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Supersymmetric Partners of the One-Dimensional Infinite Square Well Hamiltonian: Special Cases

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Abstract: In a previous paper, we used a classification of the self adjoint extensions, also called self-adjoint determinations, of the differential operator $-d^2/dx^2$ in order to obtain the whole list of Supersymmetric (SUSY) partners of those selfadjoint determinations for which the ground state has strictly positive energy. The existence of self adjoint determinations with a ground state of zero or even negative energy is a proved fact. In this paper, we analyze the possibility of constructing SUSY partners for those determinations. We also study those cases for which the ground state has a degeneracy, the study of their SUSY partners should be analyzed separately. So far, we have studied those determinations having an exactly solvable eigenvalue problem. On the present study, we also included some comments in relation to determinations not exactly solvable from this point of view. In addition, the use of self adjoint determinations for which the ground state wave function has nodes (zeroes) produces formal SUSY partners with a finite number of eigenvalues or even with a purely continuous spectrum. We give some worked examples of these situations.

Keywords: supersymmetric quantum mechanics; self-adjoint extensions; infinite square well; contact potentials; supersymmetric confluent transform

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1. Introduction

One dimensional quantum models are often useful for a better understanding of quantum systems. The infinite one dimensional square well (or particle of mass m in a one-dimensional box) is characterized by an infinite value of the potential outside a finite interval, in which this potential is zero. Elementary books in Quantum Mechanics give an exact solution to the eigenvalue problem (Schrödinger equation with boundary conditions) provided by this model, showing a countably infinite number of energy levels given by the formula $E_n = (n\pi\hbar)^2/2mL^2$, $n = 1, 2, 3, \ldots$, where m has the dimensions of a mass and L is the length of the interval in which the potential is equal to zero. These energy levels has been determined using the boundary condition that the solutions of the Schrödinger equation $-\frac{\hbar^2}{2m}\frac{d^2\phi(x)}{dx^2} = E\phi(x)$ vanishes at the edges of the interval with zero potential. We call this solution the *textbook solution*.

From the mathematical point of view, this is the same that solving the eigenvalue problem $H\psi = E\psi$ with $H = -d^2/dx^2$ as an operator on the Hilbert space $L^2[-a, a]$, where we have chosen $L = 2a$ and $\hbar^2/2m = 1$ for simplicity. So far, we always write $-D^2$ to denote the operator $-d^2/dx^2$ on the Hilbert space $L^2[-a, a]$, unless otherwise stated.

The choice of the interval is really irrelevant under the condition that it be finite. If we demand that the solutions vanish inside the region where the potential is infinite and assume that they *should be continuous at all points*, which implies that they should be zero at the edges (in our case at $x = \pm a$), we obtain the textbook solution for the energy levels. This is all well known.

However, this is not the whole story. Recall that a Hamiltonian should be a self adjoint operator (we discard other considerations such that pseudo-hermiticity or PT-symmetric Hamiltonians). In addition, although we always should require to the wave functions to be zero inside the region of infinite potential, there is no clear reason why they should be continuous at all points. They have to be square integrable and this allows for finite jumps of the wave function at some points. Thus, we may consider wave functions which do not vanish at the edges $x = \pm a$.

The operator $-D^2 := -d^2/dx^2$ on the Hilbert space $L^2[-a, a]$ is an unbounded operator, which in particular means that it is not defined on all functions of the Hilbert space, but instead on all functions belonging to the dense subspace $\mathcal{D}$ of all absolutely continuous square integrable functions on $[-a, a]$ with absolutely continuous first derivative and whose second derivative, which is well defined except possibly on a set of zero Lebesgue measure, is also square integrable on the same interval. Let us also consider the following scalar product of functions $f(x), g(x) \in L^2[-a, a]$, $D := d/dx$, and use integration by parts:

$$\begin{aligned}\langle f|(-D^2)g\rangle = \int_{-a}^{a} f^*(x)(-D^2 g)(x)\,dx = -\{f^*(a)g'(a) - f^*(-a)g'(-a)\} \\ +\{f^{*\prime}(a)g(a) - f^{*\prime}(-a)g(-a)\} + \langle(-D^2)f|g\rangle\,.\end{aligned} \tag{1}$$

Now, the objective is to characterize all self adjoints determinations (also called *extensions* [1], both expressions will be used indistinctly along the paper) of the differential operator $-D^2$ on $L^2[-a, a]$. To do it, we need to characterize their respective domains, which are dense subspaces of $L^2[-a, a]$ contained in $\mathcal{D}$. The procedure is well know, since functions $f(x)$ belong to one of these domains if and only if they fulfil the following relation:

$$\{f^{*\prime}(a)g(a) - f^{*\prime}(-a)g(-a)\} - \{f^*(a)g'(a) - f^*(-a)g'(-a)\} = 0\,. \tag{2}$$

The terminology of self adjoint determinations or self adjoint extensions of a symmetric operator, in particular the differential operator $-D^2$, is explained in Appendix C.

As proven in [2], this condition may be written in matrix form as $f(x) \in L^2[-a, a]$:

$$\begin{pmatrix} 2a\,f'(-a) - if(-a) \\ \\ 2a\,f'(a) + if(a) \end{pmatrix} = U \begin{pmatrix} 2a\,f'(-a) + if(-a) \\ \\ 2a\,f'(a) - if(a) \end{pmatrix}, \tag{3}$$

where U is an arbitrary 2×2 unitary matrix. In fact, according to a Theorem due to von Neumann [3], each unitary 2×2 matrix gives different choices of domains for $-D^2$ in which $-D^2$ is self adjoint. Each of the 2×2 unitary matrices is characterized by the particular choice of a set of four independent real parameters. The most general form of these matrices is [2]:

$$U = e^{i\psi} \begin{pmatrix} m_0 - im_3 & -m_2 - im_1 \\ \\ m_2 - im_1 & m_0 + im_3 \end{pmatrix}, \tag{4}$$

with $\psi \in [0, \pi]$, $m_0^2 + m_1^2 + m_2^2 + m_3^2 = 1$. Note that there are four independent parameters, ψ and three out of the m_i, where the dependent parameter may be conveniently chosen. Each of the choices of these parameters gives relations between the boundary values of functions and their derivatives and these relations determine the domains of each self adjoint determination of $-D^2$. A pedagogical presentation of the self-adjoint determinations of $-D^2$ is given in [2]. All them have some features in common such as having a purely discrete spectrum with an infinite number of eigenvalues and eigenfunctions, as happens for the particular case studied in textbooks. This fact is shown in [4], vol 2, p. 90.

The study of supersymmetric (SUSY) partners of a given Hamiltonian has been an object of study during the past few decades. As a result, many articles have been published. For the benefit of the reader, we have selected to list in the bibliography just a

few [5–19]. This kind of supersymmetry intends the construction of a sequence of Hamiltonians, derived from an original Hamiltonian, with a closely related energy spectrum, which could be obtained from the spectrum of this original Hamiltonian.

In a previous paper [20], we have obtained sequences of supersymmetric partners from a wide class of these self adjoint determinations of $-D^2$. In the studied cases, being given a self adjoint determination of $-D^2$ on $L^2[-a,a]$, we obtain a sequence of partners with the property that each of the Hamiltonians of the sequence loses the ground state of the previous one; all the rest of the spectrum remains the same. All potentials in one and the same sequence have the form $-D^2+V$, where V is a potential obtained with the ground state of the previous Hamiltonian using a well known procedure [17] and that we describe in Appendix A. In all cases studied in [20], the energy of the ground state of the self-adjoint determination of $-D^2$ used is positive, no degenerate and its corresponding wave function has no nodes, i.e., no zeroes.

In spite of its appearance, not all self adjoint determinations of $-D^2$ have ground state with positive energy. Some determinations have ground state with zero energy and, most surprising, with negative energy, so that these determinations are not positive definite operators. In both cases, a result due to Naimark [4] shows that we may have either one (possible with double degeneracy) or two negative energy states, or a ground state with negative energy and a first excited state with zero energy, or a zero energy ground state, possibly with double degeneracy. No other possibility exists. Since the study of SUSY partners for these determinations is somehow different than for the rest, we call *exceptional cases* to this situations. The category of exceptional cases also include those determinations for which the ground state wave function has a node (zero). Their first formal partner may have a completely different properties than the original Hamiltonian, for instance it may have a finite number of eigenvalues or even a purely continuous spectrum. By formal partner, we mean a first SUSY partner constructed following the general rules as explained in Appendix A. To close the category of exceptional cases, we include those which are not exactly solvable in the sense that we cannot give for the first partner either the exact values of all eigenvalues or their respective wave functions.

The objective of the present article is the determination and study of the first partner in these exceptional cases. We should note that this determination means the construction of the potential of the Hamiltonian partner.

The present article is organized as follows: in Section 2, we summarize the main results given in [20] and makes the present article self contained. This summary will help us to understand the problem under consideration. We say in passing that we have corrected here an imprecision in [20]. In Section 3, we make a first attempt to classify the self adjoint extensions with ground state with zero or negative energy including degeneracies, such that the wave functions of these states still have symmetry properties. In Section 4, we construct their first or second SUSY partner. Note that for degenerate ground states a different technique should be used. Finally, on a brief Section 5, we give an example on how the absence of the mentioned symmetries for the wave function of the ground state may give a non-Hermitian first SUSY partner. The paper closes with three Appendices including the construction of first and second SUSY partners, with or without degeneracy on the ground state and one more on extensions of symmetric operators. This is known material that we add for the benefit of the reader.

2. Regular Cases

By regular cases, we mean the study of the SUSY partners of those self adjoint determinations of the operator $-D^2$ on $L^2[-a,a]$ (original Hamiltonian) which are exactly solvable and have strictly positive eigenvalues only. In addition to positivity, we add the extra conditions $m_2=m_3=0$ in order to assure the solvability of both the eigenvalue problem associated to the original Hamiltonian and the analytic determination of all SUSY partners at all orders. Along this section, by positivity we mean strict positivity, in the sense that all eigenvalues are strictly positive.

In the introduction of Formula (4), we have given an operator U that characterize all self adjoint determinations of $-D^2$ on $L^2[-a,a]$. This operator gives the boundary conditions that give the different domains of all self-adjoint determinations. Now, it is convenient to provide of a notation for the domain defined by U as $\mathcal{D}_U$, so that $\mathcal{D}_U$ is the domain of the self adjoint determination of $-D^2$ fixed by U.

Why is it possible to have determinations of $-D^2$ with negative energy? Assume that $\phi(x) \in \mathcal{D}(U)$. Then,

$$\langle \phi | - D^2 \, \phi \rangle = \langle -iD \, \phi | - iD \, \phi \rangle + \phi^*(-a)\phi'(-a) - \phi^*(a)\phi'(a) \, . \tag{5}$$

Observe that the right hand side of (5) may be negative for some $\phi(x) \in \mathcal{D}_U$. We give some examples in Section 3.

Regular cases have been studied in [20] and it is interesting to give a summary of the obtained results here. First of all, we have to characterize those self adjoint determinations of $-D^2$ on $L^2[-a,a]$ corresponding to the regular cases. Since all eigenvalues are positive, their general form should be $E = s^2/(2a)^2 > 0$, where s is a real discrete parameter. Determinations of $-D^2$ satisfying this condition are *strictly positive*. Recall that all extensions have an infinite number of eigenvalues. These eigenvalues must satisfy the time independent Schrödinger equation $-D^2\phi(x) = E\phi(x)$. The general form of the eigenfunctions for all self adjoint extensions should be:

$$\phi(x) = A \cos\left(\frac{sx}{2a}\right) + B \sin\left(\frac{sx}{2a}\right) , \tag{6}$$

where the coefficients A and B are determined by the boundary conditions. Now, let us consider strictly positive determinations. Using (6) in (4), we obtain the following system of linear equations on the parameters A and B:

$$(\mathcal{L}(s) - U\mathcal{M}(s)) \begin{pmatrix} A \\ \\ B \end{pmatrix} = \mathcal{N}(s) \begin{pmatrix} A \\ \\ B \end{pmatrix} = 0 \, , \tag{7}$$

where $\mathcal{N}(s) = (\mathcal{L}(s) - U\mathcal{M}(s))$, with

$$\mathcal{L}(s) := \begin{pmatrix} s \sin\frac{s}{2} - i\cos\frac{s}{2} & s\cos\frac{s}{2} + i\sin\frac{s}{2} \\ \\ -s\sin\frac{s}{2} + i\cos\frac{s}{2} & s\cos\frac{s}{2} + i\sin\frac{s}{2} \end{pmatrix} , \tag{8}$$

and

$$\mathcal{M}(s) = \begin{pmatrix} s \sin\frac{s}{2} + i\cos\frac{s}{2} & s\cos\frac{s}{2} - i\sin\frac{s}{2} \\ \\ -s\sin\frac{s}{2} - i\cos\frac{s}{2} & s\cos\frac{s}{2} + i\sin\frac{s}{2} \end{pmatrix} . \tag{9}$$

Since $\det \mathcal{N}(s) = 0$, otherwise the solutions of (7) would have been $A = B = 0$, one of the eigenvalues of $\mathcal{N}(s)$ must be zero and the other the trace of the matrix $\mathcal{N}(s)$, $\mathrm{Tr}\,[\mathcal{N}(s)]$. The determinant of $\mathcal{N}(s)$ is easy to calculate and, after some algebra, the expression $\det \mathcal{N}(s) = 0$ yields:

$$s \sin s = \frac{m_1 - \cos s \sin\psi}{m_0 - \cos\psi} \pm \sqrt{\left(\frac{m_1 - \cos s \sin\psi}{m_0 - \cos\psi}\right) + \frac{m_0 + \cos\psi}{m_0 - \cos\psi} \sin^2 s} \, . \tag{10}$$

These are two transcendental equations, one with plus sign and the other with minus sign, which give the values of s and, therefore, the values of the energy in terms of the parameters ψ, m_0 and m_1. This expression is valid for all positive determinations of $-D^2$ including those with $m_2 \neq 0$ and $m_3 \neq 0$. However, the energy values do not depend on m_2 and m_3.

