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Solving Quaternion Linear System Based on Semi-Tensor Product of Quaternion Matrices

Xueling Fan ^{1,2} , Ying Li ^{1,2,*}, Zhihong Liu ^{1,2} and Jianli Zhao ^{1,2}
¹ College of Mathematical Sciences, Liaocheng University, Liaocheng 252000, China; 2110100316@stu.lcu.edu.cn (X.F.); 2110100304@stu.lcu.edu.cn (Z.L.); zhaojl1964@126.com (J.Z.)

² Research Center of Semi-Tensor Product of Matrices: Theory and Applications, Liaocheng 252000, China

* Correspondence: liying@lcu.edu.cn; Tel.: +86-188-6520-9166

Abstract: In this paper, we use semi-tensor product of quaternion matrices, $\mathcal{L}$ -representation of quaternion matrices, and $\mathbf{GH}$ -representation of special quaternion matrices such as quaternion (anti)-centrosymmetric matrices to solve the special solutions of quaternion matrix equation. Based on semi-tensor product of quaternion matrices and the structure matrix of the multiplication of quaternion, we propose the vector representation operation conclusion of quaternion matrices, and study the different matrix representations of quaternion matrices. Then the problem of the quaternion matrix equation is transformed into the corresponding problem in the real number fields by using vector representation and $\mathcal{L}$ -representation of quaternion matrices, combined with the special structure of (anti)-centrosymmetric matrices, the independent elements are extracted by $\mathbf{GH}$ -representation method, so as to reduce the number of variables to be calculated and improve the calculation accuracy. Finally, the effectiveness of the method is verified by numerical examples, and the time comparison with the two existing algorithms is carried out. The algorithm in this paper is also applied in a centrosymmetric color digital image restoration model.



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1. Introduction

The symbols to be used in this paper are as follows: $\mathbb{R}/\mathbb{Q}$ represent the set of all the real numbers/quaternions, respectively. $\mathbb{R}^t$ represents the set of all real column vectors with t -dimension. $\mathbb{R}^{m \times n}/\mathbb{Q}^{m \times n}$ represent the set of all $m \times n$ real matrices/quaternion matrices, respectively. $\mathbb{S}^{n \times n}/\mathbb{AS}^{n \times n}/\mathbb{S}^{n \times n}/\mathbb{AS}^{n \times n}$ represent the set of all $n \times n$ real centrosymmetric matrices/real anti-centrosymmetric matrices/quaternion centrosymmetric matrices/quaternion anti-centrosymmetric matrices, respectively. In addition, I_n represents the unit matrix with n -dimension, $\delta_n^i (i = 1, 2, \dots, n)$ represents the i th column of I_n . $\bar{A}/A^T/A^H/A^\dagger$ represent the conjugate/transpose/conjugate transpose/Moore-Penrose inverse of matrix A . $\otimes$ represents the Kronecker product of matrices, $\|\cdot\|$ represents the Frobenius norm of a matrix or Euclidean norm of a vector.

Currently the numerical computation is not only a tool for scientific calculations, but also one of the ways to discover truths. However, the traditional matrix theory also has some shortages; for example, it has dimensional restriction and noncommutativity. Semi-tensor product of matrices proposed by Cheng [1] is different from the traditional matrix product. It does not need size matching conditions and can be used for any two matrices. It is designed to deal with higher-dimensional data as well as multilinear mappings. In a computer the higher-dimensional data can easily be treated without arranging the m into a cube or even higher-dimensional cuboid. Semi-tensor product of matrices is designed in such a way that the product rule can automatically search the proper position for each

factor of multiplier. At present, semi-tensor product of matrices is widely used in biological system and life science [2,3], game theory [4,5], graph theory and formation control [6,7], fuzzy control [8,9], coding theory, and algorithm implementation [10,11]. In addition, some scholars proposed a new quaternion real vector representation method [12,13] based on semi-tensor product of matrices, and applied this method to the solution of quaternion linear system. In this paper, some new conclusions of semi-tensor product of quaternion matrices are proposed, which will be used to solve quaternion linear systems.

Quaternion is a hypercomplex number composed of a scalar and a vector, which has the dual properties of real number and complex number. Due to the rapid development of computer graphics [14], robot and other fields [15,16], quaternion has been more and more widely used in computer animation, robot trajectory planning [17], modeling [18], rendering and three-dimensional fractal display. The application of quaternion matrix in color digital images is becoming more and more important and extensive [19,20]. Color digital image restoration is usually modeled as the solution of quaternion matrix equation.

Matrix equations have wide applications in many spheres. These real, complex and quaternion matrix equations have attracted extensive attention. As a special matrix equation, quaternion matrix equation has been widely integrated into computer science [21], signals [22], statistics [23], and color image processing [24]. Because quaternion does not satisfy the commutativity of multiplication, the quaternion matrix equation is usually transformed into a familiar problem of real matrix equation or complex matrix equation by real representation or complex representation, so as to simplify the operation of matrix equation. Many scholars have discussed different solutions to different equations with the help of these methods. For example, using the real representation matrix of quaternion matrices, ref. [25] obtained the expressions of the minimal norm least squares solution for the quaternion matrix equation $AXB + CXD = E$; ref. [26] investigated the minimal norm least squares η -(anti)-Hermitian solution of quaternion matrix equation $AXB + CYD = E$; ref. [27] discussed the minimal norm least squares (anti)- j -self-conjugate solution on quaternion matrix equation $X - A\hat{X}B = C$; in addition, ref. [28] used the complex representation matrix of quaternion matrices to study the η -(anti)-Hermitian solution of quaternion matrix equation $AXB + CYD = E$; ref. [29] derive the expressions of the least squares solution, pure imaginary solution, real solution with the least norm for the quaternion matrix equation $AX = B$ by using the complex representation matrix of quaternion matrices. Some scholars have also devoted themselves to the study of quaternion matrix equations by using Cramer's rules [30–32], iterative algorithms [33–36] or rank method [37–40].

Definition 1 ([41]). If $X = (x_{ij}) \in \mathbb{Q}^{n \times n}$ satisfies:

$$x_{ij} = x_{n-i+1, n-j+1}, (i, j = 1, \dots, n),$$

then X is called a quaternion centrosymmetric matrix. If $X = (x_{ij}) \in \mathbb{Q}^{n \times n}$ satisfies:

$$x_{ij} = -x_{n-i+1, n-j+1}, (i, j = 1, \dots, n),$$

then X is called a quaternion anti-centrosymmetric matrix.

As two special kinds of matrices, (anti)-centrosymmetric matrices are applied broadly in the fields of statistical analysis and matrix countermeasures information theory, linear system theory and numerical analysis, and some matrices with special rules of elements, such as (anti)-centrosymmetric matrices. We want to extract the independent elements of the matrix to remove the redundancy and reduce the complexity of solving the matrix equation. The **H**-representation [42] method perfectly realizes our idea.

This paper presents the (anti)-centrosymmetric solutions of quaternion matrix equation

$$\sum_{i=1}^k A_i X B_i = C \quad (1)$$

by using semi-tensor product of quaternion matrices, $\mathcal{L}$ -representation and $\mathbf{GH}$ -representation.

Problem 1 Let $A_i \in \mathbb{Q}^{m \times n}$, $B_i \in \mathbb{Q}^{n \times p}$, ($i = 1, \dots, k$), $C \in \mathbb{Q}^{m \times p}$, and

$$M_S = \left\{ X \in \mathbb{S}^{n \times n} \left\| \sum_{i=1}^k A_i X B_i - C \right\| = \min \right\}.$$

Find out $X_S \in M_S$, such that

$$\|X_S\| = \min_{X \in M_S} \|X\|.$$

X_S is called the minimal norm least squares centrosymmetric solution of quaternion matrix Equation (1). If $\min = 0$, X_S is called the minimal norm centrosymmetric solution of quaternion matrix Equation (1).

Problem 2 Let $A_i \in \mathbb{Q}^{m \times n}$, $B_i \in \mathbb{Q}^{n \times p}$, ($i = 1, \dots, k$), $C \in \mathbb{Q}^{m \times p}$, and

$$M_A = \left\{ X \in \mathbb{AS}^{n \times n} \left\| \sum_{i=1}^k A_i X B_i - C \right\| = \min \right\}.$$

Find out $X_A \in M_A$, such that

$$\|X_A\| = \min_{X \in M_A} \|X\|.$$

X_A is called the minimal norm least squares anti-centrosymmetric solution of quaternion matrix Equation (1). If $\min = 0$, X_A is called the minimal norm anti-centrosymmetric solution of quaternion matrix Equation (1).

