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A New Method of Solving Special Solutions of Quaternion Generalized Lyapunov Matrix Equation

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Abstract: In this paper, we study the bisymmetric and skew bisymmetric solutions of quaternion generalized Lyapunov equation. With the help of semi-tensor product of matrices, some new conclusions on the expansion rules of row and column of matrix product on quaternion matrices are proposed and applied to the calculation of quaternion matrix equation. Using the **H**-representation method, the independent elements are extracted according to the structural characteristics of bisymmetric matrix and skew bisymmetric matrix, so as to simplify the operation process. Finally, it is compared with the real vector representation method of quaternion matrix equation to illustrate the effectiveness and superiority of the proposed method.

Keywords: quaternion; Lyapunov equation; semi-tensor product of matrices; **H**-representation; (skew) bisymmetric matrix

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1. Introduction

The notations and definitions used in this paper are summarized as follows. Let $\mathbb{R}/\mathbb{C}/\mathbb{Q}$ be the sets of the real numbers/complex numbers/quaternions, respectively. $\mathbb{R}^n$ represents the set of all real column vectors with order n. $\mathbb{R}^{m\times n}/\mathbb{C}^{m\times n}/\mathbb{Q}^{m\times n}$ represent the set of all $m\times n$ real matrices/complex matrices/quaternion matrices, respectively. $\mathbb{BR}^{n\times n}/\mathbb{SBR}^{n\times n}/\mathbb{BQ}^{n\times n}/\mathbb{SBQ}^{n\times n}$ represent the set of all $n\times n$ real bisymmetric matrices/real skew bisymmetric matrices/quaternion bisymmetric matrices/quaternion skew bisymmetric matrices, respectively. $A^{\mathrm{T}}/A^{\mathrm{H}}/A^{\dagger}$ represent the transpose/conjugate transpose/Moore–Penrose inverse of matrix A, respectively. I_n represents unit matrix with order n. For $A=(a_{ij})\in\mathbb{Q}^{m\times n}$, $V_c(A)$ is the column stacking form of the matrix A, i.e., $V_c(A)=(a_{11},\cdots,a_{m1},a_{12},\cdots,a_{m2},\cdots,a_{mn})^{\mathrm{T}}$; $V_r(A)$ is the row stacking form of the matrix A, i.e., $V_r(A)=(a_{11},\cdots,a_{1n},a_{21},\cdots,a_{2n},\cdots,a_{mn})^{\mathrm{T}}$. $\otimes$ represents the Kronecker product of matrices. $\|\cdot\|$ represents the Frobenius norm of a matrix or Euclidean norm of a vector.

With the development of science and technology, Lyapunov equation has been widely used in engineering, so its research has attracted more and more scholars' attention. The equation

$$AX+XA^{\mathrm{T}}+\sum_{i=1}^{k}C_iXC_i^{\mathrm{T}}=B \tag{1}$$

is called the generalized Lyapunov equation, which plays an important role in studying the mean square stability, precise observability and H_2/H_∞ control of linear stochastic systems [1–4]. In particular, Equation (1) is closely related to the linear Itô-type system

$$dx(t)=Ax(t)dt+\sum_{i=1}^{k}C_ix(t)d\omega_i(t),\ x(0)=x_0\in\mathbb{R}^n,$$

where $A,C_1,\cdots,C_k$ are real constant matrices of suitable dimensions, $x\in\mathbb{R}^n$ is the system state, $x_0\in\mathbb{R}^n$ is a deterministic initial state, and $\omega_i(t),i=1,\cdots,k$ are independent, standard 1-D Wiener processes defined on the filtered probability space [5].

Matrix theory is recognized as a branch of mathematics originating in China. Katz, a professor at the University of Columbia, once said, "the idea of a matrix has a long history, dated at least from its use by Chinese scholars of the Han period for solving systems of linear equations". However, the traditional matrix theory is flawed, due to the limitations of matrix dimension, which greatly limits the application of matrix method. The semi-tensor product of matrices is the development of traditional matrix theory, which overcomes the limitation of traditional matrix theory on dimension. Therefore, it is also called the matrix theory of crossing dimension. It is now widely used in biological systems and life sciences [6–8], game theory [9–11], image encryption [12,13]. In addition, some scholars applied semi-tensor product of matrices to the solution of matrix equations. For example, Li studied the least squares solution of matrix equations under semi-tensor product of matrices [14], Ding studied the triangular Toeplitz solution of complex linear system by using semi-tensor product of matrices [15], and Wang studied the least squares Hermitian solution of quaternion matrix equation by using semi-tensor product of matrices [16].

Matrix equation is one of the important research fields of numerical algebra, and the research on quaternion matrix equation also has been widely concerned by scholars. For example, Kyrchei studied the least-norm of the general solution to some system of quaternion matrix equations and its determinantal representations [17] and Cramers rules for Sylvester quaternion matrix equation and its special cases [18], Liu and Wang studied the solvability conditions and the formula of the general solution to a Sylvester-like quaternion matrix Equation [19], Mehany and Wang investigated the solvability conditions and the general solution of three symmetrical systems of coupled Sylvester-like quaternion matrix Equations [20], Jiang and Ling studied closed-form solutions of the quaternion matrix equation $A\widetilde{X} - XB = C$ in explicit forms [21], Wang studied the bisymmetric and central symmetric solutions of quaternion matrix Equations [22], Zhang studied the least squares biHermitian solutions and oblique biHermitian solutions of quaternion matrix Equations [23].

