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Some New Anderson Type h and q Integral Inequalities in Quantum Calculus

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Abstract: The calculus in the absence of limits is known as quantum calculus. With a difference operator, it substitutes the classical derivative, which permits dealing with sets of functions that are non-differentiations. The theory of integral inequality in quantum calculus is a field of mathematics that has been gaining considerable attention recently. Despite the fact of its application in discrete calculus, it can be applied in fractional calculus as well. In this paper, some new Anderson type q-integral and h-integral inequalities are given using a Feng Qi integral inequality in quantum calculus. These findings are highly beneficial for basic frontier theories, and the techniques offered by technology are extremely useful for those who can stimulate research interest in exploring mathematical applications. Due to the interesting properties in the field of mathematics, integral inequalities have a tied correlation with symmetric convex and convex functions. There exist strong correlations and expansive properties between the different fields of convexity and symmetric function, including probability theory, convex functions, and the geometry of convex functions on convex sets. The main advantage of these essential inequalities is that they can be converted into time-scale calculus. This kind of inevitable inequality can be very helpful in various fields where coordination plays an important role.

Keywords: Anderson inequality; Feng Qi inequality; quantum calculus; q-integral; h-integral

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1. Introduction

In mathematics, q-calculus is a quantum calculus that calculates without limits. We acquire q-analogue formulas of mathematics in q-calculus that can be captured as a tendency of q toward one. In the same vein of Newton's efforts on infinite series, Euler first introduced q-calculus. Therefore, the date of q-calculus can be linked back to Euler. Then, F. H. Jackson [1] defined the q-definite integral and introduced a systematic study of q-calculus, which is known as the q-Jackson integral, in 1910. In recent years, due to the high demand of mathematics in the field of quantum calculus, the interest in q-calculus has been increasing. This q-calculus has many applications in different fields of mathematics and other different areas, such as fractals, orthogonal polynomial combinatorics, mechanics, number theory, dynamical systems, special functions, and mechanics for scientific problems in many applied areas.

In numerous branches of mathematics such as differential equations, analysis, geometry, and many other fields, mathematical inequalities have been applied. One such example is Anderson type integral inequalities. Let us recall following lines from [2]. "Consider the following quotient:

$$\frac{f(x) - f(x_0)}{x - x_0}$$

As x reaches to x_0, the limit, gives the well known definition of the differentiation of a function at $x = x_0$, if it exists. However, if we take $x = qx_0$ or $x = x_0 + h$, where q is a constant different from 1, and h is a constant different from 0 and do not need to take the limit, we enter the fascinating world of Quantum Calculus and will get the definition of q derivative and h derivative using $x = qx_0$ and $x = x_0 + h$ respectively."

In [3], Feng Qi studied one of the useful inequalities, which may be stated as follows:

Theorem 1. *Let a positive real number $n \geq 1$, $Q \in C^{(n)}[m_0, m_1]$ s.t. $Q^{(i)}(m_0) \geq 0$ for $i \in \{0, 1, \ldots n-1\}$ and $Q^{(n)}(m_0) \geq n!$. Then, the following is true:*

$$\left[\int_{m_0}^{m_1} Q(\tau) d\tau\right]^{n+1} \leq \int_{m_0}^{m_1} [Q(\tau)]^{n+2} d\tau \tag{1}$$

At the end of this article, Feng Qi proposed an open problem: "What if n is replaced by any positive real number p? In what state is the inequality (1) still true?"

This inequality gained the attention of many mathematicians. Some researchers showed keen interest in this open problem and gave it a try [4,5]. Some of the researchers studied Fenq Qi type inequalities in quantum calculus involving q-integrals and h-integrals [6], and some studied these inequalities in time-scale calculus [7,8], while some researchers further generalized it using the $\Diamond_\alpha$ operator. In this paper, we shall discuss the q-integral and h-integral analogous of a Feng Qi type inequality in quantum calculus by using an Anderson inequality. On the other hand, there is considerable research in the field of q-analysis for achieving quality in quantum computing. Quantum calculus is a series between mathematics and physics and contains a wide range of applications in many fields, such as combinations, mathematical numerical theory, addition theory, and quantum theory [9,10]. Quantum calculus also combines quantum information theory, philosophy, information theory, computer science, and cryptography with several applications [11,12]. Since then, the relevant research has been steadily improving. Specifically, the left quantum integral and the left quantum differential operator were discussed in 2013 [13] by Ntouyas and Tariboon.

