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Analyzing the Jacobi Stability of Lü's Circuit System †

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Abstract: By reformulating the circuit system of Lü as a set of two second order differential equations, we investigate the nonlinear dynamics of Lü's circuit system from the Jacobi stability point of view, using Kosambi–Cartan–Chern geometric theory. We will determine the five KCC invariants, which express the intrinsic geometric properties of the system, including the deviation curvature tensor. Finally, we will obtain necessary and sufficient conditions on the parameters of the system to have the Jacobi stability near the equilibrium points.

Keywords: Lü circuit system; KCC-geometric theory; the deviation curvature tensor; Jacobi stability

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1. Introduction

In this paper, we will study the Jacobi stability of Lü's circuit system using the geometric tools of the Kosambi–Cartan–Chern theory. To find the Jacobi stability conditions, we will determine all five invariants of KCC theory which express the intrinsic geometric properties of the system, including the deviation curvature tensor which determine the Jacobi stability of the system near equilibrium points.

The Lü's dynamical system was first proposed by J. Lü and G. Chen in [1]. This system is a model of a nonlinear electrical circuit, and we want to study it from differential geometry point of view and to point out some of its geometrical and dynamical properties. The original Lü's system has the following form:

$$\left\{ \begin{array}{rcl} \dot{x} & = & a(y-x) \\ \dot{y} & = & -xz+by \\ \dot{z} & = & -cz+xy \end{array} \right. \tag{1}$$

where a, b, c are real parameters and $\dot{x} = \frac{dx}{dt}, \dot{y} = \frac{dy}{dt}, \dot{z} = \frac{dz}{dt}$. This autonomous system of ordinary differential equations, together with Lorentz's system [2] and Chua's system [3], was generally accepted then as behaving chaotically [4,5]. The general dynamics and heteroclinic orbits of the Lü's system was also studied by G. Tigan and D. Constantinescu in [6]. For these reasons, the topic of studying the stability of the Lü's system is very important both from a theoretical point of view, i.e., mathematically, and from a practical point of view, i.e., technically.

There are a lot of papers that studied the classical (linear or Lyapunov) stability of Lü's circuit system. In this paper, we will study another kind of stability for Lü's system, namely *Jacobi stability*. The Jacobi stability is a natural generalization of the stability of the geodesic flow on a differentiable manifold equipped with a Riemannian or Finslerian metrics to a manifold without a metric [7–13]. The Jacobi stability examines the robustness of a dynamical system defined by a system of second order differential equations (SODEs), where robustness is a measure of insensitivity and adaptation to a change in the system internal parameters and the environment. Jacobi stability analysis of dynamical systems has been recently studied by several authors in [8,9,14,15], using the Kosambi–Cartan–Chern (KCC) theory [16–18]. More precisely, the study of the dynamics of the system is done using the properties of geometric objects associated with the system of second order differential equations derived from the initial first order differential system.

From the mathematical point of view, KCC theory investigates the deviation of neighboring trajectories, which enables one to estimate the perturbation permitted around the steady state solutions of the second order differential equation. This theory started with the study of variation equations (of Jacobi field equations) corresponding to the given geometry on the differentiable manifold, and Jacobi stability was studied initially for geodesics in Riemannian or Finslerian geometry by P.L. Antonelli, R. Ingarden, and M. Matsumoto in [7–9]. By deviating the geodesics and using the KCC-covariant derivative, it follows a differential system in variations. Consequently, the second KCC invariant appeared, also called the *deviation curvature tensor*. This invariant is essential for establishing the Jacobi stability for geodesics and, generally, for dynamical systems defined by SODEs. In differential geometry, SODEs are also called semi-sprays and starting with a semi-spray, a nonlinear connection can be defined on the manifold and conversely, any nonlinear connection defines a semi-spray. Then, any SODEs define a geometry on the manifold by the associated geometric objects and conversely [10,19–21].

KCC theory originated from the works of D. D. Kosambi [18], E. Cartan [16] and S. S. Chern [17], and hence the abbreviation KCC (Kosambi–Cartan–Chern). This geometric theory has a lot of applications in engineering, physics, chemistry, and biology [14,15]. In addition, there are recent developments and applications of KCC theory in gravitation and cosmology [22–24]. In [25], C.G. Boehmer, T. Harko, and S.V. Sabau analyzed Jacobi stability and its relations with the linear Liapunov stability analysis of dynamical systems, and presented a comparative study of these methods in the fields of gravitation and astrophysics. In [26], they used the geometric tools of KCC theory in order to study the Jacobi stability of Lorentz's system. Furthermore, [12,27,28] studied Jacobi stability for different versions of Chua's system.

The purpose of the present paper is to study the stability of the equilibrium points of Lü's circuit system from the Jacobi stability point of view, using Kosambi–Cartan–Chern geometric theory. Reformulating the system as a set of two second order differential equations, we will investigate the nonlinear dynamics of Lü's circuit system by determining the five KCC invariants which express the intrinsic geometric properties of the system, including the deviation curvature tensor, which determine the Jacobi stability of the system near equilibrium points. This kind of approach for Lü's system is for the first time.

