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Lax Operator Algebras and Applications to τ-Symmetries for Multilayer Integrable Couplings

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Abstract: The algebraic structures of zero curvature representations are furnished for multilayer integrable couplings associated with matrix spectral problems, both discrete and continuous. The key elements are a class of matrix loop algebras consisting of block matrices with blocks of the same size. As illustrative examples, isospectral and non-isospectral integrable couplings and the corresponding commutator relations of their Lax operators are computed explicitly in the cases of the Volterra lattice hierarchy and the AKNS hierarchy, along with their τ-symmetry algebras.

Keywords: Lie algebraic structure; zero curvature representation; integrable coupling; Volterra lattice; τ-symmetry algebra

PACS: 02.10.De; 02.20.Sv; 02.30.Ik

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Citation: Li, C.-X.; Ma, W.-X.; Shen, S.-F. Lax Operator Algebras and Applications to τ-Symmetries for Multilayer Integrable Couplings. *Symmetry* **2022**, *14*, 1185. https://doi.org/10.3390/sym14061185

Academic Editor: Calogero Vetro

Received: 1 May 2022
Accepted: 2 June 2022
Published: 9 June 2022

1. Introduction

Due to the successful description and interpretation of nonlinear phenomena, integrable systems have attracted extensive attention from scientists in various fields of natural science. Matrix spectral problems associated with Lie algebras are crucial keys to constructing integrable systems [1–11]. There has recently been a growing interest in integrable couplings, which are all associated with semi-direct sums of Lie algebras or non-semisimple Lie algebras. A few methods for constructing discrete and continuous integrable couplings for given integrable systems have been presented [12–25]. Integrable couplings exhibit many interesting mathematical structures, including block matrix Lax representations and bi-Hamiltonian structures, with infinitely many symmetries and conservation laws of triangular form. The corresponding Lax airs or zero curvature representations of integrable couplings, in particular, possess interesting and beautiful algebraic structures which generate symmetries leading to exact solutions to nonlinear integrable partial differential equations [26–37].

To the best of our knowledge, there has been very little work done on $N \times N$ integrable couplings, because they are extremely complex [38,39]. In this paper, based on a kind of $N \times N$ non-semisimple Lie algebra, we explore the Lie algebraic structures of zero curvature representations for general multilayer integrable couplings and apply them to the construction of τ-symmetries (a type of time-dependent symmetry).

Let us first describe some notations in the discrete case, as follows [10,19]. Let E be the shift operator,

$$E^m f(n) = f^{(m)}(n) = f(n+m), \tag{1}$$

where $f : \mathbb{Z} \to \mathbb{R}, \quad m,n \in \mathbb{Z}$, and the inverse operator of $E - E^{-1}$ are defined as

$$(E - E^{-1})^{-1} f(n) \triangleq \frac{1}{2}\left(\sum_{k=-\infty}^{-1} f(n+1+2k) - \sum_{k=1}^{\infty} f(n-1+2k) \right). \tag{2}$$

Thus, the corresponding inverses of the forward and backward difference operators can be determined by

$$(E-1)^{-1} = (E - E^{-1})^{-1}(1 + E^{-1}), \qquad (1 - E^{-1})^{-1} = (E - E^{-1})^{-1}(E+1), \tag{3}$$

which are normally used to generate discrete integrable hierarchies (both isospectral and non-isospectral hierarchies). We assume that a pair of discrete matrix spectral problems reads

$$E\varphi = U\varphi = U(u,\lambda)\varphi, \qquad \varphi_t = V\varphi = V(u,\lambda)\varphi, \qquad \lambda_t = f(\lambda), \tag{4}$$

where $u = (u_1, u_2, \cdots, u_q)^{\mathrm{T}},\ u_i = u_i(n,t),\ i = 1,2,\cdots,q$, are complex or real functions defined over $\mathbb{Z} \times \mathbb{R}$. The Lax pair of $U \equiv U_0$ and $V \equiv V_0$, involving the spectral parameter λ, comes from a certain matrix loop algebra $\mathfrak{g}$ and determines a discrete integrable equation

$$u_t = K = K(n,t,u), \tag{5}$$

through the discrete zero curvature equation

$$U_t = (EV)U - UV. \tag{6}$$

This means that a triple (U, V, K) satisfies

$$U'(u)[K] + f(\lambda)U_\lambda = (EV)U - UV. \tag{7}$$

where $\lambda_t = f(\lambda)$, $U_\lambda = \partial U/\partial\lambda$, and $U'(u)[K]$ denotes the Gateaux derivative

$$U'(u)[K] = \frac{\partial}{\partial \epsilon} U(u + \epsilon K)\big|_{\epsilon=0}. \tag{8}$$

To generate a multilayer (N-layer) integrable coupling

$$\bar{u}_t = \bar{K}(\bar{u}) = \begin{pmatrix} K \\ K_1 \\ \vdots \\ K_N \end{pmatrix}, \quad \bar{u} = \begin{pmatrix} u \\ v_1 \\ \vdots \\ v_N \end{pmatrix}, \tag{9}$$

we use a matrix loop algebra $\bar{\mathfrak{g}}(\lambda)$ consisting of square matrices of the following block form [9]:

$$\bar{P} = M(P, P_1, \cdots, P_N) = \begin{pmatrix} P & P_1 & \cdots & P_{N-1} & P_N \\ 0 & P & \cdots & P_{N-2} & P_{N-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & P & P_1 \\ 0 & 0 & \cdots & 0 & P \end{pmatrix}, \tag{10}$$

where $P \equiv P_0$, $P_i, i = 1,2,\cdots,N$, are square submatrices of the same size as U and V. This loop algebra $\bar{g}(\lambda)$ has two sub-loop algebras

$$\tilde{\mathfrak{g}} = \{M(P, 0, \cdots, 0)\}, \quad \tilde{\mathfrak{g}}_c = \{M(0, P_1, \cdots, P_N)\}, \tag{11}$$

which form a semi-direct sum: $\tilde{\mathfrak{g}}(\lambda) = \tilde{\mathfrak{g}} \in \tilde{\mathfrak{g}}_c$. The notion of semi-direct sums means that the two subalgebras $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}_c$ satisfy $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}_c] \subseteq \tilde{\mathfrak{g}}_c$. We also require the closure property between $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}_c$ under the matrix multiplication $\tilde{\mathfrak{g}}\tilde{\mathfrak{g}}_c, \tilde{\mathfrak{g}}_c\tilde{\mathfrak{g}} \subseteq \tilde{\mathfrak{g}}_c$. Now, a multilayer discrete integrable coupling is determined by the following enlarged zero curvature representation:

$$\bar{U}_t = (E\bar{V})\bar{U} - \bar{U}\bar{V} \tag{12}$$

where

$$\bar{U} = M(U, U_1, \cdots, U_N), \qquad \bar{V} = M(V, V_1, \cdots, V_N). \tag{13}$$

This implies that an enlarged triple $(\bar{U}, \bar{V}, \bar{K})$ with $\lambda_t = f(\lambda)$ satisfies

$$\bar{U}'(\bar{u})[\bar{K}] + f(\lambda)\bar{U}_\lambda = (E\bar{V})\bar{U} - \bar{U}\bar{V}. \tag{14}$$

This precisely presents

$$\begin{aligned} &U'(u)[K] + f(\lambda)U_\lambda = (EV)U - UV, \\ &U_i'(v_i)[K_i] + f(\lambda)U_{i\lambda} = \sum_{j=0}^{i} \left[(EV_j)U_{i-j} - U_j V_{i-j}\right], \quad i = 1, 2, \cdots, N. \end{aligned} \tag{15}$$

Similarly, we can start with continuous matrix spectral problems:

$$\varphi_x = U\varphi = U(u, \lambda)\varphi, \qquad \varphi_t = V\varphi = V(u, \lambda)\varphi, \qquad \lambda_t = f(\lambda), \tag{16}$$

where $u = (u_1, u_2, \cdots, u_q)^{\mathrm{T}}$, $u_i = u_i(x, t)$, $i = 1, 2, \cdots, N$. The Lax pair of U and V, involving the spectral parameter λ, determines a continuous integrable equation

$$u_t = K = K(x, t, u), \tag{17}$$

through the continuous zero curvature equation

$$U_t - V_x + [U, V] = 0. \tag{18}$$

This means that a triple (U, V, K) satisfies

$$U'(u)[K] + f(\lambda)U_\lambda - V_x + [U, V] = 0. \tag{19}$$

Thus, the N-coupled integrable coupling can be derived from the enlarged zero curvature representation

$$\bar{U}_t - \bar{V}_x + [\bar{U}, \bar{V}] = 0, \tag{20}$$

where $\bar{U}$ and $\bar{V}$ are defined by (13). This implies that the enlarged triple $(\bar{U}, \bar{V}, \bar{K})$ with $\lambda_t = f(\lambda)$ satisfies

$$\begin{aligned} &U'(u)[K] + f(\lambda)U_\lambda - V_x + [U, V] = 0, \\ &U_i'(v_i)[K_i] + f(\lambda)U_{i\lambda} - V_{ix} + \sum_{j=0}^{i} [U_j, V_{i-j}] = 0, \quad i = 1, 2, \cdots, N. \end{aligned} \tag{21}$$

The rest of the paper is structured as follows. In Section 2, we give some definitions and the Lie algebra of enlarged vector fields for the above multilayer integrable couplings, both discrete and continuous. In Section 3, we establish Lie algebraic structures of zero curvature representations for the general integrable couplings presented above. In Section 4, we illustrate our general theory by taking the Volterra lattice hierarchy and the AKNS hierarchy as two specific examples and present the corresponding τ-symmetry algebras of the resulting integrable couplings. Section 5 is devoted to conclusions and discussion.

