symmetry

MDPI

Article

An Efficient Method for Solving Second-Order Fuzzy Order Fuzzy Initial Value Problems

Qamar Dallashi † **and Muhammed I. Syam** *,†

Department of Mathematical Sciences, College of Science, UAE University, Al-Ain P.O. Box 15551, United Arab Emirates; 202070450@uaeu.ac.ae
* Correspondence: m.syam@uaeu.ac.ae
† These authors contributed equally to this work.

Abstract: In this paper, we present an accurate numerical approach based on the reproducing kernel method (RKM) for solving second-order fuzzy initial value problems (FIVP) with symmetry coefficients such as symmetric triangles and symmetric trapezoids. Finding the exact solution of FIVP is not an easy task since the definition will produce a complicated optimization problem. To overcome this difficulty, a numerical method is developed to solve this type of problems. We start by introducing the necessary definitions and theorems about the fuzzy logic. Then, we derived the kernels for two Hilbert spaces. The RKM is derived for the second-order IVP in the Boolean sense, and then we generalize it for the fuzzy sense. Numerical and theoretical results will be given to obtain the accuracy of the developed technique. We solved four linear and non-linear fuzzy IVPs numerically using the proposed method, and we compute the error in each case to show the efficiency of the method. The absolute error was very small in the four examples.

Keywords: fuzzy initial value problems; convergence; reproducing kernel method

Citation: Dallashi, Q.; Syam, M. An Efficient Method for Solving Second-Order Fuzzy Initial Value Problems. *Symmetry* **2022**, *14*, 1218. https://doi.org/10.3390/sym14061218

Academic Editors: J. Vanterler Da C. Sousa, Jiabin Zuo, Cesar E. Torres Ledesma and Alexander Zaslavski

Received: 1 May 2022
Accepted: 6 June 2022
Published: 13 June 2022

1. Introduction

In recent years, fuzzy logic has become attractive to many researchers due to its potential applications in various fields, such as computer science [1], information science [2], mathematics [3], engineering [4], economics [5], and business and finance [6]. Fuzzy logic and fuzzy sets are powerful mathematical tools in modeling entropy systems, for example, in industry, nature, and the humanities. Several researchers have studied the fuzzy boundary value problems. For example, Sanchez et al. [7] discussed the fuzzy solution for nonlinear fuzzy boundary value problem. Gong [8] illustrated discontinuous FIVPs. Zhou et al. [9] illustrated numerous duality outcomes for fuzzy problems with fuzzy coefficients. Tapaswini et al. [10] used the Galerkin method for solving nth-order FBVPs. Gumah et al. [11] used hybrid FODE with a RKM for FDE. Patel and Desai [12] used Laplace transform to solve FIVPs. Diniz et al. [13] investigated the necessary conditions to solve a fuzzy problems using Zadeh's extension. Wu and Feng [14] used control problems to explain a boundary-disturbed uncertain beam equation. Suhhiem [15] introduced a modified method for solving second-order FDEs. Niu et al. [16] proposed a simplified RKM to solve singular boundary value problems. Shah and Wang [17] developed a numerical technique to solve fuzzy FDEs. Pradip [18] discussed a class of singular BVPs. Wasques et al. [19] illustrated a numerical technique for higher FIVPs. Wasques et al. [20] investigated FIVPs. Jeyaraj and Rajan [21] used RKM#4 to study FIVPs. Al-Refai et al. [22] used the IHBM to solve FIVPs.

Several researchers study the RKM. For example, Kashkari and Syam [23] used the RKM for solving Fredholm integro-differential equation (FIDE). Du et al. [24] developed an RKM for solving FIDEs. Akgül [25] applied the RKM to FDEs. Akgül [26] investigated the boundary layer flow of a Powell–Eyring non-Newtonian fluid over a stretching sheet by RKM. A singular kernel is implemented by Sadamoto et al. [27] Gholami et al. [28] studied the fuzzy inner product space. Mei [29] simplified the RKM to solve integral equations.

Geng [30] investigated a class of singularly perturbed delay BVPs by RKM. Li and Wu [31] constructed and applied reproducing kernels with polynomials. Moradi and Javadi [32] used the RKM for studying oscillators under the damping effect. Alvandi and Paripour [33] implemented the RKM for FIDEs. Arqub et al. [34] used RKM for studying FIDEs. Saadeh et al. [35] implemented an iterative RKM to solve BVPs. Qi et al. [36] introduced an RKM for solving FBVPs.

The purpose of this paper is to find an accurate numerical solution of the fuzzy initial value problems of a second order. The proposed methods, which are given in this article, are of high orders of precision and are very close to the exact solutions. Another advantage to these methods is that they can be implemented even when it is impossible to find the exact solution in the closed form. To reach this target, the reproducing kernel method will be applied to second-order initial value problems. Then, the suggested methods will be elaborated to solve the fuzzy type of these problems using some properties of fuzzy operations. Moreover, the convergence of the suggested method will be examined. In addition, various examples to represent the accuracy of the suggested methods are demonstrated. It is worth mentioning that these numerical methods can be used for other fuzzy problems such as fuzzy eigenvalue problems and fuzzy boundary value problems when we combine them with the shooting method.

We divide this paper into five sections. This section is devoted to the literature review, while, in Section 2, we present the necessary preliminaries which we will use. We present the RKM for solving second-order fuzzy IVPs with symmetric coefficients, such as symmetric triangles and symmetric trapezoids, in Section 3. In Section 4, we present some numerical results, and finally, we draw some conclusions in Section 5.

2. Preliminaries

In this section, the definition of fuzzy number and the differentiation of fuzzy functions with related definitions and theorems will be illustrated

Definition 1 ([37])**.** *Let $\Re$ be the set of real numbers. A fuzzy number is a function $\mu : \Re \to [0,1]$ with the following:*

1. *μ is normal, i.e., $\mu(c) = 1$ for some $c \in \Re$;*
2. *μ is a fuzzy convex, i.e., $\mu(\theta c_1 + (1-\theta)c_2) \geq \min\{\mu(c_1), \mu(c_2)\}$ for all $\theta \in [0,1], c_1, c_2 \in \Re$;*
3. *μ is upper semi-continuous on $\Re$,;*
4. *$\overline{\{x \in \Re : \mu(x) > 0\}}$ is compact.*

Let $F_{\Re}$ be the collection of all fuzzy numbers.

Next, the $\alpha - cut$ of a fuzzy number will be illustrated.

Definition 2 ([2])**.** *Let $\mu \in F_{\Re}$. Then, the $\alpha-$cut set μ_α for $\alpha \in (0,1]$ is*

$$\mu_\alpha = \{x \in \Re : \mu(x) \geq \alpha\},$$

and

$$\mu_0 = \overline{\{x \in \Re : \mu(x) > 0\}}.$$

Definition 3 ([3])**.** *Let A and B be two subsets of $\Re$. Then, the Hausdorff metric d_H is given as:*

$$d_H(A,B) = \max\left\{\sup_{x\in A}\inf_{y\in B}\|x-y\|, \sup_{y\in B}\inf_{x\in A}\|x-y\|\right\}.$$

Then, the metric d_F on $F_{\Re}$ is given as:

$$\begin{aligned} d_F(u,v) &= \sup_{\alpha\in[0,1]} \{d_H(u_\alpha, v_\alpha),\ \ u_\alpha, v_\alpha \in F_{\Re}\}, \\ &= \sup_{\alpha\in[0,1]} max\{\ |\underline{u}_\alpha - \underline{v}_\alpha|, |\overline{u}_\alpha - \overline{v}_\alpha|\ \}. \end{aligned}$$

Theorem 1 ([4]). *$(F_{\Re}, d_F)$ is a complete metric space such that for all $u, v, w, z \in F_{\Re}, \lambda \in \Re$:*

$$d_F(u \oplus w, v \oplus w) = d_F(u \oplus v),$$

$$d_F(\lambda \odot u, \lambda \odot w) = |\lambda| d_F(u, w),$$

$$d_F(u \oplus v, w \oplus z) \leq d_F(u \oplus w) + d_F(v \oplus z).$$

Definition 4 ([37]). *The Hukhara difference of two fuzzy numbers, a and b, is $c = a \ominus_H b$ if*

$$c \oplus b = a.$$

Definition 5 ([5]). *Let X be a subset of $\Re$. A fuzzy function $F : X \to F_{\Re}$ is called H-differentiable at $x_0 \in X$ if and only if the following limit exists and equals $Df(x_0)$:*

$$\begin{aligned} Df(x_0) &= \lim_{\Delta\to 0^+} \frac{1}{\Delta} \odot (f(x_0 + h) \ominus_H f(x_0)), \\ &= \lim_{\Delta\to 0^+} \frac{1}{\Delta} \odot (f(x_0) \ominus_H f(x_0 - \Delta)). \end{aligned}$$

If f is Hukuhara differentiable for all $x \in X$, then f is H-differentiable on X.