In the second section we briefly present Lü's circuit system and we will point out the equilibria of Lü's circuit system. In the third section, we review the main notions and tools of the KCC theory in order to analyze the Jacobi stability of Lü's system. We present the five invariants of the KCC theory and definition of the Jacobi stability. The new obtained results for the Lü's circuit system will be presented in Sections 3 and 4. Thus, in the fourth section, a reformulation of the Lü's system as a system of second order differential equations is obtained and the five geometrical invariants are computed. The obtained results on the Jacobi stability of the Lü's system near the equilibria are presented in Section 5. More exactly, we will find necessary and sufficient conditions in order to have the Jacobi stability of the system near the equilibrium points. Consequently, for these values of the parameters, it is not possible to have a chaotic behavior for Lü's circuit system.

Sum over crossed repeated indices is understood.

2. The Circuit System of Lü

If $a \neq 0$, $b \neq 0$, $c \neq 0$ and $bc > 0$, then the Lü's dynamical system proposed by J. Lü and G. Chen in [1],

$$\begin{cases} \dot{x} &= a(y-x) \\ \dot{y} &= -xz+by \\ \dot{z} &= -cz+xy \end{cases} \tag{2}$$

has three equilibrium points $O(0,0,0)$, $E_1(\sqrt{bc}, \sqrt{bc}, b)$ and $E_2(-\sqrt{bc}, -\sqrt{bc}, b)$.

Obviously, if $bc < 0$, then $O(0,0,0)$ is the only one equilibrium point.

According to [6], let us point out that Lü's system is invariant under the transformation $(x, y, z) \mapsto (-x - y, z)$. That means the orbits of the system are symmetrical with respect to the z-axis. Therefore, if the system has an orbit $\gamma_1(t) = (x(t), y(t), z(t))$ then it has also the orbit $\gamma_2(t) = (-x(t), -y(t), z(t))$. The orbits $\gamma_1(t)$ and $\gamma_2(t)$ are symmetric one to another with respect to the z-axis.

The Jacobi matrix at an equilibrium point (x, y, z) is

$$A = \begin{pmatrix} -a & a & 0 \\ -z & b & -x \\ y & x & -c \end{pmatrix}.$$

For the trivial equilibrium $O(0, 0, 0)$ we obtain

$$A = \begin{pmatrix} -a & a & 0 \\ 0 & b & 0 \\ 0 & 0 & -c \end{pmatrix}.$$

with eigenvalues $\lambda_1 = -a$, $\lambda_2 = b$, $\lambda_3 = -c$. Therefore, if $bc > 0$, then the origin $O(0, 0, 0)$ is a saddle point. Otherwise, if $bc < 0$, then $O(0, 0, 0)$ can be a saddle point or an attractor or a repeller, but in this case the origin remains the single equilibrium point.

For the equilibria $E_1(\sqrt{bc}, \sqrt{bc}, b)$, we have

$$A = \begin{pmatrix} -a & a & 0 \\ -b & b & -\sqrt{bc} \\ \sqrt{bc} & \sqrt{bc} & -c \end{pmatrix}.$$

and the equilibria $E_2(-\sqrt{bc}, -\sqrt{bc}, b)$, we have

$$A = \begin{pmatrix} -a & a & 0 \\ -b & b & \sqrt{bc} \\ -\sqrt{bc} & -\sqrt{bc} & -c \end{pmatrix}.$$

According to the Routh–Hurwitz criterion, the characteristic polynomial $P(X) = X^3 + a_2X^2 + a_1X + a_0$ has all roots in the open left half plane (i.e., $\lambda_i < 0$ or Re $\lambda_i < 0$, for all i) if and only if $a_2 > 0$, $a_0 > 0$ and $a_2a_1 > a_0$.

Now, we can remark that it is very difficult to establish the behavior of the system near the equilibrium points because we have a lot of parameters which are involved in computation. However, this system has been studied comprehensively in [1,4–6].

Furthermore, we are interested in the study of the Jacobi stability of Lü's circuit system.

3. Kosambi–Cartan–Chern Theory and Jacobi Stability

In this section, we will present briefly the basic notions and main results needed from Kosambi–Cartan–Chern (KCC) theory following [8,9,14–18]. The basic idea in KCC theory is that the second order differential equations (SODEs) which model the dynamical system and geodesic equations in associated Finsler space are topologically equivalent. Since the linear (or Lyapunov) stability is well established for many dynamical systems and also for Chua's circuit system or any modified version, it would be very useful to study another type of stability, namely called the Jacobi stability. The Jacobi stability is a natural generalization of the stability of the geodesic flow on a differentiable manifold endowed with a metric (Riemannian or Finslerian) to the non-metric setting. KCC theory is a modern geometric approach of the dynamical systems which associates a nonlinear connection and a Berwald type connection to the SODE's that define the dynamical system. Moreover, for every SODE, the five geometrical invariants will be obtained which determine the dynamics of the system, ε^i—the external force, P^i_j—the deviation curvature tensor, P^i_{jk}—the torsion tensor, P^i_{jkl}—the Riemann–Christoffel curvature tensor and D^i_{jkl}—the Douglas curvature tensor. Fortunately, the Jacobi stability of a dynamical system depends only on the second invariant, namely the deviation curvature tensor.

