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Fuzzy Rough Programming Models: An Expected Value Perspective

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Abstract: Usually, the quasi-normal fluctuations in practical applications are described via symmetric uncertainty variables, which is a common phenomenon in the manufacturing industry. However, it is relatively scarce in the literature to discuss two-fold uncertainty due to the its complexity. To deal with roughness and ambiguity to accommodate inherent uncertainties, fuzzy rough programming approaches are put forward. In this paper, we pay attention to exploring two kinds of programming problems, namely fuzzy rough single-objective programming and fuzzy rough multi-objective programming, in which objective functions and/or constraints involve fuzzy rough variables (FRV). In accordance with the related existing research of FRVs, such as the chance measure and the expected value (EV) operator, this paper further discusses the EV model, convexity theory, and the crisp equivalent model of fuzzy rough programming. After that, combined with the latest published NIA-S fuzzy simulation technique, a new fuzzy rough simulation algorithm is developed to calculate the EVs of complicated functions for handling the presented fuzzy rough programming problems. In the end, the two types of numerical examples are provided for demonstration.

Keywords: fuzzy rough variable; expected value model; convexity theory; fuzzy rough programming; fuzzy rough simulation

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1. Introduction

Mathematical programming is frequently used as a method employed by the stakeholders in a variety of decision-making and optimization problems. Countless academics and practitioners have conducted specific studies in a wide range of areas [1–3]. A traditional mathematical programming problem aims to maximize (minimize) an objective or several objective functions with a series of constraints, where the coefficients existing in the objective or constraint functions are all exactly known crisp values. This usually comes down to a single-objective programming (SOP) problem or a multi-objective programming (MOP) problem. Whereafter, relying on the classical mathematical programming methods or the hybrid algorithms, these models can be well settled. However, in many practical scenarios, it is difficult to offer the coefficients any accurate values due to the reality that some of the related data are incomplete, inexistent, or unavailable [4,5]. From the optimization point of view, this opens a new field of research called "uncertain programming". As an optimization theory in uncertain environments, it includes several subtopics, for instance, stochastic programming, fuzzy programming, and rough programming [6,7].

Given this uncertain information, many researchers studied the uncertain programming in different situations. Taking into account the randomization of relevant parameters in the economic dispatch of a micro-grid, Shuai et al. [8] offered a new approximate dynamic programming method to operate the micro-grid under these uncertainties, and showed its good performance in dealing with the historical forecast data to minimize the detrimental impacts of inexact prediction on the operating system with numerical analysis. In order to simplify a model's establishment in an expert system or a knowledge-based system, Chung et al. [9] put forward a new fuzzy multiple choice goal programming model to resolve a kind of linear MOP problem, and verified its effectiveness in reducing computational complexity during the solutions and practicability in providing satisfactory solutions. Hamzehee et al. [10] introduced a set of MOP problems under the rough environments and further categorized them into five types depending on the location of roughness in the decision set or the objectives. In the above models, the uncertainties, i.e., roughness, randomness and fuzziness, are treated as several single parts. Nevertheless, in a realistic decision-making process, some hybrid uncertain situations often appear.

Fuzzy set theory and rough set theory from the uncertain theory are known as two widely used tools for dealing with different kinds of ambiguous, insufficient, and inexact informational data [11,12]. Whereas, in real-life systems, sort of uncertain problems exist around us in which roughness and fuzziness take place simultaneously. For instance, the demand in a supply–demand problem may be a fuzzy variable but the EV of it is actually a rough variable, which means that the maximum and minimum average demand quantity varies within a different interval, respectively. In 1990, Dubois and Prade [13] came up with a fundamental concept of the fuzzy rough sets by structuring the upper and lower approximation operators of the fuzzy sets as regards a fuzzy min–similarity relation. Its inception provides a reliable theory base for the following research. So far, this theory has been investigated from the views of theory and practice, for example, a more general concept of a fuzzy rough set [14–17], the constructive definition of fuzzy rough approximation operators [18,19], feature selection [20–22], as well as a wide range of applications in attribute and data reduction [23,24], multi-attribute decision making [25], and so on. Unlike the fuzzy rough set, Liu [26] put forward a fuzzy rough variable (FRV) in 2002. It was considered as a measurable function from a rough space to the set of fuzzy variables. In mathematical programming, this theory offers an effective tool for disposing of the two types of uncertainty at the same time. Until now, the FRV has been explored in some academic studies and practiced applications, such as some inequalities of FRVs [27], a fuzzy rough DEA model [28], a useful way to process the fuzzy rough integer linear fractional programming problem [29], a hybrid simulation algorithm for resolving the uncertain problems with FRVs [30], and a kind of multi-objective decision-making model containing FRVs, as well as its applications to inventory problems [31]. Obviously, it is shown that the fuzzy rough set has been widely studied, especially in data mining, while the FRV focusing on solving optimization problems has been paid less attention up to now.

Realistically, for a better description and application of the optimization issues, an FRV can be effectively used to represent the uncertain phenomena in many practical decision-making problems. Consequently, this branch of mathematical research dealing with uncertainty optimization problems needs to be paid much attention to. Additionally, as single objective and multiple objectives programming models are the commonly used programming tools, in order to deal with the case where fuzzy information and rough factors arise simultaneously in objective functions and/or constraints and fill the gap existing in the prior research in which the most related one only proposes an equivalent model for Tr–Pos constrained multi-objective linear programming with FRVs, in this paper, we pay attention to two kinds of uncertain programming with FRV coefficients from the EV perspective, called fuzzy rough single-objective programming (FRSOP) and fuzzy rough multi-objective programming (FRMOP). According to the definition of FRV initiated by Liu [26], we further propose the EV models of FRSOP and FRMOP respectively. As an extension of this theory, the convexity theorem, some crisp equivalent models are discussed in detail. In addition, on account of the difficulty in computing the EVs of such uncertain programming models and the inexactness existing in the original fuzzy rough simulation in [26], this paper involves an innovative technique in simulating the EVs of fuzzy numbers [32] and puts forward a new fuzzy rough simulation for calculating the complicated EV functions, which provides an effective way to take care of the fuzzy rough programming problems with the help of the well developed solvers.

The rest of this paper is outlined below. In Section 2, some essential definitions and theorems of FRVs are reviewed with several examples. Then, an SOP problem and an MOP one with FRVs are illustrated in Sections 3 and 4, followed by their corresponding EV model, convexity theory, and crisp equivalent model respectively. Three transformation methods of FRMOP into FRSOP are put forward in detail in Section 4. For a better solution, the new fuzzy rough simulation is introduced, and some numerical examples of these two kinds of fuzzy rough programming models are offered in Section 5. In Section 6, the overall conclusions of this present paper are incorporated.

2. Fuzzy Rough Theory

Some necessary knowledge relevant to fuzzy rough theory, which provides theoretical preparations for the rest of the present paper, is introduced in this section.

Definition 1 (Liu and Liu [33])**.** *Assume that the triplet* $(\Omega, \mathcal{P}(\Omega), \mathrm{Cr})$ *is a credibility space, where* Ω *is a nonempty set,* $\mathcal{P}(\Omega)$ *is the power set of* Ω*, and* Cr *is the credibility measure. Then a fuzzy variable* $\tilde{\zeta}$ *is defined as a function from the credibility space to the real line* $\mathbb{R}$*.*

Definition 2 (Liu and Liu [33])**.** *Assume that* $\tilde{\zeta}$ *is a fuzzy variable and* $\{\tilde{\zeta} \le \sigma\}$ *is a set in* $\mathcal{P}(\Omega)$ *from the credibility space. Then the credibility measure of* $\{\tilde{\kappa} \le \sigma\}$ *is given below:*

$$\mathrm{Cr}\{\tilde{\zeta} \le \sigma\} = \frac{1}{2}(\mathrm{Pos}\{\tilde{\zeta} \le \sigma\} + \mathrm{Nec}\{\tilde{\zeta} \le \sigma\}), \tag{1}$$

where $\mathrm{Pos}\{\tilde{\zeta} \le \sigma\}$ *denotes the possibility that* $\{\tilde{\kappa} \le \sigma\}$ *occurs [34], and* $\mathrm{Nec}\{\tilde{\zeta} \le \sigma\}$ *denotes the impossibility that the opposite event* $\{\tilde{\kappa} \ge \sigma\}$ *occurs, i.e.,* $\mathrm{Nec}\{\tilde{\zeta} \le \sigma\} = 1 - \mathrm{Pos}\{\tilde{\zeta} \ge \sigma\}$ *[35].*

Intuitively, as a reasonable measurement for theoretical or real-world applications, EV has been an essential part to obtain a general evaluation. Therefore, according to a credibility measure, for a fuzzy variable $\tilde{\zeta}$, its EV could be calculated as below.

Definition 3 (Liu and Liu [33])**.** *Assume that* $\tilde{\zeta}$ *is a fuzzy variable. Then, the EV of* $\tilde{\zeta}$*,* $E[\tilde{\zeta}]$*, is given below:*

$$E[\tilde{\zeta}] = \int_0^{+\infty} \mathrm{Cr}\{\tilde{\zeta} \ge \sigma\}\mathrm{d}\sigma - \int_{-\infty}^{0} \mathrm{Cr}\{\tilde{\zeta} \le \sigma\}\mathrm{d}\sigma, \tag{2}$$

on the condition that at least one integral is finite.

In the rest of the paper, triangular fuzzy numbers are introduced and serve as illustration examples for convenience, since their linear left and right shape functions are widely applied in modeling the single or multiple programming of uncertainty decision systems. In particular, we extend the relevant definitions and results for the symmetric fuzzy number due to the quasi-normal fluctuations usually appearing in reality.