2. The Lie Algebra of Enlarged Vector Fields

Firstly, we denote by $\mathcal{B}$ all complex or real functions $P = P(n, t, u, v_1, v_2, \cdots, v_N)$ which are C^∞-differentiable with respect to $\{n, t\}$ and C^∞-Gateaux differentiable with respect to $\{u, v_1, v_2, \cdots, v_N\}$, and we set $\mathcal{B}^r = \{(P_1, P_2, \cdots, P_r)^{\mathrm{T}} | P_i \in \mathcal{B}\}$. Moreover, by $\mathcal{V}^r$, we denote all $r \times r$ matrix linear difference operators:

$$\mathcal{V}^r = \bigcup_{n=-\infty}^{\infty} \sum_{k=0}^{n} \mathcal{V}^r_{(k)}, \qquad \mathcal{V}^r_{(k)} = \left\{ \left(P_{ij} E^k \right)_{r \times r} | P_{ij} = P_{ij}(n, t, u, v_1, v_2, \cdots, v_N) \in \mathcal{B} \right\}. \tag{22}$$

Then, we define

$$\widetilde{\mathcal{V}}^r = \mathcal{V}^r \otimes C[\lambda, \lambda^{-1}], \qquad \widetilde{\mathcal{V}}^r_{(0)} = \mathcal{V}^r_{(0)} \otimes C[\lambda, \lambda^{-1}]. \tag{23}$$

We now set

$$\bar{K} = \begin{pmatrix} K \\ K_1 \\ \vdots \\ K_N \end{pmatrix}, \qquad \bar{S} = \begin{pmatrix} S \\ S_1 \\ \vdots \\ S_N \end{pmatrix} \in \mathcal{B}^{q + \sum_{i=1}^{N} q_i}, \tag{24}$$

where $K, S \in \mathcal{B}^q$, $K_i, S_i \in \mathcal{B}^{q_i}$.

The Gateaux derivative is defined as follows:

$$R'[\bar{K}] = \frac{\partial}{\partial \epsilon} R(u + \epsilon K, v_1 + \epsilon K_1, \cdots, v_N + \epsilon K_N)\big|_{\epsilon = 0}, \tag{25}$$

where $R \in \widetilde{\mathcal{V}}^r$ or $\widetilde{\mathcal{V}}^r_{(0)}$. In particular, we have

$$\begin{aligned} K_i'[\bar{S}] &= K_i'[S] + \sum_{j=1}^{i} K_i'[S_j], \\ S_i'[\bar{K}] &= S_i'[K] + \sum_{j=1}^{i} S_i'[K_j], \qquad i = 1, 2, \cdots, N. \end{aligned} \tag{26}$$

Theorem 1. *Let $P = P(n, t, u, v_1, \cdots, v_N) \in \mathcal{B}$, $\bar{K}, \bar{S} \in \mathcal{B}^{q + \sum_{i=1}^{N} q_i}$. Then, we have the relation*

$$\begin{aligned} \left(P'[\bar{K}]\right)'[\bar{S}] - \left(P'[\bar{S}]\right)'[\bar{K}] &= P'(u)\left[K'[S] - S'[K]\right] \\ &\quad + \sum_{i=1}^{N} P'(v_i) \left[K_i'[S] - S_i'[K] + \sum_{j=1}^{i} K_i'[S_j] - \sum_{j=1}^{i} S_i'[K_j] \right]. \end{aligned} \tag{27}$$

Proof. By the definition of the Gateaux derivative, we have

$$
\begin{aligned}
\left(P'[\bar{K}]\right)'[\bar{S}] &= \left(\frac{\partial}{\partial \epsilon}P(u+\epsilon K)\big|_{\epsilon=0} + \sum_{i=1}^{N}\frac{\partial}{\partial \epsilon_i}P(v_i+\epsilon_i K_i)\big|_{\epsilon_i=0}\right)'[\bar{S}] \\
&= \frac{\partial}{\partial \delta}\frac{\partial}{\partial \epsilon}P(u+\delta S+\epsilon K(u+\delta S))\big|_{\delta=\epsilon=0} \\
&\quad + \sum_{i=1}^{N}\frac{\partial}{\partial \delta}\frac{\partial}{\partial \epsilon_i}P(v_i+\epsilon_i K_i(u+\delta S))\big|_{\delta=\epsilon_i=0} \\
&\quad + \sum_{i=1}^{N}\Bigg[\frac{\partial}{\partial \delta_i}\frac{\partial}{\partial \epsilon_i}P(v_i+\delta_i S_i+\epsilon_i K_i(v_i+\delta_i S_i))\big|_{\delta_i=\epsilon_i=0} \\
&\qquad + \sum_{j=1}^{i}\frac{\partial}{\partial \delta_j}\frac{\partial}{\partial \epsilon_i}P\big(v_i+\epsilon_i K(v_j+\delta_j S_j)\big)\big|_{\delta_j=\epsilon_i=0}\Bigg] \\
&= \frac{\partial}{\partial \delta}\frac{\partial}{\partial \epsilon}P\big(u+\delta S+\epsilon K+\epsilon\delta K'[S]\big)\big|_{\delta=\epsilon=0} \\
&\quad + \sum_{i=1}^{N}\frac{\partial}{\partial \delta}\frac{\partial}{\partial \epsilon_i}P\big(v_i+\epsilon_i K_i+\epsilon_i\delta K_i'[S]\big)\big|_{\delta=\epsilon_i=0} \\
&\quad + \sum_{i=1}^{N}\Bigg[\frac{\partial}{\partial \delta_i}\frac{\partial}{\partial \epsilon_i}P\big(v_i+\delta_i S_i+\epsilon_i K_i+\epsilon_i\delta_i K_i'[S_i]\big)\big|_{\delta_i=\epsilon_i=0} \\
&\qquad + \sum_{j=1}^{i}\frac{\partial}{\partial \delta_j}\frac{\partial}{\partial \epsilon_i}P\big(v_i+\epsilon_i K_i+\epsilon_i\delta_j K'[S_j]\big)\big|_{\delta_j=\epsilon_i=0}\Bigg] \\
&= \frac{\partial}{\partial \delta}\frac{\partial}{\partial \epsilon}P(u+\delta S+\epsilon K)\big|_{\delta=\epsilon=0} + \sum_{i=1}^{N}\frac{\partial}{\partial \delta_i}\frac{\partial}{\partial \epsilon_i}P(v_i+\delta_i S_i+\epsilon_i K_i)\big|_{\delta_i=\epsilon_i=0} \\
&\quad + P'(u)\left[K'[S]\right] + \sum_{i=1}^{N}P'(v_i)\left[K_i'[S]+\sum_{j=1}^{i}K_i'[S_j]\right].
\end{aligned}
$$

At the same time, we similarly have

$$
\begin{aligned}
\left(P'[\bar{S}]\right)'[\bar{K}] &= \frac{\partial}{\partial \delta}\frac{\partial}{\partial \epsilon}P(u+\delta S+\epsilon K)\big|_{\delta=\epsilon=0} + \sum_{i=1}^{N}\frac{\partial}{\partial \delta_i}\frac{\partial}{\partial \epsilon_i}P(v_i+\delta_i S_i+\epsilon_i K_i)\big|_{\delta_i=\epsilon_i=0} \\
&\quad + P'(u)\left[S'[K]\right] + \sum_{i=1}^{N}P'(v_i)\left[S_i'[K]+\sum_{j=1}^{i}S_i'[K_j]\right].
\end{aligned}
$$

These two equalities engender our required equality. □

From the above theorem, we can easily deduce the following corollary.

Corollary 1. *For $U_i = U_i(v_i,\lambda) \in \widetilde{\mathcal{V}}^r_{(0)}$, $\quad i = 1,2,\cdots,N$, we can obtain*

$$
\left(U_i'[K_i]\right)'[\bar{S}] - \left(U_i'[S_i]\right)'[\bar{K}] = U_i'(v_i)\left[K_i'[S] - S_i'[K] + \sum_{j=1}^{i}\Big(K_i'[S_j] - S_i'[K_j]\Big)\right], \quad i = 1,2,\cdots,N. \tag{28}
$$

Here, we note that $U_i(v_i,\lambda)$ have nothing to do with the original potential vector u.

Evidently, we can also compute the commutator of two enlarged vector fields $\bar{K}, \bar{S} \in \mathcal{B}^{q+\sum_{i=1}^{N} q_i}$ as follows:

$$[\bar{K}, \bar{S}] \triangleq \bar{K}'[\bar{S}] - \bar{S}'[\bar{K}] = \begin{pmatrix} [K,S] \\ [K,S]_1 \\ \vdots \\ [K,S]_N \end{pmatrix}, \tag{29}$$

where

$$\begin{aligned} &[K,S] = K'[S] - S'[K], \\ &[K,S]_i = K_i'[S] - S_i'[K] + \sum_{j=1}^{i} K_i'[S_j] - \sum_{j=1}^{i} S_i'[K_j], \quad i = 1, 2, \cdots, N. \end{aligned} \tag{30}$$

Clearly, we can show that the above product forms a Lie algebra in $\mathcal{B}^{q+\sum_{i=1}^{N} q_i}$.