We now introduce the main ideas of KCC theory [7–9,14,15]. Let M be a real, smooth n-dimensional manifold and let TM be its tangent bundle. Usually, $M = \mathbf{R}^n$ or M is an open subset of $\mathbf{R}^n$. Let $u = (x, y)$ be a point in TM, where $x = (x^1, \ldots, x^n)$ and $y = (y^1, \ldots, y^n)$, which means $y^i = \frac{dx^i}{dt}$, $i = 1, \ldots, n$. Consider the following system of second order differential equations in normalized form [7]

$$\frac{d^2x^i}{dt^2} + 2G^i(x, y) = 0, \; i = 1, \ldots, n. \tag{3}$$

where $G^i(x, y)$ are smooth functions defined in a local system of coordinates on TM, usually an open neighborhood of some initial conditions (x_0, y_0). In fact, the system (3) is motivated by Euler–Lagrange equations of classical dynamics [7,19]

$$\begin{cases} \frac{d}{dt}\frac{\partial L}{\partial y^i} - \frac{\partial L}{\partial x^i} & = & F^i \\ y^i & = & \frac{dx^i}{dt} \end{cases}, \; i = 1, \ldots, n. \tag{4}$$

where $L(x, y)$ is a regular Lagrangian of TM and F^i are the external forces.

Generally, the system (3) has no geometrical meaning since "accelerations" $\frac{d^2x^i}{dt^2}$ or "forces" $G^i(x^j, y^j)$ is not a $(0, 1)$-type tensor under the local coordinates transformation

$$\begin{cases} \tilde{x}^i & = & \tilde{x}^i(x^1, \ldots, x^n) \\ \tilde{y}^i & = & \frac{\partial \tilde{x}^i}{\partial x^j} y^j \end{cases}, \; i = 1, \ldots, n. \tag{5}$$

The system (3) has a geometrical meaning, and it is called *a semi-spray*, if the functions $G^i(x^j, y^j)$ are changing under the local coordinates transformation (5) after the rules [7,19]

$$2\tilde{G}^i = 2G^j \frac{\partial \tilde{x}^i}{\partial x^j} - \frac{\partial \tilde{y}^i}{\partial x^j} y^j \,. \tag{6}$$

The main idea of KCC theory was to change the second order differential Equation (3) into an equivalent system (*same solutions*), but with geometrical meaning, and then to show that it defines five tensor fields called *geometrical* (or *differential*) *invariants* of KCC theory [8,9]. To find the five KCC geometrical invariants of the system (3) under the local change coordinates (5), we need to introduce the KCC-covariant differential of a vector field $\xi = \xi^i \frac{\partial}{\partial x^i}$ defined in an open set of TM (usually $TM = \mathbf{R}^n \times \mathbf{R}^n$) as follows [8,16–18].

$$\frac{D\xi^i}{dt} = \frac{d\xi^i}{dt} + N^i_j \xi^j \,, \tag{7}$$

where $N^i_j = \frac{\partial G^i}{\partial y^j}$ are the coefficients of *a nonlinear connection* N on the tangent bundle TM associated with the semi-spray (3).

For $\xi^i = y^i$ it obtains

$$\frac{Dy^i}{dt} = -2G^i + N^i_j y^j = -\varepsilon^i \,. \tag{8}$$

The contravariant vector field ε^i is called *the first invariant* of KCC theory and, from the physical point of view, this invariant plays the role of an external force [8]. Of course, the terms ε^i has geometrical character since with respect to coordinates transformation (5), we have

$$\tilde{\varepsilon}^i = \frac{\partial \tilde{x}^i}{\partial x^j} \varepsilon^j \,.$$

If the functions G^i are 2-homogeneous with respect to y^i, i.e., $\frac{\partial G^i}{\partial y^j} y^j = 2G^i$, for all $i = 1, \ldots, n$, then $\varepsilon^i = 0$, for all $i = 1, \ldots, n$. Therefore, the first invariant of the KCC theory vanishes if, and only if, the semi-spray is a spray. This always happens for the geodesic spray corresponding to a Riemannian or Finsler manifold [7,19].

One of the main goals of Kosambi–Cartan–Chern theory is to study the trajectories which are slightly deviated upon a certain trajectory of (3). More precisely, we will study the dynamics of the system in variations. According to this purpose, we will vary the trajectories $x^i(t)$ of (3) into nearby ones prescribed by

$$\tilde{x}^i(t) = x^i(t) + \eta \xi^i(t) \tag{9}$$

where $|\eta|$ is a small parameter and $\xi^i(t)$ are the components of a contravariant vector field defined along the trajectories $x^i(t)$. After substituting (9) into (3) and tacking the limit as $\eta \to 0$, it will be obtain the following variational equations [7–9]:

$$\frac{d^2\xi^i}{dt^2} + 2N^i_j \frac{d\xi^j}{dt} + 2\frac{\partial G^i}{\partial x^j}\xi^j = 0 \tag{10}$$

Using the KCC-covariant derivative from (7), the previous equations can be written in the following covariant form [7–9]:

$$\frac{D^2\xi^i}{dt^2} = P^i_j \xi^j \tag{11}$$

where the right side $(1,1)$-type tensor P^i_j is given by

$$P^i_j = -2\frac{\partial G^i}{\partial x^j} - 2G^l G^i_{jl} + y^l \frac{\partial N^i_j}{\partial x^l} + N^i_l N^l_j \tag{12}$$

where

$$G^i_{jl} = \frac{\partial N^i_j}{\partial y^l} \tag{13}$$

is called *the Berwald connection* associated with the nonlinear connection N (according to [7,19]).