We also want to obtain the values for $A(s)$ and $B(s)$ in order to obtain the wave function which corresponds to the eigenvalue E given by s, through $E = s^2/(2a)^2$. Although this coefficients have been obtained in [20], we recalculate them here as the result given in [20] was not fully correct. One interesting point is that, no matter if the energy levels do not depend on $m_2 \neq 0$ and $m_3 \neq 0$, the wave functions *do depend* on these parameters in general. To begin with, let us write the general form for the eigenvalues of the matrix $\mathcal{N}$ as:

$$\lambda_\pm = \frac{\mathrm{Tr}\,\mathcal{N}}{2} \pm \sqrt{\left(\frac{\mathrm{Tr}\,\mathcal{N}}{2}\right)^2}\,. \tag{11}$$

Since $\mathcal{N}$ is a complex matrix, its zero eigenvalue is not necessarily λ_- as claimed in [20]. Instead, the zero eigenvector is either λ_+ or λ_- depending whether the argument of $\mathrm{Tr}\,\mathcal{N}$ lies on the left, $\{z \in \mathbb{C}\ ;\ \mathrm{Real}\, z < 0\}$, or the right, $\{z \in \mathbb{C}\ ;\ \mathrm{Real}\, z > 0\}$, complex half-planes, respectively. Then, if we write the matrix $\mathcal{N}$ in simplified form as:

$$\mathcal{N} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{12}$$

the eigenvectors $(A(s), B(s))$ of $\mathcal{N}(s)$ with zero eigenvalue should have the form,

$$(A(s), B(s)) = \begin{cases} (-d, c)\,, & -\frac{\pi}{2} < \arg\left(\mathrm{Tr}\,\mathcal{N}\right) < \frac{\pi}{2}\,, \\ (a, c)\,, & \frac{\pi}{2} < \arg\left(\mathrm{Tr}\,\mathcal{N}\right) < \frac{3\pi}{2}\,. \end{cases} \tag{13}$$

This gives the following result, which has been corrected respect the result given in [20]:

$$A(s) = \begin{cases} s\cos\left(\frac{s}{2}\right)\left(-1 + e^{i\psi}(m_0 - im_1 + m_2 + im_3)\right) \\ -i\sin\left(\frac{s}{2}\right)\left(1 + e^{i\psi}(m_0 - im_1 + m_2 + im_3)\right) \qquad -\frac{\pi}{2} < \arg(\mathrm{Tr}\,\mathcal{N}) < \frac{\pi}{2}\,, \\ -i\cos\left(\frac{s}{2}\right)\left(1 + e^{i\psi}(m_0 + im_1 + m_2 - im_3)\right) \\ +s\sin\left(\frac{s}{2}\right)\left(1 - e^{i\psi}(m_0 + im_1 + m_2 - im_3)\right), \qquad \frac{\pi}{2} < \arg(\mathrm{Tr}\,\mathcal{N}) < \frac{3\pi}{2} \end{cases} \tag{14}$$

and

$$\begin{aligned} B(s) = s\Big[-1 + e^{i\psi}(m_0 + im_1 + m_2 - im_3)\Big]\sin\frac{s}{2} \\ + \Big[i + e^{i\psi}(im_0 - m_1 + im_2 + m_3)\Big]\cos\frac{s}{2}\,. \end{aligned} \tag{15}$$

From these results, one may study the behaviour of the self adjoint extensions preserving positivity, i.e., those with a ground state with positive energy. Partial results may be found in [2,20]. The form for these coefficients with $m_2 = m_3 = 0$ is obvious.

In addition to the angular parameter ψ, the relation $\sum_{i=0}^{3} m_i^2 = 1$ suggests the possibility of writing all other parameters in terms of some angular variables θ_i, $i = 0, 1, 2$ as follows:

$$m_0 = \cos\theta_1 \cos\theta_0\,,\ m_1 = \cos\theta_1 \sin\theta_0\,,\ m_2 = \sin\theta_1 \cos\theta_2\,,\ m_3 = \sin\theta_1 \sin\theta_2\,. \tag{16}$$

We have shown that the energy levels of those (strictly) positive extensions of $-D^2$ are solely determined by the values of three parameters: (ψ, m_0, m_1). Alternatively, they depend on three angular variables only, $(\psi, \theta_0, \theta_1)$, as we may infer from (16). However, due to the complicated form of the transcendental equation (10), it is not possible in general to obtain an explicit expression of the form $s = s(\psi, \theta_0, \theta_1)$ from (10) and (16). This explicit expression would have been very useful in order to obtain the spectrum of all the considered self adjoint extensions. Nevertheless, in some particular cases, simple transcendental equations can be given, which have been studied in [20]. For instance, if the wave equation

for the ground state, $\phi(x)$, is parity invariant, which means $|\phi(x)|^2 = |\phi(-x)|^2$. This happens if and only if *at least* one of the following three relations hold:

$$m_3 = 0\,, \tag{17}$$

$$\sin s = 0\,, \tag{18}$$

$$s\sin s = \frac{m_2 + \cos s\cos\psi}{m_3 - \sin\psi} \pm \sqrt{\left(\frac{m_2 + \cos s\cos\phi}{m_3 - \sin\psi}\right) + \frac{m_3 + \sin\psi}{m_3 - \sin\psi}\sin^2 s}\,. \tag{19}$$

Before going on, it is important to say that there exists a group of extensions for which the solution, $\psi(x,t)$, of the time dependent Schrödinger equation is time reversal invariant in the sense that $\psi(x,+t) = \psi(x,-t)$. These are characterized by $m_2 = 0$.

Then, the three relations (19) shows that these tractable cases may be grouped into three clases:

1. Extensions with parity and time reversal invariance. These are those for which $m_2 = m_3 = 0$ and, therefore, $m_0^2 + m_1^2 = 1$, so that $m_0 = \cos\theta_0$ and $m_1 = \sin\theta_0$. Now, the transcendental Equation (10) becomes much simpler and the sign depends on the possible values of s. Thus, the sign is positive if s lies on the intervals $(0,\pi), (2\pi,3\pi), \ldots$ and is negative if s belongs to the intervals $(\pi,2\pi), (3\pi,4\pi), \ldots$. In both cases, the transcendental equations given the values of the energy are, respectively,

$$s\tan\left(\frac{s}{2}\right) = -\cot\left(\frac{\psi+\theta_0}{2}\right), \qquad \text{plus sign in } (10) \tag{20}$$

$$s\cot\left(\frac{s}{2}\right) = \cot\left(\frac{\psi-\theta_0}{2}\right), \qquad \text{minus sign in } (10)\,. \tag{21}$$

Note that $\frac{\psi+\theta_0}{2}$ and $\frac{\psi-\theta_0}{2}$ are two independent angles varying from 0 to $\pi/2$. The solutions in s, s_n, $n = 0,1,2,\ldots$ of (20) and (21), give the energy levels, $E = s_n^2/(2a)^2$. The respective eigenfunctions are:

$$\phi_n(x) = \cos\left(\frac{s_n x}{2a}\right), \quad n = 0,2,4,\ldots,$$

$$\phi_n(x) = \sin\left(\frac{s_n x}{2a}\right), \quad n = 1,3,5,\ldots. \tag{22}$$

2. Parity preserving extensions fulfilling $\sin s = 0$. The energy levels are $E_n = (n^2\pi^2)/(2a)^2$, $n = 1,2,\ldots$, which correspond to the energy levels obtained for the extension described in textbooks.
3. Parity preserving extensions fulfilling (19).

In [20], we have obtained sequences of SUSY partners for these extensions. It is time to drop the positivity condition and consider more general situations and this is precisely the goal of the present article, to be discussed next.

3. Transcendental Equation for Other Cases

In addition to positive extensions, which are those with ground state with strictly positive energy, two other situations are possible" either the ground state has zero energy or negative energy. In the latter case, as mentioned in the Introduction, there are one or two negative energy levels, so that $E = s^2/(2a)^2$ implies $s = ir$ with r real for these levels. Zero energy means one solution with $s = 0$.

Then, let us go back to the matrix $\mathcal{N}$ in (12). This matrix has different forms depending on the three cases considered, which are: (i) Positive ground state energy $E > 0$, $\mathcal{N}_{E>0}(s)$; (ii) Negative ground state energy, $E < 0$, $\mathcal{N}_{E<0}(r)$; (iii) Zero ground state energy, $E = 0$,

$\mathcal{N}_{E=0}$. These matrices are rather complicated and their explicit form is not very interesting. The condition that their respective determinants are equal to zero gives the following respective transcendental equations:

(i) For $E = s^2 > 0$, $\det \mathcal{N}_{E>0}(s) = 0$ gives:

$$(m_0 + \cos\psi)\sin s + 2(m_1 - \cos s \sin\psi)s + \sin s(\cos\psi - m_0)s^2 = 0\,; \tag{23}$$

(ii) For $E = -r^2 < 0$, $\det \mathcal{N}_{E<0}(r) = 0$ gives,

$$(m_0 + \cos\psi)\sinh r + 2(m_1 - \cosh r \sin\psi)r + \sinh r(m_0 - \cos\psi)r^2 = 0\,; \tag{24}$$

(iii) For $E = 0$, $\det \mathcal{N}_{E=0} = 0$ gives:

$$m_0 - 2m_1 + \cos\psi - 2\sin\psi = 0\,. \tag{25}$$

While (23) gives all possible positive values of the energy in terms of s, Equation (24) gives the possible values with negative energy (two maximum). Equation (25) gives the relation that the parameters should satisfy for the existence of a ground state with zero energy.

So far, we have concentrated ourselves in positive extensions or determinations of $-D^2$ on $L^2[-a, a]$ and the calculation of their SUSY partners. After we drop the positivity condition, we have to select solvable extensions in the sense that we may, at least, find the explicit for for the wave function of the ground state. As mentioned earlier, this wave function is essential in order to obtain the first partner.

The "solvable" cases object of our next study are those preserving time reversal invariance and parity inversion, with the previously discussed senses. This implies that at the same time our determinations of $-D^2$ verify two conditions: (i) $m_2 = 0$, for the time reversal invariance and (ii) any or several of (17)–(19).

In terms of the new independent angular parameters $(\theta_0, \theta_1, \theta_2, \psi)$ as defined in (16), we are considering extensions with $\theta_1 = \theta_2 = 0$, so that these extensions are labelled by $(\theta_0, 0, 0, \psi)$.

Now, transcendental equations given the energy values are quadratic on the square root of the energy, which is given by s for the positive energy solutions and by r for the negative energies. Transcendental equations may be factorized into two equations on the respective variables s and r. They have even or odd ground state wave functions, respectively.

In the next table, we we list these transcendental equation with their respective ground state wave function.

In Table 1, we have obtained the transcendental equation given the negative energy eigenvalues (two maximum) by making the change $s \to ir$, as previously mentioned. It is convenient the use of graphics in order to describe the behaviour of the solutions of the above transcendental equations.

In Figure 1, we observe the zone of negative energies from the plane $s = 0$. We have divided the figure into several regions. Orange and blue color represent the values of (θ_0, ψ) for which there are negative eigenvalues of even or odd eigenfunctions, respectively. In the absence of color (white) the values of (θ_0, ψ) determine a ground state with positive energy. In the regions (a) and (b), we have only one state with negative energy, which is the ground state. Contrarily, regions (d) and (c) display two eigenvalues with negative energy. In the case of (d) the orange color superposes the blue color. This means that the ground state is odd and the first excited state is even. The opposite is true in the region (c). Dashed lines correspond to ground states with zero energy and the red line shows negative energy doubly degenerated.

Table 1. Transcendental equations and ground state functions for the studied extensions. Note that we two solutions for each case, depending whether the wave function is even or odd. Under the label "kind", we give the sign of the energy of the ground state and the parity of this wave function.

Kind	Transcendental Equation	Ground State Eigenfunction
Positive and even	$s\sin\left(\frac{s}{2}\right)\sin\left(\frac{\theta_0+\psi}{2}\right)+\cos\left(\frac{s}{2}\right)\cos\left(\frac{\theta_0+\psi}{2}\right)=0$	$\sqrt{\frac{s}{a(s+\sin s)}}\cos\left(\frac{sx}{2a}\right)$,
Positive and odd	$s\cos\left(\frac{s}{2}\right)\sin\left(\frac{\theta_0-\psi}{2}\right)+\sin\left(\frac{s}{2}\right)\cos\left(\frac{\theta_0-\psi}{2}\right)=0$	$\sqrt{\frac{s}{a(s-\sin s)}}\sin\left(\frac{sx}{2a}\right)$,
Zero and even	$m_0+2m_1+\cos\psi-2\sin\psi=0$	$\frac{1}{\sqrt{2a}}$,
Zero and odd	$m_0+2m_1+\cos\psi-2\sin\psi=0$	$\sqrt{\frac{3}{2a^3}}\,x$,
Negative and even	$r\sinh\left(\frac{r}{2}\right)\sin\left(\frac{\theta_0+\psi}{2}\right)-\cosh\left(\frac{r}{2}\right)\cos\left(\frac{\theta_0+\psi}{2}\right)=0$	$\sqrt{\frac{r}{a(\sinh r+r)}}\cosh\left(\frac{rx}{2a}\right)$,
Negative and odd	$r\cosh\left(\frac{r}{2}\right)\sin\left(\frac{\theta_0-\psi}{2}\right)+\sinh\left(\frac{r}{2}\right)\cos\left(\frac{\theta_0-\psi}{2}\right)=0$	$\sqrt{\frac{r}{a(\sinh r-r)}}\sinh\left(\frac{rx}{2a}\right)$.

