu(v)\exp\Big[\lambda(v+\mu+1) - \lambda(v)\Big]$$

and consequently, function

$$T_\mu(v) = a_\mu(v)\exp\Big[\lambda(v+\mu+1) - \lambda(v)\Big]$$

is strictly decreasing on $v > 0$ with $\lim\limits_{v\to\infty} T_\mu(v) = C_\mu$. Thus, we have the following.

$$a_\mu(v) > C_\mu \exp\Big[\lambda(v) - \lambda(1+v+\mu)\Big]. \tag{26}$$

In a similar way, function

$$L_\mu(v) = a_\mu(v)\exp\Big[\beta(v+\mu+1) - \beta(v)\Big]$$

is strictly increasing on $v > 0$ with $\lim\limits_{v\to\infty} L_\mu(v) = C_\mu$. Then,

$$a_\mu(v) < C_\mu \exp\Big[\beta(v) - \beta(\mu+v+1)\Big]. \tag{27}$$

Combining (26) with (27), we obtain

$$\exp\Big[\lambda(v) - \lambda(v+\mu+1)\Big] < \frac{a_\mu(v)}{C_\mu} < \exp\Big[\beta(v) - \beta(v+\mu+1)\Big],$$

which completes the proof. □

Corollary 1.

$$\Gamma_\mu(v) \sim \left(\frac{\mu^v\, v^{v-\frac{1}{2}}\, e^{\mu+1}\, \mu!}{(v+1+\mu)^{1/2+v+\mu}}\right)\exp\Big[\lambda(v) - \lambda(v+\mu+1)\Big], \quad v \longrightarrow \infty \tag{28}$$

where $\lambda(v) = \frac{1}{12v} - \frac{1}{360v^3} - \frac{1}{120v^4}$.

Proof. Using inequality (22) and relation (4) at $r = 1$, we obtain the following.

$$1 < \frac{\Gamma_\mu(v)}{\left(\frac{\mu^v\, v^{v-\frac{1}{2}}\, e^{1+\mu}\, \mu!}{(1+v+\mu)^{v+\mu+\frac{1}{2}}}\right)\exp\Big[\lambda(v) - \lambda(1+v+\mu)\Big]}$$

$$< \exp\left[\frac{1}{360v^3} + \frac{1}{120v^4} - \frac{1}{360(1+v+\mu)^3} - \frac{1}{120(1+v+\mu)^4}\right].$$

Hence, we have

$$\lim_{v\to\infty}\left[\frac{\Gamma_\mu(v)}{\left(\frac{\mu^v\, v^{v-\frac{1}{2}}\, e^{1+\mu}\, \mu!}{(v+1+\mu)^{1/2+\mu+v}}\right)\exp\Big[\lambda(v)-\lambda(v+1+\mu)\Big]}\right] = 1.$$

This completes the proof. □

Corollary 2. *Let μ and s be positive integers. Then,*

$$\ln\Gamma_\mu(v) \sim -(\mu+1)\ln(v+\mu+1) + \left(\frac{2v-1}{2}\right)\ln\left(\frac{\mu\, v}{v+1+\mu}\right) + \ln\left(\sqrt{\mu}\,\mu!\, e^{1+\mu}\right) + \Big[\lambda(v)-\lambda(v+1+\mu)\Big], \tag{29}$$

$$\psi_\mu(v) \sim -\frac{1}{2}\left(\frac{1}{v} - \frac{1}{v+1+\mu}\right) + \ln\left(\frac{\mu\, v}{v+1+\mu}\right) + \Big[\lambda'(v)-\lambda'(v+1+\mu)\Big], \quad v\longrightarrow\infty \tag{30}$$

and

$$\begin{aligned}\psi_\mu^{(s)}(v) \sim{}& (-1)^{s-1}(s-1)!\left(\frac{1}{v^s} - \frac{1}{(\mu+1+v)^s}\right) - \frac{(-1)^s s!}{2}\left(\frac{1}{v^{s+1}} - \frac{1}{(\mu+1+v)^{s+1}}\right)\\ &+ \lambda^{(s+1)}(v) - \lambda^{(s+1)}(\mu+1+v), \quad v\longrightarrow\infty\end{aligned} \tag{31}$$

where

$$\lambda^{(s)}(v) = (-1)^s\left[\frac{s!}{12v^{1+s}} - \frac{(2+s)!}{720v^{3+s}} - \frac{(3+s)!}{720v^{4+s}}\right] \qquad s\in\mathbb{N}.$$

In the next section, we will generalize some results presented by Alzer and Batir [13].