Anderson [14] discussed the following fascinating and interesting integral inequality in 1958:

Proposition 1. *If ψ_k are convex mappings on the closed interval $[0, 1]$, and $\psi_k(0) = 0$ $\forall k \in \{1, \ldots, n\}$, then*

$$\int_0^1 \psi_1(s)\psi_2(s)\ldots\psi_n(s) ds \geq \frac{2^n}{n+1}\left(\int_0^1 \psi_1(s) ds\right)\ldots\left(\int_0^1 \psi_n(s) ds\right) \tag{2}$$

where $\psi_1(\chi)\ldots\psi_n(\chi)$ are integrable functions on the closed interval $[0, 1]$.

If we let $Q = \psi_1 = \psi_2 = \cdots = \psi_n$ in Equation (2), then we obtain following Feng Qi type integral inequality:

$$\int_0^1 Q^n(\chi) d\chi \geq \frac{2^n}{n+1}\left(\int_0^1 Q(\chi) d\chi\right)^n. \tag{3}$$

2. q-Integral Inequalities of the Anderson Type

2.1. Notations and Preliminaries

For $q > 1$ we recall the notations from [1,7], where $m_0 \in \mathbb{C}$:

$$[m_0]_q = \frac{1 - q^{m_0}}{1 - q}, \qquad (m_0; q)_n = \prod_{k=0}^{n-1}(1 - m_0 q^k), \quad n \geq 1$$

$$[0]_q! = 1, \qquad [n]_q! = [1]_q[2]_q \ldots [n]_q, \quad n \geq 1$$

and

$$(\chi - m_0)_q^n = \begin{cases} 1, & \text{if} \quad n = 0, \quad \chi \in \mathbb{C}, \\ (\chi - m_0)(\chi - q m_0) \ldots (\chi - q^{n-1} m_0), & \text{if} \quad n \neq 0. \end{cases} \tag{4}$$

The "q-derivative $D_q\, Q$ of a function" Q is defined as

$$(D_q Q)(\chi) = \frac{Q(q\chi) - Q(\chi)}{(q-1)\chi}, \qquad \chi \neq 0, \tag{5}$$

where $(D_q Q)(0) = Q'(0)$, provided $Q'(0)$ exists.

The "q-Jackson integral from 0 to m_0" is defined by [15]

$$\int_0^{m_0} Q(\chi) d_q\chi = (q^{-1} - 1) m_0 \sum_{n=0}^{\infty} Q(m_0 q^{-n}) q^{-n}, \tag{6}$$

given that the sum converges absolutely.

The "q-Jackson integral in a generic interval $[m_0, m_1]$" is defined as follows (see [15]):

$$\int_{m_0}^{m_1} Q(\chi) d_q\chi = \int_0^{m_0} Q(\chi) d_q\chi - \int_0^{m_1} Q(\chi) d_q\chi. \tag{7}$$

From [2], we also have that

$$D_q\left[\int_{m_0}^{\chi} Q(\tau) d_q\tau\right] = Q(\chi). \tag{8}$$

If $m_1 > 0$ and $m_0 = m_1 q^n$, where $n \in \mathbb{N}$, we have

$$[m_0, m_1]_q = \{m_1 q^k : 0 \leq k \leq n\} \quad \textit{and} \quad (m_0, m_1]_q = [q^{-1} m_0, m_1]_q.$$

