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# Analysis of the Fractional-Order Local Poisson Equation in Fractal Porous Media

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**Abstract:** This paper investigates the fractional local Poisson equation using the homotopy perturbation transformation method. The Poisson equation discusses the potential area due to a provided charge with the possibility of area identified, and one can then determine the electrostatic or gravitational area in the fractal domain. Elliptic partial differential equations are frequently used in the modeling of electromagnetic mechanisms. The Poisson equation is investigated in this work in the context of a fractional local derivative. To deal with the fractional local Poisson equation, some illustrative problems are discussed. The solution shows the well-organized and straightforward nature of the homotopy perturbation transformation method to handle partial differential equations having fractional derivatives in the presence of a fractional local derivative. The solutions obtained by the defined methods reveal that the proposed system is simple to apply, and the computational cost is very reliable. The result of the fractional local Poisson equation yields attractive outcomes, and the Poisson equation with a fractional local derivative yields improved physical consequences.

**Keywords:** homotopy perturbation transformation method; fractional local Poisson equation; local Caputo operator; Sumudu transform



**Citation:** Alqhtani, M.; Saad, K.M.; Shah, R.; Weera, W.; Hamanah, W.M. Analysis of the Fractional-Order Local Poisson Equation in Fractal Porous Media. *Symmetry* **2022**, *14*, 1323. https://doi.org/10.3390/sym14071323

Academic Editor: Jose M. Vega-Guzman

Received: 1 June 2022
Accepted: 25 June 2022
Published: 27 June 2022



## 1. Introduction

In mathematical physics, the Poisson equation (PE) is the most useful. In his book *Partial Differential Equations*, Evans [1] explained the feature of the Poisson equation. In their contributed book [2], Elman et al. elaborated on the significance of the Poisson equation. Derriennic et al. [3] concentrated on the Poisson equation at an arbitrary order and proved several essential theorems. Griffiths and College [4] clarify their contribution to electrodynamics. Kellogg [5] investigated the Poisson equation with intersecting interfaces. Jassim [6] investigated the PE in its local sense with a fractional derivative. Chenet et al. [7] conducted an intriguing study on the fractional local PE using an analytical method, yielding very nice and significant results. The Poisson equation is an elliptic nonlinear partial differential equation with numerous implementations in quantum theory, mechanical engineering, and various other crucial and effective fields. It describes the continuous difference in the capillarity across the interface between static fluids, namely water and air, as a result of surface tension or, alternatively, wall pressure. The study of partial differential equations, particularly those derived from finance mathematics, is where the elegance of symmetry analysis is most apparent. The secret of nature is symmetry, but the majority of natural observations lack symmetry [8].

The fractional local transport equations are essential in a variety of scientific fields, including semiconductors [9], aeronautics [10], superconductivity [11], turbulence [12], plasma [13], gas mixtures [14], and biology [15]. Tarasov [16] examined fractional-order transport equations, while Zaslavsky [17] discussed anomalous fractional dynamics transport. Uchaikin and Sibatov [18] looked into the use of the application of this equation to disordered semiconductors, Lutz [19] looked into transport equations with arbitrary-order derivatives, Kadem et al. [20] looked into spectral techniques to solve fractional-order transport equations, Meng Li et al. [21] looked into the numerical results of a linear transport equation, and so on [22,23].

Because of its wide application, local fractional calculus has attracted the attention and interest of mathematicians and researchers in recent years. Rayneau-Kirkhope et al. [24] looked at ultra-light fractal architectures, the local fractional wave equation in fractal strings was established by Singh et al. [25], Yang [26] looked at how heat moves via discontinuous media, a heat conduction equation with arbitrary-order derivatives was studied by Povestenko [27], heat conduction linked with a non-integer-order derivative was investigated by Wang et al. [28], Shih [29] researched numerical heat transfer, the heat-balance integral to fractional systems was investigated by Hristov [30], Yang discussed the fractal heat conduction problem and Baleanu [31], etc. However, analytical methods are rare for most, if not all, fractional partial differential equations [32–34]. Hence, building effective numerical methods and schemes are fascinating and of relevance in practical applications and with problems related to the real world [35,36].