Theorem 2 ([6]). *Let $f(x) : I \to F_{\Re}$ be a fuzzy function defined by*

$$f(x) = u \odot h(x),$$

where u is a fuzzy number and $I = (u, v) \subset \Re$. Let $h : I \to \Re_+$ be differentiable function at $x \in I$. If $h'(x) > 0$, then:

$$f'(x) = u \odot h'(x). \tag{1}$$

Theorem 3 ([37]). *Let $f : M \to F_{\Re}$ be an H-differentiable at x_0 with derivative $f^{'}(x_0)$, where $M \subset \Re$ and $x_0 \in M$. Then, $f'_\alpha(x_0) = \left[\underline{f}^{'}(x_0), \overline{f}'(x_0)\right]$ and $\underline{f}(x), \overline{f}(x)$ are differentiable at x_0 for all $\alpha \in [0,1]$.*

Now, consider the following fuzzy differential equation:

$$y'' = f(t, y, y'), \quad a < t < b, \tag{2}$$

If Equation (2) is a linear problem, then it can be written as:

$$a(t)y'' + b(t)y' + c(t)y = r(t), \quad a < t < b,$$

where $a,\ b,\ c,\ and\ r$ are fuzzy functions. Since the functions are fuzzy, then the linear fuzzy problem can be written in the α-cut format as:

$$\begin{aligned} &\left[\underline{a}_\alpha(t), \overline{a}_\alpha(t)\right]\left[\underline{y}''(t), \overline{y}''(t)\right] + \left[\underline{b}_\alpha(t), \overline{b}_\alpha(t)\right]\left[\underline{y}'_\alpha(t), \overline{y}'_\alpha(t)\right] \\ &+ \left[\underline{c}_\alpha(t), \overline{c}_\alpha(t)\right]\left[\underline{y}_\alpha(t), \overline{y}_\alpha(t)\right] = \left[\underline{r}_\alpha(t), \overline{r}_\alpha(t)\right]. \end{aligned}$$

Thus, we will obtain two complicated optimization problems:

$$\min\left\{\underline{a}_\alpha \underline{y}'', \underline{a}_\alpha \overline{y}'', \overline{a}_\alpha \underline{y}'', \overline{a}_\alpha \overline{y}''\right\} + \min\left\{\underline{b}_\alpha \underline{y}', \underline{b}_\alpha \overline{y}'_\alpha, \overline{b}_\alpha \underline{y}'_\alpha, \overline{b}_\alpha \overline{y}'_\alpha\right\}$$
$$+ \min\left\{\underline{c}_\alpha \underline{y}_\alpha, \underline{c}_\alpha \overline{y}_\alpha, \overline{c}_\alpha \underline{y}_\alpha, \overline{c}_\alpha \overline{y}_\alpha\right\} = \underline{r}_\alpha(t),$$

and

$$\max\left\{\underline{a}_\alpha \underline{y}'', \underline{a}_\alpha \overline{y}'', \overline{a}_\alpha \underline{y}'', \overline{a}_\alpha \overline{y}''\right\} + \max\left\{\underline{b}_\alpha \underline{y}', \underline{b}_\alpha \overline{y}'_\alpha, \overline{b}_\alpha \underline{y}'_\alpha, \overline{b}_\alpha \overline{y}'_\alpha\right\}$$
$$+ \max\left\{\underline{c}_\alpha \underline{y}_\alpha, \underline{c}_\alpha \overline{y}_\alpha, \overline{c}_\alpha \underline{y}_\alpha, \overline{c}_\alpha \overline{y}_\alpha\right\} = \overline{r}_\alpha(t).$$

The above min–max problems are difficult to solve and sometimes not possible. For this reason, a numerical method to solve Problem (2) will be given.

3. Second-Order Fuzzy Initial Value Problems

In this section, linear and nonlinear second-order fuzzy initial value problems will be discussed. First, let us define the operation $\odot$ for any $a \in \Re$ and the α-cut of b as follows:

$$a \odot [b_{1,\alpha}, b_{2,\alpha}] = \begin{cases} [ab_{1,\alpha}, ab_{2,\alpha}], & a \geq 0 \\ [ab_{2,\alpha}, ab_{1,\alpha}], & a < 0 \end{cases}.$$

Consider the following linear second order FIVP:

$$y'' + a(x)y' + b(x)y = c(x), \qquad 0 \leqslant x \leqslant 1, \tag{3}$$

$$y(0) = \hat{\beta}, \tag{4}$$

$$y'(0) = \hat{\gamma}, \tag{5}$$

where $\hat{\beta}$ and $\hat{\gamma}$ are symmetric fuzzy numbers $a(x)$, $b(x)$ is a continuous functions on $[0,1]$, and $c(x)$ is fuzzy function. Let the α-cuts of $y'(x), y(x)$, $c(x)$, $\hat{\beta}$, and $\hat{\gamma}$ be given by:

$$\begin{aligned} y'_\alpha(x) &= \left[y'_{1\alpha}(x), y'_{2\alpha}(x)\right], \\ y_\alpha(x) &= [y_{1\alpha}(x), y_{2\alpha}(x)], \\ \hat{\beta} &= [\beta_1, \beta_2], \\ \hat{\gamma} &= [\gamma_1, \gamma_2], \end{aligned}$$

and

$$c_\alpha(x) = [c_{1\alpha}(x), c_{2\alpha}(x)].$$

To solve Problem (3)–(5), four cases should be implemented.

Case l: Let $a(x) \geqslant 0, b(x) \geqslant 0$ for all $x \in [0,1]$. Then,

$$\left[y''_{1\alpha}(x), y''_{2\alpha}(x)\right] + a(x) \odot \left[y'_{1\alpha}(x), y'_{2\alpha}(x)\right] + b(x) \odot [y_{1\alpha}(x), y_{2\alpha}(x)] = [c_{1\alpha}(x), c_{2\alpha}(x)],$$

which produces the following system of non-homogeneous IVPs:

$$y''_{1\alpha}(x) + a(x)y'_{1\alpha}(x) + b(x)y_{1\alpha}(x) = c_{1\alpha}(x), \quad y_1(0) = \beta_1, \;\; y'_1(0) = \gamma_1, \tag{6}$$

and

$$y''_{2\alpha}(x) + a(x)y'_{2\alpha}(x) + b(x)y_{2\alpha}(x) = c_{2\alpha}(x), \quad y_2(0) = \beta_2, \;\; y'_2(0) = \gamma_2. \tag{7}$$

