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Uniqueness Criteria for Solving a Time Nonlocal Problem for a High-Order Differential Operator Equation $l(\cdot)$—A with a Wave Operator with Displacement

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Abstract: This article presents a criterion for the uniqueness of the solution of a problem nonlocal in time for a differential-operator equation with a symmetric operator part on space variables. The symmetry of the operator part of the operator-differential equation guarantees the existence of good basic properties of its system of root elements. The spectral properties of the symmetric operator part make it possible not only to prove the necessity of the criterion formulated by us, but also to substantiate their sufficiency. In contrast to previously known works, in this work the semiboundedness of the symmetric part of the differential-operator equation can be violated. In this article, the differential-operator equation is represented as the difference of two commuting operators. The uniqueness of the solution is guaranteed when the spectra of the commuting operators do not intersect. It is important that only one of the operators should be symmetrical.

Keywords: symmetric operator part; regular boundary value problems in time; boundary value problem with displacement; uniqueness of solution; eigenfunctions; complete orthonormal systems

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1. Introduction

The article investigates the question of the uniqueness of the solution of an operator equation of the form

$$Bu = Au + f.$$

The primary purpose of this article is to establish the criterion of uniqueness of the solution of the previous for the operator equation. Various ways of proving uniqueness are known. Usually, the maximum principle [1] and its various generalizations such as [2,3] are an effective means of proving uniqueness. For the specified operator equation, the listed principles are not fulfilled. Therefore, we need another tool, different from the extremum principle.

In the work of V.A. Ilyin [4], a fairly universal method of proving the uniqueness of the solution for hyperbolic and parabolic equations is proposed. With fairly general restrictions on the domain Ω, the uniqueness theorem of the solution for hyperbolic and parabolic equations is proved in [4]. The meaning of the requirements of V.A. Ilyin's theorem [4] is that the elliptic part of a hyperbolic and parabolic operator is a symmetric operator. A symmetric operator has a complete orthogonal system of eigenfunctions in the corresponding function space.

In this article, the operator L corresponding to the operator equation represented as the difference of two commuting operators A and B. The uniqueness of the solution is guaranteed when the spectra of operators A and B do not intersect and operators A and B have complete systems of root elements in the corresponding functional spaces. In this case, it is not required that the operator B is symmetric.

We also note the work of I.V. Tikhonov [5], devoted to the uniqueness theorem in linear nonlocal problems for abstract differential equations. This work is interesting because I.V. Tikhonov proposed a new method for proving uniqueness theorems. I.V. Tikhonov's method of proving uniqueness is based on the "method of quotients" for integral functions of exponential type. In [6], the question of the uniqueness of the solution was studied for the thermal conductivity equation with a non-local condition expressed by an integral over time on a fixed interval. They managed to give a complete description of uniqueness classes in terms of the behavior of solutions at $|x| \to \infty$. In this article, I.V. Tikhonov's method is adapted for operators whose differential part is a higher-order operator.

The main issues studied in this article is the question of the uniqueness of the solution of the equation of the form

$$\frac{\partial^n u(x,y;t)}{\partial t^n} + \sum_{j=1}^{n} p_j(t) \frac{\partial^{n-j} u(x,y;t)}{\partial t^{n-j}} =$$

$$\frac{\partial^2 u(x,y;t)}{\partial x^2} - \frac{\partial^2 u(x,y;t)}{\partial y^2} + f(x,y;t), \quad (x,y) \in \Omega, \quad 0 < t < T \tag{1}$$

with regular boundary conditions on t

$$U_\nu(u(x,y;\cdot)) = 0, \quad \nu = 1,2,\ldots,n, \quad (x,y) \in \Omega \tag{2}$$

and with conditions displacement by (x,y)

$$u(\theta,0;t) = 0, \quad 0 \le \theta \le 1,$$

$$u\left(\frac{\theta}{2}, -\frac{\theta}{2}; t\right) = a\, u\left(\frac{\theta+1}{2}, \frac{\theta-1}{2}; t\right), \quad 0 \le \theta \le \frac{1}{2}, \quad 0 < t < T. \tag{3}$$

In this paper, Ω is a finite two-dimensional domain bounded by the segment $OB : 0 \le x \le 1$ of the $y = 0$ axis and the characteristics $OC : x + y = 0$, $BC : x - y = 1$.

In the work [7] in $Q = \Omega \times [0,T]$, $0 < T < \infty$ the quasi-hyperbolic equation

$$(-1)^p \frac{\partial^{2p} u(x,t)}{\partial t^{2p}} = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a^{ij} \frac{\partial u(x,t)}{\partial x_j}\right) + a(x)u(x,t) + f(x,t) \tag{4}$$

with initial conditions

$$\frac{\partial^k u(x,0)}{\partial t^k} = 0, \quad k = m,\ldots,m+p, \quad x \in \Omega, \tag{5}$$

$$\frac{\partial^k u(x,T)}{\partial t^k} = 0, \quad k = m+1,\ldots,m+p-1, \quad x \in \Omega, \tag{6}$$

and with boundary condition

$$u(x,t) = 0, \quad x \in \partial\Omega, \tag{7}$$

was considered. In this case, the right side of Equation (4) represents a semi-bounded elliptic operator. In our paper, the right-hand side of Equation (4) represents an unconstrained hyperbolic operator. Secondly, the left part of Equation (4) represents a one-dimensional differential operator with respect to t with decaying boundary conditions. At the same time, in our paper, the left side of Equation (4) represents an arbitrary one-dimensional differential operator with coefficients variable in t with general non-decaying boundary conditions (5) and (6).