Because of the uniqueness of the special matrix structure, we can use its structural characteristics to simplify the operation when solving the equation. **H**-representation is a systematic method for extracting independent elements of special matrix proposed by Zhang [5]. With the help of **H**-representation, we can reduce the number of elements involved in the operation, thereby simplifying the operation process. At present, **H**-representation has a preliminary application in the field of system and control [24,25]. For example, Zhao studied the moment stability of nonlinear discrete time delay stochastic systems based on **H**-representation [26], Sheng studied the observability of time-varying stochastic Markov jump systems based on **H**-representation [27]. **H**-representation method to simplify the linear matrix equation problem. Then,can it be used in the study of linear matrix inequality? This is a question that can be considered. For example, whether the **H**-representation method can be applied to the almost sure consensus of multi-agent systems [28] and the event-triggered $L_2 - L_\infty$ filtering for network-based neutral systems with time-varying delays by using T-S fuzzy method [29]. In this paper, we will study the bisymmetric and skew bisymmetric solutions of quaternion matrix equation by using the expansion rules of quaternion matrix product, **H**-representation of matrices and semi-tensor product of matrices.

Problem 1. *Let $A,\ B,\ C_i \in \mathbb{Q}^{n\times n}, i = 1, \cdots, n$, and*

$$T_b = \left\{ X | X \in \mathbb{BQ}^{n\times n},\ \ AX + XA^{\mathrm{T}} + \sum_{i=1}^{n} C_i X C_i^{\mathrm{T}} = B \right\}.$$

Find out $X_b \in T_b$ such that

$$\|X_b\| = \min_{X \in T_b} \|X\|.$$

X_b is called the minimal norm bisymmetric solution of (1).

Problem 2. *Let $A,\ B,\ C_i \in \mathbb{Q}^{n\times n}, i = 1,\cdots,n$*

$$T_{sb} = \left\{ X|X \in \mathbb{SBQ}^{n\times n},\ \ AX + XA^{\mathrm{T}} + \sum_{i=1}^{n} C_i X C_i^{\mathrm{T}} = B \right\}.$$

Find out $X_{sb} \in T_{sb}$ such that

$$\|X_{sb}\| = \min_{X\in T_{sb}} \|X\|.$$

X_{sb} is called the minimal norm skew bisymmetric solution of (1).

The main contributions of this paper are as follows: (i) By using the semi-tensor product of matrices, the new conclusions of the row and column expansion rules of matrix product over quaternion skew field are proposed, which can transform quaternion matrix equation into quaternion linear equations for solving. (ii) The **H**-representation method provides a systematic method to extract independent elements of special matrices.This paper applies this method to solve quaternion matrix equations, and provides a simple and feasible method to simplify the solution of quaternion matrix equations. (iii) The proposed method is compared with the real vector representation method in [16] to reflect the advantages of the proposed method in computational time and computable dimension.

This article is structured as follows. In Section 2, we introduce some basic knowledge of quaternion and semi-tensor product of matrices. In Section 3, we introduce **H**-representation, bisymmetric matrix and skew bisymmetric matrix, and give **H**-representation of these two kinds of special matrices, respectively. In Section 4, we study the minimal norm bisymmetric solution and the minimal norm skew bisymmetric solution of Equation (1) and give the necessary and sufficient conditions for the existence of solutions and the general solution expressions. In Section 5, the effectiveness of the algorithms are verified by numerical examples. Furthermore, by comparing this method with the real vector representation method, the superiority of this method is illustrated. In Section 6, a brief conclusion is given.

2. Preliminaries

In this section, we will review some basic knowledge of the quaternion and semi-tensor product of matrices.

Definition 1 ([30])**.** *The set of quaternions can be regarded as a four-dimensional algebra,that is,*

$$\mathbb{Q} = \{q = q_1 + q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k}|\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \mathbf{ijk} = -1, q_1, q_2, q_3, q_4 \in \mathbb{R}\}.$$

Quaternion q can be uniquely represented as $q = b_1 + b_2\mathbf{j}$, where $b_1 = q_1 + q_2\mathbf{i}$, $b_2 = q_3 + q_4\mathbf{i}$.

The conjugate of the quaternion q is defined as $\overline{q} = q_1 - q_2\mathbf{i} - q_3\mathbf{j} - q_4\mathbf{k} = \overline{b_1} - b_2\mathbf{j}$. The norm of a quaternion q is

$$\|q\| = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2} = \sqrt{q\overline{q}}.$$

Definition 2 ([30])**.** *Note that for any quaternion matrix $A \in \mathbb{Q}^{m\times n}$, A can be represented as*

$$A = A_{11} + A_{12}\mathbf{i} + A_{13}\mathbf{j} + A_{14}\mathbf{k} = A_1 + A_2\mathbf{j},$$

in which $A_{11}, A_{12}, A_{13}, A_{14} \in \mathbb{R}^{m\times n}$, and $A_1 = A_{11} + A_{12}\mathbf{i}, A_2 = A_{13} + A_{14}\mathbf{i}$.

