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Extended Curvatures and Lie Algebroids

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Abstract: The aim of the paper is to find conditions in which the derived almost Lie vector bundle $E^{(1)}$ of an almost Lie vector bundle E is a Lie algebroid. The conditions are that some extended curvatures on E, considered in the paper, are vanishing. Two non-trivial examples are given. One example is when E_0 is a skew symmetric algebroid; the other one is when E_1 is not a skew symmetric algebroid.

Keywords: almost Lie vector bundle; skew symmetric algebroid; Jacobiator; Lie algebroid

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1. Introduction

The Lie algebroids [1–4] are a good setting to consider some generalizations or extensions of classical constructions and notions from differential geometry, mechanics, physics, etc. Some less restrictive conditions in the definition of a Lie algebroid are studied under different names and using different forms (see [5] for an up-to-date review). Non-skew symmetric brackets can be involved, as in [6] or [7] and the references therein. In the present paper, we consider only skew symmetric brackets (as in [8–10]). Complex and/or Poisson geometry can be involved, as in [11]. An algebraic setting is studied in [12]. The Banach space setting is studied in [13–16].

Anchored vector bundles are considered in [17], where they are called *relative tangent spaces*; an associated skew symmetric bracket gives rise to an almost Lie vector bundle, called an *almost Lie structure* in [18–20]. In [19], it is proven that the anchor and bracket couple is in a one-to-one correspondence with a one-degree derivation of the exterior algebra of the given vector bundle. The Jacobiator of the bracket is linear only for skew symmetric algebroids, when the anchor compatibility of the brackets holds (see 3 of Proposition 1). The vanishing Jacobiator defines a Lie algebroid.

The first derived almost Lie vector bundle of a given almost Lie vector bundle is considered in [18], and it plays an important role in the present paper. A linear E-connection ∇ having a null torsion is used in the background. We define two extended curvatures of ∇ and we prove that if both these two extended curvatures vanish, then the derived almost Lie vector bundle $E^{(1)}$ is a Lie algebroid (the main Theorem 1).

In [10], we have considered an example of a skew symmetric algebroid E_0 whose anchor does not have a skew symmetric bracket with a vanishing Jacobiator, i.e., the anchor does not allow a Lie algebroid structure, but the derived almost Lie vector bundle $E_0^{(1)}$ is a Lie algebroid. The calculations to settle this example are direct, but too long and only stated in the paper, without an effective proof. In the present paper, we give a different effective proof of this result, also extending the case $n = 2$ to an arbitrary $n \geq 2$.

Two examples when Theorem 1 applies (involving the existence of two vanishing extended curvatures) are given. One example is when E_0 is a skew symmetric algebroid (the above mentioned example) and the other one is when E_1 is not a skew symmetric algebroid.

2. Almost Lie Vector Bundles and Skew Symmetric Algebroids

Let $\pi_E : E \to M$ be a (smooth) vector bundle and $\pi_{TM} : TM \to M$ be the tangent vector bundle of M, considering smooth structures on M and E. Let $\mathcal{F}(M)$ be the real algebra of smooth real functions on M.

If $\rho : E \to TM$ (or $\rho : \Gamma(E) \to \Gamma(TM) \overset{not.}{=} \mathcal{X}(M)$) is a vector bundle map, called an *anchor*, then (E, ρ) is called an *anchored vector bundle*.

A *skew symmetric bracket* on the anchored vector bundle (E, ρ) is a map

$$[\cdot, \cdot]_E : \Gamma(E) \times \Gamma(E) \overset{not.}{=} \Gamma(E)^2 \to \Gamma(E)$$

that fulfills the conditions

$$[Y, X]_E = -[X, Y]_E$$

and

$$[X, fY]_E = \rho(X)(f)Y + f[X, Y]_E, (\forall)X, Y, Z \in \Gamma(E), (\forall)f \in \mathcal{F}(M).$$

An *almost Lie vector bundle* is a triple $(E, \rho, [\cdot, \cdot]_E)$ (or only E, when no confusion is possible).

Consider the maps

$$\mathcal{D} : \Gamma(E)^2 \to \mathcal{X}(M), \mathcal{D}(X, Y) = [\rho(X), \rho(Y)] - \rho([X, Y]_E).$$

and

$$\mathcal{J}_E : \Gamma(E)^3 \to \Gamma(E), \mathcal{J}_E(X, Y, Z) = \sum_{circ} [X, [Y, Z]_E]_E.$$

As usual (see, for example, [21]), $\mathcal{J}_E$ is called the *Jacobiator map* of the bracket $[\cdot, \cdot]_E$.

We notice that, in general, on an almost Lie vector bundle E, $\mathcal{D}$ is $\mathcal{F}(M)$-linear in all its arguments, while $\mathcal{J}_E$ is not.

If the compatibility condition $\mathcal{D} = 0$ holds, then E is called a *skew symmetric algebroid*. A skew symmetric algebroid is a *Lie algebroid* if the Jacobiator of the bracket vanishes.

All these properties can be summarized as follows. The proofs are straightforward and follow by direct computation.

Proposition 1. *Let E be an almost Lie vector bundle with a skew-symmetric bracket. Then, the following statements hold true:*

1. *The map $\mathcal{D}$ is $\mathcal{F}(M)$-bilinear and skew symmetric;*
2. *The map $\mathcal{J}_E$ is skew symmetric, $\mathcal{J}_E(X, Y, fZ) = \mathcal{D}(X, Y)(f)Z + f\mathcal{J}_E(X, Y, Z), (\forall)f \in \mathcal{F}(M)$, and $X, Y, Z \in \Gamma(E)$;*
3. *The map $\mathcal{J}_E$ is $\mathcal{F}(M)$-linear in all its arguments if E is a skew symmetric algebroid;*
4. *The map $\mathcal{J}_E$ vanishes if E is a Lie algebroid.*

In particular, for skew symmetric algebroids, we obtain the following.