Case 2: Let $a(x) \geqslant 0, b(x) < 0$ for all $x \in [0,1]$. Then,

$$\left[y''_{1\alpha}(x), y''_{2\alpha}(x)\right] + a(x) \odot \left[y'_{1\alpha}(x), y'_{2\alpha}(x)\right] + b(x) \odot [y_{1\alpha}(x), y_{2\alpha}(x)] = [c_{1\alpha}(x), c_{2\alpha}(x)],$$

which implies that

$$y''_{1\alpha}(x) + a(x)y'_{1\alpha}(x) + b(x)y_{2\alpha}(x) = c_{1\alpha}(x), \quad y_{1\alpha}(0) = \beta_1, \;\; y'_{1\alpha} = \gamma_1, \tag{8}$$

and

$$y''_{2\alpha}(x) + a(x)y'_{2\alpha}(x) + b(x)y_{1\alpha}(x) = c_{2\alpha}(x), \quad y_{2\alpha}(0) = \beta_2, \; y'_{2\alpha} = \gamma_2. \tag{9}$$

Let

$$Y_\alpha(x) = \begin{pmatrix} y_{1\alpha}(x) \\ y_{2\alpha}(x) \end{pmatrix}, \quad C_\alpha(x) = \begin{pmatrix} c_{1\alpha}(x) \\ c_{2\alpha}(x) \end{pmatrix},$$

$$\lambda = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \quad \lambda' = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}, \quad B(x) = \begin{pmatrix} 0 & b(x) \\ b(x) & 0 \end{pmatrix}.$$

Then, Equations (8) and (9) can be written in the matrix form as:

$$Y''_\alpha(x) + a(x)Y'_\alpha(x) + B(x)Y_\alpha(x) = C_\alpha(x), \quad Y_\alpha(0) = \lambda, \; Y'_\alpha(0) = \lambda', \tag{10}$$

where B is symmetric matrix.

Case 3: Let $a(x) < 0, b(x) \geqslant 0$ for all $x \in [0,1]$. Then,

$$\left[y''_{1\alpha}(x), y''_{2\alpha}(x)\right] + a(x) \odot \left[y'_{1\alpha}(x), y'_{2\alpha}(x)\right] + b(x) \odot [y_{1\alpha}(x), y_{2\alpha}(x)] = [c_{1\alpha}(x), c_{2\alpha}(x)],$$

which implies that

$$y''_{1\alpha}(x) + a(x)y'_{2\alpha}(x) + b(x)y_{1\alpha}(x) = c_{1\alpha}(x), \; y_{1\alpha}(0) = \beta_1, \; y'_{1\alpha}(0) = \gamma_1, \tag{11}$$

$$y''_{2\alpha}(x) + a(x)y'_{1\alpha}(x) + b(x)y_{2\alpha}(x) = c_{2\alpha}(x), \; y_{2\alpha}(0) = \beta_2, \; y'_{2\alpha}(0) = \gamma_2. \tag{12}$$

Let

$$Y_\alpha(x) = \begin{pmatrix} y_1\alpha(x) \\ y_{2\alpha}(x) \end{pmatrix}, \quad C_\alpha(x) = \begin{pmatrix} c_{1\alpha}(x) \\ c_{2\alpha}(x) \end{pmatrix},$$

$$\lambda = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \quad \lambda' = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}, \quad A(x) = \begin{pmatrix} 0 & a(x) \\ a(x) & 0 \end{pmatrix}.$$

Then, Equations (11) and (12) can be written in the matrix form as:

$$Y''_\alpha(x) + A(x)Y'_\alpha(x) + b(x)Y_\alpha(x) = C_\alpha(x), \quad Y_\alpha(0) = \lambda, \; Y'_\alpha(0) = \lambda', \tag{13}$$

where A is a symmetric matrix.

Case 4: Let $a(x) < 0, b(x) < 0$ for all $x \in [0,1]$.

$$\left[y''_{1\alpha}(x), y''_{2\alpha}(x)\right] + a(x) \odot \left[y'_{1\alpha}(x), y'_{2\alpha}(x)\right] + b(x) \odot [y_{1\alpha}(x), y_{2\alpha}(x)] = [c_{1\alpha}(x), c_{2\alpha}(x)],$$

which implies that

$$y''_{1\alpha}(x) + a(x)y'_{2\alpha}(x) + b(x)y_{2\alpha}(x) = c_{1\alpha}(x), \; y_{1\alpha}(0) = \beta_1, \; y'_{1\alpha}(0) = \gamma_1, \tag{14}$$

$$y''_{2\alpha}(x) + a(x)y'_{1\alpha}(x) + b(x)y_{1\alpha}(x) = c_{2\alpha}(x), \; y_{2\alpha}(0) = \beta_2, \; y'_{2\alpha}(0) = \gamma_2. \tag{15}$$

Let

$$Y_\alpha(x) = \begin{pmatrix} y_1\alpha(x) \\ y_{2\alpha}(x) \end{pmatrix}, \quad C_\alpha(x) = \begin{pmatrix} c_{1\alpha}(x) \\ c_{2\alpha}(x) \end{pmatrix},$$

$$\lambda = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \quad \lambda' = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}, \quad A(x) = \begin{pmatrix} 0 & a(x) \\ a(x) & 0 \end{pmatrix} \quad B(x) = \begin{pmatrix} 0 & b(x) \\ b(x) & 0 \end{pmatrix}.$$

Then, Equations (14) and (15) can be written in the matrix form as:

$$Y''_\alpha(x) + A(x)Y'_\alpha(x) + B(x)Y_\alpha(x) = C_\alpha(x), \quad Y_\alpha(0) = \lambda, \;\; Y'_\alpha(0) = \lambda'. \tag{16}$$

Thus, we can summarize the four cases as follows. The linear system of non-homogeneous IVPs is:

$$Y''_\alpha(x) + A(x)Y'_\alpha(x) + B(x)Y_\alpha(x) = C_\alpha(x), \quad Y_\alpha(0) = \lambda, \;\; Y'_\alpha(0) = \lambda', \tag{17}$$

where

$$Y_\alpha(x) = \begin{pmatrix} y_1\alpha(x) \\ y_{2\alpha}(x) \end{pmatrix}, \quad C_\alpha(x) = \begin{pmatrix} c_{1\alpha}(x) \\ c_{2\alpha}(x) \end{pmatrix},$$

$$\lambda = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \quad \lambda' = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}, \quad A(x) = \begin{pmatrix} \mu_1 a(x) & \mu_2 a(x) \\ \mu_2 a(x) & \mu_1 a(x) \end{pmatrix} \quad B(x) = \begin{pmatrix} \gamma_1 b(x) & \gamma_2 b(x) \\ \gamma_2 b(x) & \gamma_1 b(x) \end{pmatrix},$$

such that $\mu_1 = 1$ and $\mu_2 = 0$ if $a(x) \geq 0$ while $\mu_1 = 0$ and $\mu_2 = 1$ if $a(x) < 0$. In addition, $\gamma_1 = 1$ and $\gamma_2 = 0$ if $b(x) \geq 0$, while $\gamma_1 = 0$ and $\gamma_2 = 1$ if $b(x) < 0$, where A and B are symmetric matrices.

Now, consider the following general form of theFIVP:

$$y'' = f(x, y, y'), \quad 0 \leqslant x \leq 1,$$

subject to

$$y(0) = \hat{\beta}, \quad y'(0) = \hat{\gamma}.$$

Let the α-cut of $y(x), y'(x), y''(x), \hat{\beta}, \hat{\gamma}$, and $f(x, y, y')$ be given by

$$y_\alpha(x) = [y_{1\alpha}(x), y_{2\alpha}(x)], \qquad y'_\alpha(x) = [y'_{1\alpha}(x), y'_{2\alpha}(x)],$$

$$y''_\alpha(x) = [y''_{1\alpha}(x), y''_{2\alpha}(x)], \quad \hat{\beta} = [\beta_{1\alpha}, \beta_{2\alpha}], \quad \hat{\gamma} = [\gamma_{1\alpha}, \gamma_{2\alpha}],$$
$$f_\alpha(x, y, y') = [f_{1\alpha}(x, y, y'), f_{2\alpha}(x, y, y')],$$

where

$$f_{1\alpha}(x, y, y') = \min\Big\{f(x, u, v) : u \in [y_{1\alpha}, y_{2\alpha}], v \in \Big[y'_{1\alpha}, y'_{2\alpha}\Big]\Big\},$$

and

$$f_{2\alpha}(x, y, y') = \max\Big\{f(x, u, v) : u \in [y_{1\alpha}, y_{2\alpha}], v \in \Big[y'_{1\alpha}, y'_{2\alpha}\Big]\Big\}.$$

Then,

$$y''_{1\alpha} = f_{1\alpha}(x, y, y'), \qquad y_{1\alpha}(0) = \beta_{1\alpha}, \qquad y'_{1\alpha}(0) = \gamma_{1\alpha},$$

and

$$y''_{2\alpha} = f_{2\alpha}(x, y, y'), \qquad y_{2\alpha}(0) = \beta_{2\alpha}, \qquad y'_{2\alpha}(0) = \gamma_{2\alpha}.$$

As we see from the previous discussion, the FIVP will produce a system of two IVPs. To solve this system, we will use the RKM.