Theorem 2. *For enlarged vector fields $\bar{K}, \bar{S} \in \mathcal{B}^{q+\sum_{i=1}^{N} q_i}$, the product (29) defines a Lie algebra in $\mathcal{B}^{q+\sum_{i=1}^{N} q_i}$.*

Proof. By setting $\bar{K} = \begin{pmatrix} K \\ K_1 \\ \vdots \\ K_N \end{pmatrix}$, $\bar{S} = \begin{pmatrix} S \\ S_1 \\ \vdots \\ S_N \end{pmatrix}$ and $\bar{L} = \begin{pmatrix} L \\ L_1 \\ \vdots \\ L_N \end{pmatrix} \in \mathcal{B}^{q+\sum_{i=1}^{N} q_i}$ and using the definition of the product (29), we have

$$\begin{aligned} \Big[[\bar{K}, \bar{S}], \bar{L}\Big] &= \left[\begin{pmatrix} [K,S] \\ [K,S]_1 \\ \vdots \\ [K,S]_N \end{pmatrix}, \begin{pmatrix} L \\ L_1 \\ \vdots \\ L_N \end{pmatrix} \right] \\ &= \left[\begin{pmatrix} K'[S] - S'[K] \\ K_1'[S] - S_1'[K] + K_1'[S_1] - S_1'[K_1] \\ \vdots \\ K_N'[S] - S_N'[K] + \sum_{j=1}^{N} K_N'[S_j] - \sum_{j=1}^{N} S_N'[K_j] \end{pmatrix}, \begin{pmatrix} L \\ L_1 \\ \vdots \\ L_N \end{pmatrix} \right] \\ &\triangleq \begin{pmatrix} \big[[K,S], L\big] \\ \big[[K,S], L\big]_1 \\ \vdots \\ \big[[K,S], L\big]_N \end{pmatrix}. \end{aligned}$$

Thus, by direct calculation, we have

$$\begin{aligned} &\big[[K,S], L\big] + \mathrm{cycle}(K, S, L) = 0, \\ &\big[[K,S], L\big]_i + \mathrm{cycle}(K, S, L) = 0, \quad i = 1, 2, \cdots, N, \end{aligned}$$

namely,

$$\Big[[\bar{K}, \bar{S}], \bar{L}\Big] + \mathrm{cycle}(\bar{K}, \bar{S}, \bar{L}) = 0.$$

This implies that (29) defines a Lie algebra in $\mathcal{B}^{q+\sum_{i=1}^{N} q_i}$. $\square$

3. The Algebraic Structure of Lax Operators

Next, we aim to discuss the algebraic structure of discrete zero curvature equations for N-coupled integrable couplings. First, the commutator of two smooth functions $f, g \in C^\infty(\mathbb{C})$ is defined as

$$[\![f,g]\!](\lambda) = f'(\lambda)g(\lambda) - f(\lambda)g'(\lambda), \qquad \lambda \in \mathbb{C}. \tag{31}$$

It is known that $(C^\infty(\mathbb{C}), [\![\cdot,\cdot]\!])$ is a Lie algebra.

3.1. The Discrete Case

In what follows, we always assume that the enlarged spectral operator $\bar{U} \in \widetilde{\mathcal{V}}_{(0)}^{(N+1)r}$ has an injective Gateaux derivative operator $\bar{U}' : \mathcal{B}^{q+\sum_{i=1}^N q_i} \to \widetilde{\mathcal{V}}_{(0)}^{(N+1)r}$. We assume that

$$P(\bar{U}) = \{(\bar{V},\bar{K},f) \text{ satisfies (14)}, \mid \bar{V} \in \widetilde{\mathcal{V}}_{(0)}^{(N+1)r},\ \bar{K} \in \mathcal{B}^{q+\sum_{i=1}^N q_i},\ f \in C^\infty(\mathbb{C})\}, \tag{32}$$

and for $f(\lambda) \in C^\infty(\mathbb{C})$, we set

$$M(\bar{U},f) = \{\bar{V} \in \widetilde{\mathcal{V}}_{(0)}^{(N+1)r} \mid \exists \bar{K} \in \mathcal{B}^{q+\sum_{i=1}^N q_i} \quad \text{s.t.} \quad (\bar{V},\bar{K},f) \in P(\bar{U})\} \tag{33}$$

and

$$EM(\bar{U},f) = \left\{\bar{K} \in \mathcal{B}^{q+\sum_{i=1}^N q_i} \middle| \exists \bar{V} \in M(\bar{U},f) \quad \text{s.t.} \quad (\bar{V},\bar{K},f) \in P(\bar{U})\right\}. \tag{34}$$

For $(\bar{V},\bar{K},f),(\bar{W},\bar{S},g) \in P(\bar{U})$, the product $[\![\bar{V},\bar{W}]\!] \in \widetilde{\mathcal{V}}_{(0)}^{(N+1)r}$ can be computed as follows:

$$\begin{aligned} [\![\bar{V},\bar{W}]\!] &= \bar{V}'[\bar{S}] - \bar{W}'[\bar{K}] + [\bar{V},\bar{W}] + g\bar{V}_\lambda - f\bar{W}_\lambda \\ &= \begin{pmatrix} [\![V,W]\!] & [\![V_1,W_1]\!] & \cdots & [\![V_{N-1},W_{N-1}]\!] & [\![V_N,W_N]\!] \\ 0 & [\![V,W]\!] & \cdots & [\![V_{N-2},W_{N-2}]\!] & [\![V_{N-1},W_{N-1}]\!] \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & [\![V,W]\!] & [\![V_1,W_1]\!] \\ 0 & 0 & \cdots & 0 & [\![V,W]\!] \end{pmatrix}, \end{aligned} \tag{35}$$

where

$$\begin{aligned} &[\![V,W]\!] = V'[S] - W'[K] + [V,W] + gV_\lambda - fW_\lambda, \\ &[\![V_i,W_i]\!] = V_i'[\bar{S}] - W_i'[\bar{K}] + \sum_{j=0}^{i}[V_j,W_{i-j}] + gV_{i\lambda} - fW_{i\lambda}, \quad i = 1,2,\cdots,N. \end{aligned} \tag{36}$$

Here, we set $V_0 \equiv V$ and $W_0 \equiv W$ for the convenience of writing. This shows a special structure of the commutator of enlarged Lax operators and plays a crucial role in our computation.

Theorem 3. *If $(\bar{V},\bar{K},f),(\bar{W},\bar{S},g) \in P(\bar{U})$, then $([\![\bar{V},\bar{W}]\!],[\bar{K},\bar{S}],[\![f,g]\!])$ also belongs to $P(\bar{U})$. That is to say,*

$$\bar{U}'[[\bar{K},\bar{S}]] + [\![f,g]\!]\bar{U}_\lambda = (E[\![\bar{V},\bar{W}]\!])\bar{U} - \bar{U}[\![\bar{V},\bar{W}]\!] \tag{37}$$

which is equivalent to the following $N+1$ equations:

$$U'[[K,S]] + [\![f,g]\!]U_\lambda = (E[\![V,W]\!])U - U[\![V,W]\!], \tag{38}$$

$$U_i'[[K,S]_i] + [\![f,g]\!]U_{i\lambda} = \sum_{j=0}^{i}\Big((E[\![V_j,W_j]\!])U_{i-j} - U_j[\![V_{i-j},W_{i-j}]\!]\Big), \qquad i = 1,2,\cdots,N. \tag{39}$$

Proof. The proof of (38) can be found in [10]. We consider (39) directly. From Equation (15), we have

$$\begin{aligned}
&\sum_{j=0}^{i}\Big((E[\![V_j,W_j]\!])U_{i-j} - U_j[\![V_{i-j},W_{i-j}]\!]\Big)\\
&= \sum_{j=0}^{i}\Bigg\{(EV_j'[\bar{S}])U_{i-j} - (EW_j'[\bar{K}])U_{i-j} + \sum_{k=0}^{j}(E[V_k,W_{j-k}])U_{i-j} + g(EV_{j\lambda})U_{i-j} - f(EW_{j\lambda})U_{i-j}\\
&\quad -U_j\Bigg(V_{i-j}'[\bar{S}] - W_{i-j}'[\bar{K}] + \sum_{k=0}^{i-j}[V_k,W_{i-j-k}] + gV_{i-j\lambda} - fW_{i-j\lambda}\Bigg)\Bigg\}.
\end{aligned}$$