The conjugate matrix A is defined as $\overline{A} = A_{11} - A_{12}\mathbf{i} - A_{13}\mathbf{j} - A_{14}\mathbf{k} = \overline{A_1} - A_2\mathbf{j}$. The Frobenius norm of quaternion matrix A is

$$\|A\| = \sqrt{\|A_{11}\|^2 + \|A_{12}\|^2 + \|A_{13}\|^2 + \|A_{14}\|^2}.$$

Lemma 1 ([30]). *Let $a, b \in \mathbb{Q}, A \in \mathbb{Q}^{m\times n}, B \in \mathbb{Q}^{n\times p}$, then*

$$\overline{ab} = \overline{b}\,\overline{a}, (\overline{A})^{\mathrm{T}} = \overline{(A^{\mathrm{T}})}, \overline{V_c(AB)} = V_c(\overline{AB})$$

Definition 3 ([31]). *Let $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{p\times q}$, $t = lcm(n,p)$ is the least common multiples of n and p, then the semi-tensor product of A and B is defined as*

$$A \ltimes B = (A \otimes I_{t/n})(B \otimes I_{t/p}).$$

Through the definition of semi-tensor product of matrices, we find that when $n = p$, it is the traditional matrix multiplication. So semi-tensor product of matrices is a generalization of traditional matrix multiplication.

Because semi-tensor product of matrices allows the expansion of the dimension of matrices, we can realize the transformation between row and column stacking form of matrix by using swap matrix.

Definition 4 ([31]). *Define the mn dimensional swap matrix as follows*

$$W_{[m,n]} = [I_n \otimes \delta_m^1, I_n \otimes \delta_m^2, \cdots, I_n \otimes \delta_m^m],$$

where δ_m^i is the ith column of I_m.

Lemma 2 ([31]). *Let $A \in \mathbb{R}^{m\times n}$, then*

$$W_{[m,n]}V_r(A) = V_c(A), W_{[n,m]}V_c(A) = V_r(A).$$

With the help of semi-tensor product of matrices, we present some new conclusions on the expansion rules of quaternion matrix product.

Theorem 1. *Suppose $B \in \mathbb{Q}^{m\times n}, X \in \mathbb{Q}^{n\times p}, Y \in \mathbb{Q}^{q\times m}$, then*

(1) $V_r(BX) = B \ltimes V_r(X), V_c(BX) = (I_p \otimes B)V_c(X)$;

(2) $V_c(\overline{YB}) = B^H \ltimes V_c(\overline{Y}), V_r(\overline{YB}) = (I_q \otimes B^H)V_r(\overline{Y})$.

Proof. (1) For $B = (b_{ij}) \in \mathbb{Q}^{m\times n}, X = (x_{ij}) \in \mathbb{Q}^{n\times p}$, and B_i is the ith row of B, then the ith block on the right side of $C = BX$ is

$$B_i \ltimes V_r(X) = (B_i \otimes I_p)V_r(X) = \begin{bmatrix} \sum_{k=1}^n b_{ik}x_{ki} \\ \vdots \\ \sum_{k=1}^n b_{ik}x_{kp} \end{bmatrix} = (C_i)^{\mathrm{T}}.$$

So $V_r(BX) = V_r(C) = B \ltimes V_r(X)$. From Lemma 2,

$$V_c(BX) = W_{[m,p]}V_r(BX) = W_{[m,p]} \ltimes B \ltimes V_r(X) = W_{[m,p]} \ltimes B \ltimes W_{[p,n]}V_c(X) = (I_p \otimes B)V_c(X).$$

(2) Note $B = (b_1, \cdots, b_n), b_i \in \mathbb{Q}^m (i = 1, \cdots, n), Y = (y_1, \cdots, y_m), y_j \in \mathbb{Q}^q (j = 1, \cdots, m)$, then

$$V_c(\overline{YB}) = V_c(\overline{Yb_1}, \cdots, \overline{Yb_n}) = \begin{bmatrix} \overline{Yb_1} \\ \vdots \\ \overline{Yb_n} \end{bmatrix}.$$

By Lemma 1,

$$\overline{Yb_i} = \overline{y_1b_{1i}} + \overline{y_2b_{2i}} + \cdots + \overline{y_mb_{mi}} = \overline{b_{1i}}\,\overline{y_1} + \overline{b_{2i}}\,\overline{y_2} + \cdots + \overline{b_{mi}}\,\overline{y_m} = \left[\overline{b_{1i}}I_q, \cdots, \overline{b_{mi}}I_q\right] V_c(\overline{Y}).$$