In Figure 2, we include a three-dimensional picture, where the third coordinate corresponds to the (negative) energy. It is clear here that the degenerate levels are in the intersection of the two latter surfaces on Table 1. Each of these surfaces are written on the form $F(\psi,\theta_0,r)=0$.

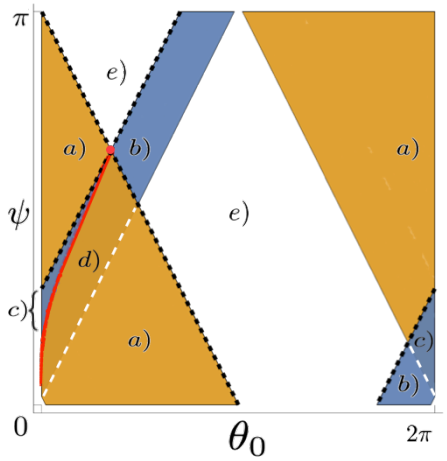

Figure 1. Regions in the parameter space (θ_0,ψ) for which we have one or two negative eigenvalue with negative energy, as explained in the text. White zones correspond to solutions with positive energy eigenvalues only.

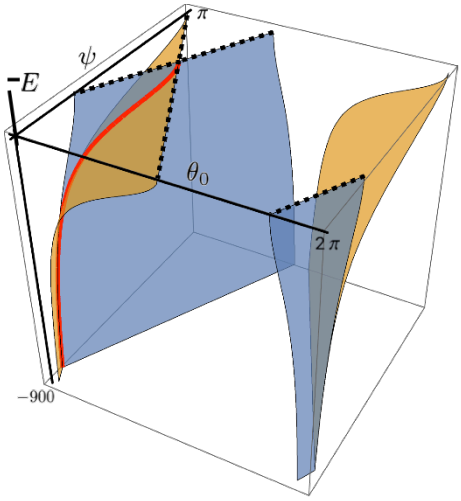

Figure 2. The two last surfaces on Table 1, corresponding to even (orange) and odd (blue) wave function. The parameter space is here $(\theta_0, \psi, -E)$ with $-E = r^2/(2a)$.

We give a summary of these results on Table 2. In the first column, a label denotes the corresponding regions in Figure 1. The last column gives the signs of the energy of the ground state and the first few excited states corresponding to the classes of self adjoint extensions described on columns 2 and 3.

Note that the parity of the first excited state having positive energy may be either odd or even, independently of the parity of the ground state. Meanwhile, if the first excited state has negative or zero energy, the parity of its wave function is the opposite to the parity of the ground state wave function.

In Figure 3, we depict the regions in the space of parameters $(\theta_0, \theta_1, \psi)$ for which there exists a ground state with negative energy. The empty regions give the values of the parameters for those extensions with ground state with positive energy. The black net covers those surfaces for which there exists zero energy levels, characterized by $m_0 + 2m_1 + \cos\psi - 2\sin\psi = 0$.

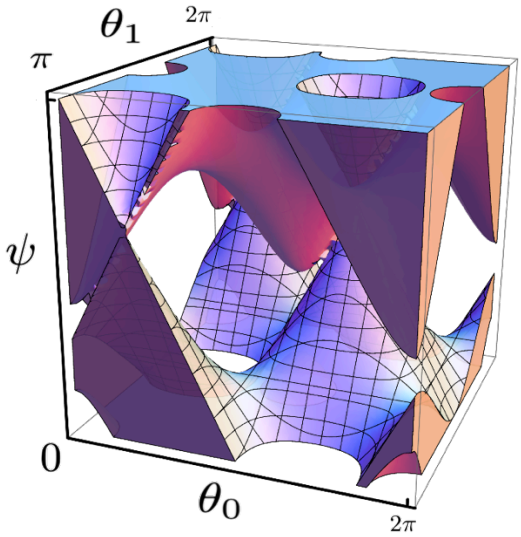

Figure 3. Values of $(\theta_0, \theta_1, \psi)$ giving extensions with ground state of negative energy, the black net represents where a state with null energy is located. Note that these regions are three dimensional.

Table 2. In the first column, we list the regions in accordance to Figure 1. In the second column, we give the sign of the energy of the first excited state and the parity of its wave function, provided that the ground state have zero or negative energy. Same for the first excited state in the third. Finally, the last column gives the signs of the forth first labels. All other levels have positive energy. The red line shows that the ground state has negative energy and is double degenerate. The red dot in Figure 1 shows a double degenerate ground state with zero energy, as indicated in the last row.

Region Label	Ground State	First Excited State	Sign of the Energy for the First Levels
(a)	Negative and even	Positive	$(-,+,+,+,\ldots)$
(b)	Negative and odd	Positive	$(-,+,+,+,\ldots)$
(c)	Negative and even	Negative and odd	$(-,-,+,+,\ldots)$
(d)	Negative and odd	Negative and even	$(-,-,+,+,\ldots)$
(e)	Positive	Positive	$(+,+,+,+,\ldots)$
(b) $\cap$ (d)	Negative and odd	Zero and even	$(-,0,+,+,\ldots)$
(a) $\cap$ (c)	Negative and even	Zero and odd	(-,0,+,+, ...)
(e) $\cap$ (b)	Zero odd	Positive	(0,+,+,+, ...)
(e) $\cap$ (a)	Zero even	Positive	(0,+,+,+, ...)
Red line	Negative and doubly degenerate	Same	$(-,-,+,+,\ldots)$
Red dot	Zero and doubly degenerate	Same	$(0,0,+,+,\ldots)$

The black dashed lines in Figure 1 mean the intersection of the plane $E = 0$ with the two last surfaces in the mid column of Table 1. The intersection of these two surfaces give the degeneration of the negative energy levels. The equations on the plane $E = 0$ of these dashed lines are as follows:

(i) The intersection of $E = 0$ with the forth surface in Table 1 is given by

$$\psi = \theta_0 + 2\,\mathrm{arccot}\,2\,. \tag{26}$$

(ii) The intersection of $E = 0$ with the third surface in Table 1 is given by

$$\psi = -\theta_0 + \pi\,. \tag{27}$$

(iii) There is only one situation in which there exists one degenerate level with zero energy. This is given by the following value of the parameters:

$$(\theta_0, \psi) = \left(\frac{\pi}{2} - \mathrm{arccot}\,2, \frac{\pi}{2} + \mathrm{arccot}\,2\right). \tag{28}$$

The corresponding even and odd eigenfunctions are given as the third and forth entry in the third column of Table 1, respectively.

When the ground state is doubly degenerate (28), we may also obtain the remaining values of the spectrum, which are all positive. For even and odd wave functions, the values of the energies are respectively given by the following transcendental equations:

$$s\sin\left(\frac{s}{2}\right) = 0\,, \qquad s\cos\left(\frac{s}{2}\right) - 2\sin\left(\frac{s}{2}\right) = 0\,. \tag{29}$$

Obviously, both equations have a solution for $s = 0$, hence the double degeneracy of the ground level. The solutions of the first equation in (29) are obvious and of the second are given by the intersections of $\cot(s/2)$ with $2/s$.

In Figure 4, we show the values of the parameters ψ and θ_0 for which the first positive excited state has wave functions with parity odd (blue) or even (orange). The point at which the dotted lines cross give the degenerate level with zero energy. Negative energy levels have been depicted in Figure 1.

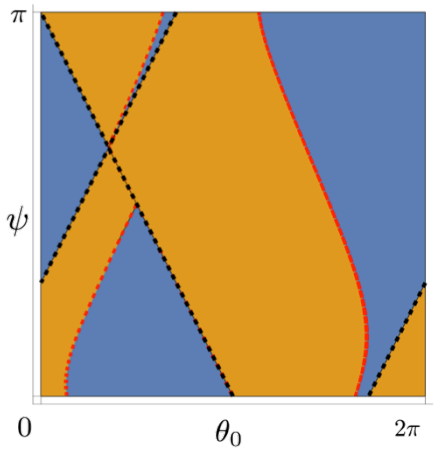

Figure 4. Regions in which the first excited state with positive energy is even, orange or odd, in blue.

In Figure 5, we represent the surfaces $f(\psi, \theta_0, s) = 0$ given the states with positive energy with wave equations with parities even (orange) or odd (blue). Contrary to the first energy levels classified in Table 2, there are plenty subcategories, so that the equivalent of Table 2 in this case would have been extremely complicated. Note that Figure 4 is nothing else than the projection into the plane $E = 0$ of Figure 5.

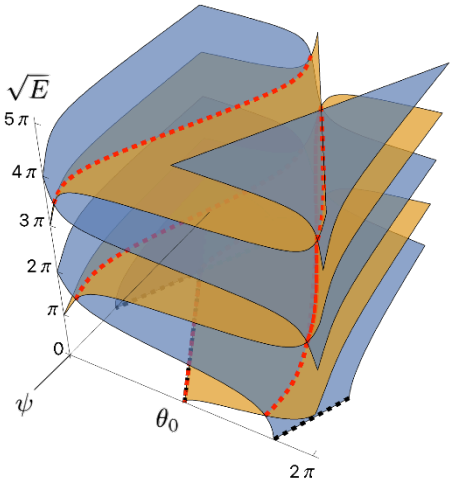

Figure 5. Surfaces corresponding the positive eigenvalues. Orange and blue colors represent even or odd states, respectively.

Concerning solutions with ground state with positive energy, we arrive to the following conclusions for spectra and eigenfunctions of extensions in terms of the parameters (θ_0, ψ):

- The energy levels and the eigenfunctions for the extensions characterized by the values $(\theta_0, \psi) = (0, 0)$, $(\theta_0, \psi) = (2\pi, 0)$ and $(\theta_0, \psi) = (\pi, \pi)$ coincide with those given in textbooks:

$$E_n := \left(\frac{n\pi}{2a}\right)^2, \quad \phi_n(x) = \begin{cases} \cos\left(\frac{n\pi x}{2a}\right), & n = 1, 3, 5 \dots, \\ \sin\left(\frac{n\pi x}{2a}\right), & n = 2, 4, 6 \dots \end{cases}, \tag{30}$$

  with $\hbar = 2m = 1$. We call this result the *textbook solution*.

- There are two extensions with a double degenerate positive spectrum, which are given by the values $(\theta_0, \psi) = \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $(\theta_0, \psi) = \left(\frac{3\pi}{2}, \frac{\pi}{2}\right)$. The former has a ground state of odd parity and its energy levels are:

$$E_n = \frac{n^2\pi^2}{a^2}, \qquad n = 0, 1, 2, \dots, \tag{31}$$

with respective wave functions:

$$\phi_0(x) = \sqrt{\frac{3}{2a^3}}\, x, \quad n = 0, \quad \phi_n(x) = \cos\frac{n\pi}{a}\, x \quad \text{and} \quad \sin\frac{n\pi}{a}\, x, \quad n = 1, 2, \dots. \tag{32}$$

From (32), we observe that excited states are doubly degenerate. The extension given by $(\theta_0, \psi) = \left(\frac{3\pi}{2}, \frac{\pi}{2}\right)$ has a ground state with even parity with $E_0 = 0$. The remaining energy values are

$$\phi(x) = \frac{1}{\sqrt{2a}}, \quad n = 0, \quad \phi_n(x) = \cos\frac{(2n-1)\pi}{2a}\, x \quad \text{and} \quad \sin\frac{(2n-1)\pi}{2a}\, x, \tag{33}$$

with $n = 0, 1, 2, \dots$. Again we observe a double degeneracy for all states save for the ground state.
- In general, it is not possible the determination of all the values of the energy, since they come after the solution of a transcendental equation. Using Mathematica or another numerical method, we may obtain a finite number of eigenvalues for each extensions, but no more. On the other hand, once we have the value of the energy, we have the corresponding eigenfunction. Nevertheless the higher the value of n the closer the energy levels are to the levels of the textbook solution given in (30).

4. Supersymmetric Partners

Along the present Section, we classify the SUSY partners of those self adjoint extensions having an exactly solvable eigenvalue problem with zero or negative energy ground state, either degenerate or not. The general procedure for the construction of SUSY partners of a given self adjoint determination of $-D^2$ is discussed in Appendix A. However, we have to consider the case in which this ground state has a double degeneracy. The construction of the first SUSY partner in this situation is somehow different as this construction works for the non degenerate ground state, as described in Appendix B. Ground states for exactly solvable determinations of $-D^2$ have a fixed parity.

In order to comply with the usual notation, let us use $H^{(0)} := -D^2$ from now on, in order to denote an arbitrary self adjoint extension.

For the construction of SUSY partners of the self adjoint extensions that are the object of our attention here, we begin with a self adjoint extension with doubly degenerate ground state. The general formalism should be adapted to this new situation as discussed in the sequel.