If all coefficients of nonlinear connection and Berwald connection are identically zero, then the deviation curvature tensor from (12) becomes $P^i_j = -2\frac{\partial G^i}{\partial x^j}$. Therefore, following [29] we can introduce the so-called *zero-connection curvature tensor* Z given by

$$Z^i_j = 2\frac{\partial G^i}{\partial x^j}\,. \tag{14}$$

For two-dimensional systems, the zero-connection curvature Z corresponds to the Gaussian curvature K of the potential surface $V(x^i) = 0$, where $\dot{x}^i = f^i(x^j) = -\frac{\partial V}{\partial x^i}(x^j)$. When the potential surface is minimal we have $P = -K$.

This tensor (P^i_j) is called *the deviation curvature tensor* and represent *the second invariant* of KCC theory. The Equation (10) are called *the deviation equations* or *Jacobi equations*. The invariant Equation (11) is called also the *Jacobi equation*. In either Riemann or Finsler geometry, when the second order system of equations describes the geodesic motion, then the above equations is exactly the Jacobi field equation corresponding to the given geometry.

In the KCC theory one can also introduce the *third*, *fourth* and *fifth invariants* of the second order system of Equation (3). These invariants are defined by

$$P^i_{jk} = \frac{1}{3}\left(\frac{\partial P^i_j}{\partial y^k} - \frac{\partial P^i_k}{\partial y^j}\right),$$
$$P^i_{jkl} = \frac{\partial P^i_{jk}}{\partial y^l},\, D^i_{jkl} = \frac{\partial G^i_{jk}}{\partial y^l}\,. \tag{15}$$

From geometrical point of view the third KCC invariant P^i_{jk} can be interpreted as a *torsion tensor*. The fourth and fifth KCC invariants P^i_{jkl} and D^i_{jkl} represent the *Riemann-Christoffel curvature tensor*, and the *Douglas tensor*, respectively.

It is important to point out that these tensors always exist [7–9,15,19].

In KCC theory, these *five invariants* are the basic mathematical quantities describing the geometrical properties and interpretation of an arbitrary system of second order differential equations [7,16,19].

A basic result of KCC theory comes from P.L. Antonelli [8]:

Theorem 1 ([8])**.** *Two second order differential systems of the type of (3)*

$$\frac{d^2x^i}{dt^2} + 2G^i(x^j, y^j) = 0,\; y^j = \frac{dx^j}{dt}$$

and

$$\frac{d^2\tilde{x}^i}{dt^2} + 2\tilde{G}^i(\tilde{x}^j, \tilde{y}^j) = 0,\; \tilde{y}^j = \frac{d\tilde{x}^j}{dt}$$

can be locally transformed one into another via changing coordinates transformation (5) if, and only if, the five invariants ε^i, P^i_j, P^i_{jk}, P^i_{jkl} and D^i_{jkl} are equivalent tensors with $\tilde{\varepsilon}^i$, $\tilde{P}^i_j$, $\tilde{P}^i_{jk}$, $\tilde{P}^i_{jkl}$ and, respectively, $\tilde{D}^i_{jkl}$.

In particular, there are local coordinates $(x^1, \ldots, x^n)$ on the base manifold M, for which $G^i = 0$, for all i, if, and only if, all five invariant tensors vanish. In this case, the trajectories of the dynamical systems are straight lines.

The term Jacobi stability within the KCC theory is justified by the fact that when (3) represents the second order differential equations for the geodesic equations in Riemannian or Finsler geometry, then (11) is the Jacobi field equation for the geodesic deviation. The Jacobi Equation (11) of the Finsler manifold (M, F) can be written in the scalar form [11]:

$$\frac{d^2v}{ds^2} + K \cdot v = 0 \tag{16}$$

where $\xi^i = v(s)\eta^i$ is the Jacobi field along the geodesic $\gamma : x^i = x^i(s)$, η^i is the unit normal vector field along the geodesic γ and K is the flag curvature of Finsler space (M, F). The sign of the flag curvature K affects the geodesic rays: if $K > 0$, then the geodesics bunch together (are Jacobi stable), and if $K < 0$, then they disperse (are Jacobi unstable). Consequently, taking account of the equivalence between (11) and (16), the results show that a positive (or negative) flag curvature is equivalent to negative (or positive) eigenvalues of the curvature deviation tensor P^i_j. We have the well-known result:

Theorem 2 (Bohmer et al., 2012)**.** *The trajectories of (3) are Jacobi stable if and only if the real parts of the eigenvalues of the deviation tensor P^i_j are strictly negative everywhere, and Jacobi unstable, otherwise.*