Definition 6 ([38,39])**.** *Let $F \neq \emptyset$. A function $Q : F \times F \to \mathbb{C}$ is called a reproducing kernel function of the Hilbert space H if and only if:*

(a) $Q(\cdot, t) \in H$, $t \in F$;
(b) $\langle \varphi, Q(\cdot, t) \rangle = \varphi(t)$, $t \in F$, $\varphi \in H$.

A Hilbert space which possesses a reproducing kernel is called a reproducing kernel Hilbert space.

Definition 7. *Let* $W_2^1[0,1] = \{w : w$ *be an absolutely continuous real-valued function on* $[0,1], w'$,
$w'' \in L^2[0,1]\}$.

The inner product in $W_2^1[0,1]$ *is symmetric and is defined as:*

$$(w(x), v(x))_{W_2^1[0,1]} = w(0)v(0) + \int_0^1 w'(x)v'(x)dx, \tag{18}$$

and its norm is defined as:

$$\|w\|_{W_2^1[0,1]} = \sqrt{(w(x), w(x))_{W_2^1[0,1]}}, \tag{19}$$

where $w, v \in W_2^1[0,1]$.

Theorem 4. *The Hilbert space* $W_2^1[0,1]$ *is a reproducing kernel, and its reproducing kernel function* $R_y(x)$ *can be defined by:*

$$R_y(x) = \begin{cases} 1+x, & x \leqslant y \\ 1+y, & x > y \end{cases}.$$

Proof. Let

$$w(y) = \big(w(x), R_y(x)\big) = w(0)R_y(0) + \int_0^1 w'(x)R_y'(x)dx. \tag{20}$$

Using integration by parts, we obtain:

$$w(y) = w(0)R_y(0) + R_y'(1)w(1) - R_y'(0)w(0) - \int_0^1 w(x)R_y^{(2)}(x)dx.$$

Therefore:

$$R_y(0) - R_y'(0) = 0, \tag{21}$$

$$R_y'(1) = 0. \tag{22}$$

Thus:

$$w(y) = \langle w(x), R_y(x) \rangle = -\int_0^1 w(x)R_y^{(2)}(x)dx.$$

Hence:

$$-R_y^{(2)}(x) = \delta(x-y) = \begin{cases} 1, \text{ if } x = y \\ 0, \text{ if } x \neq y \end{cases}.$$

Hence:

$$R_y(x) = \begin{cases} C_1(y) + C_2(y)x, & y \geq x \\ d_1(y) + d_2(y)x, & y < x \end{cases}.$$

Since $-R_y^{(2)}(x) = \delta(x-y)$, then

$$R_y(y^+) = R_y(y^-), \tag{23}$$

and

$$\frac{\partial R_y(y^+)}{\partial y} - \frac{\partial R_y(y^-)}{\partial y} = -1. \tag{24}$$

For simplicity, let $C_i(y) = C_i$ and $d_i(y) = d_i$ for $i = 1,2$. Solve system (20)–(23) to obtain:

$$\begin{aligned} C_1 - C_2 &= 0, \\ C_1 + C_2 y &= d_1 + d_2 y, \\ d_2 - C_2 &= -1, \\ d_2 &= 0. \end{aligned}$$

Then,

$$C_1(y) = 1,\ \ C_2(y) = 1,\ \ d_1(y) = 1 + y,\ \ d_2(y) = 0.$$

Then, the kernel is symmetric and given by:

$$R_y(x) = \begin{cases} 1 + x, & y \geqslant x \\ 1 + y, & x > y \end{cases}.$$

□

Definition 8. *Let* $W_2^3[0,1] = \{w : w, w', w^{(2)}$ *be absolutely continuous real-valued functions on* $[0,1], w^{(i)} \in L^2[0,1]$ *for* $i = 3,4,5,6$, $w(0) = w'(0) = 0\}$ *with the following inner product:*

$$\langle w(x), v(x)\rangle = w(0)v(0) + w'(0)v'(0) + w(1)v(1) + \int_0^1 w^{(3)}(x)v^{(3)}(x)dx, w, v \in W_2^3[0,1],$$

and the norm

$$\|w\|_{W_2^3} = \sqrt{\langle w, w\rangle_{W_2^3}}.$$

Theorem 5. *The Hilbert space* $W_2^3[0,1]$ *is a reproducing kernel space and its reproducing kernel function* $R_y(x)$ *can be defined by*

$$R_y(x) = \begin{cases} -\frac{x^2}{y^2}, & x \le y \\ \frac{1}{120}y^5 - \frac{1}{24}xy^4 + \frac{y^5-12}{12y^2}x^2 - \frac{1}{12}x^3y^2 & \\ +\frac{1}{24}x^4y^2 - \frac{1}{120}x^5y^2, & x > y \end{cases}.$$

Proof. Let

$$w(y) = \langle w(x), R_y(x)\rangle = w(0)R_y(0) + w'(0)R_y'(0) + w(1)R_y(1) + \int_0^1 w^{(3)}(x)R_y^{(3)}(x)dx.$$

Integrate by parts three times to obtain:

$$\begin{aligned} \langle w, R_y\rangle &= w(0)R_y(0) + w'(0)R_y'(0) + w(1)R_y(1) + w^{(2)}(1)R_y^{(3)}(1) - w^{(2)}(0)R_y^{(3)}(0) \\ &\quad -w'(1)R_y^{(4)}(1) + w'(0)R_y^{(4)}(0) + w(1)R_y^{(5)}(1) - w(0)R_y^{(5)}(0) - \int_0^1 w(x)R_y^{(6)}(x)dx. \end{aligned}$$

Substitute the conditions $w(0) = w'(0) = 0$ to obtain:

$$\begin{aligned} w(y) &= w(1)R_y(1) + w^{(2)}(1)R_y^{(3)}(1) - w^{(2)}(0)R_y^{(3)}(0) - w'(1)R_y^{(4)}(1) \\ &\quad +w(1)R_y^{(5)}(1) - \int_0^1 w(x)R_y^{(6)}(x)dx. \end{aligned}$$

Let

$$R_y(1) - R_y^{(5)}(1) = 0, \tag{25}$$

$$R_y^{(3)}(1) = 0, \tag{26}$$

$$R_y^{(3)}(0) = 0, \tag{27}$$

$$R_y^{(4)}(1) = 0. \tag{28}$$

Thus, under these conditions, we obtain:

$$w(y) = -\int_0^1 w(x)R_y^{(6)}(x)dx. \tag{29}$$

Then,

$$-R_y^{(6)}(x) = \delta(x-y),$$

where

$$\delta(x-y) = \begin{cases} 1 \text{ if } x = y \\ 0 \text{ if } x \neq y \end{cases}.$$

Thus,

$$R_y(x) = \begin{cases} \sum_{i=1}^{6} C_1(y)x^{i-1} & x \leqslant y \\ \sum_{i=1}^{6} d_i(y)x^{i-1} & x > y \end{cases}.$$