Example 1. *Provide that a fuzzy variable* $\tilde{\zeta} = (a, b, c)$ *is a triangular fuzzy number with membership function:*

$$\mu_{\tilde{\zeta}} = \begin{cases} \dfrac{\sigma - a}{b - a} & \text{if } a \le \sigma \le b \\ \dfrac{c - \sigma}{c - b} & \text{if } b < \sigma \le c. \end{cases} \tag{3}$$

Then, its expected value can be derived as $E[\tilde{\zeta}] = (a + 2b + c)/4$ *from Definitions 2 and 3. In addition to the special case* $b - a = c - b$*, we call* $\tilde{\zeta}$ *a symmetric fuzzy number [5] (note that the symmetric fuzzy number in [5] is defined as a fuzzy variable with not only the same left and right shape functions but same left and right spreads).*

Definition 4 (Liu [30]). *Assume that $(\Lambda, \triangle, \mathcal{A}, \pi)$ is a rough space, where $\mathcal{A}$ is a nonempty set, $\triangle$ is a subset of $\mathcal{A}$, $\mathcal{A}$ is the σ-algebra subsets over Λ, and π is a real-valued set function. Then, a rough variable $\bar{\tau}$ is defined as a function from the rough space to the real line $\mathbb{R}$.*

Definition 5 (Liu [26]). *Assume that $\bar{\tau}$ is a rough variable, and $\{\bar{\tau} \le \rho\}$ is a set in $\mathcal{A}$ from the rough space. Then the trust of the rough event $\{\bar{\tau} \le \rho\}$ is given below:*

$$\mathrm{Tr}\{\bar{\tau} \le \rho\} = \frac{1}{2}\left(\overline{\mathrm{Tr}}\{\bar{\tau} \le \rho\} + \underline{\mathrm{Tr}}\{\bar{\tau} \le \rho\}\right), \tag{4}$$

in which $\overline{\mathrm{Tr}}\{\bar{\tau} \le \rho\}$ represents the upper trust of $\{\bar{\tau} \le \rho\}$, and $\overline{\mathrm{Tr}}\{\bar{\tau} \le \rho\}$ represents the lower trust of $\{\bar{\tau} \le \rho\}$. They are given below:

$$\overline{\mathrm{Tr}}\{\bar{\tau} \le \rho\} = \frac{\pi\{\bar{\tau} \le \rho\}}{\pi\{\Lambda\}}; \ \underline{\mathrm{Tr}}\{\bar{\tau} \le \rho\} = \frac{\pi\{\{\bar{\tau} \le \rho\} \cap \Delta\}}{\pi\{\Delta\}}. \tag{5}$$

Definition 6 (Liu [30]). *Assume that $\bar{\tau}$ is a rough variable. Then, the EV of $\bar{\tau}$, $E[\bar{\tau}]$, is given below:*

$$E[\bar{\tau}] = \int_0^{+\infty} \mathrm{Tr}\{\bar{\tau} \ge \varrho\}\mathrm{d}\varrho - \int_{-\infty}^0 \mathrm{Tr}\{\bar{\tau} \le \varrho\}\mathrm{d}\varrho. \tag{6}$$

Example 2. *Provide that a rough variable $\bar{\tau} = ([b, c], [a, d])$ with*

$$\mathrm{Tr}\{\bar{\tau} \le \rho\} = \begin{cases} \dfrac{\rho - a}{2(d-a)} & \text{if } a \le \rho \le b \\ \dfrac{1}{2}\left(\dfrac{\rho - a}{d - a} + \dfrac{\rho - b}{c - b}\right) & \text{if } b < \rho \le c \\ \dfrac{\rho + d - 2a}{2(d-a)} & \text{if } c < \rho \le d. \end{cases} \tag{7}$$

Then, its expected value can be derived as $E[\bar{\tau}] = (a + b + c + d)/4$ from Definitions 5 and 6. In addition to the special case $b - a = d - c$, we may call $\bar{\tau}$ is a symmetric rough variable.

With regard to the various definitions of FRVs, we employ the definition developed by Liu [26] in this present study as below.

Definition 7 (Liu [26]). *Assume that $(\Lambda, \Delta, \mathcal{A}, \pi)$ denotes a rough space, then an FRV $\tilde{\bar{\kappa}}$ is called a function from the rough space to the set of fuzzy variables. Then, we get a measurable function of λ, i.e., $\mathrm{Cr}\{\tilde{\bar{\kappa}}(\lambda) \in C\}$ for every Borel set C of $\mathbb{R}$.*

Theorem 1 (Liu [30]). *Assume that $\tilde{\bar{\kappa}}$ is an FRV defined on the rough space $(\Lambda, \Delta, \mathcal{A}, \pi)$. If $E[\tilde{\bar{\kappa}}(\lambda)]$ has finite EV for each $\lambda \in \Lambda$, then $E[\tilde{\bar{\kappa}}(\cdot)]$ is a rough variable.*

Definition 8 (Liu [26]). *Assume that $\tilde{\bar{\kappa}}$ is an FRV. Then, the EV of $\tilde{\bar{\kappa}}$ is given below:*

$$E[\tilde{\bar{\kappa}}] = \int_0^{+\infty} \mathrm{Tr}\{\lambda \in \Lambda \mid E[\tilde{\bar{\kappa}}(\lambda)] \ge \rho\}\mathrm{d}\rho - \int_{-\infty}^0 \mathrm{Tr}\{\lambda \in \Lambda \mid E[\tilde{\bar{\kappa}}(\lambda)] \le \rho\}\mathrm{d}\rho, \tag{8}$$

in which $E[\tilde{\bar{\kappa}}(\lambda)]$ can be derived via Equation (2).

Remark 1. *By Definition 7, both fuzzy variables and rough variables are special forms of FRVs. If the non-empty set Λ contains only one element, it is easy to see that $\tilde{\bar{\kappa}}$ will naturally degenerate to a fuzzy variable, and subsequently the EV operator defined in Equation (8) is equal to Equation (2). In the same way, assuming $\tilde{\bar{\kappa}}(\lambda)$ is a real number not a fuzzy set, then $\tilde{\bar{\kappa}}$ an FRV will degenerate to a rough variable, and the EV operator in Equation (8) is correspondingly equivalent to Equation (6).*

Example 3. *Provide that* $\tilde{\kappa} = (\bar{\tau}, \bar{\tau}+1, \bar{\tau}+2)$ *with* $\bar{\tau} = ([2,4],[0,6])$, *where the triplet* (m,n,o) *denotes a fuzzy variable, and* $([m_1,m_2],[n_1,n_2])$ *denotes a rough variable. Apparently, it is a symmetric fuzzy rough variable due to its intrinsic symmetry in view of Examples 1 and 2. According to Definition 8, we have:*

$$\begin{aligned}
E[\tilde{\kappa}] &= \int_0^{+\infty} \mathrm{Tr}\{\lambda \in \Lambda \mid E[\tilde{\kappa}(\lambda)] \geq \rho\}\mathrm{d}\rho - \int_{-\infty}^{0} \mathrm{Tr}\{\lambda \in \Lambda \mid E[\tilde{\kappa}(\lambda)] \leq \rho\}\mathrm{d}\rho \\
&= \int_{-\infty}^{+\infty} E[\tilde{\kappa}(\lambda)]\mathrm{Tr}(\mathrm{d}\lambda) \\
&= \int_{-\infty}^{+\infty} \left[\int_0^{+\infty} \mathrm{Cr}\{\tilde{\kappa}(\lambda) \geq \sigma\}\mathrm{d}\sigma - \int_{-\infty}^{0} \mathrm{Cr}\{\tilde{\kappa}(\lambda) \leq \sigma\}\mathrm{d}\sigma\right]\mathrm{Tr}(\mathrm{d}\lambda) \\
&= \int_0^6 \frac{\lambda + 2(\lambda+1) + (\lambda+2)}{4}\mathrm{Tr}(\mathrm{d}\lambda) \\
&= \int_0^6 \mathrm{Tr}(\mathrm{d}\lambda) + \int_0^6 \lambda\mathrm{Tr}(\mathrm{d}\lambda) \\
&= 1 + \frac{0+2+4+6}{4} \\
&= 4.
\end{aligned}$$

Theorem 2 (Liu [26])**.** *Assume that* $\tilde{\kappa}_1$ *and* $\tilde{\kappa}_2$ *are two FRVs with finite EVs, and* $\tilde{\kappa}_1(\lambda)$ *and* $\tilde{\kappa}_2(\lambda)$ *are two independent fuzzy variables. Then, in regard to any real numbers m and n, there is:*

$$E[m\tilde{\kappa}_1 + n\tilde{\kappa}_2] = mE[\tilde{\kappa}_1] + nE[\tilde{\kappa}_2]. \tag{9}$$

Theorem 3 (Liu [30])**.** *Assume that a fuzzy rough vector* $\tilde{\boldsymbol{\kappa}} = (\tilde{\kappa}_1, \tilde{\kappa}_2, \cdots, \tilde{\kappa}_n)$ *and* $f : \mathbb{R}^n \to \mathbb{R}$ *is an n-ary real-valued function. Then,* $f(\tilde{\boldsymbol{\kappa}})$ *is an FRV.*

Theorem 4 (Liu [32])**.** *Assume that* $\tilde{\boldsymbol{\kappa}}$ *is a fuzzy rough vector and* $f : \mathbb{R}^n \to \mathbb{R}$ *is a measurable function. Then, the EV of function* $f(\tilde{\boldsymbol{\kappa}})$, $E[f(\tilde{\boldsymbol{\kappa}})]$, *is given below:*

$$E[f(\tilde{\boldsymbol{\kappa}})] = \int_0^{+\infty} \mathrm{Tr}\{\lambda \in \Lambda \mid E[f(\tilde{\boldsymbol{\kappa}}(\lambda))] \geq \rho\}\mathrm{d}\rho - \int_{-\infty}^{0} \mathrm{Tr}\{\lambda \in \Lambda \mid E[f(\tilde{\boldsymbol{\kappa}}(\lambda))] \leq \rho\}\mathrm{d}\rho. \tag{10}$$

3. Fuzzy Rough Single-Objective Programming

In conventional mathematical programming problems, an SOP model is usually referred to as a way of maximizing or minimizing a determinate objective, subject to a series of constraints with crisp data. However, it is not reasonable in some state of uncertainty. Considering the scenario of indetermination as well as hesitation existing in the practical decision process, in this section, an FRSOP is studied from several aspects.