On the other hand, by means of

$$\begin{aligned}
U_i'[K_i] + fU_{i\lambda} &= \sum_{j=0}^{i}\Big\{(EV_j)U_{i-j} - U_jV_{i-j}\Big\},\\
U_i'[S_i] + gU_{i\lambda} &= \sum_{j=0}^{i}\Big\{(EW_j)U_{i-j} - U_jW_{i-j}\Big\},
\end{aligned}$$

we immediately have

$$\begin{aligned}
&U_i'[[K,S]_i] + [\![f,g]\!]U_{i\lambda}\\
&= U_i'\Bigg(\sum_{j=0}^{i}K_i'[S_j] - \sum_{j=0}^{i}S_i'[K_j]\Bigg) + [\![f,g]\!]U_{i\lambda}\\
&= \big(U_i'[K_i]\big)'[\bar{S}] - \big(U_i'[S_i]\big)'[\bar{K}] + [\![f,g]\!]U_{i\lambda}\\
&= \Bigg(\sum_{j=0}^{i}\Big((EV_j)U_{i-j} - U_jV_{i-j}\Big) - fU_{i\lambda}\Bigg)'[\bar{S}] - \Bigg(\sum_{j=0}^{i}\Big((EW_j)U_{i-j} - U_jW_{i-j}\Big) - gU_{i\lambda}\Bigg)'[\bar{K}]\\
&\quad +[\![f,g]\!]U_{i\lambda}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\Omega \triangleq & \sum_{j=0}^{i}\Big(E[\![V_j,W_j]\!])U_{i-j}-U_j[\![V_{i-j},W_{i-j}]\!]\Big)-U_i'[[K,S]_i]-[\![f,g]\!]U_{i\lambda} \\
=& \sum_{j=0}^{i}\Bigg\{(EV_j'[\bar{S}])U_{i-j}-(EW_j'[\bar{K}])U_{i-j}+\sum_{k=0}^{j}(E[V_k,W_{j-k}])U_{i-j}+g(EV_{j\lambda})U_{i-j}-f(EW_{j\lambda})U_{i-j} \\
& -U_j\Bigg(V_{i-j}'[\bar{S}]-W_{i-j}'[\bar{K}]+\sum_{k=0}^{i-j}[V_k,W_{i-j-k}]+gV_{i-j\lambda}-fW_{i-j\lambda}\Bigg)\Bigg\} \\
& -\sum_{j=0}^{i}(EV_j'[\bar{S}])U_{i-j}-\sum_{j=0}^{i}(EV_j)U_{i-j}'[\bar{S}]+\sum_{j=0}^{i}U_j'[\bar{S}]V_{i-j}+\sum_{j=0}^{i}U_jV_{i-j}'[\bar{S}]+fU_{i\lambda}'[\bar{S}] \\
& +\sum_{j=0}^{i}(EW_j'[\bar{K}])U_{i-j}+\sum_{j=0}^{i}(EW_j)U_{i-j}'[\bar{K}]-\sum_{j=0}^{i}U_j'[\bar{K}]W_{i-j}-\sum_{j=0}^{i}U_jW_{i-j}'[\bar{K}]-gU_{i\lambda}'[\bar{K}] \\
& -[\![f,g]\!]U_{i\lambda} \\
=& \sum_{j=0}^{i}\sum_{k=0}^{j}(E[V_k,W_{j-k}])U_{i-j}+\sum_{j=0}^{i}g(EV_{j\lambda})U_{i-j}-\sum_{j=0}^{i}f(EW_{j\lambda})U_{i-j}-\sum_{j=0}^{i}\sum_{k=0}^{i-j}U_j[V_k,W_{i-j-k}] \\
& -\sum_{j=0}^{i}gU_jV_{i-j\lambda}+\sum_{j=0}^{i}fU_jW_{i-j\lambda}-\sum_{j=0}^{i}(EV_j)U_{i-j}'[S_{i-j}]+\sum_{j=0}^{i}U_j'[S_j]V_{i-j}+fU_{i\lambda}'[S_i] \\
& +\sum_{j=0}^{i}(EW_j)U_{i-j}'[K_{i-j}]-\sum_{j=0}^{i}U_j'[K_j]W_{i-j}-gU_{i\lambda}'[K_i]-[\![f,g]\!]U_{i\lambda}. \qquad (40)
\end{aligned}$$

For further calculation, by using

$$U_{i\lambda}'[K_i]+f_\lambda U_{i\lambda}+fU_{i\lambda\lambda}=\sum_{j=0}^{i}\Big\{(EV_j)_\lambda U_{i-j}+(EV_j)U_{i-j\lambda}-U_{j\lambda}V_{i-j}-U_jV_{i-j\lambda}\Big\},$$

$$U_{i\lambda}'[S_i]+g_\lambda U_{i\lambda}+gU_{i\lambda\lambda}=\sum_{j=0}^{i}\Big\{(EW_j)_\lambda U_{i-j}+(EW_j)U_{i-j\lambda}-U_{j\lambda}W_{i-j}-U_jW_{i-j\lambda}\Big\},$$

we can obtain

$$\begin{aligned}
\Omega =& \sum_{j=0}^{i}\sum_{k=0}^{j}(E[V_k, W_{j-k}])U_{i-j} + \sum_{j=0}^{i} g(EV_{j\lambda})U_{i-j} - \sum_{j=0}^{i} f(EW_{j\lambda})U_{i-j} - \sum_{j=0}^{i}\sum_{k=0}^{i-j} U_j[V_k, W_{i-j-k}] \\
& - \sum_{j=0}^{i} gU_jV_{i-j\lambda} + \sum_{j=0}^{i} fU_jW_{i-j\lambda} - \sum_{j=0}^{i}(EV_j)U'_{i-j}[S_{i-j}] + \sum_{j=0}^{i} U'_j[S_j]V_{i-j} \\
& + f\left\{-g_\lambda U_{i\lambda} - gU_{i\lambda\lambda} + \sum_{j=0}^{i}\Big((EW_j)_\lambda U_{i-j} + (EW_j)U_{i-j\lambda} - U_{j\lambda}W_{i-j} - U_jW_{i-j\lambda}\Big)\right\} \\
& + \sum_{j=0}^{i}(EW_j)U'_{i-j}[K_{i-j}] - \sum_{j=0}^{i} U'_j[K_j]W_{i-j} \\
& - g\left\{-f_\lambda U_{i\lambda} - fU_{i\lambda\lambda} + \sum_{j=0}^{i}\Big((EV_j)_\lambda U_{i-j} + (EV_j)U_{i-j\lambda} - U_{j\lambda}V_{i-j} - U_jV_{i-j\lambda}\Big)\right\} \\
& - [\![f, g]\!]U_{i\lambda} \\
=& \sum_{j=0}^{i}\sum_{k=0}^{j}\Big\{(EV_k)(EW_{j-k}) - (EW_{j-k})(EV_k)\Big\}U_{i-j} - \sum_{j=0}^{i}\sum_{k=0}^{i-j}\Big(U_jV_kW_{i-j-k} - U_jW_{i-j-k}V_k\Big) \\
& - \sum_{j=0}^{i}(EV_j)U'_{i-j}[S_{i-j}] + \sum_{j=0}^{i} U'_j[S_j]V_{i-j} + f\sum_{j=0}^{i}(EW_j)U_{i-j\lambda} - f\sum_{j=0}^{i} U_{j\lambda}W_{i-j} \\
& + \sum_{j=0}^{i}(EW_j)U'_{i-j}[K_{i-j}] - \sum_{j=0}^{i} U'_j[K_j]W_{i-j} - g\sum_{j=0}^{i}(EV_j)U_{i-j\lambda} + g\sum_{j=0}^{i} U_{j\lambda}V_{i-j} \\
=& \sum_{j=0}^{i}(EV_j)\left\{-U'_{i-j}[S_{i-j}] - gU_{i-j\lambda} + \sum_{k=0}^{i-j}\Big((EW_k)U_{i-j-k} - U_kW_{i-j-k}\Big)\right\} \\
& + \sum_{j=0}^{i}(EW_j)\left\{U'_{i-j}[K_{i-j}] + fU_{i-j\lambda} - \sum_{k=0}^{i-j}\Big((EV_k)U_{i-j-k} - U_kV_{i-j-k}\Big)\right\} \\
& + \sum_{j=0}^{i}\left\{U'_j[S_j] + gU_{j\lambda} - \sum_{k=0}^{j}\Big((EW_k)U_{j-k} - U_kW_{j-k}\Big)\right\}V_{i-j} \\
& + \sum_{j=0}^{i}\left\{-U'_j[K_j] - fU_{j\lambda} + \sum_{k=0}^{j}\Big((EV_k)U_{j-k} - U_kV_{j-k}\Big)\right\}W_{i-j} \\
=& 0.
\end{aligned}$$

□

Corollary 2. *Let* $(\bar{V}, \bar{K}, f), (\bar{W}, \bar{S}, g), (\bar{Q}, \bar{L}, h) \in P(\bar{U})$. *If* $[\![\bar{V}, \bar{W}]\!] = \bar{Q}$ *and* $[\![f, g]\!] = h$, *then* $[\bar{K}, \bar{S}] = \bar{L}$.