So

$$V_c(\overline{YB}) = \begin{bmatrix} \overline{b_{11}}I_q & \overline{b_{21}}I_q & \cdots & \overline{b_{m1}}I_q \\ \overline{b_{12}}I_q & \overline{b_{22}}I_q & \cdots & \overline{b_{m2}}I_q \\ \vdots & \vdots & \cdots & \vdots \\ \overline{b_{1n}}I_q & \overline{b_{2n}}I_q & \cdots & \overline{b_{mn}}I_q \end{bmatrix} V_c(\overline{Y}).$$

And

$$V_r(\overline{YB}) = W_{[n,q]}V_c(\overline{YB}) = W_{[n,q]} \ltimes B^H \ltimes V_c(\overline{Y}) = W_{[n,q]} \ltimes B^H \ltimes W_{[q,m]}V_r(\overline{Y}) = (I_q \otimes B^H)V_r(\overline{Y}).$$

□

Lemma 3 ([32]). *The linear system of equation $Ax = b$, with $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, has a solution $x \in \mathbb{R}^n$ if and only if $AA^\dagger b = b$. In case that it has the general solution*

$$x = A^\dagger b + (I - A^\dagger A)y,$$

where $y \in \mathbb{R}^n$ is an arbitrary vector. The minimal norm solution of the linear system of equation $Ax = b$ is $A^\dagger b$.

3. The H-Representation of Bisymmetric Matrix and Skew Bisymmetric Matrix

This section describes the **H**-representation of matrices and related properties.

Definition 5 ([22]). *For a matrix $A = (a_{ij}) \in \mathbb{Q}^{n\times n}$, if $a_{ij} = a_{n-i+1,n-j+1} = \overline{a_{ji}}$, then A is called bisymmetric matrix.*

Definition 6 ([33]). *For a matrix $A = (a_{ij}) \in \mathbb{Q}^{n\times n}$, if $a_{ij} = a_{n-i+1,n-j+1} = -\overline{a_{ji}}$, then A is called skew bisymmetric matrix.*

Next, a brief introduction to the **H**-representation is given.

Definition 7 ([5]). *Consider a p-dimensional complex matrix subspace $\mathbb{X} \subset \mathbb{C}^{n\times n}$ over the field $\mathbb{C}$. For each matrix $X = (x_{ij})_{n\times n} \in \mathbb{X}$, there always exist a map ψ:$X \in \mathbb{X} \mapsto V_c(X)$. If $dim(\mathbb{X}) = p$ and $e_1, e_2, \ldots, e_p$ $(p \leq n^2)$, form a basis of $\mathbb{X}$, define $\mathrm{H} = [V_c(e_1), V_c(e_2), \ldots, V_c(e_p)]$, there exist $x_1, x_2, \ldots, x_p \in \mathbb{C}$, such that $X = \sum_{i=1}^{p} x_i e_i$. Therefore for each $X \in \mathbb{X}$, if we express $\psi(X) = V_c(X)$ in the form of*

$$\psi(X) = V_c(X) = \mathrm{H}\widetilde{X},$$

where $\widetilde{X} = [x_1, x_2, \ldots, x_p]^{\mathrm{T}}$ is an order arrangement of independent elements in X, then $\mathrm{H}\widetilde{X}$ is called an **H***-representation of $\psi(X)$, and H is called an* **H***-representation matrix of $\psi(X)$.*

In the complex matrix subspace $\mathbb{X}$, the **H**-representation of the matrix is related to the selection of the basis. The **H**-representation of a matrix is unique when the basis are fixed.

Based on the above definitions, the following examples are given.

Example 1. *Let $\mathbb{X} = \mathbb{BR}^{n\times n}$, $X = (x_{ij})_{4\times 4} \in \mathbb{X}$, and then $dim(\mathbb{X}) = 6$. If we select a basis of $\mathbb{X}$ as*

$$e_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$e_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},\ e_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ e_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is easy to compute

$$\psi(X) = V_c(X) = [x_{11}, x_{21}, x_{31}, x_{41}, x_{21}, x_{22}, x_{32}, x_{31}, x_{31}, x_{32}, x_{22}, x_{21}, x_{41}, x_{21}, x_{31}, x_{11}]^{\mathrm{T}},$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ \widetilde{X} = [x_{11}, x_{21}, x_{31}, x_{41}, x_{22}, x_{32}]^{\mathrm{T}}.$$

Example 2. *Let $\mathbb{X} = \mathbb{SBR}^{n\times n}$, $X = (x_{ij})_{4\times 4} \in \mathbb{X}$, and then $dim(\mathbb{X}) = 2$. If we select a basis of $\mathbb{X}$ as*

$$e_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

It is easy to compute

$$\psi(X) = V_c(X) = [0, x_{21}, x_{31}, 0, -x_{21}, 0, 0, -x_{31}, -x_{31}, 0, 0, -x_{21}, 0, x_{31}, x_{21}, 0]^{\mathrm{T}},$$

$$H = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix},\ \widetilde{X} = [x_{21}, x_{31}]^{\mathrm{T}}.$$

In general, we will give the **H**-representation for $\mathbb{X} = \mathbb{BR}^{n\times n}$ and $\mathbb{X} = \mathbb{SBR}^{n\times n}$. Firstly, we select the standard basis of n-dimensional bisymmetric matrix subspace and n-dimensional skew bisymmetric matrix subspace.