3.1. General Model

In reality, an SOP model has much practical applicable value. Combined with the SOP and the fuzzy rough theory, a typical FRSOP is proposed, as shown below:

$$\begin{cases} \min\limits_{\boldsymbol{t}} \ f(\boldsymbol{t}, \tilde{\boldsymbol{\kappa}}) \\ \text{subject to:} \\ \qquad g_s(\boldsymbol{t}, \tilde{\boldsymbol{\kappa}}) \leq 0, \quad s = 1, 2, \cdots, q, \end{cases} \tag{11}$$

where $\boldsymbol{t} = (t_1, t_2, \cdots, t_m)$ is a decision vector, $\tilde{\boldsymbol{\kappa}} = (\tilde{\kappa}_1, \tilde{\kappa}_2, \cdots, \tilde{\kappa}_n)$ is a fuzzy rough vector, $f(\boldsymbol{t}, \tilde{\boldsymbol{\kappa}})$ is the objective function, and $g_s(\boldsymbol{t}, \tilde{\boldsymbol{\kappa}})$, $s = 1, 2, \cdots, q$, are a set of constraint functions.

Note that this model in (11) has no precise mathematical meaning. For the complicated uncertain programming models, the EV operator is often used to calculate the objective and constraint functions. Hence, according to the definition of FRVs [30], a fuzzy rough EV model is presented as below:

$$\begin{cases} \min\limits_{t} \ E[f(\boldsymbol{t},\tilde{\tilde{\boldsymbol{\kappa}}})] \\ \text{subject to:} \\ \qquad E[g_s(\boldsymbol{t},\tilde{\tilde{\boldsymbol{\kappa}}})] \leq 0, \quad s = 1,2,\cdots,q, \end{cases} \tag{12}$$

where $E[f(\boldsymbol{t},\tilde{\tilde{\boldsymbol{\kappa}}})]$ is the expected objective function, and $E[g_s(\boldsymbol{t},\tilde{\tilde{\boldsymbol{\kappa}}})]$, $s = 1,2,\cdots,q$, are a collection of expected constraint functions. For model (12), the goal is to acquire a decision vector $\boldsymbol{t} = (t_1,t_2,\cdots,t_m)$ with the minimum expected objective value $E[f(\boldsymbol{t},\tilde{\tilde{\boldsymbol{\kappa}}})]$ constrained by the expected constraints $E[g_s(\boldsymbol{t},\tilde{\tilde{\boldsymbol{\kappa}}})] \leq 0$, $s = 1,2,\cdots,q$. So far, the EV operator has been used to convert model (11) into a certain one, which is called the fuzzy rough single-objective programming expected value model (FRSOP-EVM).

Definition 9. *For the FRSOP-EVM in (12), the set,*

$$F = \{\boldsymbol{t} \in \mathbb{R}^m \mid E[g_s(\boldsymbol{t},\tilde{\tilde{\boldsymbol{\kappa}}})] \leq 0,\ s = 1,2,\cdots,q\}, \tag{13}$$

*is called a feasible set. Then each element **t** in F can be called a feasible solution.*

Definition 10. *For the FRSOP-EVM in (12), a feasible solution $\boldsymbol{t}^*$ can be called a global optimal solution when it satisfies the following condition:*

$$E[f(\boldsymbol{t}^*,\tilde{\tilde{\boldsymbol{\kappa}}})] \leq E[f(\boldsymbol{t},\tilde{\tilde{\boldsymbol{\kappa}}})], \tag{14}$$

for all $\boldsymbol{t} \in F$.

3.2. Convexity Theorem

Convexity plays a significant role in optimization theory, especially on seeking the global optimal solution. A mathematical programming problem can be called convex if and only if the objective function and the feasible set are all convex. Subsequently, the convexity of a general FRSOP-EVM is discussed as below.

Theorem 5. *Assume that $\tilde{\tilde{\boldsymbol{\kappa}}} = (\tilde{\tilde{\kappa}}_1,\tilde{\tilde{\kappa}}_2,\cdots,\tilde{\tilde{\kappa}}_n)$ is a fuzzy rough vector. If the objective function $f(\boldsymbol{t},\tilde{\tilde{\boldsymbol{\kappa}}})$ and constraint functions $g_s(\boldsymbol{t},\tilde{\tilde{\boldsymbol{\kappa}}}), s = 1,2,\cdots,q$, for each realization $\tilde{\tilde{\boldsymbol{\kappa}}}$, are convex in $\boldsymbol{t}$. Then the FRSOP-EVM (12) is called convex programming.*

Proof. Given the fact that $f(\boldsymbol{t},\tilde{\tilde{\boldsymbol{\kappa}}})$ is a convex function in $\boldsymbol{t}$, then for each realization $\tilde{\tilde{\boldsymbol{\kappa}}}$, we can get the inequality:

$$f(\theta\boldsymbol{t}_1 + (1-\theta)\boldsymbol{t}_2,\tilde{\tilde{\boldsymbol{\kappa}}}) \leq \theta f(\boldsymbol{t}_1,\tilde{\tilde{\boldsymbol{\kappa}}}) + (1-\theta)f(\boldsymbol{t}_2,\tilde{\tilde{\boldsymbol{\kappa}}})$$

with respect to solutions $\boldsymbol{t}_1,\boldsymbol{t}_2$ and scalar $\theta \in [0,1]$. In accordance with Theorem 2:

$$E[f(\theta\boldsymbol{t}_1 + (1-\theta)\boldsymbol{t}_2,\tilde{\tilde{\boldsymbol{\kappa}}})] \leq E[\theta f(\boldsymbol{t}_1,\tilde{\tilde{\boldsymbol{\kappa}}}) + (1-\theta)f(\boldsymbol{t}_2,\tilde{\tilde{\boldsymbol{\kappa}}})] = \theta E[f(\boldsymbol{t}_1,\tilde{\tilde{\boldsymbol{\kappa}}})] + (1-\theta)E[f(\boldsymbol{t}_2,\tilde{\tilde{\boldsymbol{\kappa}}})].$$

This demonstrates the convexity of the $E[f(\boldsymbol{t},\tilde{\tilde{\boldsymbol{\kappa}}})]$ in $\boldsymbol{t}$.

Then the convexity of the feasible set can be proven via the validation that $\theta\boldsymbol{t}_1 + (1-\theta)\boldsymbol{t}_2$ is feasible for any feasible solutions $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ with the expected constraints $E[g_s(\boldsymbol{t},\tilde{\tilde{\boldsymbol{\kappa}}})] \leq 0, s = 1,2,\cdots,q$, and any scalar $\theta \in [0,1]$. In accordance with the convexity of the constraint functions $g_s(\boldsymbol{t},\tilde{\tilde{\boldsymbol{\kappa}}})$, $s = 1,2,\cdots,q$, we can get the inequality:

$$g_s(\theta\boldsymbol{t}_1 + (1-\theta)\boldsymbol{t}_2,\tilde{\tilde{\boldsymbol{\kappa}}}) \leq \theta g_s(\boldsymbol{t}_1,\tilde{\tilde{\boldsymbol{\kappa}}}) + (1-\theta)g_s(\boldsymbol{t}_2,\tilde{\tilde{\boldsymbol{\kappa}}}), \tag{15}$$

for $s = 1,2,\cdots,q$, which yields that:

$$E[g_s(\theta\boldsymbol{t}_1 + (1-\theta)\boldsymbol{t}_2,\tilde{\tilde{\boldsymbol{\kappa}}})] \leq \theta E[g_s(\boldsymbol{t}_1,\tilde{\tilde{\boldsymbol{\kappa}}})] + (1-\theta)E[g_s(\boldsymbol{t}_2,\tilde{\tilde{\boldsymbol{\kappa}}})] \leq 0, \tag{16}$$

for $s = 1, 2, \cdots, q$. This illustrates that $\theta \boldsymbol{t}_1 + (1-\theta)\boldsymbol{t}_2$ is a feasible solution, and proves the convexity of the feasible set. Therefore, a proof of a convex programming, i.e., the fuzzy rough EV model (12) is completed. □

3.3. Crisp Equivalent Model

Generally speaking, the EV is usually adopted to denote the mean value for each rough quantity or fuzzy quantity in an uncertain programming. An EV model is helpful to express a decision-maker's preference to achieve a minimum mean objective value with a series of constraints. For the purpose of solving the EV model containing FRVs, we next discuss the crisp equivalent model of FRSOP-EVM.

Theorem 6. *Assume that $\tilde{\bar{\kappa}}_1, \tilde{\bar{\kappa}}_2, \cdots, \tilde{\bar{\kappa}}_n$ are independent FRVs. If the objective and constraint functions satisfy the following conditions that*

$$f(\boldsymbol{t}, \tilde{\bar{\boldsymbol{\kappa}}}) = f_0(\boldsymbol{t}) + f_1(\boldsymbol{t})\tilde{\bar{\kappa}}_1 + \cdots + f_n(\boldsymbol{t})\tilde{\bar{\kappa}}_n$$

and

$$g_s(\boldsymbol{t}, \tilde{\bar{\boldsymbol{\kappa}}}) = g_{s0}(\boldsymbol{t}) + g_{s1}(\boldsymbol{x})\tilde{\bar{\kappa}}_1 + \cdots + g_{sn}(\boldsymbol{t})\tilde{\bar{\kappa}}_n,$$

for all $s = 1, 2, \cdots, q$, then the FRSOP-EVM in (12) has its crisp equivalent model as below:

$$\begin{cases} \min\limits_{\boldsymbol{t}} \ f(\boldsymbol{t}, E[\tilde{\bar{\boldsymbol{\kappa}}}]) \\ \text{subject to:} \\ \qquad g_s(\boldsymbol{t}, E[\tilde{\bar{\boldsymbol{\kappa}}}]) \le 0, \quad s = 1, 2, \cdots, q, \end{cases} \tag{17}$$

where $E[\tilde{\bar{\boldsymbol{\kappa}}}] = (E[\tilde{\bar{\kappa}}_1], E[\tilde{\bar{\kappa}}_2], \cdots, E[\tilde{\bar{\kappa}}_n])$.