Proof. From the above theorem and the assumption, we have

$$\begin{aligned}
U'[[K, S]] &= -[\![f, g]\!]U_\lambda + (E[\![V, W]\!])U - U[\![V, W]\!] \\
&= -hU_\lambda + (EQ)U - UQ \\
&= U'[L],
\end{aligned}$$

$$\begin{aligned} U_i'[[K,S]_i] &= -[\![f,g]\!]U_{i\lambda} + \sum_{j=0}^{i}\Big((E[\![V_j,W_j]\!])U_{i-j} - U_j[\![V_{i-j},W_{i-j}]\!]\Big) \\ &= -hU_{i\lambda} + \sum_{j=0}^{i}\Big((EQ_j)U_{i-j} - U_jQ_{i-j}\Big) \\ &= U_i'[L_i]. \end{aligned}$$

Since $\bar{U}'$ is injective, it follows that $[K,S]=L$ and $[K,S]_i = L_i$, i.e., $[\bar{K},\bar{S}] = \bar{L}$. □

It follows from the above theorem that if two enlarged evolution equations $\bar{u}_t = \bar{K}, \bar{u}_t = \bar{S}$, $\Big(\bar{K},\bar{S} \in \mathcal{B}^{q+\sum_{i=1}^{N} q_i}\Big)$ are the compatibility conditions of the spectral problems

$$\begin{aligned} &E\bar{\varphi} = \bar{U}\bar{\varphi}, \qquad \bar{\varphi}_t = \bar{V}\bar{\varphi}, \qquad \lambda_t = a\lambda^m, \\ &E\bar{\varphi} = \bar{U}\bar{\varphi}, \qquad \bar{\varphi}_t = \bar{W}\bar{\varphi}, \qquad \lambda_t = b\lambda^n, \end{aligned} \tag{41}$$

where a, b are constants and $m, n \geq 0$, respectively, then the product equation $\bar{u}_t = [\bar{K},\bar{S}]$ is the compatibility condition of the following spectral problem:

$$E\bar{\varphi} = \bar{U}\bar{\varphi}, \qquad \bar{\varphi}_t = [\![\bar{V},\bar{W}]\!]\bar{\varphi}, \qquad \lambda_t = ab(m-n)\lambda^{m+n-1}, \tag{42}$$

where $[\![\bar{V},\bar{W}]\!]$ is defined by (35).

3.2. The Continuous Case

For the continuous case, we denote by $\mathcal{B}$ all complex or real functions $P = P(x,t,u,v_1, v_2,\cdots,v_N)$ which are C^∞-differentiable with respect to $\{x,t\}$ and C^∞-Gateaux differentiable with respect to $\{u,v_1,v_2,\cdots,v_N\}$, and we set $\mathcal{B}^r = \big\{(P_1,P_2,\cdots,P_r)^{\mathrm{T}} \big| P_i \in \mathcal{B}\big\}$. Moreover, we denote all $r \times r$ matrix integro-differential operators as

$$\mathcal{V}^r = \bigcup_{n=-\infty}^{\infty}\sum_{k=0}^{n}\mathcal{V}^r_{(k)}, \qquad \mathcal{V}^r_{(k)} = \Big\{\Big(P_{ij}\partial^k\Big)_{r\times r}\big|P_{ij} = P_{ij}(x,t,u,v_1,v_2,\cdots,v_N)\in\mathcal{B}\Big\}, \tag{43}$$

$$\widetilde{\mathcal{V}}^r = \mathcal{V}^r \otimes C[\lambda,\lambda^{-1}], \qquad \widetilde{\mathcal{V}}^r_{(0)} = \mathcal{V}^r_{(0)} \otimes C[\lambda,\lambda^{-1}]. \tag{44}$$

Assuming that

$$P(\bar{U}) = \{(\bar{V},\bar{K},f) \text{ satisfies } (20), \mid \bar{V} \in \widetilde{\mathcal{V}}^{(N+1)r}_{(0)},\ \bar{K} \in \mathcal{B}^{q+\sum_{i=1}^{N} q_i},\ f \in C^\infty(\mathbb{C})\}, \tag{45}$$

we have the following theorem, which is similar to Theorem 3.

Theorem 4. *If $(\bar{V},\bar{K},f),(\bar{W},\bar{S},g) \in P(\bar{U})$, then $([\![\bar{V},\bar{W}]\!],[\bar{K},\bar{S},[\![f,g]\!])$ also belongs to $P(\bar{U})$. That is to say,*

$$\bar{U}'[[\bar{K},\bar{S}]] + [\![f,g]\!]\bar{U}_\lambda - [\![\bar{V},\bar{W}]\!]_x + [U,[\![\bar{V},\bar{W}]\!]] = 0, \tag{46}$$

which is equivalent to the following $N+1$ equations:

$$U'[[K,S]] + [\![f,g]\!]U_\lambda - [\![V,W]\!]_x + [U,[\![V,W]\!]] = 0, \tag{47}$$

$$U_i'[[K,S]_i] + [\![f,g]\!]U_{i\lambda} - [\![V_i,W_i]\!]_x + \sum_{j=0}^{i}[U_j,[\![V_{i-j},W_{i-j}]\!]] = 0, \qquad i = 1,2,\cdots,N. \tag{48}$$

Proof. From $[\![V_i,W_i]\!] = V_i'[\bar{S}] - W_i'[\bar{K}] + \sum_{j=0}^{i}[V_j,W_{i-j}] + gV_{i\lambda} - fW_{i\lambda}$, we have

$$\llbracket V_i, W_i \rrbracket_x = V'_{ix}[\bar{S}] - W'_{ix}[\bar{K}] + \sum_{j=0}^{i}[V_{jx}, W_{i-j}] + \sum_{j=0}^{i}[V_j, W_{i-jx}] + gV_{ix\lambda} - fW_{ix\lambda},$$
$$\sum_{j=0}^{i}[U_j, \llbracket V_{i-j}, W_{i-j} \rrbracket] = \sum_{j=0}^{i}\left[U_j, V'_{i-j}[\bar{S}] - W'_{i-j}[\bar{K}] + \sum_{k=0}^{i-j}[V_k, W_{i-j-k}] + gV_{i-j\lambda} - fW_{i-j\lambda}\right].$$

On the other hand, by means of

$$U'_i[K_i] + fU_{i\lambda} - V_{ix} + \sum_{j=0}^{i}[U_j, V_{i-j}] = 0,$$
$$U'_i[S_i] + gU_{i\lambda} - W_{ix} + \sum_{j=0}^{i}[U_j, W_{i-j}] = 0,$$

we can immediately obtain

$$\begin{aligned} U'_i[[K,S]_i] + \llbracket f,g \rrbracket U_{i\lambda} &= U'_i\left[K'_i[S] + \sum_{j=1}^{i} K'_i[S_j] - S'_i[K] - \sum_{j=1}^{i} S'_i[K_j]\right] + \llbracket f,g \rrbracket U_{i\lambda} \\ &= \big(U'_i[K_i]\big)'[\bar{S}] - \big(U'_i[S_i]\big)'[\bar{K}] + \llbracket f,g \rrbracket U_{i\lambda} \\ &= \left(V_{ix} - \sum_{j=0}^{i}[U_j, V_{i-j}] - fU_{i\lambda}\right)'[\bar{S}] - \left(W_{ix} - \sum_{j=0}^{i}[U_j, W_{i-j}] - gU_{i\lambda}\right)'[\bar{K}] \\ &\quad + \llbracket f,g \rrbracket U_{i\lambda}. \end{aligned}$$

Thus, we can obtain

$$
\begin{aligned}
&[\![V_i, W_i]\!]_x - \sum_{j=0}^{i}[U_j, [\![V_{i-j}, W_{i-j}]\!]] - U_i'[[K,S]_i] - [\![f,g]\!]U_{i\lambda} \\
&= \sum_{j=0}^{i}[V_{jx}, W_{i-j}] + \sum_{j=0}^{i}[V_j, W_{i-jx}] + gV_{ix\lambda} - fW_{ix\lambda} - \sum_{j=0}^{i}U_j\sum_{k=0}^{i-j}[V_k, W_{i-j-k}] - \sum_{j=0}^{i}gU_jV_{i-j\lambda} \\
&\quad + \sum_{j=0}^{i}fU_jW_{i-j\lambda} + \sum_{j=0}^{i}\sum_{k=0}^{i-j}[V_k, W_{i-j-k}]U_j + \sum_{j=0}^{i}gV_{i-j\lambda}U_j - \sum_{j=0}^{i}fW_{i-j\lambda}U_j + \sum_{j=0}^{i}U_j'[S_j]V_{i-j} \\
&\quad - \sum_{j=0}^{i}V_{i-j}U_j'[S_j] + fU_{i\lambda}'[S_i] - \sum_{j=0}^{i}U_j'[K_j]W_{i-j} + \sum_{j=0}^{i}W_{i-j}U_j'[K_j] - gU_{i\lambda}'[K_i] - [\![f,g]\!]U_{i\lambda} \\
&= \sum_{j=0}^{i}[V_{jx}, W_{i-j}] + \sum_{j=0}^{i}[V_j, W_{i-jx}] + gV_{ix\lambda} - fW_{ix\lambda} - \sum_{j=0}^{i}U_j\sum_{k=0}^{i-j}[V_k, W_{i-j-k}] - \sum_{j=0}^{i}gU_jV_{i-j\lambda} \\
&\quad + \sum_{j=0}^{i}fU_jW_{i-j\lambda} + \sum_{j=0}^{i}\sum_{k=0}^{i-j}[V_k, W_{i-j-k}]U_j + \sum_{j=0}^{i}gV_{i-j\lambda}U_j - \sum_{j=0}^{i}fW_{i-j\lambda}U_j + \sum_{j=0}^{i}U_j'[S_j]V_{i-j} \\
&\quad - \sum_{j=0}^{i}V_{i-j}U_j'[S_j] + f\left\{W_{ix\lambda} - \sum_{j=0}^{i}[U_{j\lambda}, W_{i-j}] - \sum_{j=0}^{i}[U_j, W_{i-j\lambda}] - gU_{i\lambda\lambda} - g_\lambda U_{i\lambda}\right\} - \sum_{j=0}^{i}U_j'[K_j]W_{i-j} \\
&\quad + \sum_{j=0}^{i}W_{i-j}U_j'[K_j] - g\left\{V_{ix\lambda} - \sum_{j=0}^{i}[U_{j\lambda}, V_{i-j}] - \sum_{j=0}^{i}[U_j, V_{i-j\lambda}] - fU_{i\lambda\lambda} - f_\lambda U_{i\lambda}\right\} - [\![f,g]\!]U_{i\lambda} \\
&= \sum_{j=0}^{i}\left\{U_j'[S_j] + gU_{j\lambda} - W_{jx} + \sum_{k=0}^{j}[U_k, W_{j-k}]\right\}V_{i-j} \\
&\quad - \sum_{j=0}^{i}V_{i-j}\left\{U_j'[S_j] + gU_{j\lambda} - W_{jx} + \sum_{k=0}^{j}[U_k, W_{j-k}]\right\} \\
&\quad - \sum_{j=0}^{i}\left\{U_j'[K_j] + fU_{j\lambda} - V_{jx} + \sum_{k=0}^{j}[U_k, V_{j-k}]\right\}W_{i-j} \\
&\quad + \sum_{j=0}^{i}W_{i-j}\left\{U_j'[K_j] + fU_{j\lambda} - V_{jx} + \sum_{k=0}^{j}[U_k, V_{j-k}]\right\} \\
&= 0.
\end{aligned}
$$