Next, we will present a rigorous definition of the Jacobi stability for a geodesic on a manifold endowed with an Euclidean, Riemannian, or Finslerian metric or, more generally, for a trajectory $x^i = x^i(s)$ of the dynamical system associated with (3), following [12–15,25]:

Definition 1. *A trajectory $x^i = x^i(s)$ of (3) is said to be Jacobi stable if for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $\|\tilde{x}^i(s) - x^i(s)\| < \varepsilon$ holds for all $s \geq s_0$ and for all trajectories $\tilde{x}^i = \tilde{x}^i(s)$ for which $\|\tilde{x}^i(s_0) - x^i(s_0)\| < \delta(\varepsilon)$ and $\|\frac{d\tilde{x}^i}{ds}(s_0) - \frac{dx^i}{ds}(s_0)\| < \delta(\varepsilon)$.*

We consider the trajectories of (3) as curves in a Euclidean space $\mathbf{R}^n$ and the norm $\|\cdot\|$ is the induced norm by the canonical inner product $\langle\cdot,\cdot\rangle$ on $\mathbf{R}^n$. Moreover, we will assume that the deviation vector ξ from (11) satisfies the initial conditions $\xi(s_0) = O$ and $\dot{\xi}(s_0) = W \neq O$, where $O \in \mathbf{R}^n$ is the null vector. If we assume that $s_0 = 0$ and $\|W\| = 1$, then for $s \searrow 0$ the trajectories of (3) brunching together if and only if the real parts of all eigenvalues of $P^i_j(0)$ are strictly negative or they dispersing if and only if at least one of the real parts of the eigenvalues of $P^i_j(0)$ are positive.

This kind of stability refers to the focusing tendency (in a small neighborhood of $s_0 = 0$) of the trajectories of (3) with respect to the variation (9) that satisfy the conditions $\|\tilde{x}^i(0) - x^i(0)\| = 0$ and $\|\frac{d\tilde{x}^i}{ds}(0) - \frac{dx^i}{ds}(0)\| \neq 0$.

We remark that the system of second order differential equations or semi-spray (3) is Jacobi stable if and only if the system in variations (10) (or in the covariant form (11)) is Lyapunov (or linear) stable. Therefore, Jacobi stability analysis is based on the study of Lyapunov stability of all trajectories in a region without considering the velocity. This theory, even when derived at an equilibrium point, yields information about the behavior of the trajectories in an open region around this equilibria.

4. SODE Formulation of Lü's System

Starting with the Lü's system (1)

$$\begin{cases} \dot{x} &= a(y-x) \\ \dot{y} &= -xz+by \\ \dot{z} &= -cz+xy \end{cases}$$

by substituting

$$y = x + \frac{1}{a}\dot{x}$$

from the first equation and to the second equation, we obtain

$$\ddot{x} + (a-b)\dot{x} + axz - abx = 0\,.$$

Form the third equation, by substituting $y = x + \frac{1}{a}\dot{x}$ and taking the derivative of the first equation with respect to time t, we obtain

$$\ddot{z} + x^2 z - c^2 z + (c-b)x^2 + \left(\frac{c-b}{a} - 1\right)x\dot{x} - \frac{1}{a}(\dot{x})^2 = 0\,.$$

If we change the notations of variables as follows

$$x = x^1, \dot{x} = y^1, z = x^2, \dot{z} = y^2$$

where $y = x^1 + \frac{1}{a}y^1$, then the last two second order differential equations are equivalent to the following system of SODEs:

$$\begin{cases} \ddot{x^1} + (a-b)\dot{x^1} + ax^1x^2 - abx^1 &= 0 \\ \ddot{x^2} + \left(x^1\right)^2 x^2 - c^2x^2 + (c-b)\left(x^1\right)^2 + \left(\frac{c-b}{a} - 1\right)x^1\dot{x^1} - \frac{1}{a}\left(\dot{x^1}\right)^2 &= 0 \end{cases} \tag{17}$$

or, equivalently,

$$\left\{\begin{array}{lcl} \frac{d^2x^1}{dt^2}+(a-b)y^1+ax^1x^2-abx^1 & = & 0 \\ \frac{d^2x^2}{dt^2}+\left(x^1\right)^2x^2-c^2x^2+(c-b)\left(x^1\right)^2+\left(\frac{c-b}{a}-1\right)x^1y^1-\frac{1}{a}\left(y^1\right)^2 & = & 0 \end{array}\right. \tag{18}$$

where $\frac{dx^i}{dt}=y^i$, $i=1,2$.