Proof. It can be directly drawn from the fuzzy rough EV linearity theorem (see Theorem 2). □

4. Fuzzy Rough Multi-Objective Programming

From a practical viewpoint, many decision problems in our life involve multiple, conflicting and nondominated objectives, which need to be considered at the same time to find an optimal solution. This kind of problem arising in various fields has gained much attention. In order to extend the fuzzy rough programming to the MOP problem, in this section, an FRMOP problem is investigated in detail.

4.1. General Model

Given the actual requirements in real-life situations, another hybridization of MOP and fuzzy rough theory is studied subsequently. As a theoretical extension of the proposed FRSOP, the FRMOP is referred to as a way of optimizing a number of objectives constrained by a series of constraints, i.e.,

$$\begin{cases} \min\limits_{\boldsymbol{t}} \ (f_1(\boldsymbol{t}, \tilde{\bar{\boldsymbol{\kappa}}}), f_2(\boldsymbol{t}, \tilde{\bar{\boldsymbol{\kappa}}}), \cdots, f_l(\boldsymbol{t}, \tilde{\bar{\boldsymbol{\kappa}}})) \\ \text{subject to:} \\ \qquad g_s(\boldsymbol{t}, \tilde{\bar{\boldsymbol{\kappa}}}) \le 0, \quad s = 1, 2, \cdots, q, \end{cases} \tag{18}$$

where $\boldsymbol{t} = (t_1, t_2, \cdots, t_m)$ is a decision vector, and $\tilde{\bar{\boldsymbol{\kappa}}} = (\tilde{\bar{\kappa}}_1, \tilde{\bar{\kappa}}_2, \cdots, \tilde{\bar{\kappa}}_n)$ is a fuzzy rough vector, $f_r(\boldsymbol{t}, \tilde{\bar{\boldsymbol{\kappa}}})$, $r = 1, 2, \cdots, l$, are various objective functions, and $g_s(\boldsymbol{t}, \tilde{\bar{\boldsymbol{\kappa}}}) \le 0$, $s = 1, 2, \cdots, q$, are a series of fuzzy rough constraints.

Since the EV operator is an effective tool for the transformation of an uncertain programming problem into a deterministic one, then under the fuzzy rough environments, a novel EV model called the fuzzy rough multi-objective expected value model (FRMOP-EVM) is put forward as follows:

$$\begin{cases} \min\limits_{t} \ (E[f_1(\boldsymbol{t},\tilde{\tilde{\boldsymbol{\kappa}}})], E[f_2(\boldsymbol{t},\tilde{\tilde{\boldsymbol{\kappa}}})], \cdots, E[f_l(\boldsymbol{t},\tilde{\tilde{\boldsymbol{\kappa}}})]) \\ \text{subject to:} \\ \qquad E[g_s(\boldsymbol{t},\tilde{\tilde{\boldsymbol{\kappa}}})] \leq 0, \quad s = 1,2,\cdots,q, \end{cases} \tag{19}$$

where $E[f_r(\boldsymbol{t},\tilde{\tilde{\boldsymbol{\kappa}}})]$, $r = 1,2,\cdots,l$, are the expected objective functions, and $g_s(\boldsymbol{t},\tilde{\tilde{\boldsymbol{\kappa}}})$, $s = 1,2,\cdots,q$, are the expected constraint functions. For model (19), the goal is to get a decision vector $\boldsymbol{t} = (t_1,t_2,\cdots,t_m)$ with the minimum objective values $E[f_r(\boldsymbol{t},\tilde{\tilde{\boldsymbol{\kappa}}})]$ constrained by a series of expected constraints, i.e., $E[g_s(\boldsymbol{t},\tilde{\tilde{\boldsymbol{\kappa}}})] \leq 0$, $s = 1,2,\cdots,q$.

Likewise, the set $F = \{\boldsymbol{t} \in \mathbb{R}^m \mid E[g_s(\boldsymbol{t},\tilde{\tilde{\boldsymbol{\kappa}}})] \leq 0,\ s = 1,2,\cdots,q\}$ is called a feasible set, and each element $\boldsymbol{t} \in F$ is called the feasible solution of model (19). Additionally, for a general multi-objective decision-making problem in real world, the targets are usually contradictory to each other. Characteristically, there is no such a global optimal solution for the MOP problem that can simultaneously minimize all the objective functions $E[f_r(\boldsymbol{t},\tilde{\tilde{\boldsymbol{\kappa}}})]$ $(r = 1,2,\cdots,l)$. Therefore, the definition of a Pareto solution to this FRMOP problem is necessary.

Definition 11. *For the FRMOP-EVM in (19), a feasible solution $\boldsymbol{t}^*$ can be called the Pareto solution if there is no feasible solution $\boldsymbol{t}$ that satisfies the following condition:*

$$E[f_r(\boldsymbol{t},\tilde{\tilde{\boldsymbol{\kappa}}})] \geq E[f_r(\boldsymbol{t}^*,\tilde{\tilde{\boldsymbol{\kappa}}})],$$

for $r = 1,2,\cdots,l$, and

$$E[f_i(\boldsymbol{t},\tilde{\tilde{\boldsymbol{\kappa}}})] > E[f_i(\boldsymbol{t}^*,\tilde{\tilde{\boldsymbol{\kappa}}})],$$

for at least one index $i \in \{1,2,\cdots,l\}$.

4.2. Convexity Theorem

For an MOP problem, if the objective functions and feasible set are all convex, then it can be called convex programming. The FRMOP-EVM in (19) is also a convex programming model if it satisfies certain conditions, which are described below.

Theorem 7. *Assume that $\tilde{\tilde{\boldsymbol{\kappa}}}$ is a fuzzy rough vector. If the objective functions $f_r(\boldsymbol{t},\tilde{\tilde{\boldsymbol{\kappa}}})$, $r = 1,2,\cdots,l$, and constraint functions $g_s(\boldsymbol{t},\tilde{\tilde{\boldsymbol{\kappa}}})$, $s = 1,2,\cdots,q$, for each realization $\tilde{\tilde{\boldsymbol{\kappa}}}$, are convex in $\boldsymbol{t}$, then FRMOP-EVM (19) is called convex programming.*

Proof. Given the fact that the function $f_r(\boldsymbol{t},\tilde{\tilde{\boldsymbol{\kappa}}})$ is convex in $\boldsymbol{t}$, then for each fixed $\tilde{\tilde{\boldsymbol{\kappa}}}$, we can get the inequality:

$$f_r(\theta \boldsymbol{t}_1 + (1-\theta)\boldsymbol{t}_2,\tilde{\tilde{\boldsymbol{\kappa}}}) \leq \theta f_r(\boldsymbol{t}_1,\tilde{\tilde{\boldsymbol{\kappa}}}) + (1-\theta) f_r(\boldsymbol{t}_2,\tilde{\tilde{\boldsymbol{\kappa}}}),$$

with respect to solutions $\boldsymbol{t}_1, \boldsymbol{t}_2$ and scalar $\theta \in [0,1]$. In accordance with Theorem 2:

$$E[f_r(\theta \boldsymbol{t}_1 + (1-\theta)\boldsymbol{t}_2,\tilde{\tilde{\boldsymbol{\kappa}}})] \leq \theta E[f_r(\boldsymbol{t}_1,\tilde{\tilde{\boldsymbol{\kappa}}})] + (1-\theta)E[f_r(\boldsymbol{t}_2,\tilde{\tilde{\boldsymbol{\kappa}}})], \tag{20}$$

for $r = 1,2,\cdots,l$. This demonstrates that the objective functions $E[f_r(\boldsymbol{t},\tilde{\tilde{\boldsymbol{\kappa}}})]$ in $\boldsymbol{t}$ are convex.

Then the convexity of feasible set can be proven via the validation that $\theta \boldsymbol{t}_1 + (1-\theta)\boldsymbol{t}_2$ is feasible for any feasible solutions $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ with the expected constraints $E[g_s(\boldsymbol{t},\tilde{\tilde{\boldsymbol{\kappa}}})] \leq 0$, $s = 1,2,\cdots,q$, and any scalar $\theta \in [0,1]$. In accordance with the convexity of the constraint functions $g_s(\boldsymbol{t},\tilde{\tilde{\boldsymbol{\kappa}}})$, $s = 1,2,\cdots,q$, we can get the inequality:

$$g_s(\theta \boldsymbol{t}_1 + (1-\theta)\boldsymbol{t}_2,\tilde{\tilde{\boldsymbol{\kappa}}}) \leq \theta g_s(\boldsymbol{t}_1,\tilde{\tilde{\boldsymbol{\kappa}}}) + (1-\theta) g_s(\boldsymbol{t}_2,\tilde{\tilde{\boldsymbol{\kappa}}}), \tag{21}$$

for $s = 1,2,\cdots,q$, which yields that:

$$E[g_s(\theta \boldsymbol{t}_1 + (1-\theta)\boldsymbol{t}_2,\tilde{\tilde{\boldsymbol{\kappa}}})] \leq \theta E[g_s(\boldsymbol{t}_1,\tilde{\tilde{\boldsymbol{\kappa}}})] + (1-\theta)E[g_s(\boldsymbol{t}_2,\tilde{\tilde{\boldsymbol{\kappa}}})] \leq 0, \tag{22}$$

for $s = 1, 2, \cdots, q$. This illustrates that $\theta \boldsymbol{t}_1 + (1-\theta)\boldsymbol{t}_2$ is a feasible solution, and proves the convexity of the feasible set. Therefore, a proof of convex programming i.e., the EV programming (19), is completed. □

4.3. Compromise Model

To process the FRMOP problems, the concept of a compromise model is adopted to obtain the Pareto solutions of the FRMOP-EVM in (19). Since one cannot minimize all the objectives at the same time, establishing a reasonable value function is necessary for the decision-makers. If there is a real preference function that can aggregate all objective functions $E[f_r(\boldsymbol{t}, \tilde{\tilde{\boldsymbol{\kappa}}})]$ $(r = 1, 2, \cdots, l)$, then minimizing this special function with the same constraints can be achieved. Then the acquired SOP model is named the compromise model and the solution is known as the compromise solution.