Since

$$-R_y^{(6)}(x) = \delta(x-y),$$

then

$$\frac{\partial^k R_y(y^+)}{\partial y^k} = \frac{\partial^K R_y(y^-)}{\partial y^k}, \quad \text{for} \quad k = 0,1,2,3,4, \tag{30}$$

and

$$\frac{\partial^5 R_y(y^+)}{\partial y^5} - \frac{\partial^5 R_y(y^-)}{\partial y^5} = -1. \tag{31}$$

Since $R_y \in W_2^3[0,1]$, then

$$R_y(0) = 0, \tag{32}$$

and

$$R_y'(0) = 0. \tag{33}$$

Then, we obtain the following system:

$$\begin{aligned}
6C_4 &= 0, \\
4!d_5 + 5!d_6 &= 0, \\
6d_4 + 24d_5y + 60d_6y^2 &= 0, \\
d_1 + d_2y + d_3y^2 + d_4y^3 + d_5y^4 + d_6y^5 - 5!d_6 &= 0, \\
C_1 + C_2y + C_3y^2 + C_4y^3 + C_5y^4 + C_6y^5 &= d_1 + d_2y + d_3y^2 + d_4y^3 + d_5y^4 + d_6y^5, \\
C_2 + 2C_3y + 3C_4y^2 + 4C_5y^3 + 5C_6y^4 &= d_2 + 2d_3y + 3d_4y^2 + 4d_5y^3 + 5d_6y^4, \\
2C_3 + 6C_4y + 12C_5y^2 + 20C_6y^3 &= 2d_3 + 6d_4y + 12d_5y^2 + 20d_6y^3, \\
6C_4 + 24C_5y + 60C_6y^2 &= 6d_4 + 24d_5y + 60d_6y^2, \\
24C_5 + 120C_6y &= 24d_5 + 120d_6y, \\
120d_6 - 120C_6 &= -1, \\
C_1 &= 0, \\
C_2 &= 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
&C_1(y) = 0,\ C_2(y) = 0,\ C_3(y) = \tfrac{-1}{y^2},\ C_4(y) = 0, \\
&C_5(y) = 0,\ C_6(y) = 0,\ d_1(y) = \tfrac{1}{120}y^5, \\
&d_2(y) = -\tfrac{1}{24}y^4,\ d_3(y) = \tfrac{y^5-12}{12y^2},\ d_4(y) = -\tfrac{1}{12}y^2, \\
&d_5(y) = \tfrac{1}{24}y^2,\ d_6(y) = -\tfrac{1}{120}y^2.
\end{aligned}$$

Then,

$$R_y(x) = \begin{cases} -\frac{x^2}{y^2}, & x \leq y \\ \frac{1}{120}y^5 - \frac{1}{24}xy^4 + \frac{y^5-12}{12y^2}x^2 - \frac{1}{12}x^3y^2 & \\ +\frac{1}{24}x^4y^2 - \frac{1}{120}x^5y^2, & x > y \end{cases}.$$

□

Consider the following system of a second-order initial value problem:

$$u'' + A(x)u' + B(x)u = f(x, u, u'), \qquad 0 \leq x \leq 1, \tag{34}$$

with

$$u(0) = 0, \quad u'(0) = 0, \tag{35}$$

when A and B are symmetric matrices, $p, q \in C^2(0,1)$, and $f \in L^2[0,1]$. We can obtain homogeneous initial conditions by using a change in variables. Thus,

$$(Lu)(x) = f(x, u, u'), \qquad 0 \leq x \leq 1 \tag{36}$$

with

$$u(0) = u'(0) = 0, \tag{37}$$

where

$$Lu = \frac{d^2u}{dx^2} + A(x)\frac{du}{dx} + B(x)u', \tag{38}$$

and

$$L : W_2^3[0,1] \to W_2^1[0,1]. \tag{39}$$

Then,

$$\begin{aligned} L(\mu u + v)(x) &= \frac{d^2}{dx^2}(\mu u + v)(x) + A(x)\frac{d}{dx}(\mu u + v)(x) + B(x)(\mu u + v)(x), \\ &= \mu\frac{d^2}{dx^2}u(x) + \mu A(x)\frac{d}{dx}u(x) + \mu B(x)u(x) + \frac{d^2}{dx^2}v(x), \\ &\quad + A(x)\frac{d}{dx}v(x) + B(x)v(x), \\ &= \mu L(u)(x) + L(v)(x), \end{aligned}$$

where μ is constant. Then, L is the linear operator.

Theorem 6. *The linear operator L is a bounded linear operator.*

Proof. By Equation (19), we have:

$$\|Lw\|^2_{W_2^1} = \langle Lw, Lw\rangle_{W_2^1} = \int_0^1 \|Lw'(x)\|^2 \,\mathrm{d}x + \|Lw(0)\|^2.$$

By Theorem (5), we have:

$$w(x) = \langle w(\cdot), R_x(\cdot)\rangle_{W_2^3},$$

and

$$Lw(x) = \langle w(\cdot), LR_x(\cdot)\rangle_{W_2^1}.$$

By the Cauchy–Schwartz inequality,

$$\|Lw(x)\| \leq \|w\|_{W_2^3}\|LR_x\|_{W_2^1} = M_1\|w\|_{W_2^3},$$

where $M_1 > 0$ is a positive constant. Thus,

$$\|(Lw)(0)\|^2 \leq M_1^2\|w\|^2_{W_2^3}.$$

Since

$$(Lw)'(x) = \left\langle w(\cdot), (LR_x)'(\cdot)\right\rangle_{W_2^1},$$

then
$$\|(Lw)'(x)\| \leq \|w\|_{W_2^3} \left\|(LR_x)'\right\|_{W_2^3} = M_2 \|w\|_{W_2^3},$$
where $M_2 > 0$ is a positive constant. We have
$$\|(Lw)'(t)\|^2 \leq M_2^2 \|w\|_{W_2^3}^2,$$
and
$$\int_0^1 \|(Lw)'(x)\|^2 \,\mathrm{d}x \leq M_2^2 \|w\|_{W_2^3}^2,$$
which implies that
$$\|Lw\|_{W_2^1}^2 \leq M \|w\|_{W_2^3}^2,$$
where $M = M_1^2 + M_2^2$ □

Let $\{x_i\}_{i=1}^{\infty}$ be a countable dense subset of $[0,1]$. Let $\varphi_i(x) = R_{x_i}(x)$ and $\psi_i(x) = L^*\varphi_i(x)$, where L^* is conjugate operator of L. Let $\left\{\widehat{\Psi}_i(x)\right\}_{i=1}^{\infty} \subset W_2^3[0,1]$ be an orthonormal set of functions produced from $\{\psi_i(x)\}_{i=1}^{\infty}$. Then,
$$\widehat{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \quad \beta_{ii} > 0, i = 1, 2, \ldots.$$

Theorem 7. *The exact solution of*
$$u'' = G(x, u, u'), \quad 0 < x < 1, \tag{40}$$
with
$$u(0) = u_0, \quad u'(0) = u_1, \tag{41}$$
then
$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} G\big(x_k, u_k, u_k'\big)\widehat{\Psi}_i(x), \tag{42}$$
where $\{x_i\}_{i=1}^{\infty}$ *is dense in* $[0,1]$.

Proof. Simple calculations imply that
$$\begin{aligned}
u(x) &= \sum_{i=1}^{\infty} \left\langle u(x), \widehat{\Psi}_i(x) \right\rangle_{W_2^3} \widehat{\Psi}_i(x), \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \langle u(x), \Psi_k(x) \rangle_{W_2^3} \widehat{\Psi}_i(x), \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \langle u(x), L^*\varphi_k(x) \rangle_{W_2^3} \widehat{\Psi}_i(x), \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \langle Lu(x), \varphi_k(x) \rangle_{W_2^1} \widehat{\Psi}_i(x), \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \big\langle G(x, u, u'), T_{x_k} \big\rangle_{W_2^1} \widehat{\Psi}_i(x), \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} G\big(x_k, u_k, u_k'\big)\widehat{\Psi}_i(x).
\end{aligned}$$

□

4. Results

In this section, four examples will be discussed.