4.1. Second Order Confluent SUSY

In Appendix A, we have sketched the construction of the first SUSY partner of a given self adjoint operator. However, if the condition that the ground state of the operator $H^{(0)}$ is doubly degenerate applies, we have to use a modification of the mentioned construction. As customary, we start with a self adjoint extension $H^{(0)}$ with spectrum $\{E_0^{(0)}, E_1^{(0)}, E_2^{(0)}, \dots\}$ and respective eigenfunctions $\{\phi_0^{(0)}(x), \phi_1^{(0)}(x), \phi_2^{(0)}(x), \dots\}$. Then, we assume that $E_0^{(0)} = E_1^{(0)}$, so that the ground state is doubly degenerate. Thus, the functions $\phi_0^{(0)}(x)$ and $\phi_1^{(0)}(x)$ are different eigenfunctions of $E_0^{(0)}$, which may be taken as mutually orthogonal. Then, we use the second order confluent SUSY formalism introduced in [21]. The crucial point lies in the use of a generalized wave function defined with the help of any of the orthogonal eigenfunctions of $E_0^{(0)}$:

$$v_j(x) = u_j(x) \int \frac{w_0 - \int u_j^2(x)dx}{u_j^2(x)} dx\,. \tag{34}$$

In (34), the function $u_j(x)$ is either $\phi_0^{(0)}(x)$ or $\phi_1^{(0)}(x)$, we may choose one of them, so that $j = 0, 1$.

The constant w_0 is chosen so as to avoid the presence of nodes in the function $v_j(x)$ [21]. As seen in Appendix A, first SUSY partners are well defined for those extensions for which ground states have no zeros, also called *nodes*. In Appendix B, we have discussed the difficulties for a proper consideration of the first order SUSY partner of those extensions with ground state with degeneracy, so that we have to go directly to the second order partner and skip the first one.

Thus, the second order SUSY partner of $H^{(0)}$ is given by [21,22]:

$$H^{(2,2)} = -\partial_x^2 + V^{(2,2)}\,, \quad V^{(2,2)} = V^{(0)} - 2\partial_x^2 \log \mathcal{W}(v_j(x), \phi_j^{(0)}(x))\,, \quad j = 0, 1\,, \tag{35}$$

with $\partial_x := d/dx$, and $\mathcal{W}$ denotes the Wronskian. The eigenfunctions are:

$$\phi_n^{(2,2)}(x) = \frac{\mathcal{W}(v_j(x), \phi_j^{(0)}(x), \phi_n^{(0)})(x)}{\mathcal{W}(v_j(x), \phi_j^{(0)}(x))}\,, \quad j = 0, 1\,, \quad n \geq 2\,. \tag{36}$$

Then, we list the second order SUSY partners of the extensions with negative or zero ground state degenerate or not, which are exactly solvable. We classify these partners using the values of the parameters that fix the extension, with the additional criterium whether the ground state is either degenerate or not. Here, we do not go beyond the second partner, since the study and classification of higher order partners goes as previously studied in [20]. All the extensions depend on the parameters $(\cos\theta_0, \sin\theta_0, 0, 0, \psi)$.

4.1.1. Degenerate Ground State with Zero Energy

The only exactly solvable extension having a degenerate ground state with zero energy is characterized by the following values of the parameters:

$$(\theta_0, \psi) = \left(\frac{\pi}{2} + \operatorname{arccot} 2, \frac{\pi}{2} - \operatorname{arccot} 2\right), \tag{37}$$

being the remainder parameters equal to zero. The two dimensional eigenspace of functions with energy zero is spanned by one odd and one even eigenfunctions. The explicit values of the energy for excited states depend on whether they have even parity, with energy levels E_{2n}, $n > 1$, or odd parity, with energy levels E_{2n+1}. For even levels, E_{2n} is proportional to $(2n)^2\pi^2$, while for odd levels their energy is given by a transcendental equation. For high values of n, these levels go to $E_{2n+1} \propto (2n+1)^2\pi^2$. The list of the eigenfunctions is:

$$\left\{\phi_0^{(0)}(x), \phi_1^{(0)}(x), \phi_2^{(0)}(x), \phi_3^{(0)}(x), \dots\right\} = \left\{\frac{1}{\sqrt{2a}}, x, \cos\left(\frac{s_2 x}{2a}\right), \sin\left(\frac{s_3 x}{2a}\right), \dots\right\}, \tag{38}$$

where we have omitted the normalization constants. Their corresponding values of the energy are:

$$\left\{E_0^{(0)}, E_1^{(0)}, E_2^{(0)}, E_3^{(0)}, \dots\right\} = \left\{0, 0, \left(\frac{s_2}{2a}\right)^2, \left(\frac{s_3}{2a}\right)^2, \dots\right\}, \tag{39}$$

with $s_{2n} = 2n\pi$. The eigenfunctions are displayed at Figure 6, and the deviation of the odd labeled levers from the even are given by Figure 7.

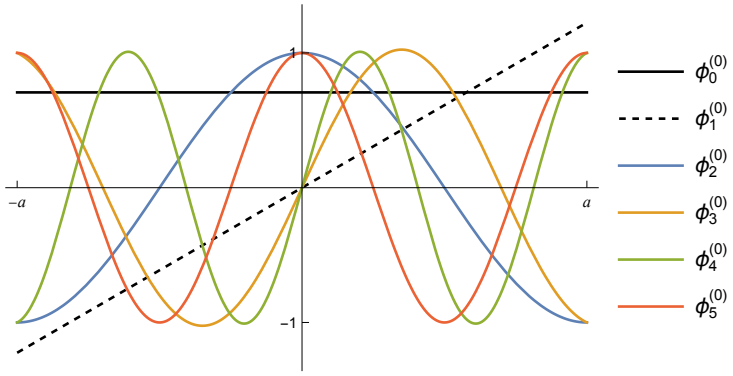

Figure 6. Eigenfunctions of the self adjoint extension of $H^{(0)}$ corresponding to the values of the parameters $(\theta_0, \psi) = \left(\frac{\pi}{2} + \text{arccot}(2), \frac{\pi}{2} - \text{arccot}(2)\right)$. Here, the gound state has zero energy and is doubly degenerate. We represent the two zero energy wave functions in black. The black continuous line represents the even wave function and the dashed line the odd wave function. Then, we depict the first four positive energy levels.

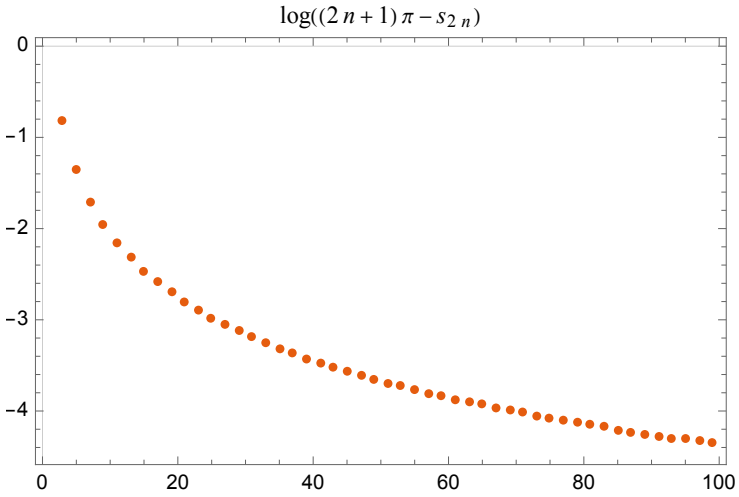

Figure 7. Deviations of the energy levels with odd parity corresponding to the extension labelled with $(\theta_0, \psi) = \left(\frac{\pi}{2} + \text{arccot}(2), \frac{\pi}{2} - \text{arccot}(2)\right)$, with respect to the odd parity energy levels corresponding to the extension with $(\theta_0, \psi) = \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$.

We recall that in Appendix B, we have justify the reasons for going directly to the second order SUSY partner, avoiding the consideration of the first order partner. There are two second order partners that are constructed with the aid of the functions $v_i(x)$, $i = 1, 2$, defined in (34). The function $v_0(x)$ comes after (34) and the even eigenfunction of the ground state, $\phi_0^{(0)}(x)$, while $v_1(x)$ is the result of using in (34) the other eigenfunction of the ground state, i.e., $\phi_1^{(0)}(x)$.

Let us find the second order partner for the even eigenfunction $\phi_0^{(0)}(x)$. After (34) and (38), the function $v_0(x)$ is:

$$v_0(x) = 2aw_0 x - \frac{x^2}{2}. \tag{40}$$

Following the procedure explained in Appendix B, see also [21,22], we obtain the following second order eigenfunctions:

$$\phi_n^{(2,2)}(x) \propto \frac{\mathcal{W}\left(v_0(x), \phi_0^{(0)}(x), \phi_n^{(0)}\right)}{\mathcal{W}\left(v_0(x), \phi_0^{(0)}(x)\right)} = \frac{\partial_x \phi_n^{(0)}}{2aw_0 - x} - E_n \phi_n^{(0)}. \tag{41}$$

The second order potential is:

$$V^{(2,2)} = -2\frac{d}{dx^2}\mathcal{W}\left(v_0, \phi_0^{(0)}\right) = \frac{2}{(x-2aw_0)^2}\,, \tag{42}$$

where we choose w_0 to avoid the singularity in the denominators of (41) and (42). Using the change on the variable given by $x = (t+1)2aw_0$, the Schrödinger equation for the partner $H^{(2,2)}$ is transformed so as to become:

$$-\frac{d^2}{dt^2}\phi_n^{(2,2)}(t) + \frac{2}{t^2}\phi_n^{(2,2)}(t) = (2aw_0)^2 E_n \phi_n^{(2,2)}(t)\,. \tag{43}$$

This is a Bessel equation with index $3/2$, for which the general solution after undoing the change of variables is:

$$\phi_n^{(2,2)}(x) = C_1\sqrt{\frac{x}{2aw_0}-1}\,J_{\frac{3}{2}}\left(\frac{s_n x}{2a} - w_0\right) + C_2\sqrt{\frac{x}{2aw_0}-1}\,J_{-\frac{3}{2}}\left(\frac{s_n x}{2a} - w_0\right), \tag{44}$$

where C_1 and C_2 are arbitrary constants, which should be fixed using (41). The spectrum of $H^{(2,2)}$ coincides with the spectrum of $H^{(0)}$, although it has lost the degeneracy in the first level: $\{0, E_1, E_2, E_3, \dots\}$.

If we begin with the odd ground eigenfunction, $\phi_1^{(0)}(x)$, we have to use the following function as given in (34):

$$v_1(x) = 4a^3w_0 + x^3\,, \tag{45}$$

while the potential is

$$V^{(2,2)}(x) = -2\frac{d}{dx^2}\mathcal{W}\left(\{v_1, \phi_1^{(0)}\}\right) = -\frac{3\left(4a^3w_0x + x^4\right)}{\left(x^3 - 2a^3w_0\right)^2}\,, \tag{46}$$

where $|w_0| > 1/2$ in order to avoid singularities. The eigenfunctions of $H^{(2,2)}$ are proportional to:

$$\phi_n^{(2,2)}(x) \propto \frac{1}{E_n}\frac{\mathcal{W}\left(\{v_1, \phi_1^{(0)}, \phi_n^{(0)}(x)\}\right)}{\mathcal{W}\left(\{v_1, \phi_1^{(0)}\}\right)} = -\frac{3x^2\partial_x\phi_n^{(0)}(x) - 3x\phi_n^{(0)}(x)}{x^3 - 2a^3w_0} - E_n\phi_n^{(0)}(x)\,, \tag{47}$$

with energy levels $\{0, E_1, E_2, E_3, \dots\}$. The potential $V^{(2,2)}$ is a quotient of polynomials and the resulting Schrödinger equation has three singular regular points located at the cubic roots of $2w_0a^3$ and a singular irregular point at the infinity, resulting of a confluence of two points. Thus, the Schrödinger equation is a Fuchsian equation of five points, taking into account the confluence of two of these points at the infinity.

We change the variable x into the variable z as $x = (2w_0)^{1/3}az$ and the function solution , with a new one $y(z)$ given by:

$$\phi_n^{(2,2)}(x) = e^{\tau x/2}(z-z_1)^{\gamma_1}(z-z_2)^{\gamma_2}(z-z_3)^{\gamma_3}\,y(z)\,. \tag{48}$$

After these changes, we arrive at the following Fuchs differential equation:

$$\frac{d^2y(z)}{dz^2} + \left(\Gamma + \frac{\gamma_1}{z-z_1} + \frac{\gamma_2}{z-z_2} + \frac{\gamma_3}{z-z_3}\right)\frac{dy(z)}{dz} + \left(\frac{q_1}{z-z_1} + \frac{q_2}{z-z_2} + \frac{q_3}{z-z_3}\right)y(z) = 0\,, \tag{49}$$

where the constants are given by:

$$\begin{cases} \gamma_1 = 2\,, & \gamma_2 = 4\,, \\ \gamma_3 = 4\,, & q_1 = -4 - 2i\sqrt{\epsilon}\,, \\ q_2 = -\dfrac{2(2+(\sqrt{3}+i)\sqrt{\epsilon})}{\sqrt[3]{-1}-1}\,, & q_3 = \dfrac{4i((\sqrt{3}-3i)\sqrt{\epsilon}+i\sqrt{3}-3)}{\sqrt{3}-3i}\,, \\ \Gamma = 2i\sqrt{\epsilon}\,, & \epsilon = 2^{3/2}a^2 w_0^{2/3} E_n^{(2)}\,, \\ z_1 = 1\,, \quad z_2 = -(-1)^{1/3}\,, & z_3 = -(-1)^{2/3}\,. \end{cases} \tag{50}$$

In Figure 8, we depict the first five energy levels for $H^{(2,2)}$ when the ground state of $H^{(0)}$ is doubly degenerate and we use the confluent SUSY using $\phi_1^{(0)}$ as initial eigenfunction. In Figure 9, we depict the same energy levels under similar conditions, this time using $\phi_0^{(0)}$ as initial eigenfunction.