3. Study of a CM Function Involving Γ_μ and ψ_μ Functions

Theorem 3. *Suppose that $\mu\in\mathbb{N}$ and $v>0$. Then, the function*

$$S_{\alpha,\mu}(v) = \ln\Gamma_\mu(v) + 1/2\,\psi_\mu(\alpha+v) - v\ln\left(\frac{\mu\, v}{v+\mu+1}\right) + (\mu+1)\ln(v+\mu+1) - \ln\left(\sqrt{\mu}\,\mu!\, e^{1+\mu}\right),\ \alpha\ge 0$$

is CM on $v>0$ if and only if $\alpha\ge\frac{1}{3}$. Moreover, $-S_{\alpha,\mu}(v)$ is CM on $v>0$ if and only if $\alpha=0$.

Proof. From (6), (7) and identity $\ln(\frac{h}{d}) = \int_0^\infty \frac{e^{-dt}-e^{-ht}}{t}dt$ for $h,d>0$ (see [1]), we have

$$S'_{\alpha,\mu}(v) = \psi_\mu(v) - \ln\left(\frac{\mu\, v}{v+1+\mu}\right) + \frac{1}{2}\psi'_\mu(\alpha+v) = \int_0^\infty \frac{e^{-(\mu+v+1)u}}{u(e^u-1)}\varphi(u)du,$$

where

$$\varphi(u) = e^{(2+\mu)u} + 1 - e^{(1+\mu)u} - e^u - u\left[e^{(2+\mu)u} - e^u\right] - \frac{1}{2}u^2\left[e^u - e^{(2+\mu)u}\right]e^{-\alpha u}.$$

Let $\alpha\ge\frac{1}{3}$, then we obtain

$$\varphi(u) \le e^{(2+\mu)u} - e^{(1+\mu)u} - e^u + 1 - u\left[e^{(2+\mu)u} - e^u\right] + \frac{1}{2}u^2\left[e^{(\mu+\frac{5}{3})u} - e^{\frac{2}{3}u}\right]$$

and hence,

$$\varphi(u) \le \sum_{r=3}^{\infty}\frac{f_\mu(r)}{(r+2)!}u^{r+2} < 0,$$

where

$$\begin{aligned} f_\mu(r) &= -(\mu+1)^{2+r} - 1 + (\mu+2)^{2+r} + \left[1 - (\mu+2)^{1+r}\right](2+r) \\ &+ \left[(\mu+5/3)^r - (2/3)^r\right]\frac{(1+r)(2+r)}{2} \\ &= \sum_{s=1}^{r}\left[-(r+2)\binom{1+r}{s} + \binom{2+r}{s} + \binom{r}{s}(2/3)^{r-s}\frac{(2+r)(1+r)}{2}\right](1+\mu)^s \\ &= \sum_{s=1}^{r}\frac{3^{r-s}\,(2+r)(1+r)}{(2+r-s)}\binom{r}{s}\left[-3^{r-s} + 2^{r-s} + 2^{r-s-1}(r-s)\right](\mu+1)^s \\ &= -\sum_{s=1}^{r}\frac{3^{r-s}\,(1+r)(2+r)}{(2+r-s)}\binom{r}{s}\left[\sum_{l=2}^{r-s}\binom{r-s}{l}2^{r-s-l}\right](1+\mu)^s. \end{aligned}$$