Several new conclusions on semi-tensor product of quaternion matrices are presented in this article. By using semi-tensor product of quaternion matrices, quaternion matrix equations can be analyzed by vector representation directly. Under the structure matrix of the multiplication of quaternion, we establish different matrix representations of quaternion matrices by semi-tensor product of quaternion matrices, in this case, we define the definition of $\mathcal{L}$ -representation. Employing vector representation of quaternion matrices and combining $\mathcal{L}$ -representation of quaternion matrices with $\mathbf{GH}$ -representation, several types of special minimal norm solutions to quaternion equation $\sum_{i=1}^k A_i X B_i = C$ are presented, along with the necessary and sufficient conditions of compatibility. Using $\mathbf{GH}$ -representation method, we can remove the redundancy and reduce the complexity of the problem by identifying the independent elements of a special matrix. It can be seen that $\mathbf{GH}$ -representation simplifies solutions to quaternion matrix equations in a simple and effective manner.

The following are the main sections of this article: In Section 2, the fundamentals of quaternion and semi-tensor product of quaternion matrices are covered. In Section 3, the vector representation conclusion of quaternion matrices is given, and combined with the structure matrix, the definition of $\mathcal{L}$ -representation of quaternion matrices is proposed. In Section 4, $\mathbf{H}$ -representation of several special matrices are given, and the definition of $\mathbf{GH}$ -representation of special quaternion matrices is proposed. In Section 5, the necessary and sufficient conditions for the minimal norm solution and compatibility of the above problems are explored. In Section 6, the corresponding algorithm and numerical examples are shown to verify the effectiveness of the method, and we give the time comparison between the algorithm in this paper and the algorithms in references [43,44]. In Section 7, the research of centrosymmetric color digital image restoration is given. In Section 8, a brief summary is made of the full text.

2. Preliminaries

2.1. Quaternion and Quaternion Matrices

This part mainly introduces the basic knowledge of quaternion. For more information, please refer to the literature [25–27].

Definition 2. A quaternion x can be uniquely expressed as

$$x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \in \mathbb{Q},$$

where $x_s \in \mathbb{R}, s = 0, 1, 2, 3$, and the three imaginary units $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfy $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \mathbf{ki} = -\mathbf{ik} = \mathbf{j}$. The conjugate of x is defined as

$$\bar{x} = x_0 - x_1\mathbf{i} - x_2\mathbf{j} - x_3\mathbf{k} \in \mathbb{Q}.$$

A quaternion matrix X can be uniquely expressed as $X = X_0 + X_1\mathbf{i} + X_2\mathbf{j} + X_3\mathbf{k} \in \mathbb{Q}^{m \times n}$, where $X_s \in \mathbb{R}^{m \times n}, s = 0, 1, 2, 3$. The conjugate of X is defined as $\bar{X} = X_0 - X_1\mathbf{i} - X_2\mathbf{j} - X_3\mathbf{k} \in \mathbb{Q}^{m \times n}$.

Definition 3 ([24]). The norm of a quaternion $x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \in \mathbb{Q}$ is defined as

$$\|x\| = \sqrt{|x_0|^2 + |x_1|^2 + |x_2|^2 + |x_3|^2} = x\bar{x},$$

and the Frobenius norm of $X = X_0 + X_1\mathbf{i} + X_2\mathbf{j} + X_3\mathbf{k} \in \mathbb{Q}^{m \times n}$ is defined as

$$\|X\| = \sqrt{\|X_0\|^2 + \|X_1\|^2 + \|X_2\|^2 + \|X_3\|^2}.$$

2.2. Semi-Tensor Product of Quaternion Matrices

In this section, some basic knowledge about semi-tensor product of quaternion matrices is given. For more details of semi-tensor product of matrices on real number fields, please refer to the literature [1,45,46].

Definition 4. Suppose $A \in \mathbb{Q}^{m \times n}, B \in \mathbb{Q}^{p \times q}$, the semi-tensor product of A and B is denoted by

$$A \ltimes B = (A \otimes I_{t/n})(B \otimes I_{t/p}),$$

where $t = \text{lcm}(n, p)$ is the least common multiple of n and p . If $n = p$, the semi-tensor product reduces to the traditional matrix product.

Example 1. Suppose $A = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & 1 & 4 & 1 \\ 5 & 1 & 1 & 1 \\ 3 & 4 & 5 & 3 \\ 1 & 1 & 2 & 2 \end{bmatrix}$. First, we block matrix A and B into

$$A = \left[\begin{array}{c|c} 3 & 0 \\ \hline 2 & 1 \end{array} \right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B = \left[\begin{array}{cc|cc} 4 & 1 & 4 & 1 \\ 5 & 1 & 1 & 1 \\ \hline 3 & 4 & 5 & 3 \\ 1 & 1 & 2 & 2 \end{array} \right] = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

Then the semi-tensor product of A and B is

$$A \ltimes B = (A \otimes I_2)B = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 1 \\ 5 & 1 & 1 & 1 \\ 3 & 4 & 5 & 3 \\ 1 & 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 3 & 12 & 3 \\ 15 & 3 & 3 & 3 \\ 11 & 6 & 13 & 5 \\ 11 & 3 & 4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} \times B_{11} + A_{12} \times B_{21} & A_{11} \times B_{12} + A_{12} \times B_{22} \\ A_{21} \times B_{11} + A_{22} \times B_{21} & A_{21} \times B_{12} + A_{22} \times B_{22} \end{bmatrix}.$$

Theorem 1. Suppose $\alpha, \beta \in \mathbb{R}$, A, B, C be quaternion matrices, then
(1) (Associative rule)

$$(A \times B) \times C = A \times (B \times C).$$

(2) (Distributive rule)

$$A \times (\alpha B + \beta C) = \alpha A \times B + \beta A \times C,$$

$$(\alpha B + \beta C) \times A = \alpha B \times A + \beta C \times A.$$

(3) (Conjugate Transpose)

$$(A \times B)^H = B^H \times A^H.$$

Definition 5 ([46]). A swap matrix $W_{[m,n]}$ is a $mn \times mn$ matrix, which is defined as

$$W_{[m,n]} = [I_n \otimes \delta_m^1, I_n \otimes \delta_m^2, \dots, I_n \otimes \delta_m^m].$$

The properties of swap matrix are as follows, which facilitates the calculation of matrix.

Theorem 2. (1) Suppose $A \in \mathbb{Q}^{m \times n}$, then

$$W_{[m,n]} V_r(A) = V_c(A); W_{[n,m]} V_c(A) = V_r(A).$$

(2) Suppose $A \in \mathbb{Q}^{s \times t}$, then for any integer $m > 0$ have

$$W_{[s,m]} \times A \times W_{[m,t]} = I_m \otimes A.$$

Example 2. Assume $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathbb{Q}^{2 \times 2}$, $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \in \mathbb{Q}^{3 \times 2}$, then $m = n = 2$,
 $s = 3, t = 2$. Hence, we have

$$W_{[3,2]} = [I_2 \otimes \delta_3^1, I_2 \otimes \delta_3^2, I_2 \otimes \delta_3^3] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix};$$

$$W_{[2,2]} = [I_2 \otimes \delta_2^1, I_2 \otimes \delta_2^2] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then

$$W_{[3,2]} \times \overline{B} \times W_{[2,2]} \times \overline{A} = (I_2 \otimes \overline{B}) \times \overline{A} = \begin{bmatrix} \overline{b_{11}} & \overline{b_{12}} & 0 & 0 \\ \overline{b_{21}} & \overline{b_{22}} & 0 & 0 \\ \overline{b_{31}} & \overline{b_{32}} & 0 & 0 \\ 0 & 0 & \overline{b_{11}} & \overline{b_{12}} \\ 0 & 0 & \overline{b_{21}} & \overline{b_{22}} \\ 0 & 0 & \overline{b_{31}} & \overline{b_{32}} \end{bmatrix} \times \begin{bmatrix} \overline{a_{11}} & \overline{a_{12}} \\ \overline{a_{21}} & \overline{a_{22}} \end{bmatrix}$$