2.2. Main Results

Lemma 1. *If we let $p \in \mathbb{R}$ s.t. $p \geq 1$ and λ be a non-negative and monotone function on $[0,1]_q$, then*

$$[p]_q \lambda^{p-1}(\chi) D_q \lambda(\chi) \leq D_q[\lambda(\chi)]^p \leq [p]_q \lambda^{p-1}(q\chi) D_q \lambda(\chi), \qquad \chi \in (0,1]_q.$$

Proof. We have

$$D_q[\lambda^p](\chi) = \frac{\lambda^p(q\chi) - \lambda^p(\chi)}{(q-1)\chi} = \frac{[p]_q}{(q-1)\chi} \int_{\lambda(\chi)}^{\lambda(q\chi)} \tau^{p-1} d_q\tau. \tag{9}$$

Since λ is a non-negative and monotonic function, therefore

$$\lambda^{p-1}(\chi)[\lambda(q\chi) - \lambda(\chi)] \leq \int_{\lambda(\chi)}^{\lambda(q\chi)} \tau^{p-1} d_q\tau \leq \lambda^{p-1}(q\chi)[\lambda(q\chi) - \lambda(\chi)].$$

From Equation (9), we obtain

$$[p]_q\lambda^{p-1}(\chi)D_q\lambda(\chi) \le D_q[\lambda]^p(\chi) \le [p]_q\lambda^{p-1}(q\chi)D_q\lambda(\chi).$$

□

Theorem 2. *If we have the function Q on $[0,1]_q$, then it satisfies*

$$Q(0) = 0, \quad D_qQ(\chi) \ge \frac{2^p}{p+1}[p]_q\chi^{p-2} \quad for \quad \chi \in (0,1]_q \quad and \quad p \ge 2, (yes,\ accordingtopower) \tag{10}$$

Then, it is true that

$$\int_0^1 Q^p(q\chi)d_q\chi \ge \frac{2^p}{p+1}\left(\int_0^1 Q(\chi)d_q\chi\right)^p.$$

Proof. Include $\lambda(\chi) = \int_0^\chi Q(u)d_qu$ and

$$F(\chi) = \int_0^\chi Q^p(qu)d_qu - \frac{2^p}{p+1}\left(\int_0^\chi Q(u)d_qu\right)^p.$$

We have

$$D_qF(\chi) = Q^p(q\chi) - \frac{2^p}{p+1}D_q[\lambda^p](\chi).$$

Since Q and F both are increasing on $[0,1]_q$, we obtain the following from Lemma 1:

$$\begin{aligned} D_qF(\chi) &\ge Q^p(q\chi) - \frac{[p]_q}{p+1}2^p\lambda^{p-1}(\chi)Q(\chi) \\ &\ge Q^p(q\chi) - \frac{[p]_q}{p+1}2^p\lambda^{p-1}(\chi)Q(q\chi) = Q(q\chi)h(\chi), \end{aligned} \tag{11}$$

where $h(\chi) = Q^{p-1}(q\chi) - \frac{[p]_q}{p+1}2^p\lambda^{p-1}(\chi)$.

We also have

$$D_qh(\chi) = D_qQ^{p-1}(q\chi) - \frac{[p]_q}{p+1}2^pD_q\lambda^{p-1}(\chi).$$

By using Lemma 1 again, we obtained

$$D_qh(\chi) \ge (p-1)Q^{p-2}(\chi)D_qQ(q\chi) - \frac{[p]_q(p-1)}{p+1}2^p\lambda^{p-2}(\chi)D_q\lambda(q\chi) \tag{12}$$

$$\ge (p-1)Q(\chi)[Q^{p-3}(\chi)D_qQ(q\chi) - \frac{[p]_q}{p+1}2^p\lambda^{p-2}(q\chi)]. \tag{13}$$

Since Q is increasing, we have

$$\int_0^\chi Q(u)d_qu \le Q(\chi)(q\chi - 0) = q\chi Q(\chi).$$

Now, using the assumptions of this theorem and the inequalities in Equations (12) and (13), we obtain

$$D_qh(\chi) \ge (p-1)Q^{p-1}(\chi)[D_qQ(q\chi) - \frac{[p]_q}{p+1}2^p(q\chi)^{p-2}] \ge 0,$$

In addition, from the fact that $h(0) = Q^p(0) = 0$, we obtain $h(\chi) \geq 0, \quad \chi \in [0,1]_q$.