For the FRMOP-EVM, we can obtain the first compromise model via weighting the objective functions $E[f_r(\boldsymbol{t}, \tilde{\tilde{\boldsymbol{\kappa}}})]$ as follows:

$$\begin{cases} \min\limits_{\boldsymbol{t}} \sum\limits_{r=1}^{l} \delta_r E[f_r(\boldsymbol{t}, \tilde{\tilde{\boldsymbol{\kappa}}})] \\ \text{subject to:} \\ \qquad E[g_s(\boldsymbol{t}, \tilde{\tilde{\boldsymbol{\kappa}}})] \le 0, \quad s = 1, 2, \cdots, q, \end{cases} \tag{23}$$

where the weights $\delta_r > 0$ and $\delta_1 + \delta_2 + \cdots + \delta_l = 1$, for instance, $\delta_r = 1/l$, $r = 1, 2, \cdots, l$.

Theorem 8. *The optimal solution of the compromise model (23) is the Pareto solution of FRMOP-EVM (19).*

Proof. Assume that $\boldsymbol{t}^*$ is an optimal solution of model (23). If $\boldsymbol{t}^*$ is not a Pareto solution of model (19), there must exist $\boldsymbol{t}'$ that satisfies $E[f_r(\boldsymbol{t}', \tilde{\tilde{\boldsymbol{\kappa}}})] \le E[f_r(\boldsymbol{t}^*, \tilde{\tilde{\boldsymbol{\kappa}}})]$ $(r = 1, 2, \ldots, l)$, and there at least exists one index in $\{1, 2, \ldots, l\}$, say i, that satisfies $E[f_i(\boldsymbol{t}', \tilde{\tilde{\boldsymbol{\kappa}}})] < E[f_i(\boldsymbol{t}^*, \tilde{\tilde{\boldsymbol{\kappa}}})]$. It follows from $\delta_r > 0$ for all $r = 1, 2, \cdots, l$ that

$$\delta_i E[f_i(\boldsymbol{t}', \tilde{\tilde{\boldsymbol{\kappa}}})] < \delta_i E[f_i(\boldsymbol{t}^*, \tilde{\tilde{\boldsymbol{\kappa}}})] \tag{24}$$

and

$$\delta_r E[f_r(\boldsymbol{t}', \tilde{\tilde{\boldsymbol{\kappa}}})] \le \delta_r E[f_r(\boldsymbol{t}^*, \tilde{\tilde{\boldsymbol{\kappa}}})], \quad \forall r \ne i. \tag{25}$$

It follows that:

$$\sum_{r=1}^{l} \delta_r E[f_r(\boldsymbol{t}', \tilde{\tilde{\boldsymbol{\kappa}}})] < \sum_{r=1}^{l} \delta_r E[f_r(\boldsymbol{t}^*, \tilde{\tilde{\boldsymbol{\kappa}}})]. \tag{26}$$

Obviously, this result conflict demonstrates that $\boldsymbol{t}^*$ is the optimal solution of model (23). Therefore, Theorem 8 holds. □

For the FRMOP-EVM, the second method is to reduce the distance between the objective vector $(E[f_1(\boldsymbol{t}, \tilde{\tilde{\boldsymbol{\kappa}}})], E[f_2(\boldsymbol{t}, \tilde{\tilde{\boldsymbol{\kappa}}})], \cdots, E[f_l(\boldsymbol{t}, \tilde{\tilde{\boldsymbol{\kappa}}})])$ and the ideal vector $(E_1^*, E_2^*, \cdots, E_l^*)$, in which E_r^* $(r = 1, 2, \cdots, l)$ is the minimum value of the rth objective function taking no account of other objectives. Then, we can obtain the second compromise model by adopting the Euclidean distance as:

$$\begin{cases} \min\limits_{\boldsymbol{t}} \sqrt{(E[f_1(\boldsymbol{t}, \tilde{\tilde{\boldsymbol{\kappa}}})] - E_1^*)^2 + \cdots + \left(E[f_l(\boldsymbol{t}, \tilde{\tilde{\boldsymbol{\kappa}}})] - E_l^*\right)^2} \\ \text{subject to:} \\ \qquad E[g_s(\boldsymbol{t}, \tilde{\tilde{\boldsymbol{\kappa}}})] \le 0, \quad s = 1, 2, \cdots, q. \end{cases} \tag{27}$$

Theorem 9. *The optimal solution of the compromise model (27) is the Pareto solution of FRMOP-EVM (19).*

Proof. The whole proof process can be dealt with in the same way as in Theorem 8. □

The third one is inspired by Liu [26], who employed goal programming to model dual uncertain decision systems. This method aims to minimize the deviations from the target levels. Combining with the priority theory and target values of each objective function, we have the following compromise model:

$$\begin{cases} \min\limits_{t} \sum\limits_{i=1}^{m} P_i \sum\limits_{r=1}^{l} (u_{ri} d_r^+ + v_{ri} d_r^-) \\ \text{subject to:} \\ \qquad E[f_r(\boldsymbol{t}, \tilde{\bar{\boldsymbol{\kappa}}})] + d_r^- - d_r^+ = b_r, \; r = 1, 2, \cdots, l \\ \qquad E[g_s(\boldsymbol{t}, \tilde{\bar{\boldsymbol{\kappa}}})] \le 0, \qquad s = 1, 2, \cdots, q \\ \qquad d_r^+, d_r^- \ge 0, \qquad r = 1, 2, \cdots, l, \end{cases} \tag{28}$$

in which P_i is the priority factor that conveys the relative importance of multiple goals, $P_i \gg P_{i+1}$, $i = 1, 2, \cdots, m$, u_{ri} and v_{ri} are the weighting factors relevant to positive and negative deviations for goal r with priority i appointed, d_r^+ and d_r^- is the positive and negative deviations from the target of goal r, b_r is the target value in accordance with goal r, l is the quantity of goal constraints, and the quantity of constraints is denoted by q.

4.4. Crisp Equivalent Model

For processing the proposed FRMOP-EVM in (19), the crisp equivalent model as a vital conversion way is also necessary. Based on the relevant theorems, we further explore the corresponding crisp equivalent models of the above-mentioned three compromise models, i.e., (23), (27) and (28). Subsequently, the converted crisp mathematical programming models can be dealt with meta-heuristic algorithms or classical numerical approaches.

Theorem 10. *Assume that $\tilde{\bar{\kappa}}_1, \tilde{\bar{\kappa}}_2, \ldots, \tilde{\bar{\kappa}}_n$ are independent FRVs. If the objective and constraint functions satisfy the conditions that:*

$$f_r(\boldsymbol{t}, \tilde{\bar{\boldsymbol{\kappa}}}) = f_{r0}(\boldsymbol{t}) + f_{r1}(\boldsymbol{t})\tilde{\bar{\kappa}}_1 + \cdots + f_{rn}(\boldsymbol{t})\tilde{\bar{\kappa}}_n,$$

for $r = 1, 2, \cdots, l$, and

$$g_s(\boldsymbol{t}, \tilde{\bar{\boldsymbol{\kappa}}}) = g_{s0}(\boldsymbol{t}) + g_{s1}(\boldsymbol{t})\tilde{\bar{\kappa}}_1 + \cdots + g_{sn}(\boldsymbol{t})\tilde{\bar{\kappa}}_n,$$

for $s = 1, 2, \cdots, q$, then the FRMOP-EVM (19) has its crisp equivalent form as below:

$$\begin{cases} \min\limits_{t} \left(f_1(\boldsymbol{t}, E[\tilde{\bar{\boldsymbol{\kappa}}}]), f_2(\boldsymbol{t}, E[\tilde{\bar{\boldsymbol{\kappa}}}]), \cdots, f_l(\boldsymbol{t}, E[\tilde{\bar{\boldsymbol{\kappa}}}])\right) \\ \text{subject to:} \\ \qquad g_s(\boldsymbol{t}, E[\tilde{\bar{\boldsymbol{\kappa}}}]) \le 0, \quad s = 1, 2, \cdots, q, \end{cases} \tag{29}$$

where $E[\tilde{\bar{\boldsymbol{\kappa}}}] = (E[\tilde{\bar{\kappa}}_1], E[\tilde{\bar{\kappa}}_2], \ldots, E[\tilde{\bar{\kappa}}_n])$.

Proof. Likewise, it can be drawn from the fuzzy rough EV linearity theorem (see Theorem 2) straightly. □

According to Theorem (10), the three compromise models of FRMOP-EVM in (19) are further explicitly equalized, relying on the same principle. The results about their crisp equivalent ones are shown below.