Consequently, $-S'_{\alpha,\mu}(v)$ is CM on $(0,\infty)$ for $\alpha \geq \frac{1}{3}$. Thus, $S_{\alpha,\mu}(v)$ is decreasing and using asymptotic (29) and (30), we have $\lim\limits_{v\to\infty} S_{\alpha,\mu}(v) = 0$ and then $S_{\alpha,\mu}(v) > 0$. Then, $S_{\alpha,\mu}(v)$ is a CM function on $v > 0$ for $\alpha \geq \frac{1}{3}$. Conversely, if function $S_{\alpha,\mu}(v)$ is CM, then we obtain for $v > 0,\ \ \mu \in \mathbb{N}$ that

$$\frac{v^2}{(1+\mu)}S_{\alpha,\mu}(v) = \frac{v^2}{(1+\mu)}\left[\ln\left(\frac{\Gamma_\mu(v)}{\frac{v^{v-\frac{1}{2}}\,\mu^v\,e^{1+\mu}\,\mu!}{(v+1+\mu)^{1/2+\mu+v}}}\right) - \frac{1}{2}\ln\left(\frac{\mu\,v}{v+1+\mu}\right) + \frac{1}{2}\psi_\mu(\alpha+v)\right] > 0. \tag{32}$$

From (29), we have

$$\lim_{v\to\infty}\frac{v^2}{(1+\mu)}\ln\left(\frac{\Gamma_\mu(v)}{\frac{\mu^v\,v^{v-\frac{1}{2}}e^{1+\mu}\,\mu!}{(\mu+v+1)^{\mu+v+\frac{1}{2}}}}\right) = \frac{1}{12}. \tag{33}$$

Using asymptotic (30), we have $\lim\limits_{v\to\infty}\frac{v^2}{(1+\mu)}\left[\ln\left(\frac{v\,\mu}{\mu+1+v}\right) - \psi_\mu(v)\right] = \frac{1}{2}$ and using (5), we obtain $\lim\limits_{v\to\infty}\frac{v^2}{(1+\mu)}\left[\psi_\mu(v) - \psi_\mu(\alpha+v)\right] = -\alpha$.

Hence, we conclude that

$$\lim_{v\to\infty}\frac{v^2}{(\mu+1)}\left[\ln\left(\frac{\mu\,v}{v+\mu+1}\right) - \psi_\mu(v+\alpha)\right] = \frac{1}{2} - \alpha.$$

From (32), we conclude that $\frac{1}{12} - \frac{1}{2}(\frac{1}{2}-\alpha) \geq 0$ and then $\alpha \geq \frac{1}{3}$. Now, for $\alpha = 0$, we obtain the following.

$$S'_{0,\mu}(v) = \int_0^\infty \frac{e^{-(v+1+\mu)u}}{u(e^u-1)}\left(\sum_{r=2}^{\infty}\left[\sum_{s=1}^{r}\frac{(r+2)(r+1)(r-s)}{2(2+r-s)}\binom{r}{s}(\mu+1)^s\right]\frac{u^{r+2}}{(r+2)!}\right)du$$

Therefore, $S'_{0,\mu}(v)$ is CM function on $u > 0$. Thus, $S_{0,\mu}(v)$ is an increasing function on $v > 0$ with $\lim\limits_{v\to\infty} S_{0,\mu}(v) = 0$ and hence, $S_{0,\mu}(v) < 0$. Then, $-S_{0,\mu}(v)$ is CM on $v > 0$. Conversely, if we assume that $-S_{\alpha,\mu}(v)$ is CM on $v > 0$ with $\alpha > 0$, then $S_{\alpha,\mu}(v) < 0$ on $v > 0$. However, this contradicts $\lim\limits_{v\to 0} S_{\alpha,\mu}(v) = \infty$; hence, $\alpha = 0$. $\square$

Corollary 3. *Let $\mu \in \mathbb{N}$. Then*

$$\lim_{v \to 0} v^{1+r} \psi_\mu^{(r)}(v) = (-1)^{1+r}\, r!, \qquad r = 0, 1, 2, \cdots \tag{34}$$

and

$$\lim_{v \to 0} v^r \psi_\mu^{(r)}(b+v) = 0, \qquad r \in \mathbb{N}, \qquad b > 0. \tag{35}$$

Some Sharp Bounds for Γ_μ and $\psi_\mu^{(r)}$ Functions

Now, we will present some sharp bounds of Γ_μ and $\psi_\mu^{(r)}$ depending on Theorem (3).