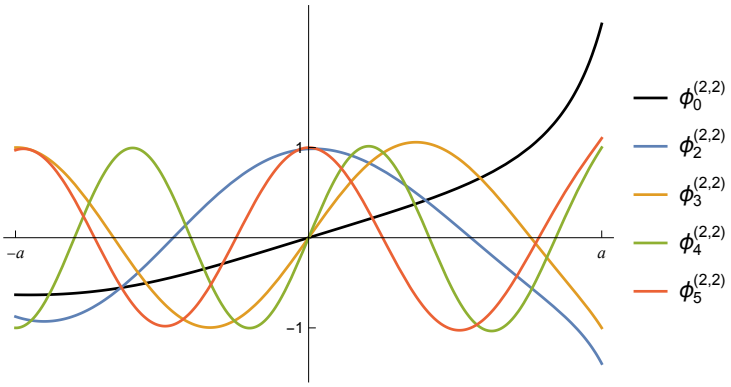

Figure 8. Eigenfunctions of $H^{(2,2)}$ when the ground state of $H^{(0)}$ is doubly degenerate with zero energy. We use $\phi_1^{(0)}$ as the point of departure for the confluent SUSY. The parameter w_0 has been chosen so that the singularity in the potential lies at $x = 1.2\,a$.

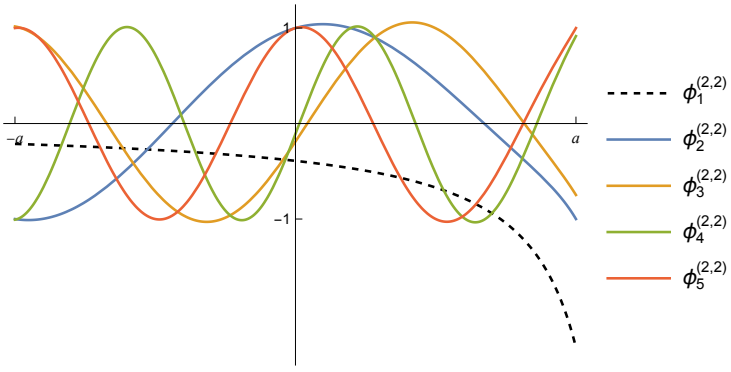

Figure 9. Same as in Figure 8, starting with $\phi_0^{(0)}$.

4.1.2. Degenerate Ground State with Negative Energy

Now, we intend to reproduce the previous analysis when the extension of $H^{(0)}$ has a doubly degenerate ground state with negative energy equal to $E_0^{(0)} = -r_0^2/(4a^2)$, $r_0 > 0$. There are several extensions with this property characterized by some values of the parameters, which are given by the red curves that appear in Figure 1. The eigenfunctions and the corresponding eigenvalues for these extensions are:

$$\left\{\phi_0^{(0)}(x), \phi_1^{(0)}(x), \phi_2^{(0)}(x), \phi_3^{(0)}(x), \dots\right\} = \left\{\cosh\left(\frac{r_0 x}{2a}\right), \sinh\left(\frac{r_0 x}{2a}\right), \cos\left(\frac{s_2 x}{2a}\right), \sin\left(\frac{s_3 x}{2a}\right), \dots\right\}, \tag{51}$$

and

$$\left\{E_0^{(0)}, E_1^{(0)}, E_2^{(0)}, E_3^{(0)}, \dots\right\} = \left\{-\left(\frac{r_0}{2a}\right)^2, -\left(\frac{r_0}{2a}\right)^2, \left(\frac{s_2}{2a}\right)^2, \left(\frac{s_3}{2a}\right)^2, \dots\right\}, \tag{52}$$

where the values of r_0 and s_n, $n > 2$, are given by transcendental equations as explained in Section 3.

Since the ground state is doubly degenerate, we obtain directly the second order partner using the technique described in the precedent subsection. To implement this technique, we begin with the even eigenfunction for the ground state $\phi_0^{(0)} = \cosh\left(\frac{r_0 x}{2a}\right)$ and determine its associated function following the recipe given in (34). This associate function is:

$$v_0(x) = (2w_0 - x)\sinh\left(\frac{r_0 x}{2a}\right). \tag{53}$$

Then, the potential for the second order partner Hamiltonian $H^{(0,0)}$ is given by

$$V^{(2,2)}(x) = -2\partial_x^2 \log \mathcal{W}\left(v_0, \phi_0^{(0)}\right) = \frac{2r_0^2\left(r_0(2w_0 - x)\sinh\left(\frac{r_0 x}{a}\right) + 2a\cosh\left(\frac{r_0 x}{a}\right) + 2a\right)}{a\left(a\sinh\left(\frac{r_0 x}{a}\right) + r_0(x - 2w_0)\right)^2}. \tag{54}$$

As always, the choice of w_0 intends to avoid singularities.

We may have began instead with the odd eigenfunction for the ground state, $\phi_1^{(0)} = \sinh\left(\frac{r_0 x}{2a}\right)$. Then we would have the following associate function:

$$v_1(x) = (2w_0 + x)\cosh\left(\frac{r_0 x}{2a}\right). \tag{55}$$

This gives the following partner potential:

$$V^{(2,2)}(x) = -2\partial_x^2 \log \mathcal{W}\left(v_1, \phi_1^{(0)}\right) = \frac{2r_0^2\left(r_0(2w_0 + x)\sinh\left(\frac{r_0 x}{a}\right) - 2a\cosh\left(\frac{r_0 x}{a}\right) + 2a\right)}{a\left(r_0(2w_0 + x) - a\sinh\left(\frac{r_0 x}{a}\right)\right)^2}, \tag{56}$$

where the choice w_0 is arbitrarily determined in order to avoid singularities.

4.1.3. Other Solvable Cases Up to Second Order

First, we look for the second SUSY partner directly for two cases in which the ground state has non degenerate negative energy and zero energy for the first excited state, respectively. As customary, we just obtain the second partner since the technique to obtain successive partners is repetitive. We look for those extensions which we have named as "exactly solvable", which are given by some determination of the parameters.

When the parameters θ_0 and ψ fulfil either one of the following relations:

$$\psi = \theta_0 + 2\,\mathrm{arccot}(2) - 2\pi \quad \text{and} \quad \psi = \theta_0 + 2\,\mathrm{arccot}(2), \tag{57}$$

then the self adjoint determination $H^{(0)}$ have a negative energy ground state with even eigenfunction and a first excited state with zero energy and odd eigenfunction, respectively. The eigenfunctions of $H^{(0)}$ are

$$\left\{\phi_0^{(0)}(x), \phi_1^{(0)}(x), \phi_2^{(0)}(x), \phi_3^{(0)}(x), \dots\right\} = \left\{\cosh\left(\frac{r_0 x}{2a}\right), x, \cos\left(\frac{s_2 x}{2a}\right), \sin\left(\frac{s_3 x}{2a}\right), \dots\right\}, \tag{58}$$

with respective eigenvalues

$$\left\{E_0^{(0)}, E_1^{(0)}, E_2^{(0)}, E_3^{(0)}, \dots\right\} = \left\{-\left(\frac{r_0}{2a}\right)^2, 0, \left(\frac{s_2}{2a}\right)^2, \left(\frac{s_3}{2a}\right)^2, \dots\right\}, \tag{59}$$

where r_0 and s_n, $n > 2$ depend on the parameters θ_0 and ψ through a transcendental equation.

The second partner potential is:

$$V^{(2,2)}(x) = \frac{r_0^2}{2a^2}\frac{\left(2a^2\cosh\left(\frac{r_0 x}{a}\right) + 2a^2 + r_0^2 x^2\right)}{2a^2\left(r_0 x\sinh\left(\frac{r_0 x}{2a}\right) - 2a\cosh\left(\frac{r_0 x}{2a}\right)\right)^2}. \tag{60}$$

The regular formalism on the construction of SUSY partners asserts that the eigenvalues of $H^{(2,2)}$ are well determined, as are those of the original Hamiltonian, marked as $H^{(0)}$, except for the two first in the list. This list in (59). However, the Schrödinger equation with potential (60) is rather complicated and so is the explicit for of its eigenfunctions. In any case, this explicit form is not especially relevant for our discussion. For the same reason, we cannot obtain $H^{(3,2)}$ and successive partners.

When the relation between the parameters is $\psi = -\theta_0 + \pi$, then the self adjoint extensions of the given $H^{(0)}$ have a non degenerate negative energy ground state with odd eigenfunction and a zero energy first excited state. Their eigenfunctions have the form:

$$\left\{\phi_0^{(0)}(x), \phi_1^{(0)}(x), \phi_2^{(0)}(x), \phi_3^{(0)}(x), \dots\right\} = \left\{\sinh\left(\frac{r_0 x}{2a}\right), \frac{1}{\sqrt{2a}}, \cos\left(\frac{s_2 x}{2a}\right), \sin\left(\frac{s_3 x}{2a}\right), \dots\right\}, \tag{61}$$

with respective eigenvalues,

$$\left\{E_0^{(0)}, E_1^{(0)}, E_2^{(0)}, E_3^{(0)}, \dots\right\} = \left\{-\left(\frac{r_0}{2a}\right)^2, 0, \left(\frac{s_2}{2a}\right)^2, \left(\frac{s_3}{2a}\right)^2, \dots\right\}, \tag{62}$$

which, again, depend on the values of the parameters θ_0 and ψ through a transcendental equation. Observe that the ground state of all these extensions has a node at the origin $x = 0$. This means that a formal construction of the first SUSY partner will give a potential that either does not have eigenvalues or has a finite number thereof. These are the singular potential partners of which we provide some examples at the end of this Section. Nevertheless, the second order SUSY potential partner, $V^{(2,2)}$, is regular and gives:

$$V^{(2,2)}(x) = -2\partial_x \log \mathcal{W}(\phi_0^{(0)}, \phi_1^{(0)}) = -\frac{r^2}{2a^2}\frac{1}{\cosh\frac{r_0 x}{2a}}. \tag{63}$$

This potential is exactly solvable, as is a hyperbolic Pöschl-Teller potential. These eigenfunctions are:

$$\begin{aligned}\phi_n^{(2,2)}(x) &= C_1 P_1^{is_n}\left(\tanh\frac{r_0 x}{2a}\right) + C_2 Q_1^{is_n}\left(\tanh\frac{r_0 x}{2a}\right)\\ &= \begin{cases} -s_n \cos\left(\frac{s_n x}{2a}\right) + r\sin\left(\frac{s_n x}{2a}\right)\tanh\left(\frac{r_0 x}{2a}\right) & \text{if} \quad n = 2,4,6,\dots \\ s_n \sin\left(\frac{s_n x}{2a}\right) + r\cos\left(\frac{s_n x}{2a}\right)\tanh\left(\frac{r_0 x}{2a}\right) & \text{if} \quad n = 3,5,7,\dots \end{cases}.\end{aligned} \tag{64}$$

Next, let us consider the extension given by the values $\{\theta_0, \psi\} = \{3\pi/2, \pi/2\}$. This extension has a ground state with zero energy and excited states with double degeneracy. The eigenfunctions and the respective eigenvalues for this extension are respectively given by:

$$\begin{aligned}\text{Eigenfunctions:} &\left\{\frac{1}{\sqrt{2a}}\right\} \cup \left\{\cos\frac{(2n-1)\pi}{2a}x, \sin\frac{(2n-1)\pi}{2a}x\right\}_{n=1}^{\infty},\\ \text{Spectrum:} &\{0\} \cup \left\{\left(\frac{(2n-1)\pi}{2a}\right)^2\right\}_{n=1}^{\infty}.\end{aligned} \tag{65}$$

We see that the eigenfunction for the ground state is a constant and, therefore, an even function. The remaining energy levels, labelled with the index n, are double degenerate as becomes clear from (69).

We may construct the first Hamiltonian partner, $H^{(1,2)} = -d^2/dx^2 + V^{(1,2)}$, for this extension, where the potential $V^{(1,2)}$ is:

$$V^{(1,2)} = E_0^{(0)} - 2\frac{d^2}{dx^2}\log\phi_0^{(0)}(x) \equiv 0. \tag{66}$$

The rule giving this potential is presented in Appendix A and obeys to Formula (A3). Here, $\phi_0^{(0)}(x)$ and $E_0^{(0)}$ are the eigenfunction of the ground state of $H^{(0)}$ and its energy value, respectively. Since $\phi_0^{(0)}(x)$ is a constant and $E_0^{(0)} = 0$, this potential vanishes. The eigenfunctions and spectrum of $H^{(1,2)}$ are the same as in (65) with the exclusion of the first one, both for eigenfunctions $(1/\sqrt{2a})$ as for eigenvalues (zero energy).