For $\mathbb{X} = \mathbb{BR}^{n\times n}$, when n is odd, we select a set of standard basis as

$$\left\{E_{11}, \cdots, E_{n1}, E_{22}, \cdots, E_{n-1,2}, \cdots, E_{\frac{n+1}{2},\frac{n+1}{2}}\right\},$$

where $E_{ij} = (e_{kl})_{n\times n}$, and $e_{kl} = e_{n-k+1,n-l+1} = e_{lk} = 1$, the other elements are 0. At this time, we have

$$\widetilde{X_b} = (x_{11}, \cdots, x_{n1}, x_{22}, \cdots, x_{n-1,2}, \cdots, x_{\frac{n+1}{2},\frac{n+1}{2}})^{\mathrm{T}}. \tag{2}$$

When n is even,we select a set of standard basis as

$$\left\{E_{11}, \cdots, E_{n1}, E_{22}, \cdots, E_{n-1,2}, \cdots, E_{\frac{n}{2},\frac{n}{2}}, E_{\frac{n}{2}+1,\frac{n}{2}}\right\},$$

where $E_{ij} = (e_{kl})_{n\times n}$, and $e_{kl} = e_{n-k+1,n-l+1} = e_{lk} = 1$, the other elements are 0. At this time, we have

$$\widetilde{X_b} = (x_{11}, \cdots, x_{n1}, x_{22}, \cdots, x_{n-1,2}, \cdots, x_{\frac{n}{2},\frac{n}{2}}, x_{\frac{n}{2}+1,\frac{n}{2}})^{\mathrm{T}}. \tag{3}$$

Similarly, for $\mathbb{X} = \mathbb{SBR}^{n\times n}$, when n is odd, we select a set of standard basis as

$$\left\{F_{21}, \cdots, F_{n-1,1}, F_{32}, \cdots, F_{n-2,2}, \cdots, F_{\frac{n+1}{2},\frac{n-1}{2}}\right\},$$

where $F_{ij} = (f_{pq})_{n\times n}$, and $f_{pq} = f_{n-p+1,n-q+1} = -f_{qp} = 1$, the other elements are 0. At this time, we have

$$\widetilde{X_{sb}} = (x_{21}, \cdots, x_{n-1,1}, x_{32}, \cdots, x_{n-2,2}, \cdots, x_{\frac{n+1}{2},\frac{n-1}{2}})^{\mathrm{T}}. \tag{4}$$

When n is even,we select a set of standard basis as

$$\left\{F_{21}, \cdots, F_{n-1,1}, F_{32}, \cdots, F_{n-2,2}, \cdots, F_{\frac{n}{2},\frac{n}{2}-1}, F_{\frac{n}{2}+1,\frac{n}{2}-1}\right\},$$

where $F_{ij} = (f_{pq})_{n\times n}$, and $f_{pq} = f_{n-p+1,n-q+1} = -f_{qp} = 1$, the other elements are 0. At this time, we have

$$\widetilde{X_{sb}} = (x_{21}, \cdots, x_{n-1,1}, x_{32}, \cdots, x_{n-2,2}, \cdots, x_{\frac{n}{2},\frac{n}{2}-1}, x_{\frac{n}{2}+1,\frac{n}{2}-1})^{\mathrm{T}}. \tag{5}$$

Remark 1. *Note that $\psi(X_b)$ is a column vector formed by all elements of X_b, while $\widetilde{X_b}$, $\widetilde{X_{sb}}$ are column vectors formed by different nonzero elements of X_b, X_{sb}, respectively. For clarity, we denote the* H*-matrix in* **H***-representation corresponding to $\mathbb{X} = \mathbb{BR}^{n\times n}$ by H_b, the* H*-matrix in* **H***-representation corresponding to $\mathbb{X} = \mathbb{SBR}^{n\times n}$ by H_{sb}.*

4. The Solutions of Problems 1 and 2

In this section, by using the properties of semi-tensor product of matrices and the **H**-representation, we study Problems 1 and 2.