Corollary 4. *Let two real numbers $a, b \geq 0$. For $\mu \in \mathbb{N}$ and $v > 0$, we have*

$$\left(\frac{v^v\, \mu^{\frac{1}{2}+v}\, \mu!\, e^{1+\mu}}{(v+1+\mu)^{v+1+\mu}} \right) \exp\left[-\frac{1}{2}\psi_\mu(a+v) \right] < \Gamma_\mu(v) < \left(\frac{v^v\, \mu^{\frac{1}{2}+v}\, \mu!\, e^{1+\mu}}{(v+1+\mu)^{v+1+\mu}} \right) \exp\left[-\frac{1}{2}\psi_\mu(b+v) \right] \tag{36}$$

with the constants $a = \frac{1}{3}$ and $b = 0$ are best possible.

Proof. In inequality (36), the left-hand side is equivalent $\frac{v^2}{(\mu+1)} S_{a,\mu}(v) > 0$, which leads to $a \geq \frac{1}{3}$ as stated in the proof of Theorem (3). Using the increasing property of the function $\psi_\mu(v)$ on $v > 0$, we have $e^{-\frac{1}{2}\psi_\mu(v+a)} \leq e^{-\frac{1}{2}\psi_\mu(v+\frac{1}{3})}$ for $a \geq \frac{1}{3}$. Then, $a = \frac{1}{3}$ is the best possible constant in (36). Moreover, Theorem (3) proves the right-hand side of the inequality (36) at $b = 0$. If there exist $b > 0$ such that the upper bound of $\Gamma_\mu(v)$ in (36) is valid for $v \in (0, \infty)$, then we would have

$$\lim_{v \to 0} \Gamma_\mu(v) \leq \left(\frac{\sqrt{\mu}\, e^{\mu+1}\, \mu!}{(\mu+1)^{\mu+1}} \right) \exp\left[-\frac{1}{2}\psi_\mu(b) \right] \lim_{v \to 0} \left(\frac{v}{v+\mu+1} \right)^v$$

and hence,

$$\lim_{v \to 0} \Gamma_\mu(v) \leq \left(\frac{\sqrt{\mu}\, \mu!\, e^{\mu+1}}{(\mu+1)^{\mu+1}} \right) \exp\left[-\frac{1}{2}\psi_\mu(b) \right]$$

which contradicts with $\lim\limits_{v \to 0} \Gamma_\mu(v) = \infty$. Then, $b = 0$ in (36) is the best possible constant. □

Remark 1. *If we let $\mu \to \infty$ in (36), then we obtain (3).*

Corollary 5. *Assume that $a, b \in [0, \infty)$ and $\mu \in \mathbb{N}$. Then, for all $v \in (0, \infty)$, we have*

$$\frac{1}{2}\psi'_\mu(a+v) < -\psi_\mu(v) + \ln\left(\frac{v\, \mu}{v+1+\mu} \right) < \frac{1}{2}\psi'_\mu(b+v), \tag{37}$$

with constants $a = \frac{1}{3}$ and $b = 0$ being the best possible.

Proof. In inequality (37), the left-hand side is equivalent

$$\frac{v^3}{(1+\mu)} S'_{a,\mu}(v) = \frac{v^3}{(1+\mu)}\sigma_\mu(v) + \frac{v^3}{2(1+\mu)} \left[\psi'_\mu(a+v) - \frac{(1+\mu)}{v(v+1+\mu)} \right] < 0, \tag{38}$$

where

$$\sigma_\mu(v) = \psi_\mu(v) - \ln\left(\frac{\mu\, v}{v+1+\mu} \right) + \frac{(1+\mu)}{2v(v+1+\mu)}.$$

Using asymptotic expansions (30) and (31), we have

$$\lim_{v\to\infty}\frac{v^3}{(\mu+1)}\sigma_\mu(v)=\frac{-1}{6}\quad\text{and}\quad\lim_{v\to\infty}\frac{v^3}{(1+\mu)}\left[\psi'_\mu(v)-\frac{(1+\mu)}{(v+1+\mu)v}\right]=1.$$

Using relation (5), we have

$$\lim_{v\to\infty}\frac{v^3}{(1+\mu)}\left[\psi'_\mu(v+a)-\psi'_\mu(v)\right]=-2a.$$

Hence, we conclude from inequality (38) that $\frac{-1}{6}+\frac{1}{2}(-2a+1)\le 0$ or $a\ge\frac{1}{3}$.