$$= \begin{bmatrix} \overline{b_{11}a_{11}} & \overline{b_{12}a_{11}} & \overline{b_{11}a_{12}} & \overline{b_{12}a_{12}} \\ \overline{b_{21}a_{11}} & \overline{b_{22}a_{11}} & \overline{b_{21}a_{12}} & \overline{b_{22}a_{12}} \\ \overline{b_{31}a_{11}} & \overline{b_{32}a_{11}} & \overline{b_{31}a_{12}} & \overline{b_{32}a_{12}} \\ \overline{b_{11}a_{21}} & \overline{b_{12}a_{21}} & \overline{b_{11}a_{22}} & \overline{b_{12}a_{22}} \\ \overline{b_{21}a_{21}} & \overline{b_{22}a_{21}} & \overline{b_{21}a_{22}} & \overline{b_{22}a_{22}} \\ \overline{b_{31}a_{21}} & \overline{b_{32}a_{21}} & \overline{b_{31}a_{22}} & \overline{b_{32}a_{22}} \end{bmatrix} = \begin{bmatrix} \overline{a_{11}b_{11}} & \overline{a_{11}b_{12}} & \overline{a_{12}b_{11}} & \overline{a_{12}b_{12}} \\ \overline{a_{11}b_{21}} & \overline{a_{11}b_{22}} & \overline{a_{12}b_{21}} & \overline{a_{12}b_{22}} \\ \overline{a_{11}b_{31}} & \overline{a_{11}b_{32}} & \overline{a_{12}b_{31}} & \overline{a_{12}b_{32}} \\ \overline{a_{21}b_{11}} & \overline{a_{21}b_{12}} & \overline{a_{22}b_{11}} & \overline{a_{22}b_{12}} \\ \overline{a_{21}b_{21}} & \overline{a_{21}b_{22}} & \overline{a_{22}b_{21}} & \overline{a_{22}b_{22}} \\ \overline{a_{21}b_{31}} & \overline{a_{21}b_{32}} & \overline{a_{22}b_{31}} & \overline{a_{22}b_{32}} \end{bmatrix} = \overline{A \otimes B}.$$

3. Main Conclusions

3.1. Vector Representation of Quaternion Matrices

As we know, quaternion multiplication does not satisfy the commutative law,

$$V_c(AXB) = (B^T \otimes A)V_c(X)$$

is not tenable on quaternion. Therefore, some scholars [24,26,28,29] mainly study the quaternion matrix equation based on the real representation matrix and complex representation matrix of quaternion matrices. However, we can find a new straightening result on quaternion according to the property of quaternion conjugation. Then some straightening conclusions of semi-tensor product of quaternion matrices are given below, which will be used to solve quaternion matrix equation.

Definition 6. For $A = (a_{ij}) \in \mathbb{Q}^{m \times n}$, the column vector representation of quaternion matrix A is defined as

$$V_c(A) = (a_{11}, \dots, a_{m1}, a_{12}, \dots, a_{m2}, \dots, a_{1n}, \dots, a_{mn})^T,$$

the row vector representation of quaternion matrix A is defined as

$$V_r(A) = (a_{11}, \dots, a_{1n}, a_{21}, \dots, a_{2n}, \dots, a_{m1}, \dots, a_{mn})^T.$$

Theorem 3. Suppose $A \in \mathbb{Q}^{m \times n}$, $X \in \mathbb{Q}^{n \times q}$, $Y \in \mathbb{Q}^{p \times m}$, then

$$\begin{aligned} (1) & V_r(AX) = A \ltimes V_r(X), V_c(AX) = (I_q \otimes A) \ltimes V_c(X). \\ (2) & V_c(\overline{YA}) = A^H \ltimes V_c(\overline{Y}), V_r(\overline{YA}) = (I_p \otimes A^H) \ltimes V_r(\overline{Y}). \end{aligned}$$

Proof. (1) For $V_r(AX) = A \ltimes V_r(X)$. Suppose $C = AX$, $a^i (i = 1, \dots, m)$ represents the i th row of matrix A , $x^j (j = 1, \dots, n)$ represents the j th row of matrix X , $c^i (i = 1, \dots, m)$ represents the i th row of matrix C , then the i th block of $A \ltimes V_r(X)$ is

$$a^i \ltimes V_r(X) = a^i \ltimes \begin{bmatrix} (x^1)^T \\ \vdots \\ (x^n)^T \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^n a_{ik}x_{k1} \\ \vdots \\ \sum_{k=1}^n a_{ik}x_{kq} \end{bmatrix} = (c^i)^T,$$

then we have $V_r(AX) = A \ltimes V_r(X)$.

By the properties of the swap matrix and $V_r(AX) = A \ltimes V_r(X)$, then,

$$\begin{aligned} V_c(AX) &= W_{[m,q]} \ltimes V_r(AX) = W_{[m,q]} \ltimes A \ltimes V_r(X) \\ &= W_{[m,q]} \ltimes A \ltimes W_{[q,n]} \ltimes V_c(X) \\ &= (I_q \otimes A) \ltimes V_c(X). \end{aligned}$$

(2) We prove $V_c(\overline{YA}) = A^H \ltimes V_c(\overline{Y})$. Let $A = [a_1, a_2, \dots, a_n]$, $a_i (i = 1, 2, \dots, n)$ represents the i th column of matrix A , $Y = [y_1, y_2, \dots, y_m]$, $y_j (j = 1, 2, \dots, m)$ represents the j th column of matrix Y , then

$$V_c(\overline{YA}) = V_c(\overline{Ya_1}, \dots, \overline{Ya_n}) = \begin{bmatrix} \overline{Ya_1} \\ \vdots \\ \overline{Ya_n} \end{bmatrix},$$

by the conjugate properties of quaternions, we have

$$\begin{aligned} \overline{Ya_i} &= \overline{y_1 a_{1i}} + \overline{y_2 a_{2i}} + \dots + \overline{y_m a_{mi}} \\ &= \overline{a_{1i}} \overline{y_1} + \overline{a_{2i}} \overline{y_2} + \dots + \overline{a_{mi}} \overline{y_m} \\ &= [\overline{a_{1i}} I_p, \dots, \overline{a_{mi}} I_p] V_c(\overline{Y}). \end{aligned}$$

So

$$\begin{aligned} V_c(\overline{YA}) &= \begin{bmatrix} \overline{a_{11}} I_p & \overline{a_{21}} I_p & \dots & \overline{a_{m1}} I_p \\ \overline{a_{12}} I_p & \overline{a_{22}} I_p & \dots & \overline{a_{m2}} I_p \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a_{1n}} I_p & \overline{a_{2n}} I_p & \dots & \overline{a_{mn}} I_p \end{bmatrix} V_c(\overline{Y}) \\ &= (A^H \otimes I_p) \ltimes V_c(\overline{Y}) = A^H \ltimes V_c(\overline{Y}). \end{aligned}$$

By the properties of the swap matrix and $V_c(\overline{YA}) = A^H \ltimes V_c(\overline{Y})$, we obtain

$$\begin{aligned} V_r(\overline{YA}) &= W_{[n,p]} \ltimes V_c(\overline{YA}) = W_{[n,p]} \ltimes A^H \ltimes V_c(\overline{Y}) \\ &= W_{[n,p]} \ltimes A^H \ltimes W_{[p,m]} \ltimes V_r(\overline{Y}) \\ &= (I_p \otimes A^H) \ltimes V_r(\overline{Y}). \end{aligned}$$

□

3.2. $\mathcal{L}$ -Representation of Quaternion Matrices

Our main work in this section is to study the matrix representation of quaternion matrices by using the structure matrix of the multiplication of quaternion.

Definition 7. [1] Let $V_i (i = 1, 2, \dots, k)$ be n_i -dimensional vector spaces with $e_1^i, \dots, e_{n_i}^i$ as the fixed bases of V_i , and $\phi : V_1 \ltimes \dots \ltimes V_k \rightarrow V_0$ be a multilinear mapping. Denote

$$\phi(e_{i_1}^1, \dots, e_{i_k}^k) = \sum_{i_0=1}^{n_0} \mu_{i_1, i_2, \dots, i_k}^{i_0} e_{i_0}^{n_0}, i_j = 1, \dots, n_j, j = 1, \dots, k.$$

Then the matrix

$$M_\phi^1 = \begin{bmatrix} \mu_{11 \dots 1}^1 & \dots & \mu_{11 \dots n_k}^1 & \dots & \mu_{n_1 n_2 \dots n_{k-1} 1}^1 & \dots & \mu_{n_1 n_2 \dots n_k}^1 \\ \mu_{11 \dots 1}^2 & \dots & \mu_{11 \dots n_k}^2 & \dots & \mu_{n_1 n_2 \dots n_{k-1} 1}^2 & \dots & \mu_{n_1 n_2 \dots n_k}^2 \\ \vdots & & \vdots & & \vdots & & \vdots \\ \mu_{11 \dots 1}^{n_0} & \dots & \mu_{11 \dots n_k}^{n_0} & \dots & \mu_{n_1 n_2 \dots n_{k-1} 1}^{n_0} & \dots & \mu_{n_1 n_2 \dots n_k}^{n_0} \end{bmatrix}$$

is defined as the right structure matrix of ϕ . The matrix

$$M_{\phi}^2 = \begin{bmatrix} \mu_{11\dots 1}^1 & \cdots & \mu_{n_1 1\dots 1}^1 & \cdots & \mu_{1n_2\dots n_{k-1}n_k}^1 & \cdots & \mu_{n_1 n_2\dots n_k}^1 \\ \mu_{11\dots 1}^2 & \cdots & \mu_{n_1 1\dots 1}^2 & \cdots & \mu_{1n_2\dots n_{k-1}n_k}^2 & \cdots & \mu_{n_1 n_2\dots n_k}^2 \\ \vdots & & \vdots & & \vdots & & \vdots \\ \mu_{11\dots 1}^{n_0} & \cdots & \mu_{n_1 1\dots 1}^{n_0} & \cdots & \mu_{1n_2\dots n_{k-1}n_k}^{n_0} & \cdots & \mu_{n_1 n_2\dots n_k}^{n_0} \end{bmatrix}$$

is defined as the left structure matrix of ϕ . The left and right structure matrices are collectively called structure matrices.