The class of operators of the form $L = B - A$ studied by us refers, according to the terminology of A.A. Dezin [8], to operators generated by differential operator equations. Boundary value problem (1)–(3) is equivalent to a differential-operator equation according to paper [8]. Questions of solvability of differential-operator equations with a symmetric operator part on space variables were studied in the works [7,9–12]. In [9], as well as in our work, it is required that the spectra of operators A and B do not intersect. In work [9], the semi-limitation of operator A is required. The given restriction in this article has been removed. In this paper, the spectrum of operator A is located on the real axis and is not bounded in both directions. Since the results of works [10,11] use the results of [9], that is, the necessary of the semiboundedness of the operators A and B, they are preserved in works [10,11]. V.V. Shelukhin [13,14] investigated the problem of predicting ocean temperature from the average data for the previous period of time, which also belongs to the class of differential operator equations. Practical applications of operators $L = B - A$ of the form can be found in the work [14]. A numerical analysis of some nonlinear partial differential equations is given in [15,16].

2. On the Spectral Properties of a Differential Operator on a Segment

Since the left part of Equation (1) represents a higher-order linear differential operator in the variable t with general boundary conditions (2), then in this paragraph we separately recall the known spectral properties of these operators.

Consider the operator B generated by the differential expression in the functional space $L_2(0, T)$

$$l(w) \equiv \frac{d^n w}{dt^n} + p_1(t)\frac{d^{n-1}w}{dt^{n-1}} + \ldots + p_n(t)w(t), \quad 0 < t < T \tag{8}$$

with regular boundary conditions

$$\sum_{k=0}^{n-1} [\alpha_{kj} w^{(k)}(0) + \beta_{kj} w^{(k)}(T)] = 0, \quad j = 1, 2, \ldots, n \tag{9}$$

where $p_j(t) \in C^{(n-j)}[0, T], \quad j = 1, 2, \ldots, n.$

Recall which boundary conditions are called regular. For this purpose, we denote by S the sector of the complex plane of the ρ –plane defined by the inequalities $0 \leq \arg \rho \leq \pi/n$, and let $\omega_1, \omega_2, \ldots, \omega_n$ be all the different roots of the n-th degree of -1, numbered so that for all $\rho \in S$ the inequalities are valid

$$\mathrm{Re}(\rho\omega_1) \leq \mathrm{Re}(\rho\omega_2) \leq \cdots \leq \mathrm{Re}(\rho\omega_n).$$

We known that [17] the boundary conditions (5) can be reduced to the form

$$U_1(w) = w^{(j_1)}(0) + \sum_{k=0}^{j_1-1} (\alpha_{1,k} w^{(k)}(0) + \beta_{1,k} w^{(k)}(T)),$$

$$U_2(w) = w^{(j_1)}(T) + \sum_{k=0}^{j_1-1} (\alpha_{2,k} w^{(k)}(0) + \beta_{2,k} w^{(k)}(T)),$$

$$\cdots$$

$$U_{2m-1}(w) = w^{(j_m)}(0) + \sum_{k=0}^{j_m-1} (\alpha_{2m-1,k} w^{(k)}(0) + \beta_{2m-1,k} w^{(k)}(T)),$$

$$U_{2m}(w) = w^{(j_m)}(T) + \sum_{k=0}^{j_m-1} (\alpha_{2m,k} w^{(k)}(0) + \beta_{2m,k} w^{(k)}(T)),$$

$$U_{2m+1}(w) = \alpha_1 w^{(\nu_1)}(0) + \beta_1 w^{(\nu_1)}(T) + \sum_{k=0}^{\nu_1-1} (\alpha_{2m+1,k} w^{(k)}(0) + \beta_{2m+1,k} w^{(k)}(T)),$$

$$\cdots$$

$$U_n(w) = \alpha_r w^{(\nu_r)}(0) + \beta_r w^{(\nu_r)}(T) + \sum_{k=0}^{\nu_r-1}(\alpha_{n,m}w^{(k)}(0) + \beta_{n,m}w^{(k)}(T)),$$

among the numbers j_k, ν_i, no two are the same $(0 \le j_k \le n-1,\ k = 1,2,\ldots,m;\ 0 \le \nu_i \le n-1,\ i = 1,2,\ldots,r; 2m+r = n,\ |\alpha_i| + |\beta_i| \ne 0)$. The regularity of the boundary conditions is determined in different ways depending on whether n is odd or even.

Requirement I. Suppose that the domain of the operator B is given by boundary conditions that are regular in the sense of Birkhoff [18]. In other words, in the case of odd $n = 2p-1$, the following two determinants are nonzero

$$\theta_0 = \begin{vmatrix} \omega_1^{j_1} & \ldots & \omega_{p-1}^{j_1} & \omega_p^{j_1} & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & \omega_{p+1}^{j_1} & \ldots & \omega_n^{j_1} \\ \cdot & \ldots & \cdot & \cdot & \cdot & \cdot & \ldots \\ \omega_1^{j_m} & \ldots & \omega_{p-1}^{j_m} & \omega_p^{j_m} & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & \omega_{p+1}^{j_m} & \ldots & \omega_n^{j_m} \\ \alpha_1\omega_1^{\nu_1} & \ldots & \alpha_1\omega_{p-1}^{\nu_1} & \alpha_1\omega_p^{\nu_1} & \beta_1\omega_{p+1}^{\nu_1} & \ldots & \beta_1\omega_n^1 \\ \cdot & \ldots & \cdot & \cdot & \cdot & \cdot & \ldots \\ \alpha_r\omega_r^{\nu_r} & \ldots & \alpha_r\omega_{p-1}^{\nu_r} & \alpha_r\omega_p^{\nu_r} & \beta_r\omega_{p+1}^{\nu_r} & \ldots & \beta_r\omega_n^r\nu_r \end{vmatrix},$$