□

It follows that if two enlarged equations $\bar{u}_t = \bar{K}, \quad \bar{u}_t = \bar{S}, \quad \left(\bar{K}, \bar{S} \in \mathcal{B}^{q+\sum_{i=1}^{N} q_i}\right)$ are the compatibility conditions of the continuous spectral problems

$$
\begin{aligned}
&\bar{\varphi}_x = \bar{U}\bar{\varphi}, \qquad \bar{\varphi}_t = \bar{V}\bar{\varphi}, \qquad \lambda_t = a\lambda^m, \\
&\bar{\varphi}_x = \bar{U}\bar{\varphi}, \qquad \bar{\varphi}_t = \bar{W}\bar{\varphi}, \qquad \lambda_t = b\lambda^n,
\end{aligned} \tag{49}
$$

where a, b are constants and $m, n \geq 0$, respectively, then the product equation $\bar{u}_t = [\bar{K}, \bar{S}]$ is the compatibility condition of the spectral problem

$$
\bar{\varphi}_x = \bar{U}\bar{\varphi}, \qquad \bar{\varphi}_t = [\![\bar{V}, \bar{W}]\!]\bar{\varphi}, \qquad \lambda_t = ab(m-n)\lambda^{m+n-1}, \tag{50}
$$

where $[\![\bar{V}, \bar{W}]\!]$ is defined by (35).

We point out that all triples of $(\bar{V}, \bar{K}, f)$ satisfying either (14) or (19) constitute a Lie algebra with the algebraic structures established above.

4. Application to the Volterra Lattice Hierarchy and the AKNS Hierarchy

4.1. The Case of the Volterra Lattice Hierarchy

Next, we establish the τ-symmetry algebra by using the above construction process for the 3-coupled Volterra lattice integrable coupling. The enlarged spectral problem of this integrable coupling reads

$$E\bar{\varphi} = \bar{U}\bar{\varphi},$$
$$\bar{U} \equiv \begin{pmatrix} U & U_1 & U_2 \\ 0 & U & U_1 \\ 0 & 0 & U \end{pmatrix} = \left(\begin{array}{cc|cc|cc} 1 & u & 0 & v_1 & 0 & v_2 \\ \lambda^{-1} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & u & 0 & v_1 \\ 0 & 0 & \lambda^{-1} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & u \\ 0 & 0 & 0 & 0 & \lambda^{-1} & 0 \end{array}\right), \tag{51}$$

where $u \equiv u(n,t)$, $v_1 \equiv v_1(n,t)$, and $v_2 \equiv v_2(n,t)$ are dependent variables. The associated enlarged temporal spectral problem is as follows:

$$\bar{\varphi}_t = \bar{V}\bar{\varphi},$$
$$\bar{V} \triangleq \begin{pmatrix} V & V_1 & V_2 \\ 0 & V & V_1 \\ 0 & 0 & V \end{pmatrix} = \left(\begin{array}{cc|cc|cc} a & b & e_1 & f_1 & e_2 & f_2 \\ c & d & g_1 & h_1 & g_2 & h_2 \\ \hline 0 & 0 & a & b & e_1 & f_1 \\ 0 & 0 & c & d & g_1 & h_1 \\ \hline 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & c & d \end{array}\right). \tag{52}$$

The compatibility condition $U_t - (EV)U + UV = 0$ in this case gives, equivalently,

$$V \triangleq \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & \lambda u c^{(1)} \\ c & -\lambda^{-1}\lambda_t - \lambda c + a^{(-1)} \end{pmatrix},$$
$$V_1 \triangleq \begin{pmatrix} e_1 & f_1 \\ g_1 & h_1 \end{pmatrix} = \begin{pmatrix} e_1 & \lambda v_1 c^{(1)} + \lambda u g_1^{(1)} \\ g_1 & e_1^{(-1)} - \lambda g_1 \end{pmatrix},$$
$$V_2 \triangleq \begin{pmatrix} e_2 & f_2 \\ g_2 & h_2 \end{pmatrix} = \begin{pmatrix} e_2 & \lambda v_2 c^{(1)} + \lambda u g_2^{(1)} \\ g_2 & e_2^{(-1)} - \lambda g_2 \end{pmatrix}. \tag{53}$$

In other words, we have a number of formulas as follows:

$$\begin{aligned}
&u_t = u\left(a^{(1)} - a^{(-1)}\right) - \lambda u\left(c^{(1)} - c\right) + u\lambda^{-1}\lambda_t, \\
&v_{1t} = v_1\left(a^{(1)} - a^{(-1)}\right) + u\left(e_1^{(1)} - e_1^{(-1)}\right) - \lambda v_1\left(c^{(1)} - c\right) - \lambda u\left(g_1^{(1)} - g_1\right) + v_1\lambda^{-1}\lambda_t, \\
&v_{2t} = v_2\left(a^{(1)} - a^{(-1)}\right) + v_1\left(e_1^{(1)} - e_1^{(-1)}\right) + u\left(e_2^{(1)} - e_2^{(-1)}\right) \\
&\qquad -\lambda v_2\left(c^{(1)} - c\right) - \lambda v_1\left(g_1^{(1)} - g_1\right) - \lambda u\left(g_2^{(1)} - g_2\right) + v_2\lambda^{-1}\lambda_t, \\
&a^{(1)} - a + u^{(1)}c^{(2)} - uc = 0, \\
&e_1^{(1)} - e_1 + v_1^{(1)}c^{(2)} + u^{(1)}g_1^{(2)} - ug_1 - v_i c = 0, \\
&e_2^{(1)} - e_2 + v_2^{(1)}c^{(2)} + v_1^{(1)}g_1^{(2)} + u^{(1)}g_2^{(2)} - ug_2 - v_1 g_1 - v_2 c = 0.
\end{aligned} \tag{54}$$

To consider these further, set

$$a = \sum_{j=0}^{m} a_j\lambda^{m-j}, \qquad c = \sum_{j=0}^{m} c_j\lambda^{m-j}, \qquad e_i = \sum_{j=0}^{m} e_{ij}\lambda^{m-j}, \qquad g_i = \sum_{j=0}^{m} g_{ij}\lambda^{m-j}, \qquad i = 1,2, \tag{55}$$

and thus Equation (54) becomes

$$
\begin{aligned}
u_t &= \sum_{j=0}^{m} u\left(a_j^{(1)} - a_j^{(-1)}\right)\lambda^{m-j} - \sum_{j=0}^{m} u\left(c_{j+1}^{(1)} - c_j\right)\lambda^{m-j+1} + u\lambda^{-1}\lambda_t, \\
v_{1t} &= \sum_{j=0}^{m}\left[v_1\left(a_j^{(1)} - a_j^{(-1)}\right) + u\left(e_{1j}^{(1)} - e_{1j}^{(-1)}\right)\right]\lambda^{m-j} - \sum_{j=0}^{m}\left[v_1\left(c_j^{(1)} - c_j\right) + u\left(g_{1j}^{(1)} - g_{1j}\right)\right]\lambda^{m-j+1} \\
&\quad + v_1\lambda^{-1}\lambda_t, \\
v_{2t} &= \sum_{j=0}^{m}\left[v_2\left(a_j^{(1)} - a_j^{(-1)}\right) + v_1\left(e_{1j}^{(1)} - e_{1j}^{(-1)}\right) + u\left(e_{2j}^{(1)} - e_{2j}^{(-1)}\right)\right]\lambda^{m-j} - \sum_{j=0}^{m}\left[v_2\left(c_j^{(1)} - c_j\right)\right. \\
&\quad \left. + v_1\left(g_{1j}^{(1)} - g_{1j}\right) + u\left(g_{2j}^{(1)} - g_{2j}\right)\right]\lambda^{m-j+1} + v_2\lambda^{-1}\lambda_t, \\
& a_j^{(1)} + u^{(1)}c_j^{(2)} - a_j - uc_j = 0, \\
& e_{1j}^{(1)} - e_{1j} + v_1^{(1)}c_j^{(2)} + u^{(1)}g_{1j}^{(2)} - ug_{1j} - v_1c_j = 0, \\
& e_{2j}^{(1)} - e_{2j} + v_2^{(1)}c_j^{(2)} + v_1^{(1)}g_{1j}^{(2)} - u^{(1)}g_{2j} - ug_{2j} - v_1g_{1j} - v_2c_j = 0.
\end{aligned} \tag{56}
$$

Hence, we can derive the isospectral and non-isospectral 3-coupled integrable couplings as follows.