From $F(0) = 0$ and $D_q F(\chi) = Q(\chi)h(\chi) \geq 0$, it is given that $F(\chi) \geq 0 \; \forall \; \chi \in [0,1]_q$, particularly

$$F(1) = \int_0^1 Q^p(qu) d_q u - \frac{2^p}{p+1} \left(\int_0^1 Q(u) d_q u \right)^p \geq 0.$$

□

Corollary 1. *Let $n \in \mathbb{N}$ and the function Q on $[0,1]_q$ satisfy*

$$Q(0) = 0, \quad D_q Q(\chi) \geq \frac{n+1}{n+2} 2^{n+1} \chi^n, \quad \chi \in (0,1]_q.$$

Then, we have

$$\int_0^1 Q^{n+1}(q\chi) d_q \chi \geq \frac{2^{n+1}}{n+2} \left(\int_0^1 Q(\chi) d_q \chi \right)^{n+1}.$$

Proof. The results follow directly by substituting $p = n+1$ into Theorem 2. □

Corollary 2. *Let $n \in \mathbb{N}$ and the function Q on $[0,1]_q$ satisfy*

$$D_q^i Q(0) \geq 0, \quad 0 \leq i \leq n-1, \quad D_q^n Q(\chi) \geq \frac{n+1}{n+2} 2^{n+1} [n]_q!, \quad \chi \in (0,1]_q.$$

Then, we have

$$\int_0^1 (Q(q\chi))^{n+2} d_q \chi \geq \frac{2^{n+2}}{n+3} \left(\int_0^1 Q(\chi) d_q \chi \right)^{n+2}.$$

Proof. Since $D_q^n Q(\chi) \geq \dfrac{n+1}{n+2} 2^{n+1} [n]_q!$, therefore, by q-integrating $(n-1)$ times on $[0,\chi]$, we obtain

$$D_q Q(\chi) \geq \frac{n+1}{n+2} 2^{n+1} \chi^n.$$

We obtain our required result by using Corollary 1. □

Theorem 3. *Let $p \in \mathbb{R}$ s.t. $p \geq 1$ and the function Q on $[0,1]_q$ satisfy*

$$Q(0) = 0, \quad p \leq D_q Q(q\chi), \forall \chi \in (0,1]_q. \tag{14}$$

Then, we have

$$\int_0^1 Q^{p+2}(q\chi) d_q \chi \geq \frac{2^{p+2}}{p+3} \left(\int_0^1 Q(\chi) d_q \chi \right)^{p+2}. \tag{15}$$

Proof. Include $\lambda(\chi) = \displaystyle\int_0^x Q(u) d_q u$ and

$$F(\chi) = \int_0^\chi Q^{p+2}(qu) d_q u - \frac{2^{p+2}}{p+3} \left(\int_0^\chi Q(u) d_q u \right)^{p+2} \quad \chi \in [0,1]_q. \tag{16}$$

We have

$$D_q F(\chi) = Q^{p+2}(q\chi) - \frac{2^{p+2}}{p+3} D_q [\lambda^{p+2}](\chi) \quad \chi \in [0,1]_q. \tag{17}$$

Since Q and λ both are increasing on $[0,1]_q$, we obtain the following from Lemma 1 for $\chi \in (0,1]_q$:

$$\begin{aligned} D_q F(\chi) &\geq Q^{p+2}(q\chi) - \frac{p+2}{p+3} 2^{p+2} \lambda^{p+1}(\chi) Q(\chi) \\ &\geq Q^{p+2}(q\chi) - \frac{p+2}{p+3} 2^{p+2} \lambda^{p+2}(\chi) Q(q\chi) = Q(q\chi) h(\chi), \end{aligned} \tag{18}$$

Here, $h(\chi) = Q^{p+1}(q\chi) - \dfrac{p+2}{p+3} 2^{p+2} \lambda^{p+1}(\chi)$.