This system can be written like SODEs from KCC theory.

$$\left\{\begin{array}{l} \frac{d^2x^1}{dt^2}+2G^1(x^1,x^2,y^1,y^2)=0 \\ \frac{d^2x^2}{dt^2}+2G^2(x^1,x^2,y^1,y^2)=0 \end{array}\right. \tag{19}$$

where $\frac{dx^i}{dt}=y^i$, $i=1,2$ and

$$\begin{array}{lcl} G^1(x^i,y^i) & = & \frac{1}{2}\left[(a-b)y^1+ax^1x^2-abx^1\right] \\ G^2(x^i,y^i) & = & \frac{1}{2}\left[\left(x^1\right)^2x^2-c^2x^2+(c-b)\left(x^1\right)^2+\left(\frac{c-b}{a}-1\right)x^1y^1\right]-\frac{1}{2a}\left(y^1\right)^2 \end{array}$$

The zero-connection curvature $Z^i_j=2\frac{\partial G^i}{\partial x^j}$ has the coefficients $Z^1_1=a\left(x^2-b\right)$, $Z^1_2=ax^1$, $Z^2_1=2x_1x_2+2(c-b)x^1+\left(\frac{c-b}{a}-1\right)y^1$, $Z^2_2=\left(x^1\right)^2-c^2$.

Since $N^1_1=\frac{\partial G^1}{\partial y^1}=\frac{1}{2}(a-b)$, $N^1_2=\frac{\partial G^1}{\partial y^2}=0$, $N^2_1=\frac{\partial G^2}{\partial y^1}=\frac{1}{2}\left(\frac{c-b}{a}-1\right)x^1-\frac{1}{a}y^1$, $N^2_2=\frac{\partial G^2}{\partial y^2}=0$, the nonlinear connection is given by

$$N=\left(\begin{array}{cc} \frac{1}{2}(a-b) & 0 \\ \frac{1}{2}\left(\frac{c-b}{a}-1\right)x^1-\frac{1}{a}y^1 & 0 \end{array}\right).$$

It follows that the coefficients of the Berwald connection $G^i_{jk}=\frac{\partial N^i_j}{\partial y^k}$ are identically zero with one exception, $G^2_{11}=\frac{\partial N^2_1}{\partial y^1}=-\frac{1}{a}$.

The first invariant of KCC theory

$$\varepsilon^i=-\left(N^i_jy^j-2G^i\right)$$

has the components

$$\begin{array}{lcl} \varepsilon^1 & = & \frac{1}{2}(a-b)y^1+ax^1x^2-abx^1 \\ \varepsilon^2 & = & \left(x^1\right)^2x^2-c^2x^2+(c-b)\left(x^1\right)^2+\frac{1}{2}\left(\frac{c-b}{a}-1\right)x^1y^1 \end{array} \tag{20}$$

Since

$$G^i_{jl}=\left\{\begin{array}{lcl} -\frac{1}{a} & , & \text{if } j=1,\ l=1,\ i=2 \\ 0 & , & \text{in rest} \end{array}\right.$$

and following (12),

$$P^i_j=-2\frac{\partial G^i}{\partial x^j}-2G^lG^i_{jl}+y^l\frac{\partial N^i_j}{\partial x^l}+N^i_lN^l_j$$

we obtain the components of the deviation curvature tensor of the Lü's circuit system (18):

$$\begin{array}{lcl} P^1_1 & = & -ax^2+\left(\frac{a+b}{2}\right)^2 \\ P^1_2 & = & -ax^1 \\ P^2_1 & = & -x^1x^2+\left(\frac{b^2-a^2+4ab-7ac-bc}{4a}\right)x^1+\left(1-\frac{c}{2a}\right)y^1 \\ P^2_2 & = & -\left(x^1\right)^2+c^2 \end{array} \tag{21}$$

Then the trace and the determinant of the deviation curvature matrix

$$P = \begin{pmatrix} P^1_1 & P^1_2 \\ P^2_1 & P^2_2 \end{pmatrix}$$

are $\mathrm{trace}(P) = P^1_1 + P^2_2$ and $\det(P) = P^1_1 P^2_2 - P^2_1 P^1_2$.

Therefore, following the results from the previous section, we have:

Theorem 3. *All roots of the characteristic polynomial of P are negative or have negative real parts (i.e., Jacobi stability) if and only if*

$$P^1_1 + P^2_2 < 0 \textit{ and } P^1_1 P^2_2 - P^2_1 P^1_2 > 0\,.$$

Taking into account that $P^i_{jk} = \frac{1}{3}\left(\frac{\partial P^i_j}{\partial y^k} - \frac{\partial P^i_k}{\partial y^j}\right)$, $P^i_{jkl} = \frac{\partial P^i_{jk}}{\partial y^l}$, $D^i_{jkl} = \frac{\partial G^i_{jk}}{\partial y^l}$, the results show that the third, fourth and fifth invariants of the Lü's circuit system are identically zero.

5. Jacobi Stability Analysis of Lü's Circuit System

Further, we will compute the first two invariants at the equilibrium points of Lü's system (1) and we will analyze the Jacobi stability of the system near each equilibria.

If $bc > 0$, then for equlibria $O(0,0,0)$, $E_1(\sqrt{bc}, \sqrt{bc}, b)$, $E_2(-\sqrt{bc}, -\sqrt{bc}, b)$ of the initial Lü's circuit system (1) we have the corresponding equilibrium points $O(0,0,0,0)$, $E_1(\sqrt{bc}, b, 0, 0)$, $E_2(-\sqrt{bc}, b, 0, 0)$ for SODEs (18).