The first to invent the homotopy perturbation technique (HPM) was the Chinese mathematician JH He, who played a crucial role, in 1998 [37]. This method is just, economical, and effective, and it eliminates unconditioned matrices, intricate integrals, and endless series. This method does not require a problem-specific parameter. In 2010, Tarig Elzaki introduced a novel integral transformation called the Ezaki transform (ET). The ET transform is a modification of the Laplace and Sumudu transforms. Remember that absolute differential equations with variable coefficients cannot be solved using the Sumudu and Laplace transforms when the ET is used [38–40]. The HPTM combines the Elzaki transformation with the homotopy perturbation method. Several scientists have utilized the HPTM to solve differential equations, including heat-like problems, Navier–Stokes problems, Fisher's equation, the gas-dynamic model, and the hyperbolic equation [41–44].

In the current research work, we implemented a hybrid method for the solution of fractional local Poisson equations. The present technique is a mixture of two well-known techniques known as the Shehu transform and the homotopy perturbation method, which is discussed in Section 4 of the paper. For the purpose of the validity of the suggested technique, some illustrative problems are presented. Moreover, the homotopy perturbation transformation method solution is determined at various fractional orders of the given equations. It has been analyzed that the fractional-order results converge toward a classical result for the problems, from a fractional-order to a classical-order approach. It is clear from the series-form solution that the homotopy perturbation transformation method has the desired degree of accuracy. Overall, the current method's discussion and numerical implementation have suggested that it can be easily extended to solve other fractional-order differential equations.

## 2. Basic Definitions

This section examines the fundamental concept of fractional local calculus, which is utilized in this study.

**Definition 1.** *For the connection $|x - x_0| < \sigma$, when $\epsilon, \sigma > 0$ and $\epsilon \in R$, we allow the functions $f(x) \in C\beta(a, b)$, while [45]*

$$|f(x) - f(x_0)| < \epsilon^{\varrho}, \;\; 0 < \varrho \leq 1, \tag{1}$$

*exists.*

**Definition 2.** *Consider the interval [a, b] and $(y_j, y_{j+1}), j = 0, \cdots, N-1, y_0 = a$, and $y_N = b$ with $\delta y_j = y_{j+1} - y_j, \delta y = max\{\Delta y_0, \Delta y_1, \Delta y_2, \cdots\}$ a partition of this interval. Then, the fractional local integral of $f(x)$ is defined as [45]*

$$I_b^{(\varrho)} f(x) = \frac{1}{\Gamma(1+\varrho)} \int_a^b f(y)(dy)^\varrho = \frac{1}{\Gamma(1+\varrho)} \lim_{\Delta y \to 0} \sum_{j=0}^{j=N-1} f(y_i)(\Delta y_j)^\varrho \tag{2}$$

**Definition 3.** *If the function $f(x)$ satisfies the conditions of Equation (1), the inverse formula of Equation (2) is described as follows [45]:*

$$\frac{d^\varrho f(x_0)}{dx^\varrho} = D_x^{(\varrho)} f(x_0) = \frac{\Delta^\varrho (f(x) - f(x_0))}{(x - x_0)^\varrho}, \tag{3}$$

*where*

$$\Delta^\varrho (f(x) - f(x_0)) \cong \Gamma(1+\varrho)[f(x) - f(x_0)]. \tag{4}$$

*In this work, the fractional local derivative is represented by the following formula:*

$$\frac{d^\varrho}{dx^\varrho} \frac{x^{n\varrho}}{\Gamma(1+n\varrho)} = \frac{x^{(n-1)\varrho}}{\Gamma(1+(n-1)\varrho)}, n \in N. \tag{5}$$