Proposition 2. *For an almost Lie vector bundle E with a skew-symmetric bracket, the following conditions are equivalent:*

1. *E is a skew symmetric algebroid;*
2. *$\mathcal{D} = 0$;*
3. *$\mathcal{J}_E$ is $\mathcal{F}(M)$-linear in all its arguments.*

The Jacobiator $\mathcal{J}_E$ vanishes if E is a Lie algebroid.

An interpretation of $\mathcal{D}$ is that it can be seen as an anchor $\rho^\wedge : E \wedge E \to TM$,

$$\rho^\wedge(X \wedge Y) = \mathcal{D}(X, Y).$$

Here, $E \wedge E$ denotes the *exterior product bundle* of E (i.e., the exterior product on each fiber).

Now consider an almost Lie vector bundle E and let $\pi_A : A \to M$ be a vector bundle over the same base M. A *linear E-connection* on A is a map $\nabla : \Gamma(E) \times \Gamma(A) \to \Gamma(A)$ that verifies Koszul conditions:

$$
\begin{aligned}
\nabla_{fX}s &= f\nabla_X s, \nabla_{(X+X')}s = \nabla_X s + \nabla_{X'}s, \\
\nabla_X(fs) &= \rho(X)(f)s + f\nabla_X s, \nabla_X(s+s') = \nabla_X s + \nabla_X s',
\end{aligned}
$$

$(\forall) X, X' \in \Gamma(E), s, s' \in \Gamma(A), f \in \mathcal{F}(M)$.

The curvature of ∇ is the map $R : \Gamma(E)^2 \times \Gamma(A) \to \Gamma(A)$, given by the formula

$$
R(X,Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]_E}s,
$$

$(\forall) X, Y \in \Gamma(E)$ and $s \in \Gamma(A)$.

Proposition 3. *Let E be an almost Lie vector bundle and A be a vector bundle over the same base M. If ∇ is a linear E-connection on A, then the formula $R_{X \wedge Y}s = R(X,Y)s$ defines a linear $E \wedge E$-connection on A, according to the anchor $\rho^\wedge$.*

Proof. One can check that for a given $s \in \Gamma(A)$, $(X,Y) \to R(X,Y)s$ is $\mathcal{F}(M)$-linear in both arguments.

However, for given $X, Y \in \Gamma(E)$, the map $s \to R_{X \wedge Y}s$ is additive, and for $f \in \mathcal{F}(M)$ we have

$R_{X \wedge Y}(fs) = R(X,Y)(fs) = \mathcal{D}(X,Y)(f)s + fR(X,Y)s = \rho^\wedge(X \wedge Y)(f)s + fR_{X \wedge Y}s$.

This implies the conclusion. $\square$

In the particular case of an E-connection ∇ on E, we can consider its torsion

$$
T : \Gamma(E)^2 \to \Gamma(E),
$$

given by the formula
$$
T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]_E
$$

and its curvature $R : \Gamma(E)^3 \to \Gamma(E)$,

$$
R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]_E}Z \stackrel{not.}{=} \nabla_{X \wedge Y}Z,
$$

$(\forall) X, Y, Z \in \Gamma(E)$. The last notation above defines a linear $E \wedge E$-connection $\nabla^\wedge$ on E, according to the anchor $\rho^\wedge$. The formula

$$
\nabla^{(\wedge)}_{X \wedge Y}(Z \wedge W) = \nabla_{X \wedge Y}Z \wedge W + Z \wedge \nabla_{X \wedge Y}W \stackrel{not.}{=} \nabla_{X \wedge Y}(Z \wedge W)
$$

defines a linear $E \wedge E$-connection $\nabla^{(\wedge)}$ on $E \wedge E$, according to the anchor $\rho^\wedge$.

Notice that when the torsion vanishes, then

$$
[X,Y]_E = \nabla_X Y - \nabla_Y X. \tag{1}
$$

If the E-connection ∇ is given, then the above Formula (1) defines the bracket $[\cdot, \cdot]_E$ on E, such that ∇ has a null torsion.

3. The Derived Almost LIE Vector Bundle

Let E be an almost Lie vector bundle and ∇ be an E-connection on E that has a null torsion.

We can consider on the vector bundle $E^{(1)} = E \oplus (E \wedge E)$ the anchor given by

$$\rho^{(1)}(X + (Y \wedge Z)) = \rho(X) + \rho^\wedge(Y \wedge Z)$$

and the $E^{(1)}$-connection $\nabla^{(1)}$ on $E^{(1)}$, according to the formulas

$$
\begin{aligned}
\nabla_X^{(1)} Y &= \nabla_X Y + \frac{1}{2} X \wedge Y, \ \nabla_X^{(1)}(Y \wedge Z) = \nabla_X(Y \wedge Z) = \nabla_X Y \wedge Z + Y \wedge \nabla_X Z, \ (2) \\
\nabla_{X \wedge Y}^{(1)} Z &= \nabla_{X \wedge Y} Z, \ \nabla_{X \wedge Y}^{(1)}(Z \wedge T) = \nabla_{X \wedge Y}(Z \wedge T).
\end{aligned}
$$

The formula

$$[U, V]_{E^{(1)}} = \nabla_U^{(1)} V - \nabla_V^{(1)} U$$

gives a bracket on $E^{(1)}$ according to the anchor $\rho^{(1)}$. More precisely, the bracket on $E^{(1)}$ is given by the formulas

$$
\begin{aligned}
[X, Y]_{E^{(1)}} &= [X, Y]_E + X \wedge Y, \\
[X \wedge Y, Z]_{E^{(1)}} &= \nabla_{X \wedge Y} Z - \nabla_Z(X \wedge Y), \quad (3) \\
[X \wedge Y, Z \wedge T]_{E^{(1)}} &= \nabla_{X \wedge Y}(Z \wedge T) - \nabla_{Z \wedge T}(X \wedge Y).
\end{aligned}
$$

We call the almost Lie vector bundle $(E^{(1)}, \rho^{(1)}, [\cdot, \cdot]_{E^{(1)}})$ the (*first*) *derived almost Lie vector bundle* of $(E, \rho, [\cdot, \cdot]_E)$, given by ∇.