We also have

$$D_q h(\chi) = D_q Q^{p+1}(q\chi) - \frac{p+2}{p+3} 2^{p+2} D_q \lambda^{p+1}(\chi).$$

By using Lemma 1 again, we obtain

$$D_q h(\chi) \geq (p+1) Q^p(\chi) D_q Q(q\chi) - \frac{(p+1)(p+2)}{p+3} 2^{p+2} \lambda^p(\chi) Q(\chi) \tag{19}$$

$$\geq (p+1) Q(\chi) [Q^{p-1}(\chi) D_q Q(q\chi) - \frac{p+2}{p+3} 2^{p+3} \lambda^p(q\chi)]. \tag{20}$$

Since the function Q is increasing, for $\chi \in [0,1]_q$, we have

$$\int_0^{q\chi} Q(u) d_q u \leq Q(q\chi)(1-0) = Q(q\chi).$$

Therefore, we have

$$D_q h(\chi) \geq (p+1) Q^p(\chi) [D_q Q(q\chi) - \frac{p+2}{p+3} 2^{p+3}] \geq 0,$$

We deduce from the conditions given in Equation (14) that h is increasing on $[0,1]_q$.

Finally, since $h(0) = Q^{p+1}(0) = 0$, it follows that F increases and $F(1) \geq F(0) = 0$, which completes the proof. □

Theorem 4. *Let Q be an increasing function such that $Q(0) = 0$. Then, for all $p \geq 0$, we have*

$$\int_0^1 [Q(q\chi)]^{2p+1} d_q \chi \geq \frac{2^{2p+1}}{2p+2} \left[\int_0^1 Q(\chi) d_q \chi \right]^{2p+1}, \tag{21}$$

provided that $D_q f \geq \frac{2p+1}{2p+2} 2^{2p+1}$.

Proof. For $\tau \in [0,1]_q$, we put forth

$$F(\tau) = \int_0^\tau [Q(q\chi)]^{2p+1} d_q \chi - \frac{2^{2p+1}}{2p+2} \left[\int_0^t Q(\chi) d_q \chi \right]^{2p+1} \quad \textit{and} \quad \lambda(\tau) = \int_0^\tau Q(q\chi) d_q \chi.$$

Then, for $\tau \in [0,1]_q$, we have

$$\begin{aligned} D_qF(\tau) &= [Q(q\tau)]^{2p+1} - \frac{2^{2p+1}}{2p+2} D_q[\lambda(\tau)]^{2p+1} \\ &\geq [Q(\tau)]^{2p+1} - \frac{2p+1}{2p+2} 2^{2p+1}\lambda^{2p}(\tau)Q(qt) \\ &\geq [Q(q\tau)]^{2p+1} - \frac{2p+1}{2p+2} 2^{2p+1}\lambda^{2p}(\tau)Q(q\tau) \quad (22) \\ &= Q(q\tau)\left([Q(q\tau)]^{2p} - \frac{2p+1}{2p+2} 2^{2p+1}\lambda^{2p}(\tau)\right) \quad (23) \\ &= Q(q\tau)G(\tau), \end{aligned}$$

where $G(\tau) = [Q(q\tau)]^{2p} - \dfrac{2p+1}{2p+2} 2^{2p+1}\lambda^{2p}(\tau)$.