The first compromise model (23) is equivalent to the model as below:

$$\begin{cases} \min\limits_{t} \sum\limits_{r=1}^{l} \delta_r f_r(\boldsymbol{t}, E[\tilde{\tilde{\boldsymbol{\kappa}}}]) \\ \text{subject to:} \\ \qquad g_s(\boldsymbol{t}, E[\tilde{\tilde{\boldsymbol{\kappa}}}]) \leq 0, \quad s = 1, 2, \cdots, q. \end{cases} \tag{30}$$

The second compromise model (27) is equivalent to the model as below:

$$\begin{cases} \min\limits_{t} \sqrt{\left(f_1(\boldsymbol{t}, E[\tilde{\tilde{\boldsymbol{\kappa}}}]) - E_1^*\right)^2 + \cdots + \left(f_l(\boldsymbol{t}, E[\tilde{\tilde{\boldsymbol{\kappa}}}]) - E_l^*\right)^2} \\ \text{subject to:} \\ \qquad g_s(\boldsymbol{t}, E[\tilde{\tilde{\boldsymbol{\kappa}}}]) \leq 0, \quad s = 1, 2, \cdots, q. \end{cases} \tag{31}$$

The third compromise model (28) is equivalent to the model as below:

$$\begin{cases} \min\limits_{t} \sum\limits_{i=1}^{m} P_i \sum\limits_{r=1}^{l} (u_{ri} d_r^+ + v_{ri} d_r^-) \\ \text{subject to:} \\ \qquad f_r(\boldsymbol{t}, E[\tilde{\tilde{\boldsymbol{\kappa}}}]) + d_r^- - d_r^+ = b_r, \quad r = 1, 2, \cdots, l \\ \qquad g_s(\boldsymbol{t}, E[\tilde{\tilde{\boldsymbol{\kappa}}}]) \leq 0, \quad s = 1, 2, \cdots, q \\ \qquad d_r^+, d_r^- \geq 0, \quad r = 1, 2, \cdots, l. \end{cases} \tag{32}$$

5. Solution

Based on the above findings, in this section, we aim to search for the optimal solutions to the above fuzzy rough programming problems. As a general rule, a two-fold uncertain model is not easier to compute directly given the existence of uncertain variables. For solving the models FRSOP-EVM in (12) and FRMOP-EVM in (19) efficiently, a new fuzzy rough simulation algorithm is proposed for calculating the various EVs occurring in the crisp equivalent models. Additionally, two kinds of numerical experiments are provided for illustrating the solution of these models via the new simulation algorithm.

5.1. Fuzzy Rough Simulation

To process the above-mentioned models, one important problem is to compute the EV $E[f(\tilde{\tilde{\boldsymbol{\kappa}}})]$. On the one hand, based on Theorem 1, for each λ in Λ, $E[f(\tilde{\tilde{\boldsymbol{\kappa}}}(\lambda))]$ is the EV of a fuzzy variable. On the other hand, $E[f(\tilde{\tilde{\boldsymbol{\kappa}}})]$ is a rough variable which can be computed according to Theorem 4. Therefore, combining the above two types of simulation techniques (i.e., fuzzy and rough simulations) to create a fuzzy rough simulation can be an efficient way [30].

5.1.1. NIA-S Based Fuzzy Simulation

Fuzzy simulation techniques have been widely adopted to simulate the EVs of various functions of fuzzy variables in the fuzzy optimization problem. In this area, Liu and Liu [33] presented a stochastic discretization-based simulation algorithm (SDA) to deal with such problems. However, the computation of the EVs by adopting this fuzzy simulation shows some flaws in computation accuracy [36]. Specifically, this algorithm is unable to provide a satisfactory approximation within the specified time, especially on computing the complex functions of fuzzy numbers. Subsequently, Li [36] put forward a numerical-integration-based fuzzy simulation algorithm (NIA-G) to improve its performance on large-size problems. Considering the unnecessary time consumption on the specified types of fuzzy numbers, Liu et al. [32] simplified the simulation procedure via employing the analytical expressions of α-optimistic values, then presented an improved fuzzy simulation algorithm called a numerical integration algorithm (NIA-S) with higher performance on the stability, convergence and accuracy. It is worth noting that this novel fuzzy simulation is suitable for simulating the EVs of strictly monotonic functions of one specified type of fuzzy number, named regular fuzzy intervals [32,37], with continuous and strictly decreasing functions, for instance, Gaussian, normal, trapezoidal and triangular fuzzy numbers. Indeed, such regular fuzzy intervals are widely adopted for factual research and application. For a better illustration, a theoretical basis of this fuzzy simulation is shown below.

Definition 12 (Liu [26])**.** *Assume that $\alpha \in [0,1]$, then the α-optimistic value of a fuzzy variable $\tilde{\zeta}$, i.e., $\tilde{\zeta}_{\sup}(\alpha)$ is defined as:*

$$\tilde{\zeta}_{\sup}(\alpha) = \sup\{\sigma \mid \mathrm{Cr}\{\tilde{\zeta} \geq \sigma\} \geq \alpha\}. \tag{33}$$

Theorem 11 (Li [32])**.** *Assume that $\tilde{\zeta}_1, \tilde{\zeta}_2, \cdots, \tilde{\zeta}_n$ are independent regular fuzzy intervals. Let $\tilde{\boldsymbol{\zeta}} = (\tilde{\zeta}_1, \tilde{\zeta}_2, \cdots, \tilde{\zeta}_n)$. If the function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and strictly increasing as regards $t_1, t_2, \cdots, t_k$, and strictly decreasing as regards $t_{k+1}, t_{k+2}, \cdots, t_n$, thus for any $\alpha \in (0,1]$, there is:*

$$f(\tilde{\boldsymbol{\zeta}})_{\sup}(\alpha) = f\big((\tilde{\zeta}_1)_{\sup}(\alpha), \cdots, (\tilde{\zeta}_k)_{\sup}(\alpha), (\tilde{\zeta}_{k+1})_{\sup}(1-\alpha), \cdots, (\tilde{\zeta}_n)_{\sup}(1-\alpha)\big). \tag{34}$$

Theorem 12 (Li [32])**.** *Assume that $\tilde{\zeta}_1, \tilde{\zeta}_2, \cdots, \tilde{\zeta}_n$ are independent regular fuzzy intervals. Let $\tilde{\boldsymbol{\zeta}} = (\tilde{\zeta}_1, \tilde{\zeta}_2, \cdots, \tilde{\zeta}_n)$. If the function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and strictly monotone function, thus for any $\alpha \in (0,1]$, the EV of $f(\tilde{\boldsymbol{\zeta}})$ is given as:*

$$E[f(\tilde{\boldsymbol{\zeta}})] = \int_0^1 f(\tilde{\boldsymbol{\zeta}})_{\sup}(\alpha)\mathrm{d}\alpha. \tag{35}$$

Relying on the above definition and theorems, the new fuzzy simulation algorithm developed by Liu et al. [32] is shown in Algorithm 1.

Algorithm 1 (NIA-S based fuzzy simulation).

Step 1: Initialize the number of sample points Q.
Step 2: Set $e = 0$ and $w = 1$.
Step 3: Set $\alpha = w/Q$. For each $1 \leq j \leq n$, compute

$$x_i = \begin{cases} (\tilde{\zeta}_i)_{\sup}(\alpha) & \text{if } 1 \leq j \leq k, \\ (\tilde{\zeta}_i)_{\sup}(1-\alpha) & \text{if } k \leq j \leq n. \end{cases}$$

Step 4: Compute $f(t_1, t_2, \cdots, t_n) = f(\tilde{\boldsymbol{\zeta}})_{\sup}(\alpha)$ via Equation (34).
Step 5: Reset $e = e + f(t_1, t_2, \cdots, t_n)/Q$ and $w = w + 1$.
Step 6: If $w \leq Q$, go back to Step 3. Or else, stop and the obtained value e is the simulated value of $E[f(\tilde{\boldsymbol{\zeta}})]$.

5.1.2. NIA-S Based Fuzzy Rough Simulation

Given the good performance of NIA-S in simulating the EVs of fuzzy variables, in this paper, we combine this newest fuzzy simulation with the rough simulation together to create a novel numerical integrated fuzzy rough simulation. The detailed procedure of this new algorithm is presented in Algorithm 2.

Algorithm 2 (NIA-S based fuzzy rough simulation)

Step 1: Initialize parameter M.
Step 2: Set $P = 0$ and $m = 1$.
Step 3: Generate $\underline{\lambda_1}, \underline{\lambda_2}, \cdots, \underline{\lambda_M}$ from the set of Δ in accordance with the measure π.
Step 4: Generate $\overline{\lambda_1}, \overline{\lambda_2}, \cdots, \overline{\lambda_M}$ from the set of Λ in accordance with the measure π.
Step 5: For $1 \leq i \leq M$, denote the value $y_i = E[f(\boldsymbol{x}, \tilde{\tilde{\boldsymbol{\kappa}}}(\underline{\lambda_i}))] + E[f(\boldsymbol{x}, \tilde{\tilde{\boldsymbol{\kappa}}}(\overline{\lambda_i}))]$.
Step 6: Reset $P = P + y_i/(2M)$ and $m = m + 1$.
Step 7: If $m \leq M$, go back to Step 4. Or else, return P as the simulated value of $E[f(\boldsymbol{x}, \tilde{\tilde{\boldsymbol{\kappa}}})]$.

In this new fuzzy rough simulation, i.e., Algorithm 2, the calculation of $E[f(\boldsymbol{x}, \tilde{\tilde{\boldsymbol{\kappa}}}(\lambda))]$ in Step 4 is achieved through the fuzzy simulation, i.e., Algorithm 1. In fact, this combination thought was initiated by Liu [30] who has made great contributions in fuzzy and rough simulation techniques, and constructed the unique fuzzy rough simulation by combining them together. However, since the part of fuzzy simulation was adopted by Liu [30], originating from the work of [33], the same issues of fuzzy simulation as stated previously have occurred in measuring the EVs of varying degrees of complexity functions of fuzzy rough variables. To make a better expression and distinction, here the fuzzy rough simulation developed by Liu [30] is referred to as SDA-FRS, which follows with its basic idea of fuzzy simulation, i.e., a stochastic discretization algorithm. Similarly, the other one put forward in this paper is referred to as NIAS-FRS based on the idea of a special numerical integration algorithm. In the next section, a comparison between these two algorithms is presented in detail.