## 3. Fractional Local Sumudu Transformation

Watugala [46] first proposed and developed the Sumudu transform, whereas Belgacem et al. [47] and Belgacem and Karaballi [48] identified and studied some of its essential features. Katatbeh and Belgacem [49] solved fractional differential equations employing the Sumudu transformation. Gupta et al. [50] solved generalized fractional kinetic equations using the Sumudu transform. The implementations of the Sumudu transformation to the Bessel function and equations were researched by Guo [51]. Srivastava [52] presented and examined more Sumudu characteristics. Using the Sumudu transform method, Gao et al. [53] discovered the analytic results to several fractional ordinary differential equations. Coupled with the HPM, the Sumudu transformation method is used to explore the fractional population biological models [54]. Srivastava et al. [55] initially introduce and define the fractional local Sumudu transformation of a function $f(x)$ as follows:

$$LFS_\varrho\{f(x)\} = F_\varrho(z) = \frac{1}{\Gamma(1+\varrho)} \int_0^\infty E_\varrho(-z^{-\varrho} x^\varrho) \frac{f(x)}{z^\varrho} (dx)^\varrho, 0 < \varrho \leq 1 \tag{6}$$

Moreover, the inverse formula is as follows:

$$LFS_\varrho^{-1}\{F_\varrho(z)\} = f(x), 0 < \varrho \leq 1. \tag{7}$$

## 4. Fractional Local Homotopy Perturbation Transformation Method

To establish the fundamental concept underlying the FLHPTM, we suppose the given with a local fractional derivative linear differential equation.

$$L_\varrho u(x,y) + R_\varrho u(x,y) = h(x,y), \tag{8}$$

where $R_\varrho$ is the remaining linear operator, $L_\varrho$ represents the linear fractional local differential derivative, and source function is $h(x,y)$.

Applying the Sumudu local transformation on Equation (8), we obtain

$$\begin{aligned} U_\varrho(x,z) =& u(x,0) + z^\varrho u^\varrho(x,0) + z^{2\varrho} u^{2\varrho}(x,0) + \cdots + z^{(k-1)\varrho} u^{(k-1)\varrho}(x,0) \\ &- z^{k\varrho} LFS_\varrho[R_\varrho u(x,y)] + z^{k\varrho} LFS_\varrho[h(x,y)]. \end{aligned} \tag{9}$$

Using the local fractional inverse Sumudu transformation on Equation (9), we obtain

$$\begin{aligned} u(x,y) =& u(x,0) + \frac{y^{\varrho}}{\Gamma(1+\varrho)} u^{\varrho}(x,0) + \frac{y^{2\varrho}}{\Gamma(1+2\varrho)}(x,0) + \cdots + \frac{y^{(k-1)\varrho}}{\Gamma(1+(k-1)\varrho)} u^{(k-1)\varrho}(x,0) \\ & - LFS_{\varrho}^{-1}[z^{k\varrho} LFS_{\varrho}[R_{\varrho} u(x,y)]] + LFS_{\varrho}^{-1}[z^{k\varrho} LFS_{\varrho}[h(x,y)]]. \end{aligned} \tag{10}$$

Now, we apply the homotopy perturbation method [24–26]

$$u(x,y) = \sum_{n=0}^{\infty} p^n u_n(x,y). \tag{11}$$

Putting Equation (11) in Equation (10), we obtain the given solution:

$$\begin{aligned} \sum_{n=0}^{\infty} p^n u_n(x,y) =& u(x,0) + \frac{y^{\varrho}}{\Gamma(1+\varrho)} u^{\varrho}(x,0) + \frac{y^{2\varrho}}{\Gamma(1+2\varrho)}(x,0) + \cdots + \frac{y^{(k-1)\varrho}}{\Gamma(1+(k-1)\varrho)} u^{(k-1)\varrho}(x,0) \\ & - LFS_{\varrho}^{-1}\left[z^{k\varrho} LFS_{\varrho}\left[R_{\varrho} \sum_{n=0}^{\infty} p^n u_n(x,y)\right]\right] + LFS_{\varrho}^{-1}\left[z^{k\varrho} LFS_{\varrho}[h(x,y)]\right]. \end{aligned} \tag{12}$$