Example 1. *Consider the following problem:*

$$y^{''} + y^{'} = x^2, \qquad 0 \le x \le 1,$$

subject to

$$\begin{aligned} y(0) &= \hat{\beta}, \\ y^{'}(0) &= \hat{\gamma}, \end{aligned}$$

where $\hat{\beta} = (0, 1, 2)$ *and* $\hat{\gamma} = (1, 2, 3)$*. Then, the* $\alpha - cut$ *of* $y, y', y'', \hat{\beta}$*, and* $\hat{\gamma}$ *are*

$$\begin{aligned} y_\alpha &= [y_{1\alpha}, y_{2\alpha}], \quad y^{'}_\alpha = [y^{'}_{1\alpha}, y^{'}_{2\alpha}], \quad y^{''}_\alpha = [y^{''}_{1\alpha}, y^{''}_{2\alpha}], \\ \hat{\beta} &= [\alpha, 2-\alpha], \quad \hat{\gamma} = [\alpha+1, 3-\alpha]. \end{aligned}$$

Then,

$$\begin{aligned} &y^{''}_{1\alpha} + y^{'}_{1\alpha} = x^2, \quad y_\alpha(0) = \alpha, \quad y'_\alpha(0) = \alpha+1, \\ &y^{''}_{2\alpha} + y'_{2\alpha} = x^2, \quad y_2\alpha(0) = 2-\alpha, \quad y'_{2\alpha}(0) = 3-\alpha. \end{aligned}$$

Then, the exact solution is

$$y_\alpha(x) = \left[2\alpha + 1 - (\alpha+1)e^{-x} - \frac{1}{3}x^3 - x^2, 5 - 2\alpha - (3-\alpha)e^{-x} - \frac{1}{3}x^3 - x^2\right].$$

Using $n = 8$*, the absolute error in* $y_{1\alpha}$ *and* $y_{2\alpha}$ *are given in Table 1.*

Table 1. The absolute errors in Example 1.

x_k	*Abs. Error of* $y_{1\alpha}$	*Abs. Error of* $y_{2\alpha}$
0	0	0
0.1	2.3×10^{-14}	2.4×10^{-14}
0.2	2.4×10^{-14}	2.5×10^{-14}
0.3	2.5×10^{-14}	2.6×10^{-14}
0.4	2.6×10^{-14}	2.7×10^{-14}
0.5	2.7×10^{-14}	2.8×10^{-14}
0.6	2.8×10^{-14}	3.1×10^{-14}
0.7	3.0×10^{-14}	3.2×10^{-14}
0.8	3.1×10^{-14}	3.4×10^{-14}
0.9	3.3×10^{-14}	3.6×10^{-14}
1	3.5×10^{-14}	3.8×10^{-14}

Example 2. *Consider the following problem:*

$$y'' + (-1) \odot y' + y = 1,$$

subject to

$$\begin{aligned} &y(0) = \hat{\beta}, \\ &y'(0) = \hat{\gamma}, \end{aligned}$$

where $\hat{\beta} = (-1, 0, 1)$ *and* $\hat{\gamma} = (0, 1, 2)$*. Then, the* α*-cut of* $y, g', y'', \hat{\beta}$*, and* $\hat{\gamma}$ *are:*

$$\begin{aligned} &y_\alpha = [y_{1\alpha}, y_{2\alpha}], \quad y'_\alpha = \left[y'_{1\alpha}, y^{'}_{2\alpha}\right], \quad y''_\alpha\left[y''_{1\alpha}, y^{''}_{2\alpha}\right], \\ &\hat{\beta} = [\alpha-1, 1+\alpha], \quad \hat{\gamma} = [\alpha, 2-\alpha]. \end{aligned}$$

Then,

$$\begin{array}{lll} y''_{1\alpha} - y'_{2\alpha} + y_{1\alpha} = 1, & y_{1\alpha}(0) = \alpha - 1, & y'_{1\alpha}(0) = \alpha, \\ y'_{2\alpha} - y'_{1\alpha} + y_{2\alpha} = 1, & y_{2\alpha}(0) = 1 + \alpha, & y'_{2}(0) = 2 - \alpha. \end{array}$$

Then, using Mathematica, the exact solution is

$$\begin{aligned} y_\alpha(x) \;=\; & \Bigg[-\frac{1}{3}e^{-x/2}\Bigg(3\cos\left[\frac{\sqrt{3}x}{2}\right] + 3e^{x}\cos\left[\frac{\sqrt{3}x}{2}\right] - 3\alpha e^{x}\cos\left[\frac{\sqrt{3}x}{2}\right] - 3e^{x/2}\cos\left[\frac{\sqrt{3}x}{2}\right]^2 \\ & +3\sqrt{3}\sin\left[\frac{\sqrt{3}x}{2}\right] - 2\sqrt{3}\alpha\sin\left[\frac{\sqrt{3}x}{2}\right] - 3\sqrt{3}e^{x}\sin\left[\frac{\sqrt{3}x}{2}\right] + \sqrt{3}\alpha e^{t}\sin\left[\frac{\sqrt{3}t}{2}\right] \\ & -3e^{x/2}\sin\left[\frac{\sqrt{3}x}{2}\right]^2\Bigg), -\frac{1}{3}e^{-x/2}\Bigg(-3\cos\left[\frac{\sqrt{3}x}{2}\right] + 3e^{x}\cos\left[\frac{\sqrt{3}x}{2}\right] - 3\alpha e^{t}\cos\left[\frac{\sqrt{3}x}{2}\right] \\ & -3e^{x/2}\cos\left[\frac{\sqrt{3}x}{2}\right]^2 - 3\sqrt{3}\sin\left[\frac{\sqrt{3}x}{2}\right] + 2\sqrt{3}\alpha\sin\left[\frac{\sqrt{3}x}{2}\right] - 3\sqrt{3}e^{x}\sin\left[\frac{\sqrt{3}x}{2}\right] \\ & +\sqrt{3}\alpha e^{x}\sin\left[\frac{\sqrt{3}x}{2}\right] - 3e^{x/2}\sin\left[\frac{\sqrt{3}x}{2}\right]^2\Bigg)\Bigg]. \end{aligned}$$

Using $n = 8$, the absolute error in $y_{1\alpha}$ and $y_{2\alpha}$ are given in Table 2.

Table 2. The absolute errors in Example 2.

x_k	*Abs. Error of* $y_{1\alpha}$	*Abs. Error of* $y_{2\alpha}$
0	0	0
0.1	3.4×10^{-13}	3.6×10^{-13}
0.2	3.6×10^{-13}	3.8×10^{-13}
0.3	3.9×10^{-13}	4.1×10^{-13}
0.4	4.2×10^{-13}	4.4×10^{-13}
0.5	4.5×10^{-13}	4.6×10^{-13}
0.6	4.8×10^{-13}	4.9×10^{-13}
0.7	5.2×10^{-13}	5.3×10^{-13}
0.8	5.6×10^{-13}	5.7×10^{-13}
0.9	5.9×10^{-13}	6.0×10^{-13}
1	6.2×10^{-13}	6.3×10^{-13}

Example 3. *Consider the following problem:*

$$y'' = -\left(y'(x)\right)^2,$$

subject to

$$y(0) = \hat{\beta}, \quad y'(0) = \hat{\gamma},$$

where

$$\hat{\beta} = [\alpha, 2-\alpha], \qquad \hat{\gamma} = [1+\alpha, 3-\alpha].$$