Depending on the fuzzy rough simulation results, the general fuzzy rough programming problems can be handled by combining the meta-heuristic algorithms, such as the particle swarm optimization algorithm (i.e., PSOA), the simulated annealing algorithm (i.e., SA) and the genetic algorithm (i.e., GA). Nevertheless, there are some special fuzzy rough optimization problems in which the objective and constraint functions involving the fuzzy rough coefficients are linear functions. Therefore, for dealing with such problems, the fuzzy rough simulation can be combined with a well developed solver such as Lingo and Gurobi to obtain the optimal solutions. In effect, this kind of programming model is extensively used for actual decision-making problems. For a clear demonstration on the effectiveness of the NIAS-FRS, in the following section, this kind of special fuzzy rough programming problem is adopted.

5.2. Numerical Experiments

In this section, some simple but classical numerical experiments are offered to verify the effectiveness of the presented fuzzy rough simulation and illustrate the solving process of the fuzzy rough programming models mentioned above. Nevertheless, it is necessary to address that the proposed methods can be extended to deal with the high-dimension monotone functions with fuzzy rough variables. With that in mind, we strongly believe it provides a valuable tool for estimating the expected value in practical problems without implementing the time-consuming simulation process as before.

Let us consider the first kind of fuzzy rough programming problem, i.e., FRSOP-EVM:

$$\begin{cases} \min\limits_{t} \ E[\tilde{\tilde{\kappa}}_1 t_1 + \tilde{\tilde{\kappa}}_2 t_2 + \tilde{\tilde{\kappa}}_3 t_3 + \tilde{\tilde{\kappa}}_4] \\ \text{subject to:} \\ \qquad E[\tilde{\tilde{\kappa}}_3 t_1 + \tilde{\tilde{\kappa}}_4 t_2 + \tilde{\tilde{\kappa}}_1 t_3] \geq 8 \\ \qquad -3t_1 - 2t_2 \leq -6 \\ \qquad t_1, t_2, t_3 \geq 0, \end{cases} \tag{36}$$

where $\tilde{\bar{\kappa}}_1$, $\tilde{\bar{\kappa}}_2$, $\tilde{\bar{\kappa}}_3$ and $\tilde{\bar{\kappa}}_4$ are independent fuzzy rough variables taking values of triangular fuzzy numbers, i.e.,

$$\tilde{\bar{\kappa}}_1 = (\bar{\tau}_1 - 1, \bar{\tau}_1, \bar{\tau}_1 + 1), \text{ with } \bar{\tau}_1 \sim ([2,3], [-1,4]),$$
$$\tilde{\bar{\kappa}}_2 = (\bar{\tau}_2 - 1, \bar{\tau}_2 + 1, \bar{\tau}_2 + 3), \text{ with } \bar{\tau}_2 \sim ([1,2], [0,5]),$$
$$\tilde{\bar{\kappa}}_3 = (\bar{\tau}_3 - 2, \bar{\tau}_3 - 1, \bar{\tau}_3), \text{ with } \bar{\tau}_3 \sim ([1,3], [0,4]),$$
$$\tilde{\bar{\kappa}}_4 = (\bar{\tau}_4, \bar{\tau}_4 + 1, \bar{\tau}_4 + 2), \text{ with } \bar{\tau}_4 \sim ([2,4], [0,6]).$$

Example 4. *The model in (36) can be then converted to its equivalent form on the basis of Theorem 6 as below:*

$$\begin{cases} \min\limits_{t} \; E[\tilde{\bar{\kappa}}_1]t_1 + E[\tilde{\bar{\kappa}}_2]t_2 + E[\tilde{\bar{\kappa}}_3]t_3 + E[\tilde{\bar{\kappa}}_4] \\ \textit{subject to:} \\ \qquad E[\tilde{\bar{\kappa}}_3]t_1 + E[\tilde{\bar{\kappa}}_4]t_2 + E[\tilde{\bar{\kappa}}_1]t_3 \geq 8 \\ \qquad -3t_1 - 2t_2 \leq -6 \\ \qquad t_1, t_2, t_3 \geq 0. \end{cases} \tag{37}$$

After that, the crisp equivalent model can be deduced via Definition 8, then we have:

$$\begin{cases} \min\limits_{t} \; 2t_1 + 3t_2 + t_3 + 4 \\ \textit{subject to:} \\ \qquad t_1 + 4t_2 + 2t_3 \geq 8 \\ \qquad -3t_1 - 2t_2 \leq -6 \\ \qquad t_1, t_2, t_3 \geq 0. \end{cases} \tag{38}$$

Finally, the exact optimal solution can be obtained as:

$$t_1^* = 2, \quad t_2^* = 0, \quad t_3^* = 3,$$

with the optimal objective value 11.

A comparison study between SDA-FRS and NIAS-FRS is conducted next. We executed Algorithm 2 by varying Q from 100 to 2000 cycles in Matlab2017 on a Windows 10 platform. We set the sample points in rough simulation M as 1000 which is usually utilized in previous research. The simulated results are obtained as illustrated in Table 1, where the relative error is the ratio of the difference between the simulated value and the exact value to the exact value.

Table 1. Comparative results on fuzzy rough simulations.

Algorithm	Sample Points in Fuzzy Simulation (Q)	Objective Value	CPU Time (s)	Relative Error (%)
SDA-FRS [30]	100	10.9612	2.8314	0.3527
	500	10.9826	29.4740	0.1582
	1000	10.9914	88.1874	0.0782
	2000	11.0057	335.2291	0.0518
NIAS-FRS	100	10.9781	0.1683	0.1991
	500	11.0156	0.2351	0.1418
	1000	11.0082	0.3243	0.0745
	2000	10.9982	0.6320	0.0164

According to the data given by Table 1, it can be easily known that both the two algorithms perform well in terms of accuracy, and NIAS-FRS is slightly better for its smaller relative error. As stated in [36], the SDA-FRS is efficient in dealing with the single fuzzy number counterpart. With respect to computing time, the NIAS-FRS is noticeably better than the SDA-FRS. Furthermore, with the increase of the number of sample points Q, the NIAS-FRS still has a good performance on the calculation accuracy, which can be drawn from the relative error in the table. This result indicates the reliability of this new algorithm. To sum up, the NIAS-FRS, on the calculation of the EVs of FRVs, is feasible, effective and has high accuracy.

In effect, the numerical experiment adopted above is a kind of special FRSOP-EVM which can be computed directly based on the former mentioned theorems and definitions to obtain its exact value. However, for a better comparison and verification of the NIAS-FRS put forward in this paper, we employed such a programming model and solved it by different methods.

Let us consider the second kind of fuzzy rough programming problem, i.e., FRMOP-EVM:

$$\begin{cases} \min\limits_{t} \ E[\tilde{\tilde{\kappa}}_2 t_1 - \tilde{\tilde{\kappa}}_1 t_2] \\ \min\limits_{t} \ E[-\tilde{\tilde{\kappa}}_4 t_1 - \tilde{\tilde{\kappa}}_2 t_2] \\ \text{subject to:} \\ \qquad 2t_1 + 3t_2 \leq 18 \\ \qquad E[2t_1 + \tilde{\tilde{\kappa}}_3 t_2] \leq 10 \\ \qquad t_1, t_2 \geq 0, \end{cases} \tag{39}$$

where $\tilde{\tilde{\kappa}}_1$, $\tilde{\tilde{\kappa}}_2$, $\tilde{\tilde{\kappa}}_3$ and $\tilde{\tilde{\kappa}}_4$ are fuzzy rough variables defined in model (36).

On account of the speciality of this fuzzy rough multi-objective programming problem, the exact optimal solutions of model (39) can also be gained via related theorems and formulas, and the solver. Therefore, for a better illustration of the effectiveness of the new proposed fuzzy rough simulation in dealing with the FRMOP problem, a comparison between the exact optimal value and simulated optimal value of model (39) is given in the first solving method.

Example 5. *For resolving model (39), the first compromise method illustrated in (23) is adopted and a new programming model is obtained as below:*

$$\begin{cases} \min\limits_{t} \ \delta E[\tilde{\tilde{\kappa}}_2 t_1 - \tilde{\tilde{\kappa}}_1 t_2] + (1-\delta) E[-\tilde{\tilde{\kappa}}_4 t_1 - \tilde{\tilde{\kappa}}_2 t_2] \\ \textit{subject to:} \\ \qquad 2t_1 + 3t_2 \leq 18 \\ \qquad E[2t_1 + \tilde{\tilde{\kappa}}_3 t_2] \leq 10 \\ \qquad t_1, t_2 \geq 0, \end{cases} \tag{40}$$

where δ indicates the weight coefficient of the two objectives, and the values of δ are between 0 and 1.

Then, on the basis of Theorem 10 and model (30), the above models are converted to a crisp equivalent model as follows:

$$\begin{cases} \min\limits_{t} \ \delta (E[\tilde{\tilde{\kappa}}_2] t_1 - E[\tilde{\tilde{\kappa}}_1] t_2) + (1-\delta)(-E[\tilde{\tilde{\kappa}}_4] t_1 - E[\tilde{\tilde{\kappa}}_2] t_2) \\ \textit{subject to:} \\ \qquad 2t_1 + 3t_2 \leq 18 \\ \qquad 2t_1 + E[\tilde{\tilde{\kappa}}_3] t_2 \leq 10 \\ \qquad t_1, t_2 \geq 0. \end{cases} \tag{41}$$

Next, the EV $E[\tilde{\tilde{\kappa}}_r]$ $(r = 1, 2, 3, 4)$ in model (41) can be computed directly via Definition 8, then we have the following deterministic model:

$$\begin{cases} \min\limits_{t} \ (7\delta - 4)t_1 + (\delta - 3)t_2 \\ \textit{subject to:} \\ \qquad 2t_1 + 3t_2 \le 18 \\ \qquad 2t_1 + t_2 \le 10 \\ \qquad t_1, t_2 \ge 0. \end{cases} \tag{42}$$

Whereafter, this model can be resolved via the well-developed solver or the classical numerical methods for an exact value.