Remark 1. For a multi-dimensional data, we can sort it by certain indices. The left structure matrix and right structure matrix given in Definition 7 are sorted according to different indexes.

Example 3. For $x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, $y = y_0 + y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k} \in \mathbb{Q}$, then fix an ordered basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, the basis is normalized to

$$1 \sim \delta_4^1, \mathbf{i} \sim \delta_4^2, \mathbf{j} \sim \delta_4^3, \mathbf{k} \sim \delta_4^4.$$

Each quaternion can be represented as a column vector:

$$x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \sim (x_0, x_1, x_2, x_3)^T = x^r.$$

Then the right structure matrix of the multiplication of quaternions can be obtained as

$$M_{\mathbb{Q}}^1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

In addition, the left structure matrix of the multiplication of quaternions can be obtained as

$$M_{\mathbb{Q}}^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

And we have

$$(xy)^r = M_{\mathbb{Q}}^1 \times x^r \times y^r = M_{\mathbb{Q}}^2 \times y^r \times x^r.$$

In the case of different basis standardization, the structure matrix of the multiplication of quaternion is also diverse. We systematically define the matrix representation of quaternion matrices by using the structure matrix of the multiplication of quaternion and semi-tensor product of quaternion matrices.

Definition 8. Suppose $X = X_0 + X_1\mathbf{i} + X_2\mathbf{j} + X_3\mathbf{k} \in \mathbb{Q}^{m \times n}$ be a quaternion matrix, where $X_t \in \mathbb{R}^{m \times n}$ ($t = 0, 1, 2, 3$), denote $\hat{X} = [\pm X_0^T \quad \pm X_1^T \quad \pm X_2^T \quad \pm X_3^T]^T$. Suppose Φ is a mapping such that $\Phi : X \mapsto \Phi(X) \in \mathbb{R}^{4m \times 4n}$, $\Phi(X)$ can be represented as

$$\Phi(X) = M_{\mathbb{Q}} \times (I_4 \otimes \hat{X}),$$

$\Phi(X)$ is called the matrix representation of quaternion matrix X . Furthermore, the first column of $\Phi(X)$ is defined as

$$\Phi_c(X) = \Phi(X) \times \delta_4^1.$$

Remark 2. It can be seen from the definition that $\Phi(X)$ and $\Phi_c(X)$ are determined by $\widehat{X}$ and M_Q , that is, when $\widehat{X}$ and M_Q are determined, $\Phi(X)$ and $\Phi_c(X)$ are also unique and certain.

Example 4. Let $X = X_0 + X_1\mathbf{i} + X_2\mathbf{j} + X_3\mathbf{k} \in \mathbb{Q}^{n \times n}$, by M_Q^1 defined in Example 3, the matrix representation of quaternion matrix X can be expressed as

$$\Phi^1(X) = M_Q^1 \times \left(I_4 \otimes \begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix} \right) = \begin{bmatrix} X_0 & -X_1 & -X_2 & -X_3 \\ X_1 & X_0 & X_3 & -X_2 \\ X_2 & -X_3 & X_0 & X_1 \\ X_3 & X_2 & -X_1 & X_0 \end{bmatrix}.$$

If we select $M_Q = M_Q^2$, then

$$\Phi^2(X) = M_Q^2 \times \left(I_4 \otimes \begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix} \right) = \begin{bmatrix} X_0 & -X_1 & -X_2 & -X_3 \\ X_1 & X_0 & -X_3 & X_2 \\ X_2 & X_3 & X_0 & -X_1 \\ X_3 & -X_2 & X_1 & X_0 \end{bmatrix}.$$

The matrix representation method in reference [47] is the matrix representation $\Phi^2(X)$ in Example 4. Furthermore, the matrix representation of quaternion matrices plays an important role in many aspects of quaternion research.

Definition 9. Suppose $X \in \mathbb{Q}^{m \times n}$, $Y \in \mathbb{Q}^{n \times p}$, $\Phi(X)$ is called $\mathcal{L}$ -representation of quaternion matrices if and only if $\Phi(X)$ satisfies the following equations,

$$\begin{aligned} (1) \Phi(XY) &= \Phi(X)\Phi(Y), \\ (2) \Phi_c(XY) &= \Phi(X)\Phi_c(Y). \end{aligned}$$

It is easy to verify that the two matrix representations given in Example 4, $\Phi^1(X)$ does not satisfy the two conditions of $\mathcal{L}$ -representation, but the matrix representation given by $\Phi^2(X)$ does. It is clear that Definition 9 has the following equivalent form.

Definition 10. Suppose $X \in \mathbb{Q}^{m \times n}$, $Y \in \mathbb{Q}^{n \times p}$, $\Phi(X)$ is called $\mathcal{L}$ -representation of quaternion matrices if and only if $\Phi(X)$ satisfies the following equations,

$$\begin{aligned} (1) (M_Q \otimes I_m)(I_4 \otimes \widehat{XY}) &= (M_Q \otimes I_m)(M_Q \otimes \widehat{X})(I_4 \otimes \widehat{Y}), \\ (2) (M_Q \otimes I_m)(\delta_4^1 \otimes \widehat{XY}) &= (M_Q \otimes I_m)(M_Q \otimes \widehat{X})(\delta_4^1 \otimes \widehat{Y}). \end{aligned}$$

4. GH-Representation of Quaternion Matrices

In this section, we will first introduce the definition of **H**-representation, and then give examples of **H**-representation of special matrices.

Definition 11 ([42]). Consider a q -dimensional real matrix subspace $\mathbb{X} \subset \mathbb{R}^{n \times n}$ over the field $\mathbb{R}$. Assume that $e_1, e_2, \dots, e_q$ form the basis of $\mathbb{X}$, and define $H = [V_c(e_1), V_c(e_2), \dots, V_c(e_q)]$. For each $X \in \mathbb{X}$, if we express $\Psi(X) = V_c(X)$ in the form of

$$\Psi(X) = H\tilde{X},$$

with a $q \times 1$ vector $\tilde{X} = (x_1, x_2, \dots, x_q)^T$ and $X = \sum_{i=1}^q x_i e_i$, then $H\tilde{X}$ is called an **H**-representation of $\Psi(X)$, and H is called an **H**-representation matrix of $\Psi(X)$.

From the definition of quaternion (anti)-centrosymmetric matrices, we can know that quaternion (anti)-centrosymmetric matrices is closely related to real (anti)-centrosymmetric

matrices. In the following, we take real (anti)-centrosymmetric matrices as examples to give their **H**-representation.

Example 5. Let $\mathbb{X} = \mathbf{S}^{3 \times 3}$, $X = (x_{ij}) \in \mathbb{X}$, and then $\dim(\mathbb{X}) = 5$. If we select a basis of $\mathbb{X}$ as

$$e_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, e_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, e_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is easy to compute

$$\Psi(X) = V_c(X) = (x_{11}, x_{21}, x_{31}, x_{12}, x_{22}, x_{12}, x_{31}, x_{21}, x_{11})^T,$$

and

$$\tilde{X} = (x_{11}, x_{21}, x_{31}, x_{12}, x_{22})^T,$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

Example 6. Let $\mathbb{X} = \mathbf{AS}^{3 \times 3}$, $X = (x_{ij}) \in \mathbb{X}$, and then $\dim(\mathbb{X}) = 4$. If we select a basis of $\mathbb{X}$ as

$$e_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, e_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

It is easy to compute

$$\Psi(X) = V_c(X) = (x_{11}, x_{21}, x_{31}, x_{12}, 0, -x_{12}, -x_{31}, -x_{21}, -x_{11})^T,$$

and

$$\tilde{X} = (x_{11}, x_{21}, x_{31}, x_{12})^T,$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}^T.$$

Then, we select the standard basis for centrosymmetric and anti-centrosymmetric matrices, and give the **H**-representation matrices, respectively.