$$\theta_1 = \begin{vmatrix} \omega_1^{j_1} & \ldots & \omega_{p-1}^{j_1} & 0 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & \omega_p^{j_1} & \omega_{p+1}^{j_1} & \ldots & \omega_n^{j_1} \\ \cdot & \ldots & \cdot & \cdot & \cdot & \cdot & \ldots \\ \omega_1^{j_m} & \ldots & \omega_{p-1}^{j_m} & 0 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & \omega_p^{j_m} & \omega_{p+1}^{j_m} & \ldots & \omega_n^{j_m} \\ \alpha_1\omega_1^{\nu_1} & \ldots & \alpha_1\omega_{p-1}^{\nu_1} & \beta_1\omega_p^{\nu_1} & \beta_1\omega_{p+1}^{\nu_1} & \ldots & \beta_1\omega_n^{\nu_1} \\ \cdot & \ldots & \cdot & \cdot & \cdot & \cdot & \ldots \\ \alpha_r\omega_r^{\nu_r} & \ldots & \alpha_r\omega_{p-1}^{\nu_r} & \beta_r\omega_p^{\nu_r} & \beta_r\omega_{p+1}^{\nu_r} & \ldots & \beta_r\omega_n^{\nu_r} \end{vmatrix}, \quad (10)$$

in this case, $i = 1,2,\ldots,r;\ j = 1,2,\ldots,n$.

When $n = 2p$ is even, then the following two determinants

$$\theta_{-1} = \begin{vmatrix} \omega_1^{j_1} & \ldots & \omega_{p-1}^{j_1} & \omega_p^{j_1} & 0 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & \omega_{p+1}^{j_1} & \omega_{p+2}^{j_1} & \ldots & \omega_n^{j_1} \\ \cdot & \ldots & \cdot & \cdot & \cdot & \cdot & \cdot & \ldots \\ \omega_1^{j_m} & \ldots & \omega_{p-1}^{j_m} & \omega_p^{j_m} & 0 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & \omega_{p+1}^{j_m} & \omega_{p+2}^{j_m} & \ldots & \omega_n^{j_m} \\ \alpha_1\omega_1^{\nu_1} & \ldots & \alpha_1\omega_1^{\nu_1} & \alpha_1\omega_p^{\nu_1} & \beta_1\omega_{p+1}^{\nu_1} & \beta_1\omega_{p+2}^{\nu_1} & \ldots & \beta_1\omega_n^{\nu_1} \\ \cdot & \ldots & \cdot & \cdot & \cdot & \cdot & \cdot & \ldots \\ \alpha_r\omega_r^{\nu_r} & \ldots & \alpha_r\omega_{p-1}^{\nu_r} & \alpha_r\omega_p^{\nu_r} & \beta_r\omega_{p+1}^{\nu_r} & \beta_r\omega_{p+2}^{\nu_r} & \ldots & \beta_r\omega_n^{\nu_r} \end{vmatrix},$$

$$\theta_1 = \begin{vmatrix} \omega_1^{j_1} & \ldots & \omega_{p-1}^{j_1} & 0 & \omega_{p+1}^{j_1} & 0 & \ldots & 0 \\ 0 & \ldots & 0 & \omega_p^{j_1} & 0 & \omega_{p+2}^{j_1} & \ldots & \omega_n^{j_1} \\ \cdot & \ldots & \cdot & \cdot & \cdot & \cdot & \cdot & \ldots \\ \omega_1^{j_m} & \ldots & \omega_{p-1}^{j_m} & 0 & \omega_{p+1}^{j_m} & 0 & \ldots & 0 \\ 0 & \ldots & 0 & \omega_p^{j_m} & 0 & \omega_{p+2}^{j_m} & \ldots & \omega_n^{j_m} \\ \alpha_1\omega_1^{\nu_1} & \ldots & \alpha_1\omega_{p-1}^{\nu_1} & \beta_1\omega_p^{\nu_1} & \alpha_1\omega_{p+1}^{\nu_1} & \beta_1\omega_{p+2}^{\nu_1} & \ldots & \beta_1\omega_n^{\nu_1} \\ \cdot & \ldots & \cdot & \cdot & \cdot & \cdot & \ldots & \\ \alpha_r\omega_r^{\nu_r} & \ldots & \alpha_r\omega_{p-1}^{\nu_r} & \beta_r\omega_p^{\nu_r} & \alpha_r\omega_{p+1}^{\nu_r} & \beta_r\omega_{p+2}^{\nu_r} & \ldots & \beta_r\omega_n^{\nu_r} \end{vmatrix} \quad (11)$$

are different from zero.

Let's introduce a fundamental system of solutions $\{w_1(t,\lambda),\ldots,w_n(t,\lambda)\}$ of the homogeneous equation

$$l(w_s) = \lambda\, w_s(t,\lambda), \quad 0 < t < T, \tag{12}$$

that satisfying the Cauchy conditions at zero

$$\frac{d^{(j-1)}}{dt^{(j-1)}} w_s(0,\lambda) = \delta_{j,s}, \quad j = 1,\ldots,n, \quad s = 1,\ldots,n, \tag{13}$$

where $\delta_{j,s}$ is the Kronecker symbol. Note that all solutions $\{w_s(t,\lambda),\, s = 1,\ldots,n\}$ are integral functions of λ. Denote by $\Delta(\lambda)$ the characteristic determinant given by the formula

$$\Delta(\lambda) = \det(U_\nu(w_j)).$$

Characteristic determinant zeros of the $\Delta(\lambda)$, taking into account their multiplicity, represent the eigenvalues of the operator B.