Proposition 4. *If the linear E-connection ∇ on the almost Lie vector bundle E has no torsion, then the following properties hold true:*

1. $\mathcal{D}^{(1)}(X, Y) = [\rho^{(1)}(X), \rho^{(1)}(Y)] - \rho^{(1)}([X, Y]_{\theta^{(1)}}) = 0, \ (\forall) X, Y \in \Gamma(E)$;
2. $\mathcal{J}^{(1)}(X, Y, Z) = 0, \ (\forall) X, Y, Z \in \Gamma(E)$, *where $\mathcal{J}^{(1)}$ denotes the Jacobiator of $[\cdot, \cdot]_{E^{(1)}}$.*

Proof. In order to prove 1, we have:
$[\rho^{(1)}(X), \rho^{(1)}(Y)] = [\rho(X), \rho(Y)] = \rho([X, Y]_E) + \mathcal{D}(X, Y) = \rho([X, Y]_E) + \rho^\wedge(X \wedge Y) = \rho^{(1)}([X, Y]_E + X \wedge Y) = \rho^{(1)}([X, Y]_{E^{(1)}})$.

In order to prove the second equality, we have:
$[X, [Y, Z]_{E^{(1)}}]_{E^{(1)}} = [X, [Y, Z]_E + Y \wedge Z]_{E^{(1)}} =$
$[X, [Y, Z]_E]_E + X \wedge [Y, Z]_E + \nabla_X(Y \wedge Z) - \nabla_{Y \wedge Z} X =$
$\nabla_X[Y, Z]_E - \nabla_{[Y, Z]_E} X + X \wedge (\nabla_Y Z - \nabla_Z Y) +$
$\nabla_X(Y \wedge Z) - \left(\nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X - \nabla_{[Y, Z]_E} X \right) =$
$\nabla_X(\nabla_Y Z - \nabla_Z Y) + X \wedge (\nabla_Y Z - \nabla_Z Y) + \nabla_X(Y \wedge Z) - (\nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X) =$
$\nabla_X(\nabla_Y Z - \nabla_Z Y) - \nabla_{[Y, Z]_E} X + X \wedge (\nabla_Y Z - \nabla_Z Y) + \nabla_X(Y \wedge Z) - \nabla_{Y \wedge Z} X =$
$\nabla_X \nabla_Y Z - \nabla_X \nabla_Z Y - \nabla_Y \nabla_Z X + \nabla_Z \nabla_Y X +$
$X \wedge \nabla_Y Z - X \wedge \nabla_Z Y - Z \wedge \nabla_X Y + Y \wedge \nabla_X Z$.

Considering the analogous expressions for $[Y, [Z, X]_{E^{(1)}}]_{E^{(1)}}$ and $[Z, [X, Y]_{E^{(1)}}]_{E^{(1)}}$, by summing, we obtain the second equality. $\square$

Besides the vanishing components, the Jacobiator on $E^{(1)}$ has other components that can be handled using two other curvatures involving ∇, which we define in the sequel.

Thus, let us consider

$$R(X, Z \wedge W)Y \overset{def.}{=} \nabla_X \nabla_{Z \wedge W} Y - \nabla_{Z \wedge W} \nabla_X Y - \nabla_{\nabla_X(Z \wedge W)} Y + \nabla_{\nabla_{Z \wedge W} X} Y,$$

$$R(Z \wedge W, X)Y \overset{def.}{=} -R(X, Z \wedge W)Y$$

and

$$R(X \wedge Y, Z \wedge W)U \overset{def.}{=} \nabla_{X \wedge Y} \nabla_{Z \wedge W} U - \nabla_{Z \wedge W} \nabla_{X \wedge Y} U - \nabla_{\nabla_{X \wedge Y}(Z \wedge W)} U + \nabla_{\nabla_{Z \wedge W}(X \wedge Y)} U.$$

We call

$$(X, Y, Z, W) \to R(X, Z \wedge W)Y$$

and

$$(X, Y, Z, W, U) \to R(X \wedge Y, Z \wedge W)U$$

the *extended curvatures* of ∇ (the *first extended curvature* and the *second extended curvature*, respectively).

If $f \in \mathcal{F}(M)$ and $X, Y, Z, W, U \in \Gamma(E)$, then we have

$$
\begin{aligned}
f(X \wedge Y) &= fX \wedge Y = X \wedge (fY), \\
R(X, Z \wedge W)Y &= -R(Z \wedge W, X)Y, \\
R(fX, Z \wedge W)Y &= R(X, f(Z \wedge W))Y = fR(X, Z \wedge W)Y, \\
R(X, Z \wedge W)(fY) &= \mathcal{D}^{(1)}(X, Z \wedge W)(f)Y + fR(X, Z \wedge W)Y
\end{aligned}
\tag{4}
$$

and also

$$
\begin{aligned}
R(fX \wedge Y, Z \wedge W)U &= R(X \wedge Y, fZ \wedge W)U = fR(X \wedge Y, Z \wedge W)U, \\
R(X \wedge Y, Z \wedge W)(fU) &= \mathcal{D}^{(1)}(X \wedge Y, Z \wedge W)(f)U + fR(X \wedge, Z \wedge W)U.
\end{aligned}
\tag{5}
$$