Furthermore, we have

$$\begin{aligned} D_qG(\tau) &\geq 2pQ^{2p-1}(qt)D_qQ(\tau) - \frac{2p(2p+1)}{2p+2} 2^{2p+1}\lambda^{2p-1}(\tau)Q(qt) \\ &\geq 2pQ(qt)\left(Q^{2p}(qt)D_qQ(\tau) - \frac{2p+1}{2p+2} 2^{2p+1}\lambda^{2p-1}(\tau)\right). \quad (24) \end{aligned}$$

Since the function Q is increasing, we have the following for $\chi \in [0,1]_q$:

$$\int_0^{q\chi} Q(qu)d_qu \leq Q(q\chi)(1-0) = Q(q\chi).$$

Therefore, we have

$$D_qG(\tau) \geq 2pQ^{2p}(qt)[D_qQ(\tau) - \frac{2p+1}{2p+2} 2^{2p+1}] \geq 0,$$

Hence, G is increasing on $[0,1]_q$. Moreover, we have

$$G(0) = [Q(0)]^{2p} - \frac{2p+1}{2p+2} 2^{2p+1}\lambda^{2p}(0) = 0,$$

for all $t \in (0,1]_q$, $G(\tau) > G(0) = 0$, implying that $D_qF(\tau) > 0$, $\forall\, \tau \in (0,1]_q$. Thus, F is increasing on $[0,1]_q$. In particular, $F(1) > F(0) = 0$, which proves our claim. □

3. *h*-Integral Inequalities of the Anderson Type

3.1. Notations and Preliminaries

Let us recall some definitions from [15].

Let $h \neq 0$. A "quantum derivative of a function" Q, denoted by D_hQ, is given by

$$D_hQ(\chi) = \frac{Q(\chi+h) - Q(\chi)}{h}. \quad (25)$$

One can easily verify that

$$\begin{aligned} D_h(Q(\chi)\lambda(\chi)) &= Q(\chi)D_h\lambda(\chi) + \lambda(\chi+h)D_hQ(\chi) \quad (26) \\ D_h\left(\frac{Q(\chi)}{\lambda(\chi)}\right) &= \frac{\lambda(\chi)D_hQ(\chi) - f(\chi)D_h\lambda(\chi)}{\lambda(\chi)\lambda(\chi+h)} \quad (27) \end{aligned}$$

If $Q'(0)$ exists, then $D_hQ(0) = Q'(0)$. As $h \to 0$, we find an ordinary derivative.

If $m_1 - m_0 \in h\mathbb{Z}$, the definite h-integral is defined by [4]

$$\int_{m_0}^{m_1} Q(\chi)d_h x = \begin{cases} h\big(Q(m_0) + Q(m_0+h) + \cdots + Q(m_1 - h)\big), & \text{if } m_0 < m_1, \\ 0, & \text{if } m_0 = m_1 \\ -h\big(Q(m_1) + Q(m_1+h) + \cdots + Q(m_0 - h)\big), & \text{if } m_0 > m_1. \end{cases} \tag{28}$$

The following theorem, whose proof can be found in [4] , justifies Equation (28):

Theorem 5. *If F is an h-antiderivative of Q and $m_1 - m_0 \in h\mathbb{Z}$, then*

$$\int_{m_0}^{m_1} Q(\chi)d_h\chi = F(m_1) - F(m_0). \tag{29}$$

By applying Theorem 5 to $D_h(Q(\chi)\lambda(\chi))$ and using Equation (26), one can find the following:

$$\int_{m_0}^{m_1} Q(\chi)d_h(\chi) = Q(m_1)\lambda(m_1) - Q(m_0)\lambda(m_0) - \int_0^1 \lambda(\chi+h)d_hQ(\chi). \tag{30}$$

For any function Q, one can easily verify

$$D_h\left[\int_a^x Q(\tau)d_h\tau\right] = Q(\chi). \tag{31}$$

3.2. Main Results

Before we proceed further, we need a lemma:

Lemma 2. *If we let $n \in \mathbb{R}$ s.t. $n \geq 1$ and the function λ be non-negative increasing on $[0,1]$, then*

$$n\lambda^n(\chi)D_h\lambda(\chi) \leq D_h[\lambda(\chi)]^n \leq n\lambda^n(\chi)D_h\lambda(\chi), \tag{32}$$

Proof. We have

$$D_h[\lambda^n](\chi) = \frac{\lambda^n(\chi+h) - \lambda^n(\chi)}{h} = \frac{n}{h}\int_{\lambda(\chi)}^{\lambda(\chi+h)} \tau^{n-1}d_h\tau. \tag{33}$$