1. If $\mathbb{X} = \mathbf{S}^{n \times n}$, we select a standard basis as

$$\{E_1, E_2, \dots, E_\alpha\},$$

where $E_i = \{(e_{pq}) | e_{l(k+1)} = e_{(n+1-l)(n-k)} = 1, i = kn + l (0 \leq k \leq l \leq n; i = 1, 2, \dots, \alpha)\}$,

$\alpha = \begin{cases} \frac{n^2+1}{2} & (\text{if } n \text{ is odd}) \\ \frac{n^2}{2} & (\text{if } n \text{ is even}) \end{cases}$. Based on above standard basis, for any $X \in \mathbb{X}$, we have

$$\tilde{X} = (x_1, x_2, \dots, x_\alpha)^T$$

and

$$H_s = [V_c(E_1), V_c(E_2), \dots, V_c(E_\alpha)] \in \mathbb{R}^{n^2 \times \alpha}.$$

2. If $\mathbb{X} = \mathbf{AS}^{n \times n}$, we select a standard basis as

$$\{F_1, F_2, \dots, F_\beta\},$$

where $F_i = \{(f_{pq}) | f_{l(k+1)} = -f_{(n+1-l)(n-k)} = 1, i = kn + l (0 \leq k \leq l \leq n; i = 1, 2, \dots, \beta)\}$,

$$\beta = \begin{cases} \frac{n^2-1}{2} & (\text{if } n \text{ is odd}) \\ \frac{n^2}{2} & (\text{if } n \text{ is even}) \end{cases}.$$
 Based on above standard basis, for any $X \in \mathbb{X}$, we have

$$\tilde{X} = (x_1, x_2, \dots, x_\beta)^T$$

and

$$H_a = [V_c(F_1), V_c(F_2), \dots, V_c(F_\beta)] \in \mathbb{R}^{n^2 \times \beta}.$$

Note that $\Psi(X)$ is a column vector formed by all elements of matrix X . For the sake of clarity, we denote the **H**-representation matrix corresponding to $\mathbb{X} = \mathbf{S}^{n \times n}$ by H_s , the **H**-representation matrix corresponding to $\mathbb{X} = \mathbf{AS}^{n \times n}$ by H_a .

Theorem 4. For an $n^2 \times 1$ vector α_1 , if $\Psi^{-1}(\alpha_1) \in \mathbf{S}^{n \times n}$, then there exists an $\alpha \times 1$ vector β_1 , such that $\alpha_1 = H_s \beta_1$. For an $n^2 \times 1$ vector α_2 , if $\Psi^{-1}(\alpha_2) \in \mathbf{AS}^{n \times n}$, then there exists an $\beta \times 1$ vector β_2 , such that $\alpha_2 = H_a \beta_2$.

H-representation prompt us to define **GH**-representation on quaternion matrices.

Definition 12. Consider a quaternion matrix subspace $\mathbb{X} \subset \mathbb{Q}^{n \times n}$, for each $X = X_0 + X_1 \mathbf{i} + X_2 \mathbf{j} + X_3 \mathbf{k} \in \mathbb{X}$, let $S = \{X_0, X_1, X_2, X_3\}$. A permutation σ on S is a one-to-one mapping from S to S , denote $\mathcal{X} = [\pm\sigma(X_0) \pm\sigma(X_1) \pm\sigma(X_2) \pm\sigma(X_3)]$. If we express $\Psi(X) = V_c(\mathcal{X})$ in the form of

$$\Psi(X) = H_G \tilde{X},$$

$$\text{where } H_G = \begin{bmatrix} H_{\sigma(X_0)} & & & \\ & H_{\sigma(X_1)} & & \\ & & H_{\sigma(X_2)} & \\ & & & H_{\sigma(X_3)} \end{bmatrix}, \tilde{X} = \widetilde{V_c(\mathcal{X})} \text{ represents a permutation of}$$

independent elements for each part of $V_c(\mathcal{X})$. Then $H_G \tilde{X}$ is called a **GH**-representation of $\Psi(X)$, and H_G is called a **GH**-representation matrix of $\Psi(X)$.

5. The Solutions of Problem 1 and Problem 2

In order to obtain the solution of the quaternion matrix Equation (1), we begin with the following Lemmas.

Lemma 1 ([48]). The least squares solution of the linear system of equations $Ax = b$, with $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ can be represented as

$$x = A^\dagger b + (I - A^\dagger A)y,$$

where $y \in \mathbb{R}^n$ is an arbitrary vector. The minimal norm least squares solution of the linear system of equations $Ax = b$ is $A^\dagger b$.

Lemma 2 ([48]). The linear system of equations $Ax = b$, with $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, has a solution $x \in \mathbb{R}^n$ if and only if

$$AA^\dagger b = b.$$

In case that it has the general solution

$$x = A^\dagger b + (I - A^\dagger A)y,$$

where $y \in \mathbb{R}^n$ is an arbitrary vector. The minimal norm solution of the linear system of equations $Ax = b$ is $A^\dagger b$.

We select $\widehat{X} = [X_0^T \ X_1^T \ X_2^T \ X_3^T]^T$ for $X = X_0 + X_1\mathbf{i} + X_2\mathbf{j} + X_3\mathbf{k} \in \mathbb{Q}^{n \times n}$ as an example in the following solving process, in this case, the matrix representation we obtain is the form of $\Phi^2(X)$ in Example 4. Based on our earlier discussion, we now turn our attention to Problem 1. We obtain the necessary and sufficient condition of the existence of centrosymmetric solutions of quaternion matrix Equation (1), we obtain the following Theorem.

Theorem 5. Suppose $A_i \in \mathbb{Q}^{m \times n}$, $B_i \in \mathbb{Q}^{n \times p}$, ($i = 1, \dots, k$), $C \in \mathbb{Q}^{m \times p}$, then the set M_S of Problem 1 can be represented as

$$M_S = \left\{ X \in \mathbb{S}^{n \times n} \mid \Phi_c(V_c(X)) = H'_s R_1^\dagger \Phi_c(V_c(C)) + H'_s (I_{4\alpha} - R_1^\dagger R_1) y \right\}, \quad (2)$$

where y is an arbitrary vector with suitable dimension. Then, the minimal norm least squares centrosymmetric solution X_S of quaternion matrix Equation (1) satisfies

$$\Phi_c(V_c(X_S)) = H'_s R_1^\dagger \Phi_c(V_c(C)), \quad (3)$$

$$\text{where } H'_s = \begin{bmatrix} H_s & & & \\ & H_s & & \\ & & H_s & \\ & & & H_s \end{bmatrix}_{4n^2 \times 4\alpha}, \quad K_n = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & -I_n & 0 & 0 \\ 0 & 0 & -I_n & 0 \\ 0 & 0 & 0 & -I_n \end{bmatrix}, \quad R_1 = \sum_{i=1}^k \Phi(I_p \otimes A_i) K_{np} \Phi(B_i^H \otimes I_n) K_{n^2} H'_s.$$

Proof. For $X = X_0 + X_1\mathbf{i} + X_2\mathbf{j} + X_3\mathbf{k} \in \mathbb{S}^{n \times n}$, from Theorem 3, Theorem 4 and the definition of GH-representation, we can obtain

$$\begin{aligned} \left\| \sum_{i=1}^k A_i X B_i - C \right\| &= \left\| \sum_{i=1}^k V_c(A_i X B_i) - V_c(C) \right\| \\ &= \left\| \sum_{i=1}^k (I_p \otimes A_i) \times \overline{V_c(\overline{X B_i})} - V_c(C) \right\| \\ &= \left\| \sum_{i=1}^k (I_p \otimes A_i) \times \overline{(B_i^H \otimes I_n) \times V_c(\overline{X})} - V_c(C) \right\| \\ &= \left\| \sum_{i=1}^k \Phi_c((I_p \otimes A_i) \overline{(B_i^H \otimes I_n) V_c(\overline{X})}) - \Phi_c(V_c(C)) \right\| \\ &= \left\| \sum_{i=1}^k \Phi(I_p \otimes A_i) K_{np} \Phi(B_i^H \otimes I_n) K_{n^2} \Phi_c(V_c(X)) - \Phi_c(V_c(C)) \right\| \\ &= \left\| \sum_{i=1}^k \Phi(I_p \otimes A_i) K_{np} \Phi(B_i^H \otimes I_n) K_{n^2} H'_s \begin{bmatrix} \widetilde{X_0} \\ \widetilde{X_1} \\ \widetilde{X_2} \\ \widetilde{X_3} \end{bmatrix} - \Phi_c(V_c(C)) \right\| \\ &= \left\| R_1 \begin{bmatrix} \widetilde{X_0} \\ \widetilde{X_1} \\ \widetilde{X_2} \\ \widetilde{X_3} \end{bmatrix} - \Phi_c(V_c(C)) \right\|. \end{aligned}$$