Notice that, according to their properties, we can consider

$$R(X, Z \wedge W)(Y \wedge U) \overset{def.}{=} R(X, Z \wedge W)Y \wedge U + Y \wedge R(X, Z \wedge W)U,$$

$$R(X \wedge Y, Z \wedge W)(U \wedge V) \overset{def.}{=} R(X \wedge Y, Z \wedge W)U \wedge V + U \wedge R(X \wedge Y, Z \wedge W)V.$$

Proposition 5. *Let E be an almost Lie vector bundle. If $X, Y, Z, W, U, V \in \Gamma(E)$, then*

$$
\begin{aligned}
\mathcal{J}^{(1)}(X, Y, Z \wedge W) &= R(Y, Z \wedge W)X - R(X, Z \wedge W)Y, \\
\mathcal{J}^{(1)}(U, X \wedge Y, Z \wedge W) &= R(U, X \wedge Y)(Z \wedge W) - \\
&\quad R(U, Z \wedge W)(X \wedge Y) + \\
&\quad R(X \wedge Y, Z \wedge W)U, \\
\mathcal{J}^{(1)}(X \wedge Y, Z \wedge W, U \wedge V) &= R(X \wedge Y, Z \wedge W)(U \wedge V) + \\
&\quad R(Z \wedge W, U \wedge V)(X \wedge Y) + \\
&\quad R(U \wedge V, X \wedge Y)(Z \wedge W).
\end{aligned}
$$

Proof. We prove only the first two relations, since the third relation can be proved analogously. We have

$\mathcal{J}^{(1)}(X, Y, Z \wedge W) =$

$\left[X, [Y, Z \wedge W]^{(1)}\right]^{(1)} + \left[Y, [Z \wedge W, X]^{(1)}\right]^{(1)} + \left[Z \wedge W, [X, Y]^{(1)}\right]^{(1)} =$

$[X, \nabla_Y(Z \wedge W)]^{(1)} - [X, \nabla_{Z \wedge W}Y]^{(1)} + [Y, \nabla_{Z \wedge W}X - \nabla_X(Z \wedge W)]^{(1)} +$

$[Z \wedge W, [X, Y] + X \wedge Y]^{(1)} =$

$\nabla_X \nabla_Y(Z \wedge W) - \nabla_{\nabla_Y(Z \wedge W)}X - \nabla_X \nabla_{Z \wedge W}Y + \nabla_{\nabla_{Z \wedge W}Y}X + X \wedge \nabla_{Z \wedge W}Y +$

$\nabla_Y \nabla_{Z \wedge W}X - \nabla_{\nabla_{Z \wedge W}X}Y + Y \wedge \nabla_{Z \wedge W}X - \nabla_Y \nabla_X(Z \wedge W) + \nabla_{\nabla_X(Z \wedge W)}Y +$

$\nabla_{Z \wedge W}(\nabla_X Y - \nabla_Y X) - \nabla_{[X,Y]}(Z \wedge W) + \nabla_{Z \wedge W}(X \wedge Y) - \nabla_{X \wedge Y}(Z \wedge W) =$

$\left(\nabla_X \nabla_Y(Z \wedge W) - \nabla_Y \nabla_X(Z \wedge W) - \nabla_{[X,Y]}(Z \wedge W)\right) +$

$$\left(\nabla_Y\nabla_{Z\wedge W}X - \nabla_{Z\wedge W}\nabla_Y X - \nabla_{\nabla_{Z\wedge W}Y}X + \nabla_{\nabla_Y(Z\wedge W)}X\right)-$$
$$\left(\nabla_X\nabla_{Z\wedge W}Y - \nabla_{Z\wedge W}\nabla_X Y - \nabla_{\nabla_X(Z\wedge W)}Y + \nabla_{\nabla_{Z\wedge W}X}Y\right) - \nabla_{X\wedge Y}(Z\wedge W) =$$
$$R(Y, Z\wedge W)X - R(X, Z\wedge W)Y.$$

Also

$$\mathcal{J}^{(1)}(U, X\wedge Y, Z\wedge W) = \left[U, [X\wedge Y, Z\wedge W]^{(1)}\right]^{(1)}+$$
$$\left[X\wedge Y, [Z\wedge W, U]^{(1)}\right]^{(1)} + \left[Z\wedge W, [U, X\wedge Y]^{(1)}\right]^{(1)} =$$
$$[U, \nabla_{X\wedge Y}(Z\wedge W) - \nabla_{Z\wedge W}(X\wedge Y)]^{(1)} + [X\wedge Y, \nabla_{Z\wedge W}U - \nabla_U(Z\wedge W)]^{(1)}+$$
$$[Z\wedge W, \nabla_U(X\wedge Y) - \nabla_{X\wedge Y}U]^{(1)} =$$
$$\nabla_U\nabla_{X\wedge Y}(Z\wedge W) - \nabla_{\nabla_{X\wedge Y}(Z\wedge W)}U - \nabla_U\nabla_{Z\wedge W}(X\wedge Y) + \nabla_{\nabla_{Z\wedge W}(X\wedge Y)}U+$$
$$\nabla_{X\wedge Y}\nabla_{Z\wedge W}U - \nabla_{Z\wedge W}U(X\wedge Y) - \nabla_{X\wedge Y}\nabla_U(Z\wedge W) + \nabla_{\nabla_U(Z\wedge W)]}(X\wedge Y)+$$
$$\nabla_{Z\wedge W}\nabla_U(X\wedge Y) - \nabla_{\nabla_U(X\wedge Y)}(Z\wedge W) - \nabla_{Z\wedge W}\nabla_{X\wedge Y}U - \nabla_{\nabla_{X\wedge Y}U}(Z\wedge W) =$$
$$R(U, X\wedge Y)(Z\wedge W) - R(U, Z\wedge W)(X\wedge Y) + R(X\wedge Y, Z\wedge W)U. \quad \square$$

Using Proposition 2 for $E^{(1)}$, we obtain the following true statement.