For $O(0,0,0,0)$, the first invariant of KCC theory ε^i has the components $\varepsilon^1 = \varepsilon^2 = 0$ and the matrix with the components of the second KCC invariant is

$$P = \begin{pmatrix} \left(\frac{a+b}{2}\right)^2 & 0 \\ 0 & c^2 \end{pmatrix}.$$

Since $P^1_1 + P^2_2 = \left(\frac{a+b}{2}\right)^2 + c^2 > 0$ and $P^1_1 P^2_2 - P^2_1 P^1_2 = \left(\frac{a+b}{2}\right)^2 c^2 > 0$, using Theorem 3 we obtain:

Theorem 4. *The trivial equilibria O is Jacobi unstable.*

For $E_1(\sqrt{bc}, b, 0, 0)$ the first invariant of KCC theory ε^i has the components $\varepsilon^1 = 0$, $\varepsilon^2 = 0$ and the components of the second KCC invariant are

$$\begin{aligned} P^1_1 &= \left(\frac{a-b}{2}\right)^2 \\ P^1_2 &= -a\sqrt{bc} \\ P^2_1 &= \frac{b^2 - a^2 - 7ac - bc}{4a} \\ P^2_2 &= -bc + c^2 \end{aligned}$$

Theorem 5. *E_1 is Jacobi stable iff $P^1_1 + P^2_2 < 0$ and $P^1_1 P^2_2 - P^2_1 P^1_2 > 0$.*

For $E_2(-\sqrt{bc}, b, 0, 0)$ the first invariant of KCC theory ε^i has the components $\varepsilon^1 = 0$, $\varepsilon^2 = 0$ and the components of the deviation curvature tensor are

$$\begin{aligned} P^1_1 &= \left(\frac{a-b}{2}\right)^2 \\ P^1_2 &= a\sqrt{bc} \\ P^2_1 &= -\frac{b^2 - a^2 - 7ac - bc}{4a} \\ P^2_2 &= -bc + c^2 \end{aligned}$$

Theorem 6. *E_2 is Jacobi stable iff $P^1_1 + P^2_2 < 0$ and $P^1_1 P^2_2 - P^2_1 P^1_2 > 0$.*

Let us remark that E_2 is Jacobi stable if and only if E_1 is Jacobi stable, because, for both equilibrium,

$$P_1^1 + P_2^2 = \left(\frac{a-b}{2}\right)^2 - bc + c^2$$

and

$$P_1^1 P_2^2 - P_1^2 P_2^1 = \frac{(a-b)^2(c^2 - bc) + \left(b^2 - a^2 - 7ac - bc\right)\sqrt{bc}}{4}.$$

In conclusion, we have the result:

Theorem 7. *The equilibrium points E_1 and E_2 are Jacobi stable if and only if are fulfilled simultaneous the following two conditions:*
$\left(\frac{a-b}{2}\right)^2 - bc + c^2 < 0$ *and* $(a-b)^2(c^2 - bc) + \left(b^2 - a^2 - 7ac - bc\right)\sqrt{bc} > 0.$

If $bc < 0$, then we have a unique trivial equlibria $O(0,0,0)$ for the initial Lü's circuit system (1) and the corresponding equilibrium point $O(0,0,0,0)$ for SODEs (18) is also Jacobi unstable.

Remark 1. *In [1], it was shown that if we fix $a = 36$, $c = 3$ and parameter b varies, then the Lü's circuit system has an attractor which is similar to the Lorenz attractor for $12.7 < b < 17.0$, the Lü's system has a transitory shape when $18.0 < b < 22.0$, and has an attractor similar to Chen's attractor when $23.0 < b < 28.5$ [4]. Moreover, Lü's system has a rather wide range of parameter values in which the system displaces a chaotic attractor of different shapes. After numerical experience, we can conclude that observable chaos exists in the following ranges at least: $12.7 < b < 17.0$, $18.0 < b < 22.0$, $23.0 < b < 28.5$, $28.6 < b < 29.0$, $29.334 < b < 29.345$. When $b < 12.6$, the system converges to a fixed point. Furthermore, when $29.1 < b < 29.334$ and $29.345 < b < 35.0$, there is at least one periodic orbit in the system, as shown in [1].*

Taking into account that the Jacobi stability excludes the chaotic behavior, it will be very interesting to make a computational analysis in order to check if the values of the parameters for which the Lü's circuit system system has a chaotic behavior correspond indeed to the unstable Jacobi conditions. Therefore, using the Jacobi stability it is possible to confirm, or not, the chaotic behavior of the system.

For $a = 36$ and $c = 3$, we have that

$$P_1^1 + P_2^2 = \frac{1}{4}\left(b - 6(7 - 2\sqrt{3})\right)\left(b - 6(7 + 2\sqrt{3})\right)$$

and

$$P_1^1 P_2^2 - P_1^2 P_2^1 = -3(b-3)(b-36)^2 + (b^2 - 2052 - 3b)\sqrt{3b}.$$

Therefore, $P_1^1 + P_2^2 < 0$ if and only if $212{,}154 < b < 627{,}846$, because $6\left(7 - 2\sqrt{3}\right) \approx 212{,}154$ and $6\left(7 + 2\sqrt{3}\right) \approx 627{,}846$.