Since λ is a non-negative increasing function, we have

$$\lambda^n(\chi)[\lambda(\chi+h) - \lambda(\chi)] \leq \int_{\lambda(\chi)}^{\lambda(\chi+h)} \tau^{n-1}d_h\tau \leq \lambda^n(\chi+h)[\lambda(\chi+h) - \lambda(\chi)]. \tag{34}$$

Therefore, through Equations (33) and (34), we obtain our required result. □

Theorem 6. *Let function Q be non-negative and increasing on $[0,1]$ and satisfy*

$$Q^{\xi-2}(\chi)D_hQ(\chi) \geq \frac{\xi}{\xi+1}2^{\xi}(\chi+2h)^{\xi-2}Q^{\xi-1}(\chi+2h) \quad for \quad \chi \geq 2 \quad and \quad (1-0) \in hz.$$

Then, we have

$$\int_0^1 [Q(\chi)]^{\xi}d_hx \geq \frac{2^{\xi}}{\xi+1}\left[\left(\int_0^1 Q(q\chi)d_hx\right)\right]^{\xi}. \tag{35}$$

Proof. For $\chi \in [0,1]$, let

$$F(\chi) = \int_0^{\chi} [Q(u)]^{\xi}d_hu - \frac{2^{\xi}}{\xi+1}\left[\left(\int_0^{\chi} Q(u)d_hu\right)\right]^{\xi}.$$

and $\lambda(\chi) = \int_0^\chi Q(u)d_h u$. By virtue of Lemma 2, it follows that

$$\begin{aligned} D_h F(\chi) &= F^{\xi}(\chi) - \frac{2^{\xi}}{\xi+1} D_h[\lambda^{\xi}](\chi). \\ &\geq Q^{\xi}(\chi) - \frac{2^{\xi}}{\xi+1}\lambda^{\xi-1}(\chi+h)Q(\chi) \\ &= Q(\chi)\left[Q^{\xi-1}(\chi) - \frac{2^{\xi}}{\xi+1}\lambda^{\xi-1}(\chi+h)\right], \end{aligned} \tag{36}$$

where

$$F_1(\chi) = Q^{\xi}(\chi) - \frac{2^{\xi}}{\xi+1}\lambda^{\xi-1}(\chi+h)Q(\chi).$$

With Lemma 2, we have

$$D_h F_1 = D_h\left(Q^{\xi-1}(\chi) - \frac{2^{\xi}}{\xi+1}\lambda^{\xi-1}(\chi+h)Q(\chi)\right) \tag{37}$$

$$\geq (\xi-1)Q^{\xi-2}(\chi)D_h Q(\chi) - \frac{\xi(\xi-1)}{\xi+1}2^{\xi}\lambda^{\xi-2}(\chi+2h)D_h\lambda(\chi+h) \tag{38}$$

$$= (n-1)Q^{\xi-2}(\chi)D_h Q(\chi) - \frac{n(\xi-1)}{\xi+1}2^{\xi}\lambda^{\xi-2}(\chi+2h)Q(\chi+h)$$

Since the function λ is increasing and non-negative, therefore

$$\lambda(\chi+2h) = \int_0^{\chi+2h} Q(u)d_h u \leq (\chi+2h-0)Q(\chi+2h) \tag{39}$$

Hence, we have

$$D_h F_1(\chi) \geq (\xi-1)Q^{\xi-2}(\chi)D_h Q(\chi) - \frac{\xi(\xi-1)}{\xi+1}2^{\xi}\lambda^{\xi-2}(\chi+2h)Q(\chi+2h).$$

$$= (\xi-1)Q^{\xi-2}(\chi)D_h Q(\chi) - \frac{\xi(\xi-1)}{\xi+1}2^{\xi}\lambda^{\xi-2}(\chi+2h-0)Q^{\xi-2}(\chi+2h) \tag{40}$$

$$= (\xi-1)\left(Q^{\xi-2}(\chi)D_h Q(\chi) - \frac{\xi}{\xi+1}2^{\xi}(\chi+2h)^{\xi-2}Q^{\xi-2}(\chi+2h)\right) \geq 0. \tag{41}$$

which assures that F_1 is increasing. Hence, $F_1(\chi) \geq F_1(0) \geq 0$ and $D_h F(\chi) \geq 0$, so F is increasing since $F(\chi) \geq F(0) = 0$. □