Finally, for completeness, we present the Lagrange formula [18]. Let $U_1,\ldots,U_n$ be linearly independent forms of variables $w(0), w'(0),\ldots,w^{(n-1)}(0), w(T), w'(T),\ldots,w^{(n-1)}(T)$. Let's add them with some linear forms $U_{n+1},\ldots,U_{2n}$ to a linearly independent system of $2n$ forms $U_1, U_2,\ldots,U_{2n}$. For any linearly independent forms $U_1, U_2,\ldots,U_{2n}$ there is a single set of $2n$ linear homogeneous forms

$$V_{2n}, V_{2n-1},\ldots,V_1$$

with respect to the variables $R(0), R'(0),\ldots,R^{(n-1)}(0), R(T),\ldots,R^{(n-1)}(T)$ for arbitrary functions $R(t) \in W_2^n[0,T]$. Then, the Lagrange formula [18] for any two functions $w(t)$ and $R(t)$ of $W_2^n[0,T]$ will be rewritten as

$$\begin{gathered}\int_0^T l(w(t))\,\overline{R(t)}\,dt - \int_0^T w(t)\,\overline{l^+(R(t))}\,dt = \\ = U_1(w)\,\overline{V_{2n}(R)} + U_2(w)\,\overline{V_{2n-1}(R)} + \cdots + U_{2n}(w)\,\overline{V_1(R)}. \quad j = 1,\ldots,n, \quad s = 1,\ldots,n.\end{gathered} \tag{14}$$

We have $l^+(\cdot)$ is a formally adjoint differential expression to the expression $l(\cdot)$ and is given by the formula

$$l^+(R) \equiv (-1)^n \frac{d^n R(t)}{dt^n} + \sum_{j=0}^{n-1} (-1)^{n-j} \frac{d^{n-j}}{dt^{n-j}}(p_j(t)R(t)), \quad 0 < t < T.$$

and domain of definition

$$D(B^*) = \{R \in W_2^n[0,T] :\ V_1(R) = 0,\,\ldots,\, V_n(R) = 0\}.$$

The adjoint operator B^* is given by the differential expression

$$B^* R(t) = l^+(R).$$

In [17], adjoint boundary conditions are calculated:

$$V_n(R) = R^{(n-1-\gamma_1)}(0) + \sum_{k=0}^{n-2-\gamma_1} (\alpha^*_{n,k} R^{(k)}(0) + \beta^*_{n,k} R^{(k)}(T)),$$

$$V_{n-1}(R) = R^{(n-1-\gamma_1)}(T) + \sum_{k=0}^{n-2-\gamma_1} (\alpha^*_{n-1,k} R^{(k)}(0) + \beta^*_{n-1,k} R^{(k)}(T)),$$

$$\cdots$$

$$V_{n-2m}(R) = R^{(n-1-\gamma_m)}(0) + \sum_{k=0}^{n-2-\gamma_m} (\alpha^*_{n-2m,k} R^{(k)}(0) + \beta^*_{n-2m,k} R^{(k)}(T)),$$

$$V_{n-2m+1}(R) = R^{(n-1-\gamma_m)}(T) + \sum_{k=0}^{n-2-\gamma_m} (\alpha^*_{n-2m+1,k} R^{(k)}(0) + \beta^*_{n-2m+1,k} R^{(k)}(T)),$$

$$V_r(R) = \overline{\beta}_1 R^{(n-1-\nu_1)}(0) + \overline{\alpha}_1 R^{(n-1-\nu_1)}(T) + \sum_{k=0}^{n-2-\nu_1} (\alpha^*_{r,k} R^{(k)}(0) + \beta^*_{r,k} R^{(k)}(T)),$$

$$\cdots$$

$$V_1(R) = \overline{\beta}_r R^{(n-1-\nu_r)}(0) + \overline{\alpha}_r R^{(n-1-\nu_r)}(T) + \sum_{k=0}^{n-2-\nu_r} (\alpha^*_{1,k} R^{(k)}(0) + \beta^*_{1,k} R^{(k)}(T)),$$

where the numbers $\gamma_1, \ldots, \gamma_m$ are determined from the relation

$$\{\gamma_1, \ldots, \gamma_m\} \cup \{j_1, \ldots, j_m\} \cup \{\nu_1, \ldots, \nu_r\} = \{0, 1, \ldots, n-1\}.$$

Let us introduce a fundamental system of solutions $\{R_1(t, \overline{\lambda}), \ldots, R_n(t, \overline{\lambda})\}$ of a homogeneous adjoint equation

$$l^+(R_s) = \overline{\lambda}\, R_s(t, \overline{\lambda}), \quad 0 < t < T \tag{15}$$

which is satisfying the Cauchy condition at zero

$$\frac{d^{(j-1)}}{dt^{(j-1)}} R_s(0, \overline{\lambda}) = \delta_{j,s}, \quad j = 1, \ldots, n, \quad s = 1, \ldots, n. \tag{16}$$

Note that all solutions $\{R_s(t, \overline{\lambda}),\ s = 1, \ldots, n\}$ are integral functions of $\overline{\lambda}$. Denote by $\Delta^*(\overline{\lambda})$ the characteristic determinant given by the formula

$$\Delta^*(\overline{\lambda}) = \det(V_\nu(R_j)). \tag{17}$$

Taking into account their multiplicity characteristic determinant zeros $\Delta^*(\overline{\lambda})$, represent the eigenvalues of the adjoint operator B^*.

We also introduce $\tau_s(t, \overline{\lambda})$ for $s = 1, \ldots, n$ solutions of homogeneous adjoint Equation (15) with inhomogeneous conditions

$$V_j(\tau_s) = \delta_{j,s} \cdot \Delta^*(\overline{\lambda}), \quad j = 1, \ldots, n. \tag{18}$$

Let λ_0 be the of the characteristic determinant zero $\Delta(\lambda)$ and its multiplicity is equal to m_0. Then, for any $s = 1, \ldots, n$ in an ordered row

$$||\tau_s(t, \overline{\lambda_0}), \frac{1}{1!}\frac{\partial}{\partial \overline{\lambda}} \tau_s(t, \overline{\lambda_0}), \ldots, \frac{1}{(m_o - 1)!}\frac{\partial^{m_0 - 1}}{\partial \overline{\lambda}^{m_0 - 1}} \tau_s(t, \overline{\lambda_0})|| \tag{19}$$

the first nonzero function represents an eigenfunction of the operator B^*, and the subsequent members of the row give a chain of associated functions generated by it.