Proposition 6. *Let E be an almost Lie vector bundle. Then, the following conditions are equivalent:*

1. $E^{(1)}$ *is a skew symmetric algebroid;*
2. $\mathcal{D}^{(1)} = 0$;
3. $\mathcal{J}^{(1)}$ *is an $\mathcal{F}(M)$-linear form.*

The map $\mathcal{J}^{(1)}$ vanishes if $(E^{(1)}, \rho^{(1)}, [\cdot, \cdot]_{E^{(1)}})$ is a Lie algebroid.

More exactly, Condition 2 can be read as follows.

Proposition 7. *Let E be an almost Lie vector bundle. The condition that $E^{(1)}$ be a skew symmetric algebroid is expressed by the relations*

$$[\rho(X), \mathcal{D}(Z, W)] - \mathcal{D}(\nabla_X(Z\wedge W)) + \rho(\nabla_{Z\wedge W}X) \quad = \quad 0, \qquad (6)$$
$$[\mathcal{D}(X, Y), \mathcal{D}(Z, W)] - \mathcal{D}(\nabla_{X\wedge Y}(Z\wedge W) - \nabla_{Z\wedge W}(X\wedge Y)) \quad = \quad 0. \qquad (7)$$

Proof. Using 1 of Proposition 4, it follows that the condition $\mathcal{D}^{(1)} = 0$ reads $\mathcal{D}^{(1)}(X, Z\wedge W)$ $= \mathcal{D}^{(1)}(X\wedge Y, Z\wedge W) = 0$, where

$$\mathcal{D}^{(1)}(X, Z\wedge W) = \left[\rho^{(1)}(X), \rho^{(1)}(Z\wedge W)\right] - \rho^{(1)}(\nabla_X(Z\wedge W) - \nabla_{Z\wedge W}X) =$$
$$[\rho(X), \mathcal{D}(Z, W)] - \mathcal{D}(\nabla_X(Z\wedge W)) + \rho(\nabla_{Z\wedge W}X),$$
$$\mathcal{D}^{(1)}(X\wedge Y, Z\wedge W) = \left[\rho^{(1)}(X\wedge Y), \rho^{(1)}(Z\wedge W)\right]-$$
$$\rho^{(1)}(\nabla_{X\wedge Y}(Z\wedge W) - \nabla_{Z\wedge W}(X\wedge Y)) =$$
$$[\mathcal{D}(X, Y), \mathcal{D}(Z, W)] - \mathcal{D}(\nabla_{X\wedge Y}(Z\wedge W) - \nabla_{Z\wedge W}(X\wedge Y)).$$
The conclusion follows. $\quad \square$

In the particular case when E is a skew symmetric algebroid, we obtain the following statement proven in [10] (Proposition 2.5).

Corollary 1. *If E is a skew symmetric algebroid, then $E^{(1)}$ is a skew symmetric algebroid if*

$$\rho(R(X, Y)Z) = 0.$$

Proposition 8. *Consider an almost Lie vector bundle E. If $E^{(1)}$ is a skew symmetric algebroid, then both extended curvatures are $\mathcal{F}(M)$-linear in their arguments.*

Proof. Using 2 of Proposition 6, it follows that $\mathcal{D}^{(1)} = 0$; then, using Formulas (4) and (5), the conclusion follows. $\quad \square$

We can prove now the main result of the paper.

Theorem 1. *Let E be an almost Lie vector bundle. If $E^{(1)}$ is a skew symmetric algebroid and both its extended curvatures vanish, then $E^{(1)}$ is a Lie algebroid.*

Proof. Using Proposition 8, it follows that the extended curvatures are $\mathcal{F}(M)$-linear in all arguments; thus, the vanishing conditions have sense. Moreover, using 2 of Proposition 4 and Proposition 5, we have $\mathcal{J}^{(1)} = 0$; thus, the conclusion follows. □

Effectively, the conditions in the hypothesis of the above theorem are expressed by Relations (6) and (7), and

$$R(X, Z \wedge W)Y = R(X \wedge Y, Z \wedge W)U = 0. \tag{8}$$

4. Some Examples

We consider below two relevant examples.

In the first example, we consider a skew-symmetric algebroid E_0 on $\mathbb{R}^n$ that is not a Lie algebroid, and we prove that its derived almost Lie vector bundle $E_0^{(1)}$ is a Lie algebroid. It is an extension of an example in [10], where the case $n = 2$ is considered and where it is proven that in this case, there is no algebroid Lie bracket associated with the anchor [10] (Theorem 2.3).

In the second example, we consider an almost Lie vector bundle E_1 that is not a skew symmetric algebroid, but its derived almost Lie vector bundle $E_1^{(1)}$ is a Lie algebroid.

We proceed now with the first example. Let us consider the vector bundle $E_0 = \mathbb{R}^n \times \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}^n$ on the base manifold $M = \mathbb{R}^n$, where $\mathcal{M}_n(\mathbb{R})$ is the set of square n-matrices with real entries. The anchor ρ on E_0 is defined as follows. In every point $\bar{x} = (x^1, \dots, x^n)$,

$$\rho\left(X_i^j\right) = \left(x^j\right)^2 \frac{\partial}{\partial x^j}. \tag{9}$$

It is easy to see that the image by ρ of the sections of E_0 generates the whole tangent space $T_{\bar{x}}\mathbb{R}^n$ for $\bar{x} \neq \bar{0}$ and $\{\bar{0}\} \subset T_{\bar{0}}\mathbb{R}^n$ for $\bar{x} = \bar{0} = (0, \dots, 0)$. A section on E_0 is in $\ker \rho$ if it is an $\mathcal{F}(\mathbb{R}^n)$-combination of sections $\mathcal{X}_{ijk} = \left(x^j\right)^2 X_k^i - \left(x^i\right)^2 X_k^j$, where $1 \leq i < j, k \leq n$. We notice that these $\frac{n^2(n-1)}{2}$ sections do not generate a (regular) vector sub-bundle of E_0.