After some numerical evaluations, for $b = 1.69$, we obtain $P_1^1 P_2^2 - P_1^2 P_2^1 = 0.89837 > 0$, but for $b = 1.691$, $P_1^1 P_2^2 - P_1^2 P_2^1 = -4.27 < 0$. More that, $P_1^1 P_2^2 - P_1^2 P_2^1$ remains strictly negative for all $b \geq 1.691$, and then, for $a = 36$ and $c = 3$, $P_1^1 P_2^2 - P_1^2 P_2^1 < 0$, whenever $P_1^1 + P_2^2 < 0$.

In conclusion, we do not have Jacobi stability for $a = 36$, $c = 3$ and any $b > 0$, which was as expected.

Let us point out that for $a = b$, $P_1^1 + P_2^2 = c(c-b)$ and $P_1^1 P_2^2 - P_1^2 P_2^1 = -2bc\sqrt{bc} < 0$, i.e., we do not have Jacobi stability in this case. The same situation is also true for the particular case $a = b = c$.

If $b = c$, then $P_1^1 + P_2^2 = \left(\frac{a-b}{2}\right)^2 > 0$ and $P_1^1 P_2^2 - P_1^2 P_2^1 = \frac{1}{4}\left(-a^2 - 7ab\right)\sqrt{b^2} < 0$, i.e., we do not have Jacobi stability in this case.

If $a = c$, then $P_1^1 + P_2^2 = \frac{c^2}{4}\left(\left(\frac{b}{c}\right)^2 - 6\frac{b}{c} + 5\right) < 0$ iff $c < b < 5c$ and $P_1^1 P_2^2 - P_1^2 P_2^1 = \frac{1}{4}(c-b)^2\left(-bc + c^2\right) + \frac{1}{4}\left(b^2 - 8c^2 - bc\right)\sqrt{bc}$

If $b = 2c$, then $P_1^1 P_2^2 - P_1^2 P_2^1 = -\frac{1}{4}c^4 - \frac{3}{2}c^3\sqrt{2} < 0$

If $b = 3c$, then $P_1^1 P_2^2 - P_1^2 P_2^1 = -2c^4 - \frac{1}{2}c^3\sqrt{3} < 0$

If $b = 4c$, then $P_1^1 P_2^2 - P_1^2 P_2^1 = -\frac{27}{4}c^4 + 2c^3 = \frac{c^3}{4}(8 - 27c) > 0$ iff $c < \frac{8}{27}$

Therefore, we have Jacobi stability at the equilibria E_1 and E_2 for $a = c \in \left(0, \frac{8}{27}\right)$ and $b = 4c \in \left(0, \frac{32}{27}\right)$. For example, $a = c = \frac{1}{9}$, $b = 1$.

Taking into account $P_1^1 P_2^2 - P_1^2 P_2^1 = -\frac{1}{4}c(c-b)^3 + \frac{1}{4}\left(b^2 - 8c^2 - bc\right)\sqrt{bc}$, we can conclude that we have Jacobi stability at the equilibria E_1 and E_2 for $a = c$ and $3,37c < b < 5c$, because $\frac{1+\sqrt{33}}{2}c \approx 3,37$.

Finally, we can announce the result:

Theorem 8. *For any $a = c > 0$, we have Jacobi stability for Lü's system at the equilibria E_1 and E_2 if and only if $3c < b < 5c$ and $\left(b^2 - 8c^2 - bc\right)\sqrt{bc} > c(c-b)^3$.*

In particular, for $a = c \in \left(0, \frac{8}{27}\right)$ and $b \in \left(0, \frac{32}{27}\right)$ we have Jacobi stability at the equilibria E_1 and E_2.

For example, for $a = c = \frac{1}{9}$, $b = 1$.

Of course, in this situation, a chaotic behavior is not possible.

6. Discussion

In this paper, we studied the Jacobi stability of Lü's circuit system using the geometric tools of the KCC theory. We reformulated the first order nonlinear differential system into a system of second order differential equations, in order to determine the five geometrical invariant of the KCC theory and we obtained that the third, fourth and fifth invariant are identically zero, but the Berwald connection is not identically zero. Moreover, we determined the nonlinear connection associated with the semi-spray (SODEs) and we computed the deviation curvature tensor at each equilibrium point to find the Jacobi stability conditions. A future work is to perform a computational study of the time variation of the deviation vector and of their curvature in order to illustrate the chaotic nature of the system for some values of parameters.

7. Conclusions

By computing the five invariant of Kosambi–Cartan–Chern (KCC) theory corresponding to the associated second order system of Lü's circuit system and by studying the Jacobi stability, we performed a comprehensive study of the geometric objects associated with the vector field generated by the dynamical system defined by Lü's system. This is very important because Jacobi stability is a natural generalization of the stability of the geodesic flow on a differentiable manifold endowed with a metric (Riemannian or Finslerian) or non-metric setting. Then we can conclude that using the KCC theory we performed a deep study of the geometry of Lü's system and we obtained new results about the stability and non-chaotic behavior of this famous dynamical system.

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