In what follows, the eigenvalues of the operator B^* will be denoted by $\overline{\lambda}_\nu$, $\nu \geq 1$, and the corresponding eigenfunctions and associated functions via $R_\nu(t)$, $\nu \geq 1$.

The following statement is proved in [17].

Theorem 1 ([17])**.** *Let the domain of the operator B be given by the boundary conditions regular in the sense of Birkhoff. Then, the domain of the operator of the adjoint B^* is also given by the boundary conditions regular in the sense of Birkhoff.*

We will also need the following statement [18].

Theorem 2 ([18])**.** *Let the operator B be generated by the boundary conditions regular in the sense of Birkhoff. Then, the system of root functions of operator B is a complete system in the space $L_2(0, T)$.*

Applying theorems A and B to the adjoint operator B^*, we can formulate the statement.

Theorem 3. *Let the requirement I be fulfilled. Then, the system of roots functions of the operator B^* is a complete system in the space $L_2(0,T)$.*

3. Preliminaries

Since the right-hand side of Equation (1) represents a wave equation with a shift in the variable (x,y), then in this paragraph we will separately recall the known spectral properties of the specified operator.

Let $\Omega \in \mathbb{R}^2$ be the finite region bounded by the segment $OB : 0 \leq x \leq 1$ axes $y = 0$ and characteristics $OC : x + y = 0$, $BC : x - y = 1$ equations

$$Av = v_{xx}(x,y) - v_{yy}(x,y) = f(x,y). \tag{20}$$

Problem 1 (S_a)**.** *Find a solution to Equation* (20) *that satisfies the conditions*

$$\begin{gathered} v(\theta,0) = 0,\ 0 \leq \theta \leq 1, \\ v\left(\frac{\theta}{2}, -\frac{\theta}{2}\right) = a\, v\left(\frac{\theta+1}{2}, \frac{\theta-1}{2}\right),\ 0 \leq \theta \leq \frac{1}{2}, \end{gathered} \tag{21}$$

where a is an arbitrary complex number.

The operator, corresponding to the boundary value problem S_a, is denoted by A. The eigenvalues of the operator A will be numbered by a pair of integer indices $\eta_{k,m}$. The eigenfunctions of the operator A are denoted by $v_{k,m}(x,y)$ corresponding to the eigenvalue $\eta_{k,m}$.

In [19], the eigenvalues and eigenfunctions of the operator A are explicitly calculated:

$$\eta_{k,m} = -4(i\ln(-a) + 2\pi k)(i\ln(-a) + 2\pi m),\ k,m = 0, \pm 1, \ldots, \tag{22}$$

$$v_{k,m}(x,y) = (-a)^{-2x}\left(e^{\{2\pi i[(k+m)x+(k-m)y]\}} - e^{\{2\pi i[(k+m)x-(k-m)y]\}}\right), \tag{23}$$

and when $k_1 \neq k_2, m_1 \neq m_2$ there can be

$$\int_\Omega |a|^{2x} v_{k_1,m_1}(x,y)\overline{v_{k_2,m_2}(x,y)}\,dxdy = 0. \tag{24}$$

The following statement is proved in [19].

Theorem 4 ([19])**.** *For $a = 0$ the operator A is Voltaire and for $a(a+1) \neq 0$ has a complete in the space $L_2(\Omega)$ system of eigenfunctions $\{v_{k,m}(x,y)\}$, given by equality* (23).

4. On the Uniqueness of the Solution of a Time Nonlocal Problem for a High-Order Differential Operator Equation $l(\cdot)$—A with the Wave Operator A

Boundary value problem (1)–(3) has the next operator form

$$Bu = Au(x,y;t) + f(x,y;t), \quad (x,y;t) \in Q. \tag{25}$$

The operator B applies on the variable t and its properties are given in paragraph 1. The operator A applies on variables (x,y) and its spectral properties are given in paragraph 2.

In this section, we prove the criterion of uniqueness of the solution of the homogeneous operator Equation (25).

Theorem 5. *Let the requirement I be fulfilled and $a(a+1) \neq 0$. Then, the operator*

$$Bu = Au \tag{26}$$

has only a trivial solution $u \in D(B) \cap D(A)$ if and only if

$$\sigma(B) \cap \sigma(A) = \varnothing, \tag{27}$$

where $\sigma(B)$ and $\sigma(A)$ are the spectra of operators B and A, respectively.

Remark 1. *The meaning of Theorem 1 is that it is sufficient to explicitly calculate the eigenvalues $\eta_{k,m}$ of the operator A. The eigenvalues of the operator B cannot be explicitly calculated. From the dispersion relation $\Delta(\lambda) = 0$ for operator B, it is sufficient to check the condition $\Delta(\eta_{k,m}) \neq 0$ for all $\eta_{k,m}$.*

Remark 2. *In [9–11], the requirements concerned the arrangement of the spectra of operators A and B on the entire complex plane. In particular, in ref. [9], the operator A must be semi-bounded from below, and the operator B must be semi-bounded from above. According to Theorem 1, there are no such restrictions.*

Example 1. *Let $n = 1$. In this case, the regular boundary condition (2) will take the for*

$$u(x, y; 0) - u(x, y; T) = 0, \; (x, y) \in \Omega,$$

where $b \neq 0$. In this case, the characteristic determinant has the form

$$\Delta(\lambda) = b - e^{\lambda T}.$$

Spectrum B operator's consists of eigenvalues having the form

$$\Delta(\lambda)\lambda_s = \frac{\ln b}{T} + \frac{2\pi i s}{T}, \; s \in \mathbb{Z}.$$

The requirement of Theorem 1 is satisfied if

$$\frac{\ln b}{T} + \frac{2\pi i s}{T} \neq -4(i \ln -a + 2\pi k)(i \ln -a + 2\pi n),$$

for all $s, k, m \in \mathbb{Z}$.