Then,

$$y''_{1\alpha}(x) = f_{1\alpha}(x, y, y'), \qquad y_{1\alpha}(0) = \alpha, \qquad y'_{1\alpha}(0) = 1+\alpha,$$

and

$$y''_{2a}(x) = f_{2\alpha}(x_1 y, y'), \qquad y_{2\alpha}(0) = 2-\alpha, \qquad y'_{2\alpha}(0) = 3-\alpha,$$

where

$$f(x, y, y') = -\left(y'(x)\right)^2.$$

Then, the exact solution is:

$$y_\alpha(x) = [\ln((\alpha e^\alpha + e^\alpha)x + e^x), \ln\left((3e^{2-\alpha} - \alpha e^{2-\alpha}x) + e^{2\alpha}\right)].$$

Then, using the method proposed in the previous section, one obtains:

$$y_{1\alpha n}(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik}\hat{\psi}_i(x) f_{1\alpha}(x_k, y_{\alpha k}, y'_{\alpha k}),$$

$$y_{2\alpha n}(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik}\hat{\psi}_i(x) f_{2\alpha}(x_k, y_{\alpha k}, y'_{\alpha k}),$$

where

$$f_{1\alpha}\left(x_k, y_{\alpha_k}, y'_{\alpha_k}\right) = \min\left\{-\nu^2 : \nu \in [y'_{1\alpha}(x_k), y'_{2\alpha}(x_k)]\right\},$$

and

$$f_{2\alpha}\left(x_k, y_{\alpha_k}, y'_{\alpha_k}\right) = \max\left\{-\nu^2 : \nu \in [y'_{1\alpha}(x_k), y'_{2\alpha}(x_k)]\right\}.$$

Let $n = 8$. Let $E_1(x_k)$ and $E_2(x_k)$ be the absolute error in $y_{1\alpha}$ and $y_{2\alpha}$, respectively. The results are reported in Table 3.

Table 3. The absolute error of Example 3.

x_k	$E_1(x_k)$	$E_2(x_k)$
0	0	0
0.1	3.1×10^{-12}	2.9×10^{-12}
0.2	3.3×10^{-12}	3.2×10^{-12}
0.3	3.7×10^{-12}	3.5×10^{-12}
0.4	4.1×10^{-12}	3.9×10^{-12}
0.5	4.5×10^{-12}	4.3×10^{-12}
0.6	4.8×10^{-12}	4.7×10^{-12}
0.7	5.2×10^{-12}	5.1×10^{-12}
0.8	5.5×10^{-12}	5.4×10^{-12}
0.9	5.8×10^{-12}	5.7×10^{-12}
1	6.2×10^{-12}	6.0×10^{-12}

Example 4. *Consider the following problem:*

$$y'' = x^2 \odot y'(x) \oplus 2x \odot y(x) \oplus x \odot \hat{\beta},$$

subject to

$$y(0) = \hat{\beta} \quad , \quad y'(0) = \hat{\gamma},$$

where

$$\hat{\beta} = [1+\alpha, 3-\alpha], \quad \hat{\gamma} = [0,0].$$

Then,

$$y_1''(x) = f_{1\alpha}(x, y, y'), \qquad y_{1\alpha}(0) = 1+\alpha, \qquad y'_{1\alpha}(0) = 0,$$

and

$$y_{2\alpha}''(x) = f_{2\alpha}(x, y, y'), \qquad y_{2x}(0) = 3-\alpha, \qquad y_{2\alpha}(0) = 0,$$

where

$$f(x, y, y') = x^2 \odot y'(x) \oplus 2x \odot y(x) \oplus x \odot \hat{\beta}.$$

Then, the exact solution is given by

$$\left[\left(e^{(x^3/3)} - 1\right)(1-\alpha), \left(2e^{(x^3/3)} - 1\right)(3-\alpha)\right].$$

Then using the method which proposed in the previous section, one obtains:

$$y_{1\alpha n}(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik}\hat{\psi}_i(x) f_1\alpha\left(x_k, y_{\alpha k}, y'_{\alpha k}\right),$$

$$y_{2an}(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik}\hat{\psi}_i(x) f_{2\alpha}\left(x_k, y_{\alpha k}, y'_{\alpha k}\right),$$

where

$$f_{1\alpha}\left(x_k, y_{\alpha k}, y'_{\alpha k}\right) = x^2 y'_{2\alpha} + 2xy_{1\alpha} + x\beta_1,$$
$$f_{2\alpha}\left(x_k, y_{\alpha k}, y'_{\alpha k}\right) = x^2 y'_{2\alpha} + 2xy_{1\alpha} + x\beta_2.$$

Let $n = 8$. Let $E_1(x_k)$ and $E_2(x_k)$ be the absolute error in $y_{1\alpha}$ and $y_{2\alpha}$, respectively. The results are reported in Table 4.

Table 4. The absolute error of Example 4.

x_k	$E_1(x_k)$	$E_2(x_k)$
0	0	0
0.1	2.7×10^{-14}	2.5×10^{-14}
0.2	2.9×10^{-14}	2.7×10^{-14}
0.3	3.1×10^{-14}	2.8×10^{-14}
0.4	3.4×10^{-14}	3.0×10^{-14}
0.5	3.5×10^{-14}	3.2×10^{-14}
0.6	3.7×10^{-14}	3.4×10^{-14}
0.7	3.9×10^{-14}	3.7×10^{-14}
0.8	4.2×10^{-14}	3.9×10^{-14}
0.9	4.4×10^{-14}	4.1×10^{-14}
1	4.7×10^{-14}	4.4×10^{-14}

5. Conclusions

In this article, the reproducing kernel method has been presented for second-order fuzzy initial value problems. It started with preliminaries about fuzzy numbers and differentiation and then highlighted the direct method for solving fuzzy problems. When the direct method was used to solve the problem, complicated optimization problems that are difficult to solve appeared. The proposed method was based on an RKM and the Gram Schmidt process. The structure of the RKM was explained and supported by several examples. The numerical results showed the efficiency of the proposed method. The absolute errors were computed using Mathematica. For future work, the fuzzy boundary value problems should be investigated using RKM by implementing the shooting method. Moreover, several applications for this method should be investigated, such as Fuzzy Strum–Liouville problems and the delay fuzzy initial value problems. In addition, this approach can be used to solve several fuzzy integral and differential equations such as fuzzy integro-differential equations, fractional IVPs, fuzzy Cahn–Allen equations, and fuzzy duffing modesl.

Author Contributions: Conceptualization, Q.D. and M.I.S.; methodology, M.I.S.; software, Q.D.; validation, M.I.S.; investigation, Q.D.; writing—original draft preparation, Q.D.; writing—review and editing, M.I.S.; supervision, M.I.S.; project administration. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not Applicable.

Data Availability Statement: All data are included in the paper.