For convenience, we denote $B = B_{11} + B_{12}\mathbf{i} + B_{13}\mathbf{j} + B_{14}\mathbf{k} = B_1 + B_2\mathbf{j}$, $\overrightarrow{B} = \begin{bmatrix} V_c(B_{11}) \\ V_c(B_{13}) \\ V_c(B_{12}) \\ V_c(B_{14}) \end{bmatrix}$, $I_B = \begin{bmatrix} I_{n^2} & 0 & 0 & 0 \\ 0 & 0 & I_{n^2} & 0 \\ I_{n^2} & 0 & 0 & 0 \\ 0 & 0 & I_{n^2} & 0 \\ 0 & I_{n^2} & 0 & 0 \\ 0 & 0 & 0 & I_{n^2} \\ 0 & -I_{n^2} & 0 & 0 \\ 0 & 0 & 0 & -I_{n^2} \end{bmatrix}$, $D = I_n \otimes A = D_1 + D_2\mathbf{j}$, $E = \overline{A} \otimes I_n = E_1 + E_2\mathbf{j}$, $F_i = I_n \otimes C_i = F_{1i} + F_{2i}\mathbf{j}$, $G_i = \overline{C_i} \otimes I_n = G_{1i} + G_{2i}\mathbf{j}$, $M_{11} = D_1 + \overline{E_1} + \sum_{i=1}^n F_1\overline{G_{1i}}$, $M_{21} = -E_2 - \sum_{i=1}^n F_1 G_{2i}$, $M_{12} = \overline{E_2} + \sum_{i=1}^n F_1\overline{G_{2i}}$, $M_{22} = D_1 + E_1 + \sum_{i=1}^n F_1 G_{1i}$, $M_{13} = \sum_{i=1}^n F_2\overline{G_{2i}}$, $M_{23} = D_2 + \sum_{i=1}^n F_2 G_{1i}$, $M_{14} = -D_2 - \sum_{i=1}^n F_2\overline{G_{1i}}$, $M_{24} = \sum_{i=1}^n F_2 G_{2i}$. Then $M = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \end{bmatrix} = M_1 + M_2\mathbf{i}$.

Theorem 2. *Suppose A, B, $C_i \in \mathbb{Q}^{n\times n}$ $(i = 1, \cdots, n)$, and denote* $H_1 = \begin{bmatrix} H_b & & & \\ & H_{sb} & & \\ & & H_{sb} & \\ & & & H_{sb} \end{bmatrix}$, $L_1 = \begin{bmatrix} M_1 & -M_2 \\ M_2 & M_1 \end{bmatrix} I_B H_1$. *Hence (1) has a bisymmetric solution if and only if*

$$\left(L_1 L_1^\dagger - I_{4n^2}\right)\overrightarrow{B} = 0. \tag{6}$$

Moreover, if (6) holds, the solution set of (1) can be represented as

$$H_b = \left\{ X = X_{11} + X_{12}\mathbf{i} + X_{13}\mathbf{j} + X_{14}\mathbf{k} \,\middle|\, \begin{bmatrix} \widetilde{X_{11}} \\ \widetilde{X_{12}} \\ \widetilde{X_{13}} \\ \widetilde{X_{14}} \end{bmatrix} = \begin{cases} L_1^\dagger\overrightarrow{B} + (I_{n^2-n+1} - L_1^\dagger L_1)y, \ \forall y \in \mathbb{R}^{n^2-n+1}, & \text{if } n \text{ is odd} \\ L_1^\dagger\overrightarrow{B} + (I_{n^2-n} - L_1^\dagger L_1)y \ \forall y \in \mathbb{R}^{n^2-n}, & \text{if } n \text{ is even} \end{cases} \right\},$$

where $\widetilde{X_{1p}}$ is the independent elements of X_{1p}, $p = 1, 2, 3, 4$.

Then, the minimal norm bisymmetric solution $\widehat{X}_b$ satisfies

$$\begin{bmatrix} \widetilde{X_{11}} \\ \widetilde{X_{12}} \\ \widetilde{X_{13}} \\ \widetilde{X_{14}} \end{bmatrix} = L_1^\dagger\overrightarrow{B}. \tag{7}$$

Proof. For $X = X_1 + X_2\mathbf{j} = X_{11} + X_{12}\mathbf{i} + X_{13}\mathbf{j} + X_{14}\mathbf{k} \in \mathbb{BQ}^{n\times n}$, using Theorem 1 and $\mathbf{H}$-representation of bisymmetric matrix, we can obtain

$$\begin{aligned} & \left\| AX + XA^{\mathrm{T}} - \sum_{i=1}^n C_i X C_i^{\mathrm{T}} - B \right\| \\ = & \left\| V_c(AX + XA^{\mathrm{T}} - \sum_{i=1}^n C_i X C_i^{\mathrm{T}} - B) \right\| \\ = & \left\| V_c(AX) + V_c(XA^{\mathrm{T}}) - V_c(\sum_{i=1}^n C_i X C_i^{\mathrm{T}}) - V_c(B) \right\| \\ = & \left\| (I_n \otimes A)V_c(X) + \overline{(\overline{A} \otimes I_n)V_c(\overline{X})} + \sum_{i=1}^n (I_n \otimes C_i)\overline{(\overline{C_i} \otimes I_n)V_c(\overline{X})} - V_c(B) \right\| \end{aligned}$$