5. Proof of the Main Theorem (Theorem 1)

Proof of Necessary. Let λ_ν— some eigenvalue of operator B (with eigenfunction $w_\nu(t)$), also an eigenvalue of operator A, that is, $\lambda_\nu = \eta_{k,m}$ (with eigenfunction $v_{k,m}(x, y)$). Then, the function $u(x, t) = w_\nu(t) \cdot v_{k,m}(x, y)$ will be a nontrivial solution to the homogeneous problem (26). The necessity of the requirements of Theorem 1 is proved. □

Proof of Sufficient. Let none of the $\{\lambda_k, \, k \geq 1\}$ eigenvalues of operator B be the eigenvalue of operator A. In other words, a number of the form (22) is not an eigenvalue of operator B, i.e.,

$$\Delta(\eta_{k,m}) \neq 0.$$

We show that $u(x, y; t)$—the solution of the homogeneous boundary value problem (22) that is identically equal to zero in the space $L_2(Q)$.

To do this, for a fixed $(x, y) \in \Omega$ and $j = 1, \ldots, n$ we introduce the functions

$$F_j(x, y; \overline{\lambda}) = \int_0^T \overline{\tau_j(t, \overline{\lambda})}\, u(x, y; t) dt. \tag{28}$$

The $F_j(x, y; \overline{\lambda})$ functions represent integrals of functions from $\overline{\lambda}$. According to the Lagrange formula (14), the function $AF_j(x, y; \overline{\lambda})$ at $j = 1, \ldots, n$ can be rewritten as:

$$AF_j(x,y;\overline{\lambda}) = \int_0^T \overline{\tau_j(t,\overline{\lambda})} \cdot Au(x,y;t)dt = \int_0^T \overline{\tau_j(t,\overline{\lambda})} \cdot Bu(x,y;t)dt =$$

$$\int_0^T u(x,y;t)\, \overline{l^+(\tau_j)}dt + U_{n+1}(u)\, \overline{V_n(\tau_j)} + \cdots + U_{2n}(u)\, \overline{V_1(\tau_j)} =$$

$$\lambda \int_0^T u(x,y;t)\, \overline{\tau_j(t,\overline{\lambda})}dt + U_{n+1}(u)\, \overline{V_n(\tau_j)} + \cdots + U_{2n}(u)\, \overline{V_1(\tau_j)} =$$

$$\lambda F_j(x,y;\lambda) + \Delta^*(\overline{\lambda}) \cdot U_{2n-j+1}(u). \tag{29}$$

If λ_0 is the of the characteristic determinant $\Delta(\lambda)$ zero of the multiplicity m_0, then the equalities follow from the relations (29)

$$AF_j(x,y;\overline{\lambda_0}) = \lambda_0\, F_j(x,y;\overline{\lambda_0}),$$

$$A\frac{dF_j(x,y;\overline{\lambda_0})}{d\overline{\lambda}} = \lambda_0\, \frac{dF_j(x,y;\overline{\lambda_0})}{d\overline{\lambda}} + F_j(x,y;\overline{\lambda_0}),$$

$$\cdots$$

$$A\frac{d^{m_0-1}}{d\overline{\lambda}^{m_0-1}} F_j(x,y;\overline{\lambda_0}) = \lambda_0\, \frac{d^{m_0-1}F_j(x,y;\overline{\lambda_0})}{d\overline{\lambda}^{m_0-1}} + \frac{d^{m_0-2}F_j(x,y;\overline{\lambda_0})}{d\overline{\lambda}^{m_0-2}}.$$

Because of $\lambda_0 \overline{\in} \sigma(A)$, the equalities follow from the relations (28)

$$\frac{d^s F_j(x,y;\overline{\lambda_0})}{d\overline{\lambda}^s} \equiv 0 \quad \text{for} \quad s = 0,1,\ldots,m_0-1. \tag{30}$$

Then $j = 1,\ldots,n$ relation $\dfrac{F_j(x,y;\overline{\lambda})}{\Delta^*(\overline{\lambda})}$ are integral functions of λ, since at the point $\lambda = \lambda_0$ the relations $\dfrac{F_j(x,y;\overline{\lambda})}{\Delta^*(\overline{\lambda})}$ has only removable singularities .

Now, we move on to the second step of the proof. According to the methodology of V. A. Ilyin [4], we multiply the function $F_j(x,y;\overline{\lambda})$ scalar by the eigenfunction $v_{k,m}(x,y)$ and denote them by

$$G_j^{k,m}(\overline{\lambda}) \equiv \int_\Omega F_j(x,y;\overline{\lambda})\, v_{k,m}(x,y)\, dx\, dy, \quad k,m = 0,\pm 1,\ldots, \quad j = 1,\ldots,n. \tag{31}$$

Multiplicities of functional $G_j^{k,m}(\lambda)$ zeros at least not bigger than multiplicities of functions $F_j(x,y;\lambda)$ zeros. Hence, the relation

$$Q_j^{k,m}(\overline{\lambda}) \equiv \frac{G_j^{k,m}(\overline{\lambda})}{\Delta^*(\overline{\lambda})} \tag{32}$$

are define integral functions of $\overline{\lambda}$.

Further analysis of integral functions $Q_j^{k,m}(\overline{\lambda})$ is based on the technique of estimating the order of growth and the type of integral functions.