Then, the left-hand side of the inequality (37) is satisfied only if $a\ge\frac{1}{3}$. Using the decreasing property of $\psi'_\mu(v)$ on $(0,\infty)$, we obtain $a=\frac{1}{3}$ in (37), and it is the best possible constant. Moreover, Theorem (3) proves the right-hand side of the inequality (37) for $b=0$. If there exist $b>0$ such that the upper bound of (37) valid for $v>0$, then we obtain

$$\lim_{v\to 0} v\left[\ln\left(\frac{\mu\, v}{v+\mu+1}\right)-\psi_\mu(v)\right]<\frac{1}{2}\lim_{v\to 0} v\psi'_\mu(v+b)$$

which leads to

$$-\lim_{v\to 0} v\ \psi_\mu(v)<\frac{1}{2}\lim_{v\to 0} v\ \psi'_\mu(v+b). \tag{39}$$

From (34) and (35), we have $\lim\limits_{v\to 0} v\ \ \psi_\mu(v)=-1$ and $\lim\limits_{v\to 0} v\ \ \psi'_\mu(v+b)=0$, which contradict inequality (39). Hence, $b=0$ is the best possible constant in (37). □

Remark 2. *Substituting $k=1$ in (8), we obtain*

$$0\le-\psi_\mu(v)+\ln\left(\frac{v\,\mu}{v+1+\mu}\right).$$

Using the completely monotonicity property of $\psi'_\mu(v)$ and inequality (37), we have

$$0<\frac{1}{2}\psi'_\mu(1/3+v)<-\psi_\mu(v)+\ln\left(\frac{v\,\mu}{v+1+\mu}\right).$$

Then, the lower bound of (37) improves its counterpart in (8) at $k=1$ for all $v\in(0,\infty)$.

Corollary 6. *Suppose that $a,b\in[0,\infty)$ and $\mu\in\mathbb{N}$. Then, for all $v\in(0,\infty)$ and $r=2,3,\cdots$, we have*

$$\frac{(-1)^{r+1}}{2}\psi^{(r)}_\mu(a+v)<\left[\frac{(r-2)!}{(1+v+\mu)^{r-1}}-\frac{(r-2)!}{v^{r-1}}\right]+(-1)^r\psi^{(r-1)}_\mu(v)<\frac{(-1)^{1+r}}{2}\psi^{(r)}_\mu(b+v), \tag{40}$$

with the constants $a=\frac{1}{3}$ and $b=0$ being the best possible.

Proof. In inequality (40), the left-hand side is equivalent

$$\begin{aligned}\frac{v^{r+2}}{(1+\mu)}\,(-1)^r S^{(r)}_{a,\mu}(v)=\frac{v^{2+r}}{(1+\mu)}F_\mu(v)+\frac{v^{2+r}}{2(1+\mu)}\Big[&(-1)^r\psi^{(r)}_\mu(a+v)\\ &+\left(\frac{1}{v^r}-\frac{1}{(1+v+\mu)^r}\right)(r-1)!\Big]>0,\end{aligned} \tag{41}$$

where

$$F_\mu(v)=(-1)^r\psi^{(r-1)}_\mu(v)-\frac{(r-1)!}{2}\left[\frac{1}{v^r}-\frac{1}{(v+1+\mu)^r}\right]+\left[\frac{(r-2)!}{(v+1+\mu)^{r-1}}-\frac{(r-2)!}{v^{r-1}}\right].$$

Using the asymptotic expansion (31), we have

$$\lim_{v\to\infty}\frac{v^{r+2}}{(1+\mu)}F_\mu(v)=\frac{(1+r)!}{12}$$

and

$$\lim_{v\to\infty}\frac{v^{r+2}}{(1+\mu)}\left[(-1)^r\psi_\mu^{(r)}(v)+\left(\frac{1}{v^r}-\frac{1}{(v+1+\mu)^r}\right)(r-1)!\right]=-\frac{(1+r)!}{2}.$$