Case 1: Isospectral 3*-coupled Volterra integrable coupling*

When $\lambda_t = 0$, we let $a_0 = -u, c_0 = 1, g_{10} = 0, g_{20} = 0, e_{10} = -v_1$, and $e_{20} = -v_2$ be constants, and from (56) we have

$$
\begin{aligned}
\bar{u}_t &= \begin{pmatrix} u \\ v_1 \\ v_2 \end{pmatrix}_t = \bar{K}_m = \begin{pmatrix} K_{0m} \\ K_{1m} \\ K_{2m} \end{pmatrix} \\
&= \begin{pmatrix} u\left(a_m^{(1)} - a_m^{(-1)}\right) \\ v_1\left(a_m^{(1)} - a_m^{(-1)}\right) + u\left(e_{1m}^{(1)} - e_{1m}^{(-1)}\right) \\ v_2\left(a_m^{(1)} - a_m^{(-1)}\right) + v_1\left(e_{1m}^{(1)} - e_{1m}^{(-1)}\right) + u\left(e_{2m}^{(1)} - e_{2m}^{(-1)}\right) \end{pmatrix} \\
&= \bar{\Phi}^m(u, v_1, v_2)\bar{K}_0,
\end{aligned} \tag{57}
$$

where

$$
\begin{aligned}
& \bar{\Phi}(u, v_1, v_2) \triangleq \begin{pmatrix} \Phi(u) & 0 & 0 \\ \Phi_1(u, v_1) & \Phi(u) & 0 \\ \Phi_2(u, v_1, v_2) & \Phi_1(u, v_1) & \Phi(u) \end{pmatrix}, \\
& \Phi = u\left(1 + E^{-1}\right)\left(u - u^{(-1)}E^2\right)(E-1)^{-1}u^{-1}, \\
& \Phi_1 = -u\left(1 + E^{-1}\right)\left(u - u^{(1)}E^2\right)(E-1)^{-1}u^{-1}v_1u^{-1} + u\left(1 + E^{-1}\right)\left(v_1 - v_1^{(1)}E^2\right)(E-1)^{-1}u^{-1} \\
& \qquad + v_1\left(1 + E^{-1}\right)\left(u - u^{(1)}E^2\right)(E-1)^{-1}u^{-1}, \\
& \Phi_2 = u\left(1 + E^{-1}\right)\left(v_2 - v_2^{(1)}E^2\right)(E-1)^{-1}u^{-1} - u\left(1 + E^{-1}\right)\left(v_1 - v_1^{(1)}E^2\right)(E-1)^{-1}u^{-2}v_1 \\
& \qquad + v_1\left(1 + E^{-1}\right)\left(v_1 - v_1^{(1)}E^2\right)(E-1)^{-1}u^{-1} + u\left(1 + E^{-1}\right)\left(u - u^{(1)}E^2\right)(E-1)^{-1}u^{-1}\left(v_1^2u^{-2} - v_2u^{-1}\right) \\
& \qquad - v_1\left(1 + E^{-1}\right)\left(u - u^{(1)}E^2\right)(E-1)^{-1}u^{-1}v_1u^{-1},
\end{aligned} \tag{58}
$$

$$
\bar{K}_0 = \begin{pmatrix} K_{00} \\ K_{10} \\ K_{20} \end{pmatrix} = \begin{pmatrix} u\left(u^{(-1)} - u^{(1)}\right) \\ v_1\left(u^{(-1)} - u^{(1)}\right) + u\left(v_1^{(-1)} - v_1^{(1)}\right) \\ v_2\left(u^{(-1)} - u^{(1)}\right) + v_1\left(v_1^{(-1)} - v_1^{(1)}\right) + u\left(v_2^{(-1)} - v_2^{(1)}\right) \end{pmatrix}. \tag{59}
$$

This enlarged isospectral 3-coupled Volterra lattice hierarchy has the recursion relation

$$\bar{u}_t = \bar{K}_m = \bar{\Phi}(u, v_1, v_2)\bar{K}_{m-1}, \qquad m \geq 1. \tag{60}$$

Case 2: Non-isospectral 3*-coupled Volterra integrable coupling*

When $\lambda_t = \lambda^{m+2}$, we let $a_0 = -u, c_0 = n, g_{10} = g_{20} = 0$, and

$$e_{10}^{(1)} - e_{10}^{(-1)} = -(n+2)v_1^{(1)} - v_1 + (n-1)v_1^{(-1)},$$
$$e_{20}^{(1)} - e_{20}^{(-1)} = -(n+2)v_2^{(1)} - v_2 + (n-1)v_2^{(-1)}$$

from Equation (56). Thus, we have

$$\begin{aligned} \bar{u}_t = \bar{\rho}_m &= \begin{pmatrix} \rho_{0m} \\ \rho_{1m} \\ \rho_{2m} \end{pmatrix} \\ &= \begin{pmatrix} u\left(a_m^{(1)} - a_m^{(-1)}\right) \\ v_1\left(a_m^{(1)} - a_m^{(-1)}\right) + u\left(e_{1m}^{(1)} - e_{1m}^{(-1)}\right) \\ v_2\left(a_m^{(1)} - a_m^{(-1)}\right) + v_1\left(e_{1m}^{(1)} - e_{1m}^{(-1)}\right) + u\left(e_{2m}^{(1)} - e_{2m}^{(-1)}\right) \end{pmatrix} \\ &= \bar{\Phi}^m(u, v_1, v_2)\bar{\rho}_0, \end{aligned} \tag{61}$$

where

$$\begin{aligned} \bar{\rho}_0 &= \begin{pmatrix} \rho_{00} \\ \rho_{10} \\ \rho_{20} \end{pmatrix}, \\ \rho_{00} &= u\left(a_0^{(1)} - a_0^{(-1)}\right) \\ &= u\left[(n-1)u^{(-1)} - u - (n+2)u^{(1)}\right], \\ \rho_{10} &= v_1\left(a_0^{(1)} - a_0^{(-1)}\right) + u\left(e_{10}^{(1)} - e_{10}^{(-1)}\right) \\ &= (n-1)\left(u^{(-1)}v_1 + uv_1^{(-1)}\right) - (n+2)\left(v_1u^{(1)} + uv_1^{(1)}\right) - 2uv_1 \\ \rho_{20} &= v_2\left(a_0^{(1)} - a_0^{(-1)}\right) + v_1\left(e_{10}^{(1)} - e_{10}^{(-1)}\right) + u\left(e_{20}^{(1)} - e_{20}^{(-1)}\right) \\ &= (n-1)\left(u^{(-1)}v_2 + v_1v_1^{(-1)} + uv_2^{(-1)}\right) - (n+2)\left(v_2u^{(1)} + v_1v_1^{(1)} + uv_2^{(1)}\right) - 2uv_2 - v_1^2 \end{aligned} \tag{62}$$

and $\bar{\Phi}(u, v_1, v_2)$ is defined by (58). Therefore, this enlarged non-isospectral Volterra lattice hierarchy has a recursion relation

$$\bar{u}_t = \bar{\rho}_m = \bar{\Phi}(u, v_1, \cdots, v_N)\bar{\rho}_{m-1}, \qquad m \geq 1. \tag{63}$$