which combines the fractional local Sumudu transformation method with homotopy perturbation method. Evaluating coefficients of identical powers of $p$ yields

$$\begin{aligned} p^0 : u_0(x,y) =& u(x,0) + \frac{y^{\varrho}}{\Gamma(1+\varrho)} u^{\varrho}(x,0) + \frac{y^{2\varrho}}{\Gamma(1+2\varrho)}(x,0) + \cdots \\ & + \frac{y^{(k-1)\varrho}}{\Gamma(1+(k-1)\varrho)} u^{(k-1)\varrho}(x,0) + + LFS_{\varrho}^{-1}\left[z^{k\varrho} LFS_{\varrho}[h(x,y)]\right], \\ p^1 : u_1(x,y) =& - LFS_{\varrho}^{-1}\left[z^{k\varrho} LFS_{\varrho}\left[R_{\varrho} u_0(x,y)\right]\right], \\ p^2 : u_2(x,y) =& - LFS_{\varrho}^{-1}\left[z^{k\varrho} LFS_{\varrho}\left[R_{\varrho} u_1(x,y)\right]\right], \\ & \vdots \end{aligned} \tag{13}$$

Therefore, the solution of Equation (8) is defined as

$$u(x,y) = \lim_{N\to\infty} \sum_{n=0}^{N} u_n(x,y) \tag{14}$$

## 5. Non-Differential Solutions for the Fractional Local Poisson Equation

In this section, we show the results for the local Poisson equation for the local fractional derivative arising in fractal transonic flow with local fractional operator with the initial condition.

**Example 1.** *Consider the local fractional PE is given as [56]*

$$\frac{\partial^{\varrho} u(x,y)}{\partial y^{\varrho}} + \frac{\partial^{\varrho} u(x,y)}{\partial x^{\varrho}} = 0, \quad 0 < \varrho \le 1, \tag{15}$$

*with the initial condition*

$$u(x,0) = \frac{x^{\varrho}}{\Gamma(1+\varrho)}. \tag{16}$$

*Using the fractional local Sumudu transformation on Equation (15), we have*

$$U_{\varrho}(x,z) = u(x,0) + z^{2\varrho} LFS_{\varrho}\left[-\frac{\partial^{\varrho} u(x,y)}{\partial x^{\varrho}}\right], \tag{17}$$

*which implies*

$$U_\varrho(x,z) = \frac{x^\varrho}{\Gamma(1+\varrho)} + z^{2\varrho} LFS_\varrho\left[-\frac{\partial^\varrho u(x,y)}{\partial x^\varrho}\right]. \tag{18}$$

*Using the local fractional inverse Sumudu transformation to Equation (18), we achieved as*

$$u(x,y) = \frac{x^\varrho}{\Gamma(1+\varrho)} + LFS_\varrho^{-1}\left[z^{2\varrho} LFS_\varrho\left[-\frac{\partial^\varrho u(x,y)}{\partial x^\varrho}\right]\right]. \tag{19}$$

*Now, applying homotopy perturbation method, we obtain*

$$\sum_{n=0}^{\infty} p^n u_n(x,y) = \frac{x^\varrho}{\Gamma(1+\varrho)} + LFS_\varrho^{-1}\left[z^{2\varrho} LFS_\varrho\left[-\frac{\partial^\varrho \sum_{n=0}^{\infty} p^n u_n(x,y)}{\partial x^\varrho}\right]\right]. \tag{20}$$

*We obtain the following component of the series result by comparing the like powers of p.*

$$\begin{aligned} p^0 : u_0(x,y) =& \frac{x^\varrho}{\Gamma(1+\varrho)}, \\ p^1 : u_1(x,y) =& \frac{y^\varrho}{\Gamma(1+\varrho)}, \\ &\vdots \end{aligned} \tag{21}$$