Thus

$$\left\| \sum_{i=1}^k A_i X B_i - C \right\| = \min \iff \left\| R_1 \begin{bmatrix} \widetilde{X}_0 \\ \widetilde{X}_1 \\ \widetilde{X}_2 \\ \widetilde{X}_3 \end{bmatrix} - \Phi_c(V_c(C)) \right\| = \min.$$

For the real matrix equation

$$R_1 \begin{bmatrix} \widetilde{X}_0 \\ \widetilde{X}_1 \\ \widetilde{X}_2 \\ \widetilde{X}_3 \end{bmatrix} = \Phi_c(V_c(C)).$$

Using Lemma 1, its least squares solution can be represented as

$$\begin{bmatrix} \widetilde{X}_0 \\ \widetilde{X}_1 \\ \widetilde{X}_2 \\ \widetilde{X}_3 \end{bmatrix} = R_1^\dagger \Phi_c(V_c(C)) + (I_{4\alpha} - R_1^\dagger R_1)y, \forall y \in \mathbb{R}^{4\alpha}.$$

Then we have

$$\Phi_c(V_c(X)) = H_s' \begin{bmatrix} \widetilde{X}_0 \\ \widetilde{X}_1 \\ \widetilde{X}_2 \\ \widetilde{X}_3 \end{bmatrix} = H_s' R_1^\dagger \Phi_c(V_c(C)) + H_s' (I_{4\alpha} - R_1^\dagger R_1)y, \forall y \in \mathbb{R}^{4\alpha}.$$

And then, Equation (3) can be obtained. $\square$

Theorem 6. Suppose $A_i \in \mathbb{Q}^{m \times n}$, $B_i \in \mathbb{Q}^{n \times p}$, $(i = 1, \dots, k)$, $C \in \mathbb{Q}^{m \times p}$. Hence quaternion matrix Equation (1) has a solution $X \in \mathbb{S}^{n \times n}$ if and only if

$$(R_1 R_1^\dagger - I_{4mp}) \Phi_c(V_c(C)) = 0, \quad (4)$$

where R_1 is denoted in Theorem 5. Moreover, if (4) holds, the centrosymmetric solution set of quaternion matrix Equation (1) can be represented as

$$M_S = \left\{ X \in \mathbb{S}^{n \times n} \mid \Phi_c(V_c(X)) = H_s' R_1^\dagger \Phi_c(V_c(C)) + H_s' (I_{4\alpha} - R_1^\dagger R_1)y \right\},$$

where y is an arbitrary vector suitable for dimension. Then, the minimal norm centrosymmetric solution X_S satisfies

$$\Phi_c(V_c(X_S)) = H_s' R_1^\dagger \Phi_c(V_c(C)). \quad (5)$$

Proof. Quaternion matrix Equation (1) has a solution $X \in \mathbb{S}^{n \times n}$ if and only if

$$\left\| \sum_{i=1}^k A_i X B_i - C \right\| = 0.$$

By means of Theorem 5 and the properties of the Moore–Penrose inverse, we obtain

$$\left\| \sum_{i=1}^k A_i X B_i - C \right\| = \left\| R_1 \begin{bmatrix} \widetilde{X}_0 \\ \widetilde{X}_1 \\ \widetilde{X}_2 \\ \widetilde{X}_3 \end{bmatrix} - \Phi_c(V_c(C)) \right\|$$

$$\begin{aligned}
&= \left\| R_1 R_1^\dagger R_1 \begin{bmatrix} \widetilde{X}_0 \\ \widetilde{X}_1 \\ \widetilde{X}_2 \\ \widetilde{X}_3 \end{bmatrix} - \Phi_c(V_c(C)) \right\| \\
&= \left\| R_1 R_1^\dagger \Phi_c(V_c(C)) - \Phi_c(V_c(C)) \right\| \\
&= \left\| (R_1 R_1^\dagger - I_{4mp}) \Phi_c(V_c(C)) \right\|.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\left\| \sum_{i=1}^k A_i X B_i - C \right\| = 0 &\iff \left\| (R_1 R_1^\dagger - I_{4mp}) \Phi_c(V_c(C)) \right\| = 0 \\
&\iff (R_1 R_1^\dagger - I_{4mp}) \Phi_c(V_c(C)) = 0.
\end{aligned}$$

In case that quaternion matrix Equation (1) is compatible, its solution $X \in \mathbb{S}^{n \times n}$ satisfies

$$R_1 \begin{bmatrix} \widetilde{X}_0 \\ \widetilde{X}_1 \\ \widetilde{X}_2 \\ \widetilde{X}_3 \end{bmatrix} = \Phi_c(V_c(C)).$$

Moreover, by Lemma 2, we can obtain the centrosymmetric solution X satisfies

$$\begin{bmatrix} \widetilde{X}_0 \\ \widetilde{X}_1 \\ \widetilde{X}_2 \\ \widetilde{X}_3 \end{bmatrix} = R_1^\dagger \Phi_c(V_c(C)) + (I_{4\alpha} - R_1^\dagger R_1) y, \forall y \in \mathbb{R}^{4\alpha}.$$

Then we have

$$\Phi_c(V_c(X)) = H_s' R_1^\dagger \Phi_c(V_c(C)) + H_s' (I_{4\alpha} - R_1^\dagger R_1) y, \forall y \in \mathbb{R}^{4\alpha}.$$

And the minimal norm centrosymmetric solution X_S satisfies

$$\Phi_c(V_c(X_S)) = H_s' R_1^\dagger \Phi_c(V_c(C)).$$

□

For Problem 2, we can also obtain the necessary and sufficient condition for the existence of anti-centrosymmetric solutions of quaternion matrix Equation (1) through vector representation of quaternion matrices, $\mathcal{L}$ -representation and $\mathbf{GH}$ -representation method. Similar to the analysis procedure of Problem 1, we obtain the following conclusions.

Theorem 7. Suppose $A_i \in \mathbb{Q}^{m \times n}$, $B_i \in \mathbb{Q}^{n \times p}$, ($i = 1, \dots, k$), $C \in \mathbb{Q}^{m \times p}$, then the set M_A of Problem 2 can be represented as

$$M_A = \left\{ X \in \mathbb{AS}^{n \times n} \mid \Phi_c(V_c(X)) = H_a' R_2^\dagger \Phi_c(V_c(C)) + H_a' (I_{4\beta} - R_2^\dagger R_2) y \right\}, \quad (6)$$

where $\forall y \in \mathbb{R}^{4\beta}$. Then, the minimal norm least squares anti-centrosymmetric solution X_A of quaternion matrix Equation (1) satisfies

$$\Phi_c(V_c(X_A)) = H_a' R_2^\dagger \Phi_c(V_c(C)), \quad (7)$$

$$\text{where } H'_a = \begin{bmatrix} H_a & & & \\ & H_a & & \\ & & H_a & \\ & & & H_a \end{bmatrix}_{4n^2 \times 4\beta}, K_n = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & -I_n & 0 & 0 \\ 0 & 0 & -I_n & 0 \\ 0 & 0 & 0 & -I_n \end{bmatrix}, R_2 = \sum_{i=1}^k \Phi(I_p \otimes A_i) K_{np} \Phi(B_i^H \otimes I_n) K_{n^2} H'_a.$$

Theorem 8. Suppose $A_i \in \mathbb{Q}^{m \times n}$, $B_i \in \mathbb{Q}^{n \times p}$, ($i = 1, \dots, k$), $C \in \mathbb{Q}^{m \times p}$. Hence quaternion matrix Equation (1) has a solution $X \in \mathbb{AS}^{n \times n}$ if and only if

$$(R_2 R_2^\dagger - I_{4mp}) \Phi_c(V_c(C)) = 0, \quad (8)$$

where R_2 is denoted in Theorem 7. Moreover, if (8) holds, the anti-centrosymmetric solution set of quaternion matrix Equation (1) can be represented as

$$M_A = \left\{ X \in \mathbb{AS}^{n \times n} \mid \Phi_c(V_c(X)) = H'_a R_2^\dagger \Phi_c(V_c(C)) + H'_a (I_{4\beta} - R_2^\dagger R_2) y, \forall y \in \mathbb{R}^{4\beta} \right\}.$$

And then, the minimal norm anti-centrosymmetric solution X_A satisfies

$$\Phi_c(V_c(X_A)) = H'_a R_2^\dagger \Phi_c(V_c(C)). \quad (9)$$

6. Algorithms and Numerical Examples

Numerical experiments are used to verify the effectiveness of the above algorithms.