For purposes of comparison, in the software MATLAB, taking the parameters $M = 1000$ and $Q = 1000$, the simulated results can be obtained through the simulation algorithm running, as listed in Table 2. With the variation of weight coefficient, the simulated optimal solutions of t_1 and t_2, as well as the corresponding values of the first and second objective functions, i.e., $E[\tilde{\tilde{\kappa}}_2 t_1 - \tilde{\tilde{\kappa}}_1 t_2]$ and $E[-\tilde{\tilde{\kappa}}_4 t_1 - \tilde{\tilde{\kappa}}_2 t_2]$ are shown in the 2nd–5th rows. Moreover, the 6th and 7th rows are the weighted sum of simulated objective values and the exact objective values respectively. For a better comparison, the relative error between them is offered in the 8th row, which clearly illustrates the effectiveness of the NIAS-FRS technique in dealing with FRMOP problems. Additionally, the CPU time in the last row indicates the running efficiency of the NIAS-FRS.

Table 2. Results of the simulation values and exact values for the first compromise model.

Weight Coefficient: δ	**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1**
Optimal solution: t_1^*	3.0108	3.0014	0.0000	0.0000	0.0000	0.0000
Optimal solution: t_2^*	3.9928	3.9991	6.0000	6.0000	6.0000	6.0000
E_1: $E[\tilde{\tilde{\kappa}}_2 t_1^* - \tilde{\tilde{\kappa}}_1 t_2^*]$	1.1825	1.1558	−12.0227	−11.7577	−11.8826	−11.9587
E_2: $E[-\tilde{\tilde{\kappa}}_4 t_1^* - \tilde{\tilde{\kappa}}_2 t_2^*]$	−24.1820	−23.9437	−17.8993	−18.1397	−18.0652	−17.9578
$\delta E_1 + (1-\delta)E_2$	−24.1820	−18.9238	−15.5486	−14.3105	−13.1191	−11.9587
Exact sum of weighted objective values	−24.0000	−19.0000	−15.6000	−14.4000	−13.2000	−12.0000
Relative error (%)	0.0882	0.1470	0.1338	0.1514	0.1949	0.0807
CPU time (s)	0.4537	0.4536	0.4492	0.4407	0.4686	0.4521

Example 6. *According to model (27), the second compromise model is presented as follows:*

$$\begin{cases} \min\limits_{t} \ \sqrt{(E[\tilde{\tilde{\kappa}}_2 t_1 - \tilde{\tilde{\kappa}}_1 t_2] - E_1^*)^2 + (E[-\tilde{\tilde{\kappa}}_4 t_1 - \tilde{\tilde{\kappa}}_2 t_2] - E_2^*)^2} \\ \textit{subject to:} \\ \qquad 2t_1 + 3t_2 \le 18 \\ \qquad E[2t_1 + \tilde{\tilde{\kappa}}_3 t_2] \le 10 \\ \qquad t_1, t_2 \ge 0, \end{cases} \tag{43}$$

in which E_1^* and E_2^* represent the optimal values of the first and second objectives. Depending on the results listed in Table 2, it is easy to know that $E_1^* = -12$ and $E_2^* = -24$.

In accordance with Theorem 10 and model (31), the crisp equivalent model is obtained as below:

$$\begin{cases} \min\limits_{t} \ \sqrt{(E[\tilde{\tilde{\kappa}}_2]t_1 - E[\tilde{\tilde{\kappa}}_1]t_2 + 12)^2 + (-E[\tilde{\tilde{\kappa}}_4]t_1 - E[\tilde{\tilde{\kappa}}_2]t_2 + 24)^2} \\ \textit{subject to:} \\ \qquad 2t_1 + 3t_2 \le 18 \\ \qquad 2t_1 + E[\tilde{\tilde{\kappa}}_3]t_2 \le 10 \\ \qquad t_1, t_2 \ge 0. \end{cases} \tag{44}$$

Next, the value of $E[\tilde{\bar{\kappa}}_r]$ $(r = 1,2,3,4)$ *in model (44) can be obtained through the NFRS, and then a traditional single-objective programming model can be further settled via the well developed solver. Finally, the optimal solution is:*

$$t_1^* = 0.6067, \quad t_2^* = 5.5955,$$

whose objective value is 5.6830. Furthermore, we have:

$$E[\tilde{\bar{\kappa}}_2 t_1 - \tilde{\bar{\kappa}}_1 t_2] = -9.6650,$$
$$E[-\tilde{\bar{\kappa}}_4 t_1 - \tilde{\bar{\kappa}}_2 t_2] = -18.8188.$$

Example 7. *For solving the FRMOP problem illustrated in (39), the third transformation method is available, and then a fuzzy rough EV programming problem is obtained as below:*

$$\begin{cases} \operatorname*{lexmin}_{t} \{d_1^-, d_2^-\} \\ \textit{subject to:} \\ \qquad E[\tilde{\bar{\kappa}}_2 t_1 - \tilde{\bar{\kappa}}_1 t_2] + d_1^- - d_1^+ = b_1 \\ \qquad E[-\tilde{\bar{\kappa}}_4 t_1 - \tilde{\bar{\kappa}}_2 t_2] + d_2^- - d_2^+ = b_2 \\ \qquad 2t_1 + 3t_2 \leq 18 \\ \qquad E[2t_1 + \tilde{\bar{\kappa}}_3 t_2] \leq 10 \\ \qquad t_1, t_2 \geq 0 \\ \qquad d_r^+, d_r^- \geq 0, \ r = 1,2, \end{cases} \tag{45}$$

in which d_r^- *and* d_r^+ *are the negative and positive deviations from the target of goal* r $(r = 1,2)$*, and* b_r $(r = 1,2)$ *is the target value in accordance with goal* r*. Note that the "lexmin" represents lexicographically minimizing the objective vector, which reflects the hierarchy of importance among these incompatible objectives.*

Then model (45) can be further transformed into a crisp one in accordance with Theorem 10 and model (32) as:

$$\begin{cases} \operatorname*{lexmin}_{t} \{d_1^-, d_2^-\} \\ \textit{subject to:} \\ \qquad E[\tilde{\bar{\kappa}}_2] t_1 - E[\tilde{\bar{\kappa}}_1] t_2 + d_1^- - d_1^+ = b_1 \\ \qquad -E[\tilde{\bar{\kappa}}_4] x_1 - E[\tilde{\bar{\kappa}}_2] t_2 + d_2^- - d_2^+ = b_2 \\ \qquad 2t_1 + 3t_2 \leq 18 \\ \qquad 2t_1 + E[\tilde{\bar{\kappa}}_3] t_2 \leq 10 \\ \qquad t_1, t_2 \geq 0 \\ \qquad d_r^+, d_r^- \geq 0, \ r = 1,2. \end{cases} \tag{46}$$

After that, the values of $E[\tilde{\bar{\kappa}}_r]$ $(r = 1,2,3,4)$ *can be calculated by the new presented NIAS-FRS. Assume that the target value of the first goal (i.e.,* $E[\tilde{\bar{\kappa}}_2 t_1 - \tilde{\bar{\kappa}}_1 t_2]$*) is* -8*, and the second one (i.e.,* $E[-\tilde{\bar{\kappa}}_4 t_1 - \tilde{\bar{\kappa}}_2 t_2]$*) is* -22*. Then solver in Matlab is utilized to obtain the optimal solution:*

$$t_1^* = 0.9549, \quad t_2^* = 5.3634.$$

This result satisfies the first goal, but it is off by 2.1347 for the second one.

6. Conclusions

It is believed that fuzzy rough SOP and MOP can be applied much more widely in real-world applications, since single/multiple objectives and fuzzy rough phenomena extensively appear in all kinds of real-life decision-making problems. Given the practical needs, this paper introduced two kinds of uncertain programming with fuzzy rough variables, referred to as FRSOP and FRMOP. For the sake of exploring such fuzzy rough programmings more specifically, the general EV models (i.e., FRSOP-EVM and FRMOP-EVM) are put forward respectively. Based on these two EV models, the convexity theorems are discussed further, which expands the research on fuzzy rough theory. As for the solutions to these two types of fuzzy rough mathematical programming problems, a crisp equivalent model of FRSOP-EVM and three compromise models and their corresponding crisp equivalent ones of FRMOP-EVM are presented in detail. Particularly, a new fuzzy rough simulation technique (i.e., NIAS-FRS) for computing the EV is designed, since the computation may be difficult in some complicated functions. To verify the accuracy and efficiency of the presented methods, numerical examples are shown at the end. Obviously, this paper provides an efficient solution method for dealing with the common fuzzy rough programming problems, which are scant in the existing research.

The main contributions of our work are concluded below: (1) Two kinds of fuzzy rough programming models are proposed for modeling the single-objective and multi-objective optimization problems with fuzzy rough parameters, which provide a useful theoretical framework for dealing with uncertain programming problems in practice; (2) Some crisp equivalent mathematical programming models of the EV models are put forward, which can be easily handled through the classical numerical methods or the well developed solver; (3) For the computation of EVs, a reliable and efficient fuzzy rough simulation is designed, offering a sufficient tool for dealing with fuzzy rough programming models.

Frankly, the numerical examples given in Section 5 are some simple fuzzy rough programming models where the objective functions and constraint functions involving fuzzy rough parameters are linear. Only this kind of problem is taken into account in this paper owing to its wide applications in actual life. Whereas, considering the generality in dealing with the complicated problems, combining the fuzzy rough simulation with the classical mathematical programming methods or the meta-heuristic algorithms is desirable to be designed in a future study, just as Liu and Liu [38] developed a hybrid intelligent algorithm to resolve the general fuzzy random programming problem. Furthermore, given that the fuzzy rough programming studied in this paper is from an EV perspective, the other research perspectives can be further explored in the future, such as from a chance-constrained programming view, or a dependent-chance programming view.

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