*we obtain the series-form solution of Equation (15), given as*

$$\begin{aligned} u(x,y) =& \lim_{N\to\infty} \sum_{n=0}^{\infty} u_n(x,y), \\ =& \frac{x^\varrho}{\Gamma(1+\varrho)} + \frac{y^\varrho}{\Gamma(1+\varrho)}. \end{aligned} \tag{22}$$

**Example 2.** *Consider the local fractional PE is given as*

$$\frac{\partial^\varrho u(x,y)}{\partial y^\varrho} + \frac{\partial^\varrho u(x,y)}{\partial x^\varrho} = 0, 0 < \varrho \le 1, \tag{23}$$

*with the initial condition*

$$u(x,0) = E_\varrho(x^\varrho). \tag{24}$$

*Using the fractional local Sumudu transformation on Equation (23), we have*

$$U_\varrho(x,z) = u(x,0) + z^{2\varrho} LFS_\varrho\left[-\frac{\partial^\varrho u(x,y)}{\partial x^\varrho}\right], \tag{25}$$

*which implies*

$$U_\varrho(x,z) = E_\varrho(x^\varrho) + z^{2\varrho} LFS_\varrho\left[-\frac{\partial^\varrho u(x,y)}{\partial x^\varrho}\right]. \tag{26}$$

*Using the local fractional inverse Sumudu transformation to Equation (26), given as*

$$u(x,y) = E_\varrho(x^\varrho) + LFS_\varrho^{-1}\left[z^{2\varrho} LFS_\varrho\left[-\frac{\partial^\varrho u(x,y)}{\partial x^\varrho}\right]\right]. \tag{27}$$

*Now, applying homotopy perturbation method, we obtain*

$$\sum_{n=0}^{\infty} p^n u_n(x,y) = E_\varrho(x^\varrho) + LFS_\varrho^{-1}\left[z^{2\varrho} LFS_\varrho\left[-\frac{\partial^\varrho \sum_{n=0}^{\infty} p^n u_n(x,y)}{\partial x^\varrho}\right]\right]. \tag{28}$$

*We obtain the following component of the series result by comparing the like powers of p.*

$$\begin{aligned} p^0 : u_0(x,y) =& E_\varrho(x^\varrho), \\ p^1 : u_1(x,y) =& E_\varrho(x^\varrho)\frac{y^\varrho}{\Gamma(1+\varrho)}, \\ p^2 : u_2(x,y) =& E_\varrho(x^\varrho)\frac{y^{2\varrho}}{\Gamma(1+2\varrho)}, \\ & \vdots \end{aligned} \tag{29}$$

*We obtain the series-form solution of Equation (23), given as*

$$\begin{aligned} u(x,y) =& \lim_{N\to\infty}\sum_{n=0}^{\infty} u_n(x,y), \\ =& E_\varrho(x^\varrho)\left[1 - \frac{y^\varrho}{\Gamma(1+\varrho)} + \frac{y^{2\varrho}}{\Gamma(1+2\varrho)} - \frac{y^{3\varrho}}{\Gamma(1+3\varrho)} + \frac{y^{4\varrho}}{\Gamma(1+4\varrho)} - \cdots\right]. \end{aligned} \tag{30}$$

**Example 3.** *Consider local fractional PE is given as [56]*

$$\frac{\partial^\varrho u(x,y)}{\partial y^\varrho} + \frac{\partial^\varrho u(x,y)}{\partial x^\varrho} = 0, \ \ 0 < \varrho \leq 1, \tag{31}$$

*with the initial condition*

$$u(x,0) = \cos_\varrho(x^\varrho). \tag{32}$$

*Using the fractional local Sumudu transformation on Equation (31), we have*

$$U_\varrho(x,z) = u(x,0) + z^{2\varrho} LFS_\varrho\left[-\frac{\partial^\varrho u(x,y)}{\partial x^\varrho}\right], \tag{33}$$