Using relation (5), we obtain

$$\lim_{v\to\infty}\frac{(-1)^r v^{2+r}}{(1+\mu)}\left[\psi_\mu^{(r)}(a+v)-\psi_\mu^{(r)}(v)\right]=(1+r)!\,a.$$

Then, we conclude from inequality (3) that $\left[\frac{1}{12}+\frac{1}{2}\left(a-\frac{1}{2}\right)\right](r+1)!\geq 0$ or $a\geq\frac{1}{3}$. Since $\psi'_\mu(v)$ is strictly CM function on $v>0$, then for $r=0,1,2,\ldots$, function $(-1)^r\psi_\mu^{(r)}(v)$ is increasing on $v>0$; hence, the best possible constant in (40) is $a=\frac{1}{3}$. Moreover, Theorem (3) proves the right-hand side of inequality (40) for $b=0$. If there exist $b>0$ such that the upper bound of (40) is valid for $v>0$, then we obtain the following.

$$\lim_{v\to 0}v^r\left[(-1)^r\psi_\mu^{(r-1)}(v)-\left(\frac{(r-2)!}{v^{r-1}}-\frac{(r-2)!}{(v+1+\mu)^{r-1}}\right)\right]<\lim_{v\to 0}\left[-\frac{(-1)^r v^r}{2}\psi_\mu^{(r)}(b+v)\right]=0. \tag{42}$$

Using (34), we have

$$\lim_{v\to 0}v^r\left[(-1)^r\psi_\mu^{(r-1)}(v)+\left(\frac{(r-2)!}{(v+1+\mu)^{r-1}}-\frac{(r-2)!}{v^{r-1}}\right)\right]=(r-1)!,\qquad r=2,3,\cdots$$

which contradicts with inequality (42); hence, $b=0$ is the best possible constant. □

Remark 3. *Inserting $k=1$ in (9), we obtain the following.*

$$0\leq\psi'_\mu(v)-\left(\frac{1}{v}-\frac{1}{(1+v+\mu)}\right).$$

Using the completely monotonicity property of $\psi'_\mu(v)$ and inequality (40), we have the following.

$$0<\frac{-1}{2}\psi''_\mu(1/3+v)<\psi'_\mu(v)-\left(\frac{1}{v}-\frac{1}{(1+v+\mu)}\right).$$

Then, for $r=2$, the lower bound of (40) improves its counterpart in (9) at $k=1$ for all $v\in(0,\infty)$.

4. Conclusions

The main conclusions of this paper are stated in Theorems 1–3. We deduced the following asymptotic expansions for the generalized gamma

$$\Gamma_\mu(r+1)\sim\frac{\mu^{r+1}\,r^{r+\frac{1}{2}}\,e^{1+\mu}\,\mu!}{(r+\mu+1)^{\frac{3}{2}+\mu+r}},\quad \mu\in\mathbb{N};\ r\longrightarrow\infty.$$

and the bounds

$$\exp\left[\frac{1}{360}\left(\frac{30v^3-v-3}{v^4}+\frac{\mu-30(\mu+v+1)^3+v+4}{(\mu+v+1)^4}\right)\right]$$

$$< \frac{\Gamma_\mu(1+v)}{\left(\frac{\mu^{1+v} v^{\frac{1}{2}+v} e^{\mu+1}\,\mu!}{(\mu+v+1)^{\mu+v+3/2}}\right)} < \exp\left[\frac{\mu+1}{12v(\mu+v+1)}\right], \quad \mu \in \mathbb{N};\ v > 0.$$

Moreover, we proved that the function

$$S_{\alpha,\mu}(v) = \ln\Gamma_\mu(v) + \frac{\psi_\mu(\alpha+v)}{2} - v\ln\left(\frac{\mu\, v}{v+\mu+1}\right) + (\mu+1)\ln(v+\mu+1) - \ln\left(\sqrt{\mu}\,\mu!\,e^{1+\mu}\right),\ \mu \in \mathbb{N}$$

is CM on $v > 0$ if and only if $\alpha \geq \frac{1}{3}$, and $-S_{\alpha,\mu}(v)$ is the CM function on $v > 0$ if and only if $\alpha = 0$.

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