Next, let us consider how to compute the corresponding τ-symmetry algebra for the obtained N-coupled integrable couplings. The procedure below is an application of the idea in [10] and can be applied to other cases. We first perform the following computation at $u = 0$:

$$\bar{K}_m\big|_{\bar{u}=0} = \begin{pmatrix} K_{0m} \\ K_{1m} \\ K_{2m} \end{pmatrix}\Bigg|_{\bar{u}=0} = \bar{\Phi}^m(u, v_1, v_2)\bar{K}_0\big|_{\bar{u}=0} = 0, \tag{64}$$

$$\bar{\rho}_n\big|_{\bar{u}=0} = \begin{pmatrix} \rho_{0n} \\ \rho_{1n} \\ \rho_{2n} \end{pmatrix}\Bigg|_{\bar{u}=0} = \bar{\Phi}^n(u, v_1, v_2)\bar{\rho}_0\big|_{\bar{u}=0} = 0, \tag{65}$$

where $m, n \geq 0$. We denote by $\bar{V}_m$ and $\bar{W}_n$ the Lax operators corresponding to the vector fields $\bar{K}_m$ and $\bar{\rho}_n$, respectively. Then, we can find by Definition (36) of the product of Lax operators that

$$\begin{aligned}
&[\![\bar{V}_m, \bar{V}_n]\!]\big|_{\bar{u}=0} = 0, \\
&[\![\bar{V}_m, \bar{W}_n]\!]\big|_{\bar{u}=0} = \left[\bar{V}_m\big|_{\bar{u}=0}, \bar{W}_n\big|_{\bar{u}=0}\right] + \lambda^{n+2}\bar{V}_{m\lambda}\big|_{\bar{u}=0} \\
&\qquad = (m+1)\bar{V}_{m+n+1}\big|_{\bar{u}=0}, \\
&[\![\bar{W}_m, \bar{W}_n]\!]\big|_{\bar{u}=0} = \left[\bar{W}_m\big|_{\bar{u}=0}, \bar{W}_n\big|_{\bar{u}=0}\right] + \lambda^{n+2}\bar{W}_{m\lambda}\big|_{\bar{u}=0} - \lambda^{m+2}\bar{W}_{n\lambda}\big|_{\bar{u}=0} \\
&\qquad = (m-n)\bar{W}_{m+n+1}\big|_{\bar{u}=0}, \qquad m, n \geq 0.
\end{aligned} \tag{66}$$

Since $[\![\bar{V}_m, \bar{V}_n]\!]$, $[\![\bar{V}_m, \bar{W}_n]\!] - (m+1)\bar{V}_{m+n+1}$, and $[\![\bar{W}_m, \bar{W}_n]\!] - (m-n)\bar{W}_{m+n+1}$ are all isospectral Lax operators belonging to $\widetilde{\mathcal{V}}_{(0)}^{2(N+1)}$, we obtain a Lax operator algebra by the uniqueness property of the enlarged spectral problem:

$$\begin{aligned}
&[\![\bar{V}_m, \bar{V}_n]\!] = 0, \\
&[\![\bar{V}_m, \bar{W}_n]\!] = (m+1)\bar{V}_{m+n+1}, \\
&[\![\bar{W}_m, \bar{W}_n]\!] = (m-n)\bar{W}_{m+n+1}, \qquad m, n \geq 0.
\end{aligned} \tag{67}$$

Further, due to the injective property of $\bar{U}'$, we finally obtain a vector field algebra of the enlarged isospectral and non-isospectral Volterra lattice hierarchies:

$$\begin{aligned}
&[\bar{K}_m, \bar{K}_n] = 0, \\
&[\bar{K}_m, \bar{\rho}_n] = (m+1)\bar{K}_{m+n+1}, \\
&[\bar{\rho}_m, \bar{\rho}_n] = (m-n)\bar{\rho}_{m+n+1}, \qquad m, n \geq 0.
\end{aligned} \tag{68}$$

This implies that $\bar{\rho}_n, n \geq 0$ are all common master symmetries of degree 1 for the whole isospectral hierarchy, and for each lattice equation $\bar{u}_t = \bar{K}_l, l \geq 0$, there are two hierarchies of symmetries,

$$\begin{aligned}
&\bar{K}_m, \quad m \geq 0, \\
&\tau_n^l = t[\bar{K}_l, \bar{\rho}_n] + \bar{\rho}_n = t(l+1)\bar{K}_{n+l+1} + \bar{\rho}_n, \quad n, l \geq 0,
\end{aligned} \tag{69}$$

which constitute an infinite-dimensional τ-symmetry algebra, the commutator of which is

$$\begin{aligned}
&[\bar{K}_m, \bar{K}_n] = 0, \\
&[\bar{K}_m, \tau_n^l] = (m+1)\bar{K}_{m+n+1}, \\
&[\tau_m^l, \tau_n^l] = (m-n)\tau_{m+n+1}^l, \qquad m, n, l \geq 0.
\end{aligned} \tag{70}$$

4.2. The Case of the AKNS Hierarchy

In the following, we present the enlarged spectral problem of the AKNS integrable coupling by semi-direct sums of Lie algebras, as follows:

$$\begin{aligned}
&\bar{\varphi}_x = \bar{U}\bar{\varphi} \\
&\bar{U} = \begin{pmatrix} U & U_1 & \cdots & U_{N-1} & U_N \\ 0 & U & \cdots & U_{N-2} & U_{N-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & U & U_1 \\ 0 & 0 & \cdots & 0 & U \end{pmatrix}, \\
&U = \begin{pmatrix} -\lambda & u \\ v & \lambda \end{pmatrix}, \quad U_i = \begin{pmatrix} -i & v_i \\ w_i & i \end{pmatrix}, \quad i = 1, 2, \cdots, N,
\end{aligned} \tag{71}$$

where u, v and v_i, w_i, $i = 1, 2, \cdots, N$, are dependent variables. The associated enlarged temporal spectral problem is assumed to be

$$\bar{\varphi}_t = \bar{V}\bar{\varphi},$$
$$\bar{V} = \begin{pmatrix} V & V_1 & \cdots & V_{N-1} & V_N \\ 0 & V & \cdots & V_{N-2} & V_{N-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & V & V_1 \\ 0 & 0 & \cdots & 0 & V \end{pmatrix},$$
$$V = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad V_i = \begin{pmatrix} e_i & f_i \\ g_i & -e_i \end{pmatrix}, \quad i = 1, 2, \cdots, N. \tag{72}$$

Then, the corresponding enlarged zero curvature equation $\bar{U}_t - \bar{U}_x + [\bar{U}, \bar{V}] = 0$ generates the following isospectral and non-isospectral ($\lambda_t = \lambda^n$, $n \geq 0$) AKNS integrable couplings:

$$\bar{u}_t = \bar{K}_m, \quad m \geq 0; \qquad \bar{u}_t = \bar{\rho}_n, \quad n \geq 0. \tag{73}$$

Thus, the corresponding vector field algebra

$$\begin{aligned} &[\bar{K}_m, \bar{K}_n] = 0, \\ &[\bar{K}_m, \bar{\rho}_n] = m\bar{K}_{m+n}, \\ &[\bar{\rho}_m, \bar{\rho}_n] = (m-n)\bar{\rho}_{m+n}, \qquad m, n \geq 0 \end{aligned} \tag{74}$$

can also be calculated, which automatically gives rise to the τ-symmetry algebra. Because the process of calculation is completely similar, we recommend that the interested reader refers to the discrete case above to see all hierarchies of τ-symmetries. In addition, similar τ-symmetry algebras could be generated through the above scheme for the vector AKNS soliton hierarchy [32,33].

5. Conclusions and Discussion

In this paper, we applied a type of non-semisimple matrix loop algebra to construct the algebraic structures of zero curvature representations for multilayer integrable couplings, both continuous and discrete. We furnished the commutator relations of Lax operators for isospectral ($\lambda_t = 0$) and non-isospectral ($\lambda_t = \lambda^m$, $m \geq 0$) hierarchies. Finally, τ-symmetry algebras for the multilayer Volterra lattice and AKNS integrable couplings were presented. Our theories supplement the existing theories and the corresponding results, particularly those from [19,23,25].

Inspired by the research related to the Frobenius algebra, the authors in [39] constructed the following non-semisimple Lie algebra, which consists of an $N \times N$ square matrix of the form

$$\bar{M}(A_1, A_2, \cdots, A_N) = \begin{pmatrix} A_1 & \epsilon A_N & \epsilon A_{N-1} & \cdots & \epsilon A_2 \\ A_2 & A_1 & \epsilon A_N & \cdots & \epsilon A_3 \\ A_3 & A_2 & A_1 & \cdots & \epsilon A_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{N-1} & A_{N-2} & A_{N-3} & \cdots & \epsilon A_N \\ A_N & A_{N-1} & A_{N-2} & \cdots & A_1 \end{pmatrix}, \tag{75}$$

and obtained a Z_N^ϵ isospectral mKdV integrable coupling and a Z_N^ϵ non-isospectral ANKS integrable coupling. We believe that the method presented above can be also applied to the above non-semisimple matrix loop algebra (75). Considering that the method is similar, we omit the construction process for convenience.

For integrable couplings, Hamiltonian structures could be presented by applying the variational identities associated with non-semisimple Lie algebras. Moreover, symmetries of non-isospectral hierarchies could be constructed in terms of series of vector fields [36]. An interesting question for us is how to construct new non-semisimple Lie algebras and how to classify these types of Lie algebras. Addressing this question will lead to a classification theory of integrable couplings.

Author Contributions: C.-X.L.: formal analysis, writing—original draft. W.-X.M.: supervision. S.-F.S.: writing—review and editing. All authors have read and agreed to the published version of the manuscript.

Funding: This work was in part supported by the National Natural Science Foundation of China (Grant Nos. 11871336, 11771395, 11975145, and 11972291).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Acknowledgments: We would like to express our sincere thanks to the referees for their useful comments and timely help.

Conflicts of Interest: The authors declare that they have no known competing financial interest or personal relationships that could have appeared to influence the work reported in this paper.

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