$$
\begin{aligned}
&= \left\| (D_1 + D_2\mathbf{j})V_c(X_1 + X_2\mathbf{j}) + \overline{(E_1 + E_2\mathbf{j})V_c(\overline{X_1} - X_2\mathbf{j})} + \sum_{i=1}^{n}(F_1 + F_2\mathbf{j})\overline{(G_{1i} + G_{2i}\mathbf{j})V_c(\overline{X_1} - X_2\mathbf{j})} - V_c(B_1 + B_2\mathbf{j}) \right\| \\
&= \| D_1V_c(X_1) - D_2V_c(\overline{X_2}) + D_1V_c(X_2)\mathbf{j} + D_2V_c(\overline{X_1})\mathbf{j} + \overline{E_1}V_c(X_1) + \overline{E_2}V_c(X_2) + E_1V_c(X_2)\mathbf{j} - E_2V_c(X_1)\mathbf{j} \\
&\quad + \sum_{i=1}^{n} F_1\overline{G_{1i}}V_c(X_1) + \sum_{i=1}^{n} F_1\overline{G_{2i}}V_c(X_2) + \sum_{i=1}^{n} F_2\overline{G_{1i}}V_c(\overline{X_2}) + \sum_{i=1}^{n} F_2\overline{G_{2i}}V_c(\overline{X_1}) \\
&\quad + \sum_{i=1}^{n} F_1G_{1i}V_c(X_2)\mathbf{j} - \sum_{i=1}^{n} F_1G_{2i}V_c(X_1)\mathbf{j} + \sum_{i=1}^{n} F_2G_{1i}V_c(\overline{X_1})\mathbf{j} - \sum_{i=1}^{n} F_2G_{2i}V_c(\overline{X_2})\mathbf{j} - V_c(B_1) - V_c(B_2)\mathbf{j} \| \\
&= \| [M_{11}V_c(X_1) + M_{12}V_c(X_2) + M_{13}V_c(\overline{X_1}) + M_{14}V_c(\overline{X_2})] - V_c(B_1) \\
&\quad + [M_{21}V_c(X_1) + M_{22}V_c(X_2) + M_{23}V_c(\overline{X_1}) + M_{24}V_c(\overline{X_2})]\mathbf{j} - V_c(B_2)\mathbf{j} \| \\
&= \left\| \begin{bmatrix} M_{11}V_c(X_1) + M_{12}V_c(X_2) + M_{13}V_c(\overline{X_1}) + M_{14}V_c(\overline{X_2}) \\ M_{21}V_c(X_1) + M_{22}V_c(X_2) + M_{23}V_c(\overline{X_1}) + M_{24}V_c(\overline{X_2}) \end{bmatrix} - \begin{bmatrix} V_c(B_1) \\ V_c(B_2) \end{bmatrix} \right\| \\
&= \left\| \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \end{bmatrix} \begin{bmatrix} V_c(X_1) \\ V_c(X_2) \\ V_c(\overline{X_1}) \\ V_c(\overline{X_2}) \end{bmatrix} - \begin{bmatrix} V_c(B_1) \\ V_c(B_2) \end{bmatrix} \right\| \\
&= \left\| (M_1 + M_2\mathbf{i}) \begin{bmatrix} V_c(X_{11} + X_{12}\mathbf{i}) \\ V_c(X_{13} + X_{14}\mathbf{i}) \\ V_c(X_{11} - X_{12}\mathbf{i}) \\ V_c(X_{13} - X_{14}\mathbf{i}) \end{bmatrix} - \begin{bmatrix} V_c(B_{11} + B_{12}\mathbf{i}) \\ V_c(B_{13} + B_{14}\mathbf{i}) \end{bmatrix} \right\| \\
&= \left\| (M_1 + M_2\mathbf{i}) \left(\begin{bmatrix} V_c(X_{11}) \\ V_c(X_{13}) \\ V_c(X_{11}) \\ V_c(X_{13}) \end{bmatrix} + \begin{bmatrix} V_c(X_{12}) \\ V_c(X_{14}) \\ -V_c(X_{12}) \\ -V_c(X_{14}) \end{bmatrix} \mathbf{i} \right) - \left(\begin{bmatrix} V_c(B_{11}) \\ V_c(B_{13}) \end{bmatrix} + \begin{bmatrix} V_c(B_{12}) \\ V_c(B_{14}) \end{bmatrix} \mathbf{i} \right) \right\| \\
&= \left\| \begin{bmatrix} M_1 & -M_2 \\ M_2 & M_1 \end{bmatrix} \begin{bmatrix} V_c(X_{11}) \\ V_c(X_{13}) \\ V_c(X_{11}) \\ V_c(X_{13}) \\ V_c(X_{12}) \\ V_c(X_{14}) \\ -V_c(X_{12}) \\ -V_c(X_{14}) \end{bmatrix} - \begin{bmatrix} V_c(B_{11}) \\ V_c(B_{13}) \\ V_c(B_{12}) \\ V_c(B_{14}) \end{bmatrix} \right\| \\
&= \left\| \begin{bmatrix} M_1 & -M_2 \\ M_2 & M_1 \end{bmatrix} \begin{bmatrix} I_{n^2} & 0 & 0 & 0 \\ 0 & 0 & I_{n^2} & 0 \\ I_{n^2} & 0 & 0 & 0 \\ 0 & 0 & I_{n^2} & 0 \\ 0 & I_{n^2} & 0 & 0 \\ 0 & 0 & 0 & I_{n^2} \\ 0 & -I_{n^2} & 0 & 0 \\ 0 & 0 & 0 & -I_{n^2} \end{bmatrix} \begin{bmatrix} H_b & & & \\ & H_{sb} & & \\ & & H_{sb} & \\ & & & H_{sb} \end{bmatrix} \begin{bmatrix} \widetilde{X_{11}} \\ \widetilde{X_{12}} \\ \widetilde{X_{13}} \\ \widetilde{X_{14}} \end{bmatrix} - \begin{bmatrix} V_c(B_{11}) \\ V_c(B_{13}) \\ V_c(B_{12}) \\ V_c(B_{14}) \end{bmatrix} \right\|.
\end{aligned}
$$