Associated with the above anchor, we consider the bracket $[\cdot, \cdot]_{E_0}$ defined on generators by

$$\left[X_j^i, X_v^u\right]_{E_0} = 2x^u \delta_j^u X_v^i - 2x^i \delta_v^i X_j^u$$

and the linear E_0-connection ∇ on E_0 defined on generators by

$$\nabla_{X_j^i} X_v^u = 2x^u \delta_j^u X_v^i.$$

It is easy to see that $\rho\left(\left[X_j^i, X_l^k\right]_{E_0}\right) = \left[\rho\left(X_j^i\right), \rho\left(X_l^k\right)\right]$; thus, $\left(E_0, \rho, [\cdot, \cdot]_{E_0}\right)$ is a skew symmetric algebroid.

The curvature of ∇ is linear in all arguments and

$$\nabla_{X_j^i \wedge X_l^k} X_v^u = 2(u, j, l)\left(\left(x^i\right)^2 X_v^k - \left(x^k\right)^2 X_v^i\right) = 2(u, j, l) X_{kiv},$$

where $(i, j, k) = \delta_j^i \delta_k^i$ and $X_{kiv} = \left(x^i\right)^2 X_v^k - \left(x^k\right)^2 X_v^i$.

Since $\rho\left(\nabla_{X_j^i \wedge X_l^k} X_v^u\right) = 0$, then, using Corollary 1, it follows that the derived bundle $\left(E_0^{(1)}, \rho^{(1)}, [\cdot, \cdot]_{E_0}^{(1)}\right)$ is a skew symmetric algebroid as well. Using Proposition 8, it follows that the extended curvatures are $\mathcal{F}(\mathbb{R}^n)$-linear in their arguments.

Proposition 9. *The Jacobiator of* $[\cdot, \cdot]_{E_0}$ *is*

$$\mathcal{J}_{E_0}\left(X_j^i, X_l^k, X_v^u\right) = 2(j, u, l)X_{kiv} + 2(j, v, k)X_{iul} + 2(i, v, l)X_{ukj}.$$

Proof. We have

$$\mathcal{J}_{E_0}\left(X_j^i, X_l^k, X_v^u\right) = \left[X_j^i, \left[X_l^k, X_v^u\right]_{E_0}\right]_{E_0} + \left[X_l^k, \left[X_v^u, X_j^i\right]_{E_0}\right]_{E_0} + \left[X_v^u, \left[X_j^i, X_l^k\right]_{E_0}\right]_{E_0} =$$

$$\left[X_j^i, 2x^u\delta_l^u X_v^k - 2x^k\delta_v^k X_l^u\right]_{E_0} + \left[X_l^k, 2x^i\delta_v^i X_j^u - 2x^u\delta_j^u X_v^i\right]_{E_0} + \left[X_v^u, x^k\delta_j^k X_l^i - 2x^i\delta_l^i X_j^k\right]_{E_0} =$$

$$2(j, u, l)\left(x^i\right)^2 X_v^k + 2x^u\delta_l^u\left[X_j^i, X_v^k\right]_{E_0} - 2(j, v, k)\left(x^i\right)^2 X_l^u + 2x^k\delta_v^k\left[X_j^i, X_l^u\right]_{E_0} +$$

$$2(i, v, l)\left(x^k\right)^2 X_j^u + 2x^i\delta_v^i\left[X_l^k, X_j^u\right]_{E_0} - 2(j, u, l)\left(x^k\right)^2 X_v^i + 2x^u\delta_j^u\left[X_l^k, X_v^i\right]_{E_0} +$$

$$2(j, v, k)(x^u)^2 X_l^i + 2x^k\delta_j^k\left[X_v^u, X_l^i\right]_{E_0} - 2(i, v, l)(x^u)^2 X_j^k + 2x^i\delta_l^i\left[X_v^u, X_j^k\right]_{E_0} =$$

$$2(j, u, l)\left(x^i\right)^2 X_v^k + 2x^u\delta_l^u\left(2x^k\delta_j^k X_v^i - 2x^i\delta_v^i X_j^k\right)_{E_0} - 2(j, v, k)\left(x^i\right)^2 X_l^u +$$

$$2x^k\delta_v^k\left(2x^u\delta_j^u X_l^i - 2x^i\delta_l^i X_j^u\right) + 2(i, v, l)\left(x^k\right)^2 X_j^u + 2x^i\delta_v^i\left(2x^u\delta_l^u X_j^k - 2x^k\delta_j^k X_l^u\right) -$$

$$2(j, u, l)\left(x^k\right)^2 X_v^i + 2x^u\delta_j^u\left(2x^i\delta_l^i X_v^k - 2x^k\delta_v^k X_l^i\right) + 2(j, v, k)(x^u)^2 X_l^i +$$

$$2x^k\delta_j^k\left(2x^i\delta_v^i X_l^u - 2x^u\delta_l^u X_v^i\right) - 2(i, v, l)(x^u)^2 X_j^k + 2x^i\delta_l^i\left(2x^k\delta_v^k X_j^u - 2x^u\delta_j^u X_v^k\right) =$$

$$2(j, u, l)\left(\left(x^i\right)^2 X_v^k - \left(x^k\right)^2 X_v^i\right) + 2(j, v, k)\left((x^u)^2 X_l^i - \left(x^i\right)^2 X_l^u\right) +$$

$$2(i, v, l)\left(\left(x^k\right)^2 X_j^u - (x^u)^2 X_j^k\right) = 2(j, u, l)X_{kiv} + 2(j, v, k)X_{iul} + 2(i, v, l)X_{ukj}. \quad \square$$

It follows that E_0 is not a Lie algebroid.