*which implies*

$$U_\varrho(x,z) = \cos_\varrho(x^\varrho) + z^{2\varrho} LFS_\varrho\left[-\frac{\partial^\varrho u(x,y)}{\partial x^\varrho}\right]. \tag{34}$$

*Using the local fractional inverse Sumudu transformation to Equation (34), we obtain*

$$u(x,y) = \cos_\varrho(x^\varrho) + LFS_\varrho^{-1}\left[z^{2\varrho} LFS_\varrho\left[-\frac{\partial^\varrho u(x,y)}{\partial x^\varrho}\right]\right]. \tag{35}$$

*Now, applying homotopy perturbation method, we obtain*

$$\sum_{n=0}^{\infty} p^n u_n(x,y) = E_\varrho(x^\varrho) + LFS_\varrho^{-1}\left[z^{2\varrho} LFS_\varrho\left[-\frac{\partial^\varrho \sum_{n=0}^{\infty} p^n u_n(x,y)}{\partial x^\varrho}\right]\right]. \tag{36}$$

*We obtain the following component of the series result by comparing the like powers of p.*

$$\begin{aligned}
p^0 &: u_0(x,y) = \cos_\varrho(x^\varrho), \\
p^1 &: u_1(x,y) = \sin_\varrho(x^\varrho)\frac{y^\varrho}{\Gamma(1+\varrho)}, \\
p^2 &: u_2(x,y) = \sin_\varrho(x^\varrho)\frac{y^\varrho}{\Gamma(1+\varrho)} - \cos_\varrho(x^\varrho)\frac{y^{2\varrho}}{\Gamma(1+2\varrho)}, \\
&\vdots
\end{aligned} \tag{37}$$

*We obtain the series-form solution of Equation (31), given as*

$$\begin{aligned}
u(x,y) &= \lim_{N\to\infty}\sum_{n=0}^{\infty} u_n(x,y), \\
&= \cos_\varrho(x^\varrho) + \sin_\varrho(x^\varrho)\frac{y^\varrho}{\Gamma(1+\varrho)} - \cos_\varrho(x^\varrho)\frac{y^{2\varrho}}{\Gamma(1+2\varrho)} + \cdots .
\end{aligned} \tag{38}$$

*Equation (38) can be presented in the following way*

$$\begin{aligned}
u(x,y) &= \sin_\varrho(x^\varrho)\sum_{l=0}^{\infty}(-1)^l\frac{y^{(2l+1)\varrho}}{\Gamma(1+(2l+1)\varrho)} + \cos_\varrho(x^\varrho)\sum_{l=0}^{\infty}(-1)^l\frac{y^{2l\varrho}}{\Gamma(1+2l\varrho)}, \\
&= \sin_\varrho(x^\varrho)\sin_\varrho(y^\varrho) + \cos_\varrho(x^\varrho)\cos_\varrho(y^\varrho).
\end{aligned} \tag{39}$$

**Example 4.** *Consider the local fractional PE equation is given as [56]*

$$\frac{\partial^\varrho u(x,y)}{\partial y^\varrho} + \frac{\partial^\varrho u(x,y)}{\partial x^\varrho} = 0, \ \ 0 < \varrho \le 1 \tag{40}$$

*with the initial condition*

$$u(x,0) = \sin_\varrho(x^\varrho). \tag{41}$$

*Using the fractional local Sumudu transformation on Equation (40), we have*

$$U_\varrho(x,z) = u(x,0) + z^{2\varrho} LFS_\varrho\left[-\frac{\partial^\varrho u(x,y)}{\partial x^\varrho}\right], \tag{42}$$

*which implies*

$$U_\varrho(x,z) = \sin_\varrho(x^\varrho) + z^{2\varrho} LFS_\varrho\left[-\frac{\partial^\varrho u(x,y)}{\partial x^\varrho}\right]. \tag{43}$$