We observe that the ground state corresponds to $n = 1$ and is twice degenerate. Due to this double degeneracy, we may use the confluent SUSY technique here taken one of the eigenfunctions of the ground state. Thus,

$$u_0(x) = \phi_1^{(1,2)}(x) = \cos\left(\frac{\pi x}{2a}\right), \quad v_0(x) = (2w_0 - x)\sin\left(\frac{\pi x}{2a}\right), \tag{67}$$

$$V^{(3,2)} = \frac{2\pi^2\left(\pi(x-2w_0)\sin\left(\frac{\pi x}{a}\right) + 2a\cos\left(\frac{\pi x}{a}\right) + 2a\right)}{a\left(a\sin\left(\frac{\pi x}{a}\right) + \pi(x-2w_0)\right)^2}. \tag{68}$$

The potential $V^{(3,2)}$ produces a Schrödinger equation, which may be explicitly written. However, this equation is too complicated and does not match with any studied differential equation. We depict this potential in Figure 10. As usual, the implicit form of the wave functions is given by:

$$\phi_n^{(3,2)}(x) = \frac{\mathcal{W}\left(v_0, \phi_1^{(1,2)}, \phi_n^{(1,2)}\right)}{\mathcal{W}\left(v_0, \phi_1^{(1,2)}\right)}. \tag{69}$$

Their final form is a bit too long and not quite interesting, so we omit it and just ass that the term $w(x) := \mathcal{W}(v_0, \phi_0^{(0)})$ always appears in the denominator .

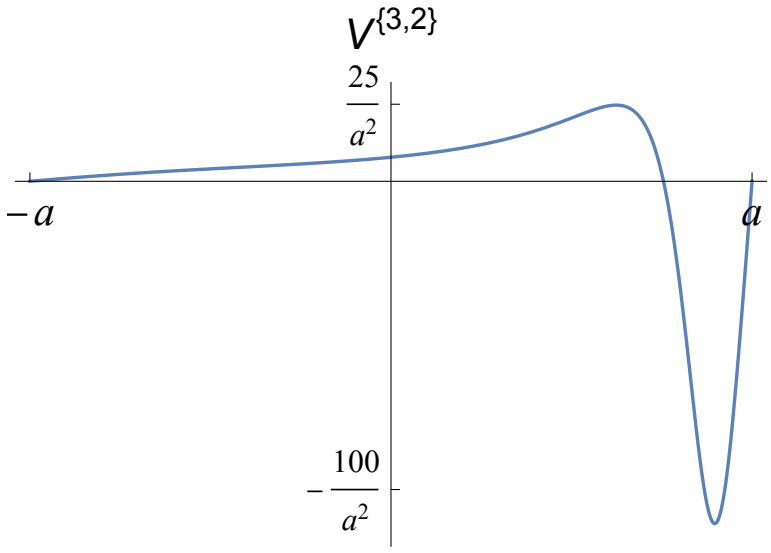

Figure 10. Potential for the confluent first order SUSY corresponding to the self adjoint extension of $H^{(0)}$ determined by the parameter values $\{\theta_0, \psi\} = \{3\pi/2, \pi/2\}$.

Similarly, the extension given by the $\{\theta_0, \psi\} = \{\pi/2, \pi/2\}$ has zero energy ground state and doubly degenerate excited states. Its eigenfunctions and respective eigenvalues are:

$$\begin{aligned} \text{Eigenfunctions: } &\left\{\sqrt{\frac{3}{2a^3}}x\right\} \cup \left\{\cos\frac{n\pi}{a}x, \sin\frac{n\pi}{a}x\right\}, \\ \text{Spectrum: } &\{0\} \cup \left\{\left(\frac{n\pi}{a}\right)^2\right\}_{n=1}^{\infty}. \end{aligned} \tag{70}$$

Note that the bound state wave function is $\phi_0^{(0)} \equiv x$, which is odd and has a *node* at the origin. This has some consequences. First of all, the first order SUSY Hamiltonian is singular at the origin:

$$H^{(1,2)} = -\frac{d^2}{dx^2} - 2\partial_x^2 \log x = -\frac{d^2}{dx^2} + \frac{2}{x^2}\,. \tag{71}$$

The potential of this hamiltonian is displayed at Figure 11. The eigenfunctions of $H^{(1,2)}$ are the following:

$$\phi_n^{(1,2)}(x) = \frac{\mathcal{W}\left(x, \phi_n^{(0)}(x)\right)}{x} = \begin{cases} \sqrt{x}\, J_{-3/2}\left(\frac{\pi n x}{a}\right) \propto \frac{\pi n \sin\left(\frac{\pi n x}{a}\right)}{a} + \frac{\cos\left(\frac{\pi n x}{a}\right)}{x}\,, \\ \sqrt{x}\, J_{3/2}\left(\frac{\pi n x}{a}\right) \propto \frac{\pi n \cos\left(\frac{\pi n x}{a}\right)}{a} - \frac{\sin\left(\frac{\pi n x}{a}\right)}{x}\,. \end{cases} \tag{72}$$

These solutions including the Bessel functions $J_{-3/2}$ and $J_{3/2}$ come from even and odd eigenfunctions of $H^{(0)}$, respectively. Analogously, these solutions are even and odd, so that the parity is preserved after the transformation. However, both solutions are not square integrable, which is absolutely not a surprise, since $H^{(1,2)}$ in (71) cannot have bound states and only have continuous spectrum. This is a consequence of the presence of a node in the wave function of the odd ground state.

The use of the technique described in 4.1 allows for a direct construction of $V^{(2,2)}$. This is given by:

$$V^{(2,2)} = -2\partial_x^2 \log \mathcal{W}\left(\left(\{x, \cos\frac{\pi x}{2a}\}\right)\right) = \frac{\pi^2}{2a^2} \frac{\left(\pi^2 x^2 - 2\cos\left(\frac{\pi x}{a}\right) - 2\right)}{\left(\pi x \sin\left(\frac{\pi x}{2a}\right) + 2\cos\left(\frac{\pi x}{2a}\right)\right)^2}\,. \tag{73}$$

This potential is depicted in Figure 12. The energy eigenvalues for this potential are:

$$E_n^{(2)} = E_n^{(1)} - E_1^{(1)} = E_n^{(1)} - E_1^{(0)} = \left(\frac{n(n-1)\pi}{2a}\right)^2, \qquad n = 2, 3, \ldots\,. \tag{74}$$

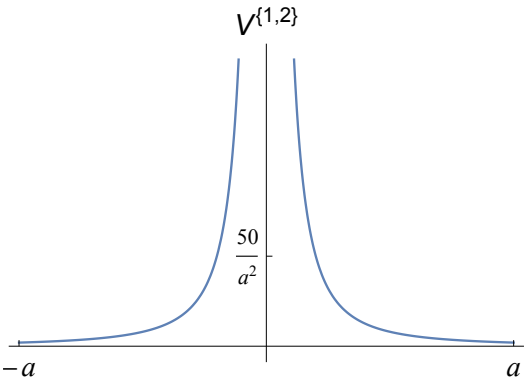

Figure 11. Potential $V^{(1,2)}$ for the self adjoint extension of $H^{(0)}$ fixed by $\{\theta_0, \psi\} = \{\pi/2, \pi/2\}$. This potential has continuous spectrum only. This anomaly is due to the presence a node at the origin in the odd wave function of the ground state.

Observe that $E_1^{(2)}$ is not degenerate, although all other energy levels are degenerate.

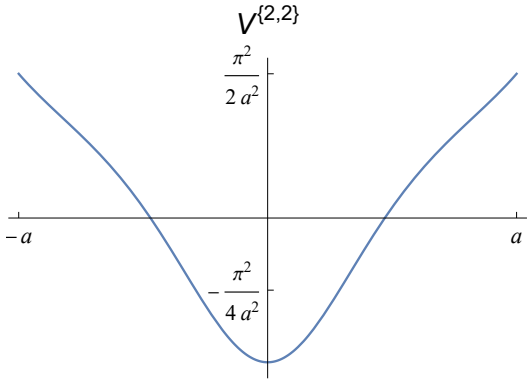

Figure 12. Potential $V^{(2,2)}$ corresponding to the self adjoint extension of $H^{(0)}$ with $\{\theta_0, \psi\} = \{\pi/2, \pi/2\}$.

5. A Comment and Example on General Cases

So far, we have studied the exactly solvable cases here and in [20], all others admit numerical solutions only. In addition, we have considered situations in which the ground state wave function is real. What happens if this wave function were complex? Then, the first SUSY Hamiltonian $H^{(1,2)}$ is not even Hermitian. Let us see an example thereof.

We determine a self adjoint extension were the wave function of the ground state have not a defined parity, so that $m_2 \neq 0$, and is not "time reversal invariant" in the sense given in [2], which means that $m_3 \neq 0$. We may use either one of the two kinds of parameters, with equivalence given in (16), in order to fix a generic self adjoint extension without this two mentioned symmetries. We choose to fix the angles θ_i, $i = 0, 1, 2$, and ψ, so that:

$$\theta_0 = \theta_1 = \theta_2 = \psi = \frac{\pi}{10} \Longleftrightarrow \{m_0, m_1, m_2, m_3, \psi\} \approx \{0.90, 0.29, 0.29, 0.095, 0.31\}\,. \tag{75}$$

The determination of the wave functions for the ground level and the first excited levels is possible . One sees that these wave functions are complex, so that we may depict their probability density, which we do in Figure 13. In Figure 14, we depict the logarithmic deviation of the square root of the energy levels with respect to the standard extension discussed in the textbooks.

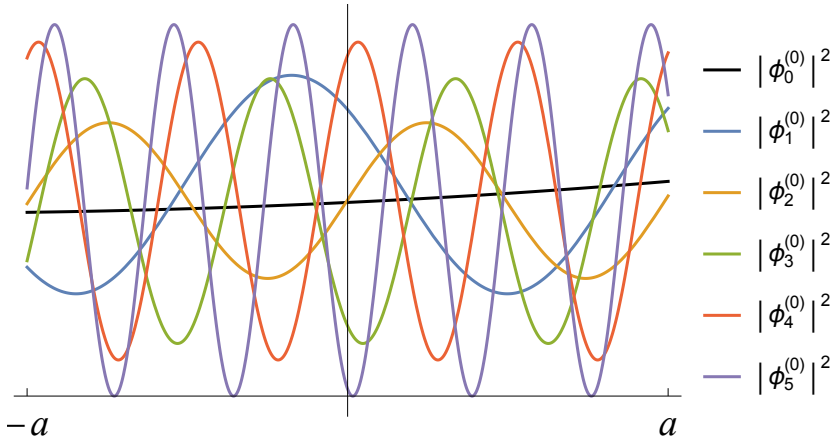

Figure 13. Probability densities of the first six bound states corresponding to the self adjoint extension under our study.

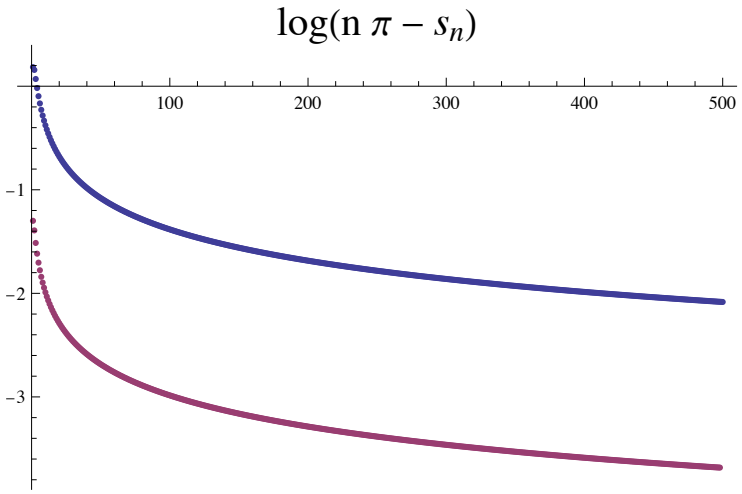

Figure 14. Logarithmic deviation of the square root of the energy levels with respect to the standard textbook solution. Blue and red colours represent odd and even labels, respectively.

The wave function of the ground state is:

$$\phi_0^{(0)}(x) = (0.22 - 0.669i)\cosh\left(0.111\frac{x}{a}\right) - (0.507 + 0.373i)\sinh\left(0.111\frac{x}{a}\right). \tag{76}$$

Observe that this ground state wave function is complex, contrarily to the cases previously studied. The ground state energy is now,

$$E_0^{(0)} \approx -0.012\frac{1}{a^2}, \tag{77}$$

which is negative. In the present example, the only level having a negative energy is the ground level.

With these ingredients, we may obtain the Hamiltonian for the first SUSY partner, which is not Hermitian due to the fact that $\phi_0^{(0)}(x)$ is not real. The potential for $H^{(1,2)}$ is:

$$\begin{aligned} V^{(1,2)}(x) &= E_0^{(0)} - 2\partial_x^2 \log\phi_0^{(0)}(x) \\ &= -0.012\frac{1}{a^2} - \frac{1}{a^2}\frac{0.041 + 0.012i}{(1.\cosh\left(0.111\frac{x}{a}\right) + (0.276 - 0.845i)\sinh\left(0.111\frac{x}{a}\right))^2}. \end{aligned} \tag{78}$$

In Figures 15 and 16, we represent the real and the imaginary parts of the potential in (78) and the graphics for the probability density corresponding to the first six eigenfunctions of $H^{(1,2)}$.