Lemma 1. *For* E_0, *both extended curvatures of* ∇ *vanish.*

Proof. We have

$$R\left(X_j^i, X_l^k \wedge X_q^p\right)X_v^u = \nabla_{X_j^i}\nabla_{X_l^k \wedge X_q^p}X_v^u - \nabla_{X_l^k \wedge X_q^p}\nabla_{X_j^i}X_v^u - \nabla_{\nabla_{X_j^i}\left(X_l^k \wedge X_q^p\right) - \nabla_{X_l^k \wedge X_q^p}X_j^i}X_v^u =$$

$$\nabla_{X_j^i}2\left(\delta_i^v\delta_q^v\right)\left(\left(x^k\right)^2 X_u^p - (x^p)^2 X_u^k\right) - \nabla_{X_l^k \wedge X_q^p}\left(2x^u\delta_j^u X_v^i\right) -$$

$$\nabla_{2x^k\delta_j^k X_l^i \wedge X_q^p + X_l^k \wedge \left(2x^p\delta_j^p X_q^i\right)}X_v^u + \nabla_{\left(\left(\delta_l^i\delta_q^i\right)2\left(\left(x^k\right)^2 X_j^p - (x^p)^2 X_j^k\right)\right)}X_v^u =$$

$$4\left(\delta_l^u\delta_q^u\right)x^k\delta_j^k\left(x^i\right)^2 X_v^p + 2\left(\delta_l^u\delta_q^u\right)\left(x^k\right)^2 2x^p\delta_j^p X_v^i - 4\left(\delta_l^u\delta_q^u\right)x^p\delta_j^p\left(x^i\right)^2 X_v^k -$$

$$2\left(\delta_l^u\delta_q^u\right)(x^p)^2 2x^k\delta_j^k X_v^i - 2x^u\delta_j^u 2\left(\delta_l^i\delta_q^i\right)\left(\left(x^k\right)^2 X_v^p - (x^p)^2 X_v^k\right) -$$

$$2x^k\delta_j^k 2\left(\delta_l^u\delta_q^u\right)\left(\left(x^i\right)^2 X_v^p - (x^p)^2 X_v^i\right) - 2x^p\delta_j^p 2\left(\delta_l^u\delta_q^u\right)\left(\left(x^k\right)^2 X_v^i - \left(x^i\right)^2 X_v^k\right) +$$

$$+2\left(\delta_l^i\delta_q^j\right)\left(x^k\right)^2 2x^u\delta_j^u X_v^p - 2\left(\delta_l^i\delta_q^j\right)(x^p)^2 2x^p\delta_j^p X_v^k =$$

$$(u, l, q)x^k\delta_j^k\left(\left(x^i\right)^2 X_v^p - (x^p)^2 X_v^i\right) + (u, l, q)x^p\delta_j^p\left(\left(x^k\right)^2 X_v^i - \left(x^i\right)^2 X_v^k\right) -$$

$$(i, l, q)x^u\delta_j^u\left(\left(x^k\right)^2 X_v^p - (x^p)^2 X_v^k\right) - (u, l, q)x^k\delta_j^k\left(\left(x^i\right)^2 X_v^p - (x^p)^2 X_v^i\right) -$$

$$(u, l, q)x^p\delta_j^p\left(\left(x^k\right)^2 X_v^i - \left(x^i\right)^2 X_v^k\right) + (i, l, q)x^u\delta_j^u\left(\left(x^k\right)^2 X_v^p - (x^p)^2 X_v^k\right) = 0$$

We also have

$$R\left(X_j^i \wedge X_l^k, X_q^p \wedge X_s^r\right)X_v^u = \nabla_{X_j^i \wedge X_l^k}\nabla_{X_q^p \wedge X_s^r}X_v^u - \nabla_{X_q^p \wedge X_s^r}\nabla_{X_j^i \wedge X_l^k}X_v^u -$$

$$\nabla_{\nabla_{X_j^i \wedge X_l^k}\left(X_q^p \wedge X_s^r\right) - \nabla_{X_q^p \wedge X_s^r}\left(X_j^i \wedge X_l^k\right)}X_v^u = \nabla_{X_j^i \wedge X_l^k}\left(2(u,q,s)\left((x^p)^2 X_v^r - (x^r)^2 X_v^p\right)\right) -$$

$$\nabla_{X_q^p \wedge X_s^r}\left(2(u,j,l)\left((x^i)^2 X_v^k - \left(x^k\right)^2 X_v^i\right)\right) - \nabla_{\left(2(p,j,l)\left((x^i)^2 X_q^k - (x^k)^2 X_q^i\right)\right) \wedge X_s^r}X_v^u -$$

$$\nabla_{X_q^p \wedge \left(2(r,j,l)\left((x^i)^2 X_s^k - (x^k)^2 X_s^i\right)\right)}X_v^u + + \nabla_{\left(2(i,q,s)\left((x^p)^2 X_j^r - (x^r)^2 X_j^p\right)\right) \wedge X_l^k}X_v^u +$$

$$\nabla_{X_j^i \wedge \left(2(k,q,s)\left((x^p)^2 X_l^r - (x^r)^2 X_l^p\right)\right)}X_v^u = 2(u,q,s)(x^p)^2 \nabla_{X_j^i \wedge X_l^k}X_v^r - 2(u,q,s)(x^r)^2 \nabla_{X_j^i \wedge X_l^k}X_v^p -$$