Example 7. Suppose $m = n = p$, $A_i, B_i \in \mathbb{Q}^{n \times n}$ be generated randomly for $n = 5K, K = 1 : 11$. Randomly generate centrosymmetric matrix X_S or anti-centrosymmetric matrix X_A , respectively. Then for the left side of quaternion matrix Equation (1), replace X with X_S or X_A , let $k = 2$, calculate $C = A_1 X_S B_1 + A_2 X_S B_2$ or $C = A_1 X_A B_1 + A_2 X_A B_2$. For the quaternion matrix Equation (1) with A_i, B_i and C above, its computational solutions can be obtained by using Algorithms 1 and 2 and denoted as $\check{X}_S, \check{X}_A$, respectively. Denote $\varepsilon_1 = \log_{10} \|\Phi_c(X_S) - \Phi_c(\check{X}_S)\|$, $\varepsilon_2 = \log_{10} \|\Phi_c(X_A) - \Phi_c(\check{X}_A)\|$. As the dimension changes, ε_t ($t = 1, 2$) is shown in Figure 1.

It can be seen from Figure 1 that the order of magnitude of error between the exact solution and the numerical solution in Problem 1 and 2 increases with the increase in dimension. However, for Problem 1, the order of magnitude of error of the centrosymmetric solution is always less than -11 ; for Problem 2, the order of magnitude of error of the anti-centrosymmetric solution is always less than -12 , which indicates that the order of magnitude of error between the numerical solution and the exact solution is very small, that is, the algorithm in this paper is effective.

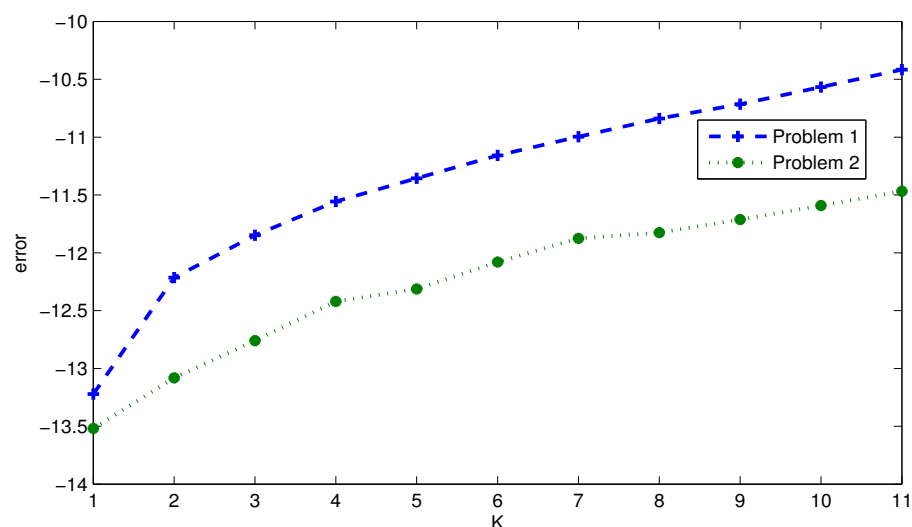


Figure 1. Errors in different dimensions.

Algorithm 1 Calculate the minimal norm centrosymmetric solution of quaternion matrix Equation (1).

Input: Quaternion matrix $A_i \in \mathbb{Q}^{m \times n}$, $B_i \in \mathbb{Q}^{n \times p}$, ($i = 1, 2, \dots, k$), $C \in \mathbb{Q}^{m \times p}$;

Output: Output the minimal norm centrosymmetric solution $\tilde{X}_S$ of quaternion matrix Equation (1) according to (5);

- 1: Compute $\Phi_c(V_c(C))$;
 - 2: Input $H_s, K_{np}, K_{n^2}, \Phi(I_p \otimes A_i), \Phi(B_i^H \otimes I_n)$;
 - 3: Compute $H'_s, R_1 = \sum_{i=1}^k \Phi(I_p \otimes A_i) K_{np} \Phi(B_i^H \otimes I_n) K_{n^2} H'_s$;
 - 4: **if** (4) hold **then**
 - 5: Calculate the minimal norm solution of quaternion matrix equation according to (5);
 - 6: **end if**
-

Algorithm 2 Calculate the minimal norm anti-centrosymmetric solution of quaternion matrix Equation (1).

Input: Quaternion matrix $A_i \in \mathbb{Q}^{m \times n}$, $B_i \in \mathbb{Q}^{n \times p}$, ($i = 1, 2, \dots, k$), $C \in \mathbb{Q}^{m \times p}$;

Output: Output the minimal norm centrosymmetric solution $\tilde{X}_A$ of quaternion matrix Equation (1) according to (9);

- 1: Compute $\Phi_c(V_c(C))$;
 - 2: Input $H_a, K_{np}, K_{n^2}, \Phi(I_p \otimes A_i), \Phi(B_i^H \otimes I_n)$;
 - 3: Compute $H'_a, R_2 = \sum_{i=1}^k \Phi(I_p \otimes A_i) K_{np} \Phi(B_i^H \otimes I_n) K_{n^2} H'_a$;
 - 4: **if** (8) hold **then**
 - 5: Calculate the minimal norm solution of quaternion matrix equation according to (9);
 - 6: **end if**
-

Next, taking the centrosymmetry solution as an example, we compare the method of solving the special solution of quaternion matrix equation in this paper with the method of in references [43,44].

The method in reference [43] used the real representation of quaternion matrices to process quaternion matrix equation firstly, the transformation from quaternion matrix equation to real matrix equation is realized, then the straighten operator is used to transform the real matrix equation into real vector matrix equation.

Remark 3. The symbols appearing in Algorithm 3 follow the symbol representation in reference [43], J and K are defined in reference [43]. H_s is the $\mathbf{H}$ -representation matrix of the centrosymmetric matrix in this paper, and H'_s is also defined in Theorem 4.

Algorithm 3 Calculate the minimal norm centrosymmetric solution of quaternion matrix Equation (1) according to the method of reference [43].

Input: Quaternion matrix $A_i \in \mathbb{Q}^{m \times n}$, $B_i \in \mathbb{Q}^{n \times p}$, ($i = 1, 2, \dots, k$), $C \in \mathbb{Q}^{m \times p}$;

Output: Output the minimal norm centrosymmetric solution $\tilde{X}_s$ of quaternion matrix Equation (1);

- 1: Compute $\text{vec}(\vec{C}_c)$;
 - 2: Input H_s, J, K ;
 - 3: Compute $H'_s, R_3 = \sum_{i=1}^k (\vec{B}_{ic}^T \otimes \vec{A}_i) J K H'_s$;
 - 4: Calculate the minimal norm solution of quaternion matrix equation according to $\tilde{X}_s = H'_s R_3^+ \text{vec}(\vec{C}_c)$.
-

The real vector representation method in reference [44] is to represent a quaternion as a 4×1 dimension vector, and then establish the relationship between quaternion matrix real vector representation operations through semi-tensor product of matrices.

Remark 4. The symbols appearing in Algorithm 4 follow the symbol representation in reference [44],

$$\text{and } J_n = \begin{cases} \begin{bmatrix} I_{8k^2} \\ V_{2k^2} \otimes I_4 \end{bmatrix} & (\text{if } n \text{ is even}) \\ \begin{bmatrix} I_{4(2k^2+2k+1)} \\ V'_{2k^2+2k+1} \otimes I_4 \end{bmatrix} & (\text{if } n \text{ is odd}) \end{cases}, \text{ where } V_n = \begin{bmatrix} 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & 1 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 1 & \dots & 0 & 0 & 0 \end{bmatrix}_{n \times n} \text{ and } V'_n = \begin{bmatrix} 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 1 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 1 & \dots & 0 & 0 & 0 \end{bmatrix}_{(n-1) \times n}.$$

Algorithm 4 Calculate the minimal norm centrosymmetric solution of quaternion matrix Equation (1) according to the method of reference [44].

Input: Quaternion matrix $A_i \in \mathbb{Q}^{m \times n}$, $B_i \in \mathbb{Q}^{n \times p}$, $(i = 1, 2, \dots, k)$, $C \in \mathbb{Q}^{m \times p}$;

Output: Output the minimal norm centrosymmetric solution $\tilde{X}_s$ of quaternion matrix Equation (1);

- 1: Compute $\vec{A}_i^r, \vec{B}_i^c, \vec{C}^c$;
- 2: Compute G, G', J_n ;
- 3: Compute $R_4 = \sum_{i=1}^k G \times G' \times \vec{A}_i^r \times W_{[4np, 4n^2]} \times \vec{B}_i^c \times J_n$;
- 4: Calculate the minimal norm solution of quaternion matrix equation according to $\tilde{X}_s = J_n R_4^+ \vec{C}^c$.