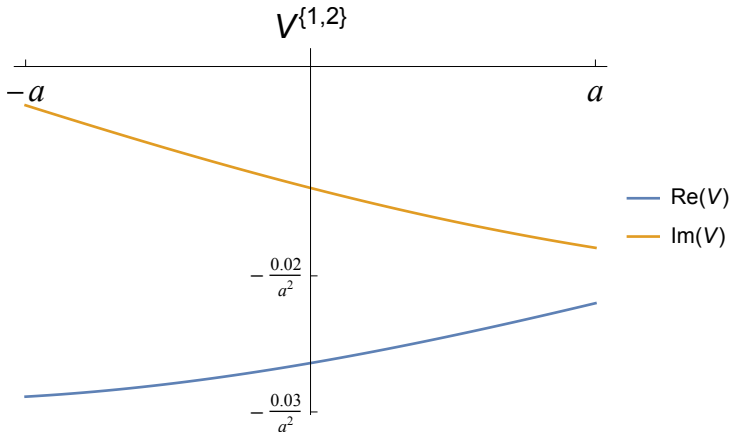

Figure 15. Real and imaginary parts of $V^{(1,2)}$.

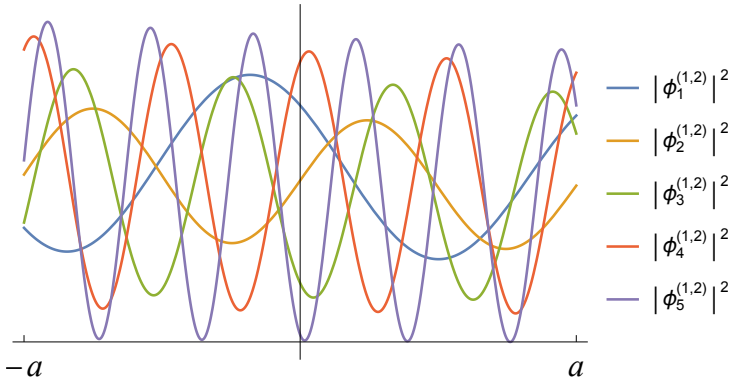

Figure 16. Probability densities of the first five bound states corresponding to the first SUSY partner constructed with the negative energy ground state (76).

SUSY partners with complex potentials have previously been obtained. For instance, it typically appear when, instead a bound state, we use a decaying state or Gamow state function for a resonance to construct the partner instead the ground state of a self adjoint operator with discrete spectrum [23].

6. Discussion and Conclusions

Supersymmetric (SUSY) is a technique that may serve to construct Hamiltonians with discrete spectrum, similar to the spectrum of a given Hamiltonian. One of the most celebrated models in non-relativistic quantum mechanics is the one dimensional infinite square well, which is mathematically modelled by the operator $-D^2 := -d^2/dx^2$ on $L^2[-a,a]$, $a > 0$ being finite. However, there is not a unique self adjoint operator defined by $-D^2$ on $L^2[-a,a]$, but instead a family of independent self adjoint determination of this operator depending on the values of four real parameters.

The objective of the present research is the investigation of the SUSY partners for these self adjoint determinations (also called extensions). For a large number of these extensions, under certain conditions such as a ground state with positive energy, no zeroes (also called nodes) and some symmetry conditions, all these SUSY partners have been classified in a previous paper [20]. For each of these determinations, we have obtained an infinite chain of partners, each one differing from the previous one by an eigenvalue. The shape of the potential and the eigenfunctions may differ from one partner to another.

The objective of the present article is the investigation of the SUSY partners of those extensions that do not satisfy the previous conditions. Some of them may have a ground state with negative energy that may have double degeneracy. Others have a ground state with zero energy that may also be doubly degenerate. Still keeping symmetry conditions for the wave function of the ground state, such as having a definite parity, we may find the exact form of the first SUSY partner Hamiltonian, and, hence, following a technique explained in [20], of all partners. When a double degeneracy on the ground state appears, we need to use a different technique that goes directly to the second partner. We show that extensions with a ground state with nodes in the wave function may lead to formal partners which do not satisfy the condition of having a similar discrete spectrum than of the original Hamiltonian and that may even have no eigenvalue.

Partners of extensions without symmetry conditions are difficult to classify. Here, the ground state wave function may even be complex and this gives rise to a complex partner potential, as to a non-Hermitian partner Hamiltonian. It would be of some interest to search for the possibility of finding PT-symmetric Hamiltonians among these non-Hermitian Hamiltonians obtained by this method.

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Appendix A. SUSY Partners

In this Appendix, we briefly review the foundations of the construction of the Supersymmetric (SUSY) partners of a given self adjoint operator with discrete spectrum. For a more complete account, see the bibliography on the subject listed at the end of the paper and particularly [10]. We use the notation introduced in [20].

To begin with, let us consider an initial one dimensional Hamiltonian $H^{(0)} = -d^2/dx^2 + V^{(0)}(x)$, where for simplicity we always assume that $\hbar = 1$ and $m = 1/2$. Assume that $H^{(0)}$ has a non-degenerate discrete spectrum $\left\{E_i^{(0)}\right\}_{i=0}^{n}$ with respective eigenfunctions $\left\{\phi_i^{(0)}\right\}_{i=0}^{n}$, where n is either finite or infinite.

According to the standard formalism [10], the operator $H^{(0)}$ has two SUSY partners:

$$H^{(1,1)} = -\frac{d^2}{dx^2} + V^{(1,1)}(x)\,, \qquad H^{(1,2)} = -\frac{d^2}{dx^2} + V^{(1,2)}(x)\,, \tag{A1}$$

The superscripts of a given operator $O^{(p,q)}$ mean: p is the SUSY order, to be clarified later, and $q = 1, 2$. Here the indices $(1, 1)$ and $(1, 2)$ means: first order SUSY partner and partner one or partner two, respectively. Here, we need to define the partner potentials $V^{(1,i)}(x)$, $i = 1, 2$, in (A1). To this end, let us take the ground state, $\phi_0^{(0)}(x)$, of $H^{(0)}$. Then,

$$V^{(1,1)}(x) := V^0(x) + \frac{\partial_x^2 \phi_0^{(0)}(x)}{\phi_0^{(0)}(x)} = V^0(x) - E_0^{(0)}\,, \tag{A2}$$

where ∂_x and ∂_x^2 stand for the first and second derivative with respect to the variable x. The second identity in (A2) is obvious, since $\phi_0^{(0)}(x)$ satisfies the Schrödinger equation for $V_0(x)$ and $E_0^{(0)}$. The second partner potential is

$$V^{(1,2)}(x) = V^{(1,1)}(x) - 2\frac{d^2}{dx^2}\,\log \phi_0^{(0)}(x)\,. \tag{A3}$$

It is interesting to consider the following two operators on $L^2(\mathbb{R})$:

$$A^{(1)} := \partial_x - \frac{\partial_x \phi_0^{(0)}(x)}{\phi_0^{(0)}(x)}\,, \qquad (A^{(1)})^\dagger := -\partial_x - \frac{\partial_x \phi_0^{(0)}(x)}{\phi_0^{(0)}(x)}\,, \tag{A4}$$

These operators are well defined provided that $\phi_0^{(0)}(x)$ does not have zeroes, that are usually called *nodes*. They intertwine the Hamiltonian (A1) in the following sense:

$$H^{(1,1)}\,A^{(1)} = (A^{(1)})^\dagger\,H^{(1,2)}\,, \qquad H^{(1,2)}\,(A^{(1)})^\dagger = A^{(1)}\,H^{(1,1)}\,. \tag{A5}$$

The Hamiltonians $H^{(1,1)}$ and $H^{(1,2)}$ have the following properties:

- Both have an infinite number of eigenvalues.
- The eigenfunctions of $H^{(1,1)}$, $\{\phi_i^{(1,1)}(x)\}_{i=1}^\infty$, coincide with the eigenfunctions of the original Hamiltonian $H^{(0)}$, $\{\phi_i^{(0)}(x)\}_{i=1}^\infty$, although the spectrum of $H^{(0)}$ is of the form $\{\epsilon_i^{(0)} := E_i^{(0)} - E_0^{(0)}\}_{i=0}^\infty$, where $E_i^{(0)}$ are the energy levels of $H^{(0)}$ and $E_0^{(0)}$ its ground state. Thus, zero is an eigenvalue of $H^{(1,1)}$.
- The eigenfunctions of $H^{(1,2)}$, $\{\phi_i^{(1,2)}(x)\}_{i=1}^\infty$ are different from those of $H^{(0)}$, while its spectrum coincides with the spectrum of $H^{(1,1)}$ with the ground level removed: $\{\epsilon_i^{(0)} := E_i^{(0)} - E_0^{(0)}\}_{i=1}^\infty$. Note that $\epsilon_i^{(0)} > 0$ for $i = 1, 2, \ldots$.
- Since the eigenfunctions of $H^{(1,1)}$ are known for being those of $H^{(0)}$, we need a receipt so as to obtain $\{\phi_i^{(1,2)}(x)\}_{i=1}^\infty$, the eigenfunctions of $H^{(1,2)}$. This is

$$\phi_n^{(1,2)}(x) = \epsilon_n^{-1/2}\,A^{(1)}\,\phi_n^{(1,1)}(x)\,, \qquad n = 1, 2, \ldots\,. \tag{A6}$$

In addition, we have the following inverse relation:

$$\phi_n^{(1,1)}(x) = \epsilon_n^{-1/2}\,(A^{(1)})^\dagger\,\phi_n^{(1,2)}(x)\,, \qquad n = 1, 2, \ldots\,. \tag{A7}$$

From (A4), (A6) and (A7), we understand the importance of the condition that $\phi_0^{(0)}(x)$ has no nodes.

This is the origin of a chain of SUSY partners of the given $H^{(0)}$. Take $H^{(1,2)}$, and consider a new pair of potentials, $H^{(2,1)}$ and $H^{(2,2)}$, for which their potentials have the following form:

$$V^{(2,1)}(x) := V^{(1,2)}(x) + \frac{\partial_x^2\,\phi_1^{(1,2)}(x)}{\phi_1^{(1,2)}(x)}\,, \tag{A8}$$

and

$$V^{(2,2)}(x) := V^{(2,1)}(x) - 2\,\frac{d^2}{dx^2}\,\log\phi_1^{(1,2)}(x)\,. \tag{A9}$$

The operators (A4) are here replaced by

$$A^{(2)} := \partial_x - \frac{\partial_x\,\phi_1^{(1,2)}(x)}{\phi_1^{(1,2)}(x)}\,, \qquad (A^{(2)})^\dagger := -\partial_x - \frac{\partial_x\,\phi_1^{(1,2)}(x)}{\phi_1^{(1,2)}(x)}\,. \tag{A10}$$

We recall that $\partial_x := d/dx$.

Thus, the Hamiltonian $H^{(2,2)}$ has a spectrum equal to $\epsilon_n - \epsilon_1 = E_n^{(0)} - E_0^{(0)} - E_1^{(0)} + E_0^{(0)} = E_n^{(0)} - E_1^{(0)}$, $n = 2, 3, \ldots$.

Thus, we have started the construction of a chain of Hamiltonian SUSY partners of $H^{(0)}$, where the first two Hamiltonians are $H^{(1,2)}$ and $H^{(2,2)}$, first and second SUSY partners, and whose respective spectrum can be easily obtain form $H^{(0)}$. We can proceed with this chain of SUSY partners of $H^{(0)}$ up to infinite. At the ℓ-th, we have another pair of Hamiltonians:

$$H^{(\ell,1)} := -\frac{d^2}{dx^2} + V^{(\ell,1)}(x)\,, \qquad H^{(\ell,2)} := -\frac{d^2}{dx^2} + V^{(\ell,2)}(x)\,, \tag{A11}$$

with

$$V^{(\ell,1)}(x) = V^{(0)}(x) - E^{(0)}_{\ell-1} + \partial_x^2 \log \mathcal{W}\{\phi^{(0)}_0, \phi^{(0)}_1, \ldots, \phi^{(0)}_{\ell-2}\}\,, \tag{A12}$$

$$V^{(\ell,2)}(x) = V^{(0)}(x) - E^{(0)}_{\ell-1} - \partial_x^2 \log \mathcal{W}\{\phi^{(0)}_0, \phi^{(0)}_1, \ldots, \phi^{(0)}_{\ell-1}\}\,, \tag{A13}$$

where $\mathcal{W}$ denotes the Wronskian of the functions under brackets.

The eigenfunctions of the ℓ-th SUSY partner $H^{(\ell,2)}$ are

$$\phi^{(\ell,2)}_n(x) = C^{(\ell,2)}_n \, \frac{\mathcal{W}\{\phi^{(0)}_0, \phi^{(0)}_1, \ldots, \phi^{(0)}_{\ell-1}, \phi^{(0)}_n\}}{\mathcal{W}\{\phi^{(0)}_0, \phi^{(0)}_1, \ldots, \phi^{(0)}_{\ell-1}\}}\,, \tag{A14}$$

where $C^{(\ell,2)}_n$ stand for normalization constants. This general formula allows us to have the eigenfunctions of all SUSY partner Hamiltonians of the chain without having to solve the Schrödinger equation.

This is the procedure that we have to apply in Section 4.

Appendix B. Second Order SUSY Partners: Some Peculiarities for the Confluent Case

The purpose of this Appendix is the review on the construction of the second order partner directly from the original Hamiltonian. Although we essentially follow the presentation in [21] for the general case, we consider that we need to justify the reasons why we jump from the original Hamiltonian to the second order partner in the confluent case mentioned in Section 4.1. At the same time, we intend a pedagogical presentation for the benefit of the reader.