Note that the whole function $Q_j^{k,m}(\overline{\lambda})$ does not depend on the choice of the fundamental system of solutions of the homogeneous equation

$$l^+\,(R(t,\overline{\lambda})) = \overline{\lambda} \cdot R(t,\overline{\lambda}), \quad 0 < t < T. \tag{33}$$

Denote by $\omega_1, \omega_2, \ldots, \omega_n$ all the different roots of the n-th degree of $(-1)^{n+1}$. Let us first analyze the case of odd $n = 2p-1$. Let ρ be an arbitrary complex number from the sector $S_0 = \{\rho \in \mathbb{C} | \, 0 < \arg\rho < \frac{\pi}{n}\}$. Renumber the numbers $\omega_1, \omega_2, \ldots, \omega_n$ in the following order

$$\varepsilon_1 = \omega_{i_1}, \, \varepsilon_2 = \omega_{i_2}, \ldots \, \varepsilon_n = \omega_{i_n}. \tag{34}$$

Then $\forall \rho \in S_0$ inequalities are fulfilled

$$\mathrm{Re}(\rho\varepsilon_1) \leq \mathrm{Re}(\rho\varepsilon_2) \leq \cdots \leq \mathrm{Re}(\rho\varepsilon_{p-1}) < 0,$$

$$\mathrm{Re}(\rho\varepsilon_{p-1}) \leq \mathrm{Re}(\rho\varepsilon_p) \leq \mathrm{Re}(\rho\varepsilon_{p+1}),$$

$$0 < \mathrm{Re}(\rho\varepsilon_{p+1}) \leq \mathrm{Re}(\rho\varepsilon_{p+2}) \leq \cdots \leq \mathrm{Re}(\rho\varepsilon_n). \tag{35}$$

Instead of solutions$\{R_1(t,\rho^n), \ldots, R_n(t,\rho^n)\}$ a fundamental system we will consider another fundamental system of solutions of a homogeneous adjoint equation

$$l^+ \left(R(t,\overline{\lambda})\right) = \overline{\lambda} \cdot R(t,\overline{\lambda}), \quad 0 < t < T, \quad \overline{\lambda} = (-\rho)^n.$$

Let choose it according to Theorem 1 [18]

$$h_1(t,\rho) = e^{\rho\varepsilon_1 t}[1 + o(1/\rho)], \ldots, h_n(t,\rho) = e^{\rho\varepsilon_n t}[1 + o(1/\rho)], \, \rho \in S_0, \, \rho \to \infty. \tag{36}$$

Owing to [18] $\forall \rho \in S_0$ we have an asymptotic representation of the characteristic determinant $\widetilde{\Delta}(\rho)$ for $\rho \to \infty$, written through the fundamental system of solutions $\{h_1(t,\rho), \ldots, h_n(t,\rho)\}$.

When $\rho \in S_0$, $\rho \to \infty$, we have

$$V_n(h_j) = (\rho\varepsilon_j)^{(n-1-\gamma_1)}[1], \, V_{n-1}(h_j) = (\rho\varepsilon_j)^{(n-1-\gamma_1)}[0], \ldots,$$

$$V_{n-2m+2}(h_j) = (\rho\varepsilon_j)^{(n-1-\gamma_m)}[1], \, V_{n-2m+1}(h_j) = (\rho\varepsilon_j)^{(n-1-\gamma_m)}[0], \ldots,$$

$$V_r(h_j) = (\rho\varepsilon_j)^{(n-1-\nu_1)}[\overline{\alpha}_1], \, V_1(h_j) = (\rho\varepsilon_j)^{(n-1-\nu_r)}[\overline{\alpha}_r], \quad \text{at} \quad j < p.$$

Just as if $j > p$ we have $\rho \in S_0$, $\rho \to \infty$

$$V_n(h_j) = (\rho\varepsilon_j)^{(n-1-\gamma_1)}e^{\rho\varepsilon_j T}[0], \, V_{n-1}(h_j) = (\rho\varepsilon_j)^{(n-1-\gamma_1)}e^{\rho\varepsilon_j T}[1], \ldots,$$

$$V_{n-2m+2}(h_j) = (\rho\varepsilon_j)^{(n-1-\gamma_m)}e^{\rho\varepsilon_j T}[0], \, V_{n-2m+1}(h_j) = (\rho\varepsilon_j)^{(n-1-\gamma_m)}e^{\rho\varepsilon_j T}[1], \ldots,$$

$$V_r(h_j) = (\rho\varepsilon_j)^{(n-1-\nu_1)}e^{\rho\varepsilon_j T}[\overline{\beta}_1], \, V_1(h_j) = (\rho\varepsilon_j)^{(n-1-\nu_r)}e^{\rho\varepsilon_j T}[\overline{\beta}_r].$$

We denote for simplicity

$$[a] = a + o(1/\rho).$$

All these expressions substitute in the relation (16)

$$\widetilde{\Delta}^*(\overline{\lambda}) = \det(V_\nu(h_j)) = \rho^{\widehat{\alpha}} e^{\rho(\omega_{p+1} + \ldots + \omega_n)T} \Delta_0^*, \tag{37}$$

where

$$\widehat{\alpha} = 2[n - 1 - \gamma_1 + \ldots + n - 1 - \gamma_m] + n - 1 - \nu_1 + \ldots + n - 1 - \nu_r,$$

$$\Delta_0^* = [\theta_0^*] + e^{\rho\omega_p}[\theta_1^*].$$

The numbers θ_0^* and θ_1^* are determined by adjoint forms $\{V_1, \ldots, V_n\}$ and are analogous to the numbers θ_0 and θ_1, defined by the forms $\{U_1, \ldots, U_n\}$. These numbers are nonzero according to Theorem 1.