*Using the local fractional inverse Sumudu transformation on Equation (43) is obtained as*

$$u(x,y) = \sin_\varrho(x^\varrho) + LFS_\varrho^{-1}\left[z^{2\varrho} LFS_\varrho\left[-\frac{\partial^\varrho u(x,y)}{\partial x^\varrho}\right]\right]. \tag{44}$$

*Now, applying homotopy perturbation method, we obtain*

$$\sum_{n=0}^{\infty} p^n u_n(x,y) = \sin_\varrho(x^\varrho) + LFS_\varrho^{-1}\left[z^{2\varrho} LFS_\varrho\left[-\frac{\partial^\varrho \sum_{n=0}^{\infty} p^n u_n(x,y)}{\partial x^\varrho}\right]\right]. \tag{45}$$

*We obtain the following component of the series result by comparing the like powers of p.*

$$
\begin{aligned}
p^0 &: u_0(x,y) = \sin_\varrho(x^\varrho), \\
p^1 &: u_1(x,y) = \cos_\varrho(x^\varrho)\frac{y^\varrho}{\Gamma(1+\varrho)}, \\
p^2 &: u_2(x,y) = \cos_\varrho(x^\varrho)\frac{y^\varrho}{\Gamma(1+\varrho)} - h^2\sin_\varrho(x^\varrho)\frac{y^{2\varrho}}{\Gamma(1+2\varrho)}, \\
&\vdots
\end{aligned}
\tag{46}
$$

*We obtain the series-form solution of Equation (40), given as*

$$
\begin{aligned}
u(x,y) &= \lim_{N\to\infty}\sum_{n=0}^{\infty} u_n(x,y), \\
&= \sin_\varrho(x^\varrho) - \cos_\varrho(x^\varrho)\frac{y^\varrho}{\Gamma(1+\varrho)} - \sin_\varrho(x^\varrho)\frac{y^{2\varrho}}{\Gamma(1+2\varrho)} + \cdots .
\end{aligned}
\tag{47}
$$

*Equation (47) can be presented in the following way*

$$
\begin{aligned}
u(x,y) &= \sin_\varrho(x^\varrho)\sum_{l=0}^{\infty}(-1)^l\frac{y^{(2l+1)\varrho}}{\Gamma(1+(2l+1)\varrho)} - \cos_\varrho(x^\varrho)\sum_{l=0}^{\infty}(-1)^l\frac{y^{2l\varrho}}{\Gamma(1+2l\varrho)}, \\
&= \sin_\varrho(x^\varrho)\sin_\varrho(y^\varrho) - \cos_\varrho(x^\varrho)\cos_\varrho(y^\varrho).
\end{aligned}
\tag{48}
$$

## 6. Conclusions

This paper investigates the fractional-order local Poisson equation using the homotopy perturbation transformation method. The Poisson equation explains the potential area resulting from a given charge, and if the potential area is known, one can compute the electrostatic or gravitational area in the fractal domain. There is frequent usage of elliptic partial differential equations in the modeling of electromagnetic mechanisms. This paper investigates the Poisson equation in the context of a local fractional operator. Several illustrative difficulties are given concerning the fractional local Poisson equation. The needed results illustrate the well-organized and uncomplicated nature of the homotopy perturbation transformation approach for partial differential equations with fractional derivatives in the local fractional operator sense. The mentioned methodologies' results demonstrate that the suggested system is simple to implement and computationally precise.

**Author Contributions:** Data curation, M.A.; formal analysis, K.M.S.; funding acquisition, W.W.; methodology, R.S.; project administration, K.M.S.; resources, W.M.H. and W.W.; supervision, W.W.; writing—original draft, R.S. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** The numerical data used to support the findings of this study are included within the article.

**Acknowledgments:** The authors are thankful to the Deanship of Scientific Research at Najran University for funding this work under the General Research Funding program grant code (NU/RG/SERC/11/3).

**Conflicts of Interest:** The authors declare that there is no conflict of interest regarding the publication of this article.

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