We begin with the original self adjoint Hamiltonian $H^{(0)}$ with its eigenfunctions $\{\phi^{(0)}_n(x)\}$ and respective eigenvalues $\{E^{(0)}_n\}$. The first partner potential $H^{(1,1)}$ has potential $V^{(1,1)} = V^{(0)} - E^{(0)}_0$, same eigenfunctions as $H^{(0)}$ and eigenvalues those of $H^{(0)}$ sifted by $-E^{(0)}_0$ as described in the previous Appendix.

The usual procedure to obtain directly the second partner uses the construction of a second order differential operator B such that

$$B\, H^{(1,1)} = H^{(2,2)}\, B\,. \tag{A15}$$

Compare (A15) with (A5). In the latter, the operator $A^{(1)}$ and its adjoint are first order operators as we remain in the first order. In (A15), we jump from first order to second order so that we need add one more order of differentiation to the intertwining operator. Thus, the operator B should have the form [21]:

$$B(x) := \frac{d^2}{dx} + \eta(x)\,\frac{d}{dx} + \gamma(x)\,, \tag{A16}$$

where $\eta(x)$ and $\gamma(x)$ are functions to be determined, by using (A16) into (A15). We obtain the following relations (recall that $\partial_x := d/dx$ and the prime also means derivation with respect to x):

$$\partial_x^2 V^{(1,1)}(x) + \eta(x)\partial_x V^{(1,1)}(x) + \gamma(x)V^{(1,1)}(x) + \gamma''(x) = \gamma(x)V^{(2,2)}(x)\,, \tag{A17}$$

$$2\partial_x V^{(1,1)}(x) + \eta(x)V^{(1,1)}(x) + 2\gamma'(x) + \eta''(x) = \eta(x)V^{(2,2)}(x)\,, \tag{A18}$$

$$V^{(2,2)}(x) = V^{(1,1)}(x) + 2\eta'(x)\,. \tag{A19}$$

Equation (A18) permits to write $\gamma(x)$ in terms of $\eta(x)$ as

$$\gamma(x) = \gamma_0 - V^{(1,1)}(x) - \frac{\eta'(x)}{2} + \frac{\eta(x)^2}{2}\,, \tag{A20}$$

where γ_0 is an integration constant. Then, we need to obtain $\eta(x)$. Let us multiply (A18) by $\eta(x)$ and integrate. The result is

$$4\eta(x)^2\Big(\gamma_0 - V^{(1,1)}(x) - \eta'(x)\Big) + 2\eta(x)\eta''(x) - \eta'(x)^2 + \eta(x)^4 + \eta_0 = 0\,, \tag{A21}$$

where η_0 is a new integration constant. Equation (A21) is second order and non-linear and looks intractable at first sight.

Nevertheless, there is a way out. The operator B should annihilate the ground state, $\phi_0^{(1,1)}$, of $H^{(1,1)}$. This gives the following differential equation to be satisfied by $\eta(x)$:

$$B(\phi_0^{(1,1)}) = 0 \Longrightarrow \eta'(x) = 2\gamma_0 + \frac{2\eta(x)\partial_x\phi_0^{(1,1)}(x)}{\phi_0^{(1,1)}(x)} + \eta(x)^2\,. \tag{A22}$$

Then, if we use (A22) in (A21), we obtain a relation between both integration constants, which is $\eta_0 = 4\gamma_0^2$. If we choose the particular solution $\eta_0 = 0$, we obtain the following solution [21,22]:

$$\eta(x) = -\frac{d}{dx}\log(w(x))\,, \quad w(x) = w_0 - \int^x \omega^2(y)dy\,, \quad \omega(x) = \phi_0^{(1,1)}(x)\,, \tag{A23}$$

where $\int^x f(y)\,dy$ denotes the primitive of the function $f(x)$.

Since B is a second order operator, it should exists a second function $v(x)$ linearly independent with $\phi_0^{(1,1)}(x)$ and such that is annihilated by B: $B(v(x)) = 0$. This implies that [21,22]:

$$\eta(x) = -\partial_x \log \mathcal{W}(v(x), \phi_0^{(1,1)}(x))\,. \tag{A24}$$

If we compare (A24) to the first relation in (A23), we have that

$$\mathcal{W}(v(x), \phi_0^{(1,1)}(x)) = w(x)\,. \tag{A25}$$

This is a first order differential equation on $v(x)$. Choosing the value of the arbitrary constant equal to one, we have the following solution:

$$v(x) = \phi_0^{(1,1)}(x)\int^x \frac{w(y)}{(\phi_0^{(1,1)}(y))^2}dy\,, \tag{A26}$$

which is well defined in absence of nodes for $\phi_0^{(1,1)}(x)$. The function $v(x)$ is not an eigenfunction of $H^{(1,1)}$. Instead,

$$H^{(1,1)}(v(x)) = \phi_0^{(1,1)}(x) \Longrightarrow [H^{(1,1)}]^2 v(x) = H^{(1,1)}(\phi_0^{(1,1)}(x)) = 0\,. \tag{A27}$$

Thus, $v(x)$ is an eigenfunction of the square of the Hamiltonian $H^{(1,1)}$.

In Section 4.1, we have gone directly from the original Hamiltonian $H^{(0)}$ to the second partner $H^{(2,2)}$ without considering the first partner $H^{(1,2)}$. This has a justification which is not given in [21,22]. To begin with this, we recall that $H^{(1,1)}$ and $H^{(1,2)}$ should be interconnected via a first order differential operator A_1. Another first order differential operator, A_2 interconnects $H^{(2,1)}$ and $H^{(2,2)}$. We have the following relations:

$$A_1\,H^{(1,1)} = H^{(1,2)}\,A_1\,, \quad A_2\,H^{(2,1)} = H^{(2,2)}\,A_2\,, \quad B\,H^{(1,1)} = H^{(2,2)}\,B\,, \tag{A28}$$

and $H^{(1,2)} = H^{(2,1)} + E_1^{(1)}$. From (A28), it comes that

$$B = A_2\,A_1\,. \tag{A29}$$

In summary, operators A_1, A_2 and B interconnect the mentioned Hamiltonian as follows:

$$H^{(1,1)} \xrightarrow{A_1} H^{(1,2)} \equiv H^{(2,1)} \xrightarrow{A_2} H^{(2,2)}\,, \quad H^{(1,1)} \xrightarrow{B} H^{(2,2)}\,. \tag{A30}$$

So far, the treatment is quite general. Now, let us focus in the confluent case, where the ground state is doubly degenerate. With no degeneracy, B annihilates both $\phi_0^{(1,1)}$ and $\phi_1^{(1,1)}$. With double degeneracy for the ground state, B annihilates $\phi_0^{(1,1)}$ and $v(x)$. The former is an eigenvalue of $H^{(1,1)}$, although this is not the case for $v(x)$. It is not difficult to finds the pair of first order differential operators A_1 and A_2 verifying (A28) and (A29). In fact, two solutions are possible for this pair of operators. The first solution is

$$A_1 = \partial_x - \frac{\partial_x \phi_0^{(1,1)}(x)}{\phi_0^{(1,1)}(x)}\,, \quad A_2 = \partial_x - \partial_x \frac{\mathcal{W}(\phi_0^{(1,1)}, v)(x)}{\phi_0^{(1,1)}(x)}\,, \tag{A31}$$

and the second solution is

$$A_1 = \partial_x - \frac{v'(x)}{v(x)}\,, \quad \partial_x - \partial_x \frac{\mathcal{W}(v, \phi_0^{(1,1)})(x)}{v(x)}\,. \tag{A32}$$

The first solution does not add anything new. For this solution, $A_1(\phi_0^{(1,1)}) = 0$ and then, $H^{(1,2)}$ is the first order partner without the ground state of $H^{(0)}$. In addition, the state $A_1(v)$ is not an eigenfunction of $H^{(1,2)}$, since $v(x)$ is not an eigenfunction of $H^{(1,1)}$.

Take the second solution, which is the solution that we should take into account for the confluent case. Here $A_1(v) = 0$, although A_1 does not annihilate an eigenstate as in the previous case, since $v(x)$ is not ein eigenfunction of $H^{(1,1)}$. According to the general theory, the eigenfunctions of $H^{(1,2)}$ should have the form:

$$\phi_n^{(1,2)} = A_1(\phi_n^{(1,1)}) = A_1(\phi_n^{(0)}) = \frac{\mathcal{W}(\{v(x), \phi_n^{(0)}(x)\})}{v(x)}\,. \tag{A33}$$

Let us see whether the form of these functions is compatible with the properties they should have as eigenfunctions of $H^{(1,2)}$. In fact,

$$H^{(1,2)}\,\phi_n^{(1,2)} = H^{(1,2)}\,A_1\,\phi_n^{(1,1)} = A_1\,H^{(1,1)}\,\phi_n^{(1,1)} = E_n^{(0)}\,A_1(\phi_n^{(1,1)}) = E_n^{(1)}\,A_1(\phi_n^{(0)})\,. \tag{A34}$$

According to (A32) and (A33), the eigenfunctions $\phi_n^{(1,1)}(x)$, $n = 1, 2, \ldots$, should depend on $v(x)$, which is not possible, since

$$V^{(1,2)} = V^{(1,1)} - \frac{1}{\int \frac{w_0 - \int \phi_0^{(1,1)}(x)^2\,dx}{\phi_0^{(1,1)}(x)^2}\,dx}\,, \tag{A35}$$

which does not depend on $v(x)$. Thus, this second solution is spurious. Therefore, the first partner $H^{(1,2)}$ is ill defined. Thus is the motivation that we should go directly to the second partner when the bound state shows degeneracy.

Appendix C. Self Adjoint Extensions of Symmetric Operators

The terminology self adjoint determinations or self adjoint extensions comes from the following: The *domain*, $\mathcal{D}_A$, of a (linear) operator A is the subspace of the Hilbert space, $\mathcal{H}$, of all vectors on which A acts. Bounded operators have the whole Hilbert space as domain. However, this is not true for the unbounded operators such as the differential operator $-D^2$. Most of operators used in quantum mechanics, position, momentum, angular momentum, Hamiltonians, are unbounded operators. An unbounded operator has two parts, the prescription on how it acts on the vector of its domain, for instance $(-D^2)f(x) = -f''(x)$,

and the domain itself. Two operators with analogous action on their vectors, for instance take the second derivative, having different domains are different, so that an unbounded operator is a pair $(A, \mathcal{D}_A)$. Two operators $(A, \mathcal{D}_A)$ and $(B, \mathcal{D}_B)$ are equal if and only if their domains are equal, $\mathcal{D}_A \equiv \mathcal{D}_B$, and being given an arbitrary $f \in \mathcal{D}_A \equiv \mathcal{D}_B$, $Af = Bf$.

Let $(A, \mathcal{D}_A)$ be an unbounded operator such that the domain $\mathcal{D}_A$ is dense in the Hilbert space $\mathcal{H}$. This means that for any $g \in \mathcal{H}$ and for any neighbourhood V of g, there exists at least one $f \in \mathcal{D}_A$ such that $f \in V$. The adjoint $A^\dagger$ of A is an operator with domain

$$\mathcal{D}_{A^\dagger} = \{g \in \mathcal{H} \text{ such that } \exists h \in \mathcal{H} \,;\, \langle h|f\rangle = \langle g|Af\rangle \ \forall f \in \mathcal{D}_A\}\,. \tag{A36}$$

In this case,

$$A^\dagger g = h\,, \qquad \forall g \in \mathcal{D}_{A^\dagger}\,. \tag{A37}$$

We call $A^\dagger$ the *adjoint operator* of A. It is well defined if and only if $\mathcal{D}_A$ is dense in $\mathcal{H}$.

An operator $(B, \mathcal{D}_B)$ *extends* another operator $(A, \mathcal{D}_A)$, $A \prec B$, if and only if $\mathcal{D}_A \subset \mathcal{D}_B$ and for any $g \in \mathcal{D}_A$, $Ag = Bg$.

An operator $(A, \mathcal{D}_A)$ is *symmetric* (also called Hermitian) if for all $f, g \in \mathcal{D}_A$, one has that

$$\langle f|Ag\rangle = \langle Af|g\rangle\,. \tag{A38}$$

After (A38), we note that for all symmetric operator A, $A \prec A^\dagger$. A symmetric operator A is *self adjoint* if $A^\dagger = A$. This means in particular that $\mathcal{D}_A \equiv \mathcal{D}_{A^\dagger}$.

Let I be the identity operator, i.e., $If = f$, $\forall f \in \mathcal{H}$ and $N(A)$ the nucleus (kernel) of the operator A. The *deficiency indices* of an unbounded symmetric operator A are $n_\pm := \dim N(A \pm iI)$, where $\dim V$ means the dimension of the subspace V.

A well know theorem [3] establishes that A has self adjoint extensions if and only if $n_+ = n_- \neq 0$. If $n_+ = n_- = 0$, then A admits one and only one self adjoint extension, that coincides with A if this is self adjoint.

In the case of the operator $-D^2$ on $L^2[-a, a]$, we take as its domain, $\mathcal{D}$, the functions $f(x) \in L^2[-a, a]$ satisfying the following properties:

(i). The function $f(x)$ is absolutely continuous on the interval $[-a, a]$, having a square integrable second derivative, on the same interval, defined almost everywhere.

(ii). For any $f(x) \in \mathcal{D}$, one has that

$$f(-a) = f'(-a) = f(a) = f(a) = 0\,. \tag{A39}$$

This operator is symmetric with deficiency indices $n_\pm = 2$ [1]. It has an infinite number of self adjoint extensions (also called determinations along the present paper) depending on four real parameters as described in the Introduction, see (4) and comment thereafter.

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