Example 8. Suppose $m = n = p$, $A_i, B_i \in \mathbb{Q}^{n \times n}$ be generated randomly for $n = 4K, K = 1 : 10$. Randomly generate centrosymmetric matrix X_s . Then for the left side of quaternion matrix Equation (1), let $k = 1$, calculate $C = A_1 X_s B_1$. For the quaternion matrix Equation (1) with A_i, B_i and C above, its computational solutions can be obtained by using Algorithms 1, 3 and 4. As the dimension changes, time consumed by the algorithms is shown in Figure 2.

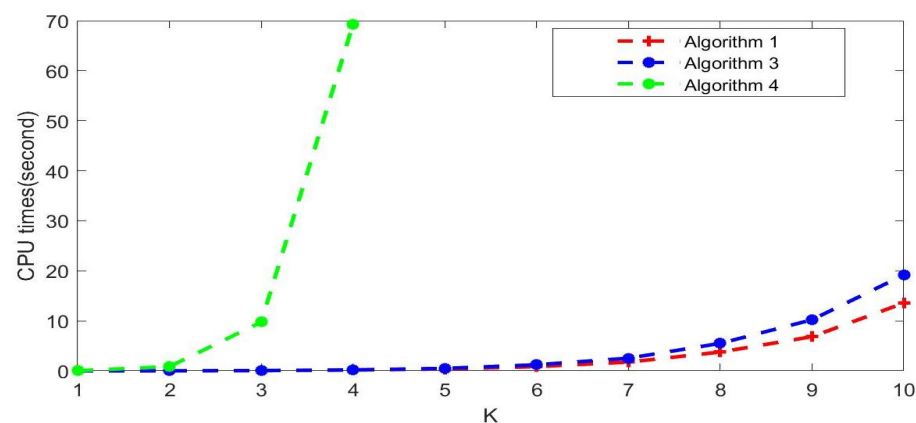


Figure 2. Time comparison results.

For the method in reference [44], because the matrix dimension is too large, we only choose $K = 1 : 4$. If the form of the solution obtained by the algorithm in reference [43] wants to be consistent with the form of the solution obtained by the algorithm in this paper, it needs to be transformed with the help of a large matrix. The method of expressing quaternion as real vector in reference [44] makes the calculation process of quaternion matrix equation have a large dimension, which is not conducive to the improvement of calculation efficiency. As can be seen from Figure 2, the algorithm in this paper takes less time than the algorithm in references [43,44].

7. Application in Color Digital Image Restoration

We know that a color digital image consists of three primary colors: red, green and blue, and these three primary colors can correspond to the three imaginary parts

of quaternion, respectively. That is, a color digital image can be represented by a pure imaginary quaternion matrix. One of the most basic applications in color digital image is color digital image restoration, and the process of color digital image restoration is the solution process of the minimal norm least squares solution of quaternion matrix equation. For an $n \times n$ pixel observation image $g = g_r\mathbf{i} + g_g\mathbf{j} + g_b\mathbf{k}$, we know its blurring phenomena K , where K is a real matrix, then the color digital image restoration model is established as

$$g = Kf + N.$$

But in general, the noise N is unknown. In this section, we will work with the centrosymmetric color digital image restoration model. The centrosymmetric color image restoration problem is transformed into the least squares pure imaginary centrosymmetric solution problem of quaternion matrix equation $Kf = g$.

Example 9. Given two ideal centrosymmetric color digital image (see Figures 3a and 4a), $f = (f_r, f_g, f_b)$ is the image matrix, f can be represented as $f = f_r\mathbf{i} + f_g\mathbf{j} + f_b\mathbf{k}$. By using $LEN = 15$; $THETA = 30$; $PSF = fspecial('motion', LEN, THETA)$ disturb the image f_g , and obtain the disturb image g_g . Obviously, $K = g_g f_g^\dagger$ is a singular matrix. By using the matrix K , we can obtain the disturb image $g = (g_r, g_g, g_b)$ (see Figures 3b and 4b). The minimal norm least squares pure imaginary centrosymmetric solution $\vec{F}$ can be obtained by Algorithm 5. Through the "reshape" command of MATLAB, we obtain the corresponding color digital restored image $F = (F_r, F_g, F_b)$ (see Figures 3c and 4c).

Finally, we give the mean-square error of each channel which is defined as

$$MSE = \frac{1}{mn} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} [I(i, j) - K(i, j)]^2.$$

The mean-square error of each channel is represented by $\epsilon_r, \epsilon_g, \epsilon_b$, respectively, and the results are shown in Table 1.

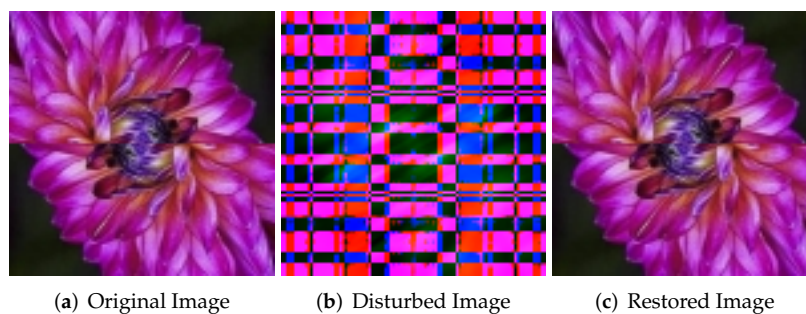


Figure 3. Image 1: 100 × 100 Pixel Centrosymmetric Color Digital .

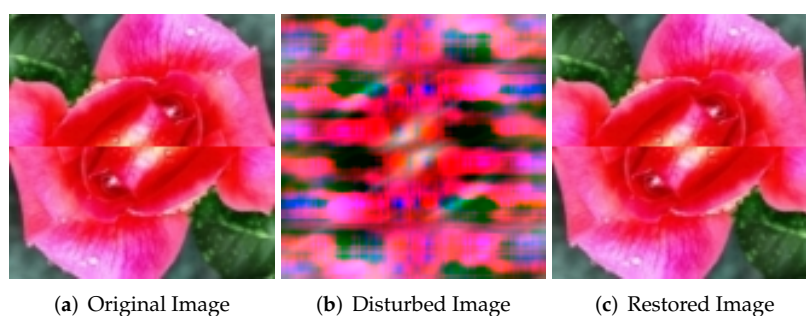


Figure 4. Image 2: 110 × 110 Pixel Centrosymmetric Color Digital .

Algorithm 5 Calculate the minimal norm least squares pure imaginary centrosymmetric solution of color digital image model $Kf = g$.

Output: Output the minimal norm least squares pure imaginary centrosymmetric solution of quaternion matrix equation $Kf = g$;

- 1: Compute $K' = \begin{bmatrix} I_n \otimes K & & \\ & I_n \otimes K & \\ & & I_n \otimes K \end{bmatrix}, H_S = \begin{bmatrix} H_s & & \\ & H_s & \\ & & H_s \end{bmatrix};$
- 2: Compute $\vec{g} = \begin{bmatrix} V_c(g_r) \\ V_c(g_g) \\ V_c(g_b) \end{bmatrix};$
- 3: Calculate the minimal norm least squares pure imaginary centrosymmetric solution of quaternion matrix equation $Kf = g$ according to $\vec{f} = H_S(K'H_S)^+ \vec{g}$.

Table 1. Mean-square error (MSE).

	ε_r	ε_g	ε_b
Figure 3	4.9586×10^{-18}	2.4722×10^{-19}	1.9076×10^{-18}
Figure 4	1.4071×10^{-20}	4.0846×10^{-22}	1.2557×10^{-21}

8. Conclusions

The new conclusions of vector representation and $\mathcal{L}$ -representation of quaternion matrices makes semi-tensor product of quaternion matrices have a new application in solving quaternion matrix equation. Starting from these new conclusions of semi-tensor product of quaternion matrices, combined $\mathcal{L}$ -representation with $\mathbf{H}$ -representation method, the special solution of quaternion matrix equation $\sum_{i=1}^k A_i X B_i = C$ are solved. Furthermore, numerical examples show that the method is effective. Through a time comparison, it is found that the algorithm in this paper is relatively efficient compared with the algorithm in references [43,44]. The application of centrosymmetric color digital image restoration is also considered.

Notes:

- The images used are from the MATLAB image processing toolbox or USC-SIPI image database image library of the University of Southern California (<http://sipi.usc.edu/database/>, accessed on 1 June 2022).
- All computations are performed on an Intel(R) core(TM) i9-10940U @3.30 GHz/64 GB computer using MATLAB R2019b software.

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