In any ρ from the sector S_0 asymptotic representation $\overline{\widetilde{\tau}_1}(t,\rho)$ at $\rho \to \infty$ has the following form, written through the fundamental system of solutions (36) $\overline{\widetilde{\tau}_1}(t,\rho)$ $\rho \to \infty$

$$\overline{\widetilde{\tau}_1}(t,\rho) = \frac{1}{(\rho\varepsilon_p)^{(n-1-\gamma_1)}} \rho^{\widehat{\alpha}} e^{\rho(\omega_{p+1} + \cdots + \omega_n)T} e^{\rho\varepsilon_p t} \left([\xi_0^*]e^{-\rho\varepsilon_p t} + e^{\rho\omega_p(T-t)}[\xi_1^*]\right), \tag{38}$$

where ζ_0^*, ζ_1^* are some numerical determinants. Similar asymptotic representations we receive for $\widetilde{\overline{\tau_j}}(t,\rho)$ at $j > 1$. Here, we see

$$Q_1^{k,m}(\overline{\lambda}) = \int_\Omega \left(\int_0^T \frac{\widetilde{\overline{\tau_1}}(t,\overline{\lambda})}{\widetilde{\Delta^*}(\rho)} u(x,y;t)dt \right) v_{k,m}(x,y)\,dxdy =$$

$$\int_\Omega \int_0^T \frac{e^{\rho\varepsilon_p t}([\zeta_0^*]e^{-\rho\varepsilon_p t} + e^{\rho\omega_p(T-t)}[\zeta_1^*])}{(\rho\varepsilon_p)^{(n-1-\gamma_1)}([\theta_0^*] + e^{\rho\omega_p}[\theta_1^*])} u(x,y;t)\,v_{k,m}(x,y)\,dt\,dx\,dy. \tag{39}$$

By Riemann's lemma [20] in the case of $\mathrm{Re}(\rho\varepsilon_p) = \mathrm{Re}(\rho\varepsilon_{p+1}) = 0$, we easily get

$$\lim_{|\rho|\to\infty} Q_1^{k,m}(\overline{\lambda}) = 0, \quad \rho \in S_0.$$

If $\mathrm{Re}(\rho\varepsilon_p) > 0$, then $\mathrm{Re}(\rho\varepsilon_{p+1}) < 0$. Next, we get

$$\lim_{\rho\to\infty} Q_1^{k,m}(\overline{\lambda}) = 0, \quad \text{at}\, \rho \in S_0.$$

Therefore, along all rays $\rho \in S_0$ and $\rho \to \infty$, then we have the limit equality

$$\lim_{\rho\to\infty} Q_1^{k,m}(\overline{\lambda}) = 0.$$

Similar asymptotic representations are obtained for $Q_j^{k,m}(\overline{\lambda})$ at $j > 1$ with all possible (k,m).

Exactly the same analysis can be carried out for the sector $\rho \in S_1$, where $S_1 = \{\rho \in \mathbb{C} |\, \frac{\pi}{2p} < \arg\rho < \frac{\pi}{p}\}$. Consequently, by the Fragmen–Lindelof and Liouville theorem [21] for functions of finite order, we obtain that

$$Q_j^{k,m}(\overline{\lambda}) \equiv 0 \quad \text{at all} \quad \overline{\lambda} \in \mathbb{C}.$$

For any valid (k,m) and for any $j = 1,\dots,n$ we have

$$\int_\Omega v_{k,m}(x,y)F_j(x,y;\overline{\lambda})\,dxdy \equiv 0, \quad \forall\lambda \in \mathbb{C}.$$

Then, from the completeness of the system $\{v_{k,m}(x,y),\, k,m = 0,\pm1,\dots\}$ in $L_2(\Omega)$ follows that

$$F_j(x,y;\overline{\lambda}) \equiv 0, \quad \forall x,y \in \Omega, \quad \forall\lambda \in \mathbb{C}, \quad j = 1,\dots,n.$$

Therefore

$$\int_0^T \widetilde{\overline{\tau_j}}(t,\overline{\lambda})\,u(x,y;t)\,dt \equiv 0, \quad \forall x,y \in \Omega, \quad \forall\lambda \in \mathbb{C}, \quad j = 1,\dots,n.$$

Hence, we have:

$$\frac{1}{\nu!}\frac{\partial^\nu}{\partial\lambda^\nu}\int_0^T \widetilde{\overline{\tau_j}}(t,\overline{\lambda})\,u(x,t)\,dt \equiv 0, \quad \forall x,y \in \Omega, \quad \forall\lambda \in \mathbb{C}, \quad j = 1,\dots,n, \quad \forall\nu \geq 0. \tag{40}$$

Now, instead of λ into equality (40) we substitute λ_τ—an arbitrary eigenvalue of operator B. Eigenvalue multiplicity of λ_τ is considered as equal to m_τ. Let the parameter ν in formula (40) take the values $1,2,\dots,m_\tau - 1$. Using by the formulas (19) and (40), we find that for any fixed $x,y \in \Omega$ function, $u(x,y;t)$ is orthogonal to all root functions of the operator B^*. Since the system of root functions of the operator B^* is a complete system in $L_2(0,T)$, then we have

$$u(x,y;t) \equiv 0 \quad \text{at all} \quad t \in (0,T), \quad x,y \in \Omega.$$

The statement of Theorem 1 is proved similarly for even $n = 2p$. Thus, the sufficiency of Theorem 1 is completely proved. □

6. Conclusions

In this paper, the operator L is investigated. The operator L represented the difference of two commuting operators, A and B. The operator B is generated by a high-order linear differential expression and regular two-point boundary conditions $[0, T]$. The symmetric operator A corresponds to a wave equation in a plane domain bounded by two characteristics and their coinciding segment. The operator A considered by us is not semi-bounded and has a complete orthogonal system of eigenfunctions in the corresponding functional space. With the specified choice of operators B and A, the operator equation $Lu = Bu - Au = 0$ has a trivial solution if and only if the spectra of operators B and A do not intersect.

Let us pay attention to the method of proving the uniqueness theorem proposed in the article. The method is a hybrid of M.G. Krein's method of direct functionals [18] and V.A. Ilyin's method [4].

Further generalizations of the main result of this article can be carried out in the following directions:

1. Weakening of the regularity conditions of the boundary conditions of operator B;
2. Extension of Theorem 1 for multidimensional operators B depending on several variables.

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