$$2(u,j,l)(x^i)^2 \nabla_{X_q^p \wedge X_s^r}X_v^k + 2(u,j,l)\left(x^k\right)^2 \nabla_{X_q^p \wedge X_s^r}X_v^i - 2(p,j,l)(x^i)^2 \nabla_{X_q^k \wedge X_s^r}X_v^u +$$

$$2(p,j,l)\left(x^k\right)^2 \nabla_{X_q^i \wedge X_s^r}X_v^u - 2(r,j,l)(x^i)^2 \nabla_{X_q^p \wedge X_s^k}X_v^u + 2(r,j,l)\left(x^k\right)^2 \nabla_{X_q^p \wedge X_s^i}X_v^u +$$

$$+2(i,q,s)(x^p)^2 \nabla_{X_j^r \wedge X_l^k}X_v^u - 2(i,q,s)(x^r)^2 \nabla_{X_j^p \wedge X_l^k}X_v^u +$$

$$2(k,q,s)(x^p)^2 \nabla_{X_j^i \wedge X_l^r}X_v^u - 2(k,q,s)(x^r)^2 \nabla_{X_j^i \wedge X_l^p}X_v^u = 0,$$

where in the last step we have used the definition of ∇.   □

Proposition 10. *The derived almost Lie vector bundle $E_0^{(1)}$ is a Lie algebroid.*

Proof. Using Lemma 1 and Theorem 1, the conclusion follows.   □

We proceed now with the next example, where Theorem 1 can also be used also in the case when $E = E_1$ is not a skew symmetric algebroid.

Consider $M = \mathbb{R}^{2n+1}$ with coordinates $\left\{x^i, y^i, z\right\}_{i=\overline{1,n}}$ and the vector fields

$$X_i = \frac{\partial}{\partial x^i} - y^i \frac{\partial}{\partial z}, Y_i = \frac{\partial}{\partial y^i}, i = \overline{1,n}. \tag{10}$$

Their Lie brackets are given by

$$\left[X_i, Y_j\right] = \delta_{ij}\frac{\partial}{\partial z}, \left[X_i, X_j\right] = \left[Y_i, Y_j\right] = 0, i, j = \overline{1,n}.$$

Let E_1 be the vector bundle generated by $\left\{X_i, Y_j\right\}_{i,j=\overline{1,n}}$. The anchor $\rho_1 : E_1 \to T\mathbb{R}^{2n+1}$ is the natural inclusion. The corresponding bracket $[\cdot, \cdot]'$ on E_1 extends the following values on generators:

$$\left[X_i, Y_j\right]' = \left[X_i, X_j\right]' = \left[Y_i, Y_j\right]' = 0, i, j = \overline{1,n}.$$

We consider also the linear E_1-connection ∇ on E_1, which extends the following values on generators:

$$\nabla_{X_i}Y_j = \nabla_{X_i}X_j = \nabla_{Y_i}Y_j = 0, i, j = \overline{1,n}.$$

Notice that $\mathcal{D}'\left(X_i, Y_j\right) = \left[\rho_1(X_i), \rho_1(Y_j)\right] - \rho_1\left(\left[X_i, Y_j\right]'\right) = \left[X_i, Y_j\right] = \delta_{ij}\frac{\partial}{\partial z}$; thus, $\left(E_1, \rho_1, [\cdot, \cdot]'\right)$ is not a skew symmetric algebroid and the curvature R is not $\mathcal{F}(\mathbb{R}^{2n+1})$-linear in all arguments.

Proposition 11. *The derived almost Lie vector bundle $E_1^{(1)}$ is a Lie algebroid.*

Proof. Let us denote by ρ'' and $[\cdot, \cdot]''$ the anchor and the bracket on $E'' = E_1^{(1)}$, respectively.

For this anchor, we have $\rho''(X_i) = X_i$, $\rho''(Y_i) = Y_i$, $\rho''(X_i \wedge X_j) = \rho''(Y_i \wedge Y_j) = 0$, $\rho''(X_i \wedge Y_j) = \delta_{ij}\frac{\partial}{\partial z}$. The corresponding bracket gives:

$$\left[X_i, Y_j\right]'' = X_i \wedge Y_j, \ \left[X_i, X_j\right]'' = X_i \wedge X_j, \ \left[Y_i, Y_j\right]'' = Y_i \wedge Y_j, \ i, j = \overline{1,n},$$

and the other brackets that involve generators

$$\left\{ X_i, Y_j, X_i \wedge Y_j, X_i \wedge X_j, Y_i \wedge Y_j \right\} \tag{11}$$

vanish.

Considering $\mathcal{D}''(U, V) = [\rho''(U), \rho''(V)] - \rho''\left([U, V]''\right)$, $(\forall)U, V \in \Gamma(E'')$, one can check on the generators (11) that $\mathcal{D}'' = 0$; thus, E'' is a skew symmetric algebroid. Since both extended curvatures vanish on generators, by using Theorem 1, it follows that $E'' = E_1^{(1)}$ is a Lie algebroid. $\square$

5. Conclusions

A new construction providing Lie algebroids is considered in the paper.

Some relaxed conditions in the Lie algebroid definition give rise to other kinds of structures. For a general almost Lie vector bundle, the Jacobiator can be non-null or nonlinear. In this paper, we consider the derived almost Lie vector bundle $E^{(1)}$ of a given almost Lie vector bundle E, and we define two extended curvatures of a given linear E-connection ∇ with null torsion. We prove that if these two extended curvatures vanish, then $E^{(1)}$ is a Lie algebroid. Two given examples show that the result can be applied not only when E is a skew-symmetric algebroid, but also when E is not.

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