Acknowledgments: We would like to thank the referees for their valuable comments.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Fullér, R.; Giove, S.; Masulli, S. Fuzzy Logic and Applications. In Proceedings of the 12th International Workshop, WILF 2018, Genoa, Italy, 6–7 September 2018; Springer: Cham, Switzerland, 2018. [CrossRef]
2. de Barros, L.C.; Santo Pedro, F. Fuzzy differential equations with interactive derivative. *Fuzzy Sets Syst.* **2017**, *309*, 64–80. [CrossRef]
3. Baidosov, V.A. Fuzzy differential inclusions. *J. Appl. Math. Mech.* **1990**, *54*, 8–13. [CrossRef]
4. Hüllermeier, E. Numerical methods for fuzzy initial value problems. *Int. J. Uncertain. Fuzziness Knowl.-Based Syst.* **1999**, *7*, 439–461. [CrossRef]
5. Diamond, P. Stability and periodicity in fuzzy differential equations. *IEEE Trans. Fuzzy Syst.* **2000**, *8*, 583–590. [CrossRef]
6. Lakshmikantham, V.; Mohapatra, R.N. Theory of Fuzzy Differential Equations and Inclusions. Taylor & Francis: Abingdon, UK, 2003.
7. Sanchez, D.; de Barros, L.; Esmi, E. *On interactive Fuzzy boundary value problems*; *Fuzzy Sets Syst.* **2019**, *358*, 84–96. [CrossRef]
8. Gong, Z. Fuzzy Laplace transform based on the Henstock integral and its applications in discontinuous fuzzy systems. *Fuzzy Sets Syst.* **2019**, *358*, 1–28. [CrossRef]
9. Zhou, X.; Cao, B.; Nasseri, S. Optimality Conditions for Fuzzy Number Quadratic Programming with Fuzzy Coefficients. *J. Appl. Math.* **2014**, *2014*, 489893. [CrossRef]
10. Tapaswini, S.; Chakraverty, S.; Nieto, J. Numerical solution of fuzzy boundary value problems using Galerkin method. *Indian Acad. Sci.* **2017**, *42*, 45–61. [CrossRef]
11. Gumah, G.; Nasera, M.; Al-Smadi, M.; Al-Omari, S.; Baleanu, D. Numerical solutions of hybrid fuzzy differential equations in a Hilbert space. *Appl. Numer. Math.* **2020**, *151*, 402–412. [CrossRef]
12. Patel, K.; Desai, N. Solution Of Fuzzy Initial Value Problems By Fuzzy Laplace Transform. *Kalpa Publ. Comput.* **2017**, *2*, 25–37.
13. Diniz, M.; Gomes, L.; Bassanezi, R. Solution to Convex Variational Problems with Fuzzy Initial Condition Using Zadeh's Extension. In Proceedings of the North American Fuzzy Information Processing Society Annual Conference, Fortaleza, Brazil, 4–6 July 2018; Springer: Berlin/Heidelberg, Germany, 2018; Volume 831, pp. 431–438.
14. Wu, H.; Feng, S. Mixed Fuzzy Boundary Control Design for Nonlinear Coupled Systems of ODE and Boundary-Disturbed Uncertain Beam. *IEEE Trans. Fuzzy Syst.* **2018**, *26*, 3379–3390. [CrossRef]
15. Suhhiem, M. New Numerical Approach for Solving Fuzzy Boundary Value Problems. *Ann. Pure Appl. Math.* **2019**, *19*, 147–159. [CrossRef]
16. Niu, J.; Xu, M.; Lin, Y.; Xue, Q. Numerical solution of nonlinear singular boundary value problems. *J. Comput. Appl. Math.* **2018**, *331*, 42–51. [CrossRef]
17. Shah, K.; Wang, J. A numerical scheme based on non-discretization of data for boundary value problems of fractional order differential equations. *Rev. Real Acad. Cienc. Exactas Físicas Nat. Ser. A Mat.* **2019**, *113*, 2277–2294. [CrossRef]
18. Roul, P. A fast and accurate computational technique for efficient numerical solution of nonlinear singular boundary value problems. *Int. J. Comput. Math.* **2019**, *96*, 51–72. [CrossRef]
19. Wasques, V.; Esmi, E.; Barros, L.; Bede, B. Comparison Between Numerical Solutions of Fuzzy Initial-Value Problems via Interactive and Standard Arithmetics. *Int. Fuzzy Syst. Assoc. World Congr.* **2019**, *1000*, 704–715.
20. Wasques, V.; Esmi, E.; Barros, L.; Sussner, P. Numerical Solution for Fuzzy Initial Value Problems via Interactive Arithmetic: Application to Chemical Reactions. *Int. J. Comput. Intell.* **2020**, *13*, 1517–1529. [CrossRef]
21. Jeyaraj, T.; Rajan, D. Explicit Runge Kutta Method in Solving Fuzzy Intial Value Problem. *Adv. Appl. Math.* **2021**, *20*, 663–674.
22. Al-Refai, S.; Syam, M.I.; Al-Refai, M. Optimization of One-Step Block Method for Solving SecondOrder Fuzzy Initial Value Problems.*Complexity* **2021**, *2021*, 6650413. [CrossRef]
23. Kashkari, B.; Syam, M.I. Reproducing Kernel Method for Solving Nonlinear Fractional Fredholm integro-differential Equation. *Complexity* **2018**, *2018*, 2304858. [CrossRef]
24. Du, H.; Zhao, G.; Zhao, C. Reproducing kernel method for solving Fredholm integro-differential equations with weakly singularity. *J. Comput. Appl. Math.* **2014**, *255*, 122–132. [CrossRef]
25. Akgül, A. Reproducing Kernel Method for Fractional Derivative with Non-local and Non-singular Kernel. In *Fractional Derivatives with Mittag-Leffler Kernel*; Springer: Berlin/Heidelberg, Germany, 2019; Volume 194, pp. 1–12.
26. Akgül, A. Reproducing kernel Hilbert space method based on reproducing kernel functions for investigating boundary layer flow of a Powell-Eyring non-Newtonian fluid. *J. Taibah Univ. Sci.* **2019**, *13*, 858–863. [CrossRef]
27. Sadamoto, S.; Ozdemir, M.; Tanaka, S.; Taniguchi, K.; Yu, T.; Bui, T. An effective meshfree reproducing kernel method for buckling analysis of cylindrical shells with and without cutouts. *Comput. Mech.* **2017**, *59*, 919–932. [CrossRef]
28. Gholami, N.; Allahviranloo, T.; Abbasbandy, S.; Karamikabir, N. Fuzzy reproducing kernel space method for solving fuzzy boundary value problems. *Math. Sci.* **2019**, *13*, 97–103. [CrossRef]
29. Mei, L.; Lin, Y. Simplified reproducing kernel method and convergence order for linear Volterra integral equations with variable coefficients. *J. Comput. Appl. Math.* **2019**, *346*, 390–398. [CrossRef]
30. Geng, F.; Qian, S. Modified reproducing kernel method for singularly perturbed boundary value problems with a delay. *Appl. Math. Model.* **2015**, *39*, 5592–5597. [CrossRef]

31. Li, X.; Wu, B. A new reproducing kernel method for variable order fractional boundary value problems for functional differential equations. *J. Comput. Appl. Math.* **2017**, *311*, 387–393. [CrossRef]
32. Moradi, F.; Javadi, S. Reproducing Kernel Method for Solving Nonlinear Oscillators Under Damping Effect. *Iran. J. Sci. Technol. Trans. A Sci.* **2020**, *44*, 763–772. [CrossRef]
33. Alv, ia, A.; Paripour, M. The combined reproducing kernel method and Taylor series for handling nonlinear Volterra integro-differential equations with derivative type kernel. *Appl. Math. Comput.* **2019**, *355*, 151–160. [CrossRef]
34. Arquba, O.; Al-Smadi, M.; Shawagfeh, N. Solving Fredholm integro-differential equations using reproducing kernel Hilbert space method. *Appl. Math. Comput.* **2013**, *219*, 8938–8948. [CrossRef]
35. Saadeh, R.; Smadi, M.; Gumah, G.; Khalil, H.; Khan, R. Numerical Investigation for Solving Two-Point Fuzzy Boundary Value Problems by Reproducing Kernel Approach. *Appl. Math. Inf. Sci.* **2016**, *10*, 1–13. [CrossRef]
36. Qi, M.; Yang, Z.P.; Xu, T.Z. A reproducing kernel method for solving nonlocal fractional boundary value problems with uncertainty. **2017**, *21*, 4019–4028. [CrossRef]
37. Allahviranloo, T.; Noeiaghdam, Z.; Noeiaghdam, S.; Nieto, J. A fuzzy Method for solving fuzzy fractional differential equations based on the generalized fuzzy Taylor expansion. *Mathematics* **2020**, *8*, 2166. [CrossRef]
38. Inc, M.; Akgül, A.; Geng, F. Reproducing Kernel Hilbert Space Method for Solving Bratu's Problem. *Bull. Malays. Math. Sci. Soc.* **2015**, *38*, 271–287. [CrossRef]
39. Arqub, O. Numerical solutions of systems of first-order, two-point BVPs based on the reproducing kernel algorithm. *Calcolo* **2018**, *55*, 31. [CrossRef]