Theorem 7. *Let $n \geq 1$. If the function Q is non-negative and increasing on $[0,1]$ and satisfies*

$$Q^{n-1}(\chi)D_h Q(\chi) \geq 2^n(\chi+2h)^{n-1}Q^n(\chi+2h) \tag{42}$$

then

$$\int_0^1 Q^{n+1}(\chi)d_h(\chi) \geq \frac{2^n}{n+1}\left(\int_0^1 Q(\chi)d_h(\chi)\right)^{n+1} \tag{43}$$

Proof. For $\chi \in [0,1]$, let

$$F(\chi) = \int_0^\chi Q^{n+1}(u)d_h(\tau) - \frac{2^n}{n+1}\left(\int_0^\chi Q(u)d_h(u)\right)^{n+1}$$

and

$$\lambda(\chi) = \int_0^\chi Q(u)d_h(u).$$

Utilizing Lemma 2 gives

$$\begin{aligned} D_h F(\chi) &= Q^{n+1}(\chi) - \frac{2^n}{n+1} D_h(\lambda^{n+1}(\chi)) \\ &\geq Q^{n+1}(\chi) - \frac{2^n}{n+1}(n+1)\lambda^n(\chi+h)Q(\chi) \\ &= Q^{n+1}(\chi) - 2^n\lambda^n(\chi+h)Q(\chi) \\ &= Q(\chi)[Q^n(\chi) - 2^n\lambda^n(\chi+h)] \end{aligned}$$

where

$$F_1(\chi) = Q^n(\chi) - 2^n\lambda^n(\chi+h).$$

From Lemma 2, it follows that

$$\begin{aligned} D_h F_1(\chi) &= D_h(Q^n(\chi)) - 2^n D_h(\lambda^n(\chi+h)) \\ &\geq nQ^{n-1}(\chi)D_h Q(\chi) - 2^n.n\lambda^{n-1}(\chi+2h)Q(\chi+h). \end{aligned}$$

Since Q is increasing and non-negative, then

$$\lambda(\chi+2h) = \int_0^{\chi+2h} Q(u)d_h(u) \leq (\chi+2h)Q(\chi+2h) \tag{44}$$

Hence, we have

$$\begin{aligned} D_h F_1(\chi) &\geq nQ^{n-1}(\chi)D_h Q(\chi) - n.2^n(\chi+2h)^{n-1}Q^{n-1}(\chi+2h)Q(\chi+h) \\ &\geq nQ^{n-1}(\chi)D_h Q(\chi) - n.2^n(\chi+2h)^{n-1}Q^n(\chi+2h) \end{aligned}$$

$$D_h F_1(\chi) = n\Big(Q^{n-1}(\chi)D_n Q(\chi) - 2^n(\chi+2h)^{n-1}Q^n(\chi+2h)\Big) \geq 0 \tag{45}$$

We conclude that F_1 is an increasing function. Hence, $F_1(\chi) \geq F_1(0) = 0$ and $D_h F(\chi) \geq 0$. Therefore, F increases since $F(\chi) \geq F(0) = 0$. □

4. Conclusions

- In the current study, we analyzed a Feng Qi type q-integral inequality and h-integral inequality and then discussed an Anderson type integral inequality in quantum calculus;
- We considered how to translate some Anderson type integral inequalities into quantum calculus (i.e., q-integral and h integral inequalities);
- By transforming these Anderson type integral inequalities into the equivalent q-integral inequalities and h-integral inequalities, we established a solution method for the corresponding fractional integral inequalities. Our work further develops the solution of time-scale calculus.

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