By means of the properties of the Moore–Penrose inverse, we get

$$
\begin{aligned}
&\left\|AX + XA^{\mathrm{T}} - \sum_{i=1}^{n} C_i X C_i^{\mathrm{T}} - B\right\| \\
=&\left\|\begin{bmatrix} M_1 & -M_2 \\ M_2 & M_1 \end{bmatrix} I_B H_1 \begin{bmatrix} \widetilde{X_{11}} \\ \widetilde{X_{12}} \\ \widetilde{X_{13}} \\ \widetilde{X_{14}} \end{bmatrix} - \begin{bmatrix} V_c(B_{11}) \\ V_c(B_{13}) \\ V_c(B_{12}) \\ V_c(B_{14}) \end{bmatrix}\right\| \\
=&\left\|L_1 \begin{bmatrix} \widetilde{X_{11}} \\ \widetilde{X_{12}} \\ \widetilde{X_{13}} \\ \widetilde{X_{14}} \end{bmatrix} - \overrightarrow{B}\right\| = \left\|L_1 L_1^{\dagger} L_1 \begin{bmatrix} \widetilde{X_{11}} \\ \widetilde{X_{12}} \\ \widetilde{X_{13}} \\ \widetilde{X_{14}} \end{bmatrix} - \overrightarrow{B}\right\| \\
=&\left\|L_1 L_1^{\dagger} \overrightarrow{B} - \overrightarrow{B}\right\| = \left\|\left(L_1 L_1^{\dagger} - I_{4n^2}\right) \overrightarrow{B}\right\|.
\end{aligned}
$$

Therefore, for $X \in \mathbb{BQ}^{n\times n}$, we obtain

$$
\begin{aligned}
&\left\|AX + XA^{\mathrm{T}} - \sum_{i=1}^{n} C_i X C_i^{\mathrm{T}} - B\right\| = 0 \\
\iff& \left\|\left(L_1 L_1^{\dagger} - I_{4n^2}\right) \overrightarrow{B}\right\| = 0 \\
\iff& \left(L_1 L_1^{\dagger} - I_{4n^2}\right) \overrightarrow{B} = 0.
\end{aligned}
$$

In case that (1) is compatible, its solution $X \in \mathbb{BQ}^{n\times n}$ satisfies

$$
L_1 \begin{bmatrix} \widetilde{X_{11}} \\ \widetilde{X_{12}} \\ \widetilde{X_{13}} \\ \widetilde{X_{14}} \end{bmatrix} = \overrightarrow{B}.
$$

Moreover, we can get the bisymmetric solution $\widehat{X}_b$ satisfies

$$
\begin{bmatrix} \widetilde{X_{11}} \\ \widetilde{X_{12}} \\ \widetilde{X_{13}} \\ \widetilde{X_{14}} \end{bmatrix} = L_1^{\dagger} \overrightarrow{B} + (I_p - L_1^{\dagger} L_1) y, \quad \forall y \in \mathbb{R}^p,
$$

when n is odd, $p = n^2 - n + 1$; when n is even, $p = n^2 - n$. □

Similarly, we can obtain the minimum norm skew bisymmetric solution of Problem 2.

Theorem 3. *Suppose $A,\ B,\ C_i \in \mathbb{Q}^{n\times n}$ $(i = 1, \cdots, n)$, and denote* $H_2 = \begin{bmatrix} H_{sb} & & & \\ & H_b & & \\ & & H_b & \\ & & & H_b \end{bmatrix}$,
$L_2 = \begin{bmatrix} M_1 & -M_2 \\ M_2 & M_1 \end{bmatrix} I_B H_2$. *Hence (1) has a skew bisymmetric solution if and only if*

$$
\left(L_2 L_2^{\dagger} - I_{4n^2}\right) \overrightarrow{B} = 0. \tag{8}
$$

Moreover, if (8) holds, the solution set of (1) can be represented as

$$H_{sb}=\left\{X=X_{11}+X_{12}\mathbf{i}+X_{13}\mathbf{j}+X_{14}\mathbf{k}\left|\begin{bmatrix}\widetilde{X_{11}}\\\widetilde{X_{12}}\\\widetilde{X_{13}}\\\widetilde{X_{14}}\end{bmatrix}=\begin{cases}L_2^{\dagger}\overrightarrow{B}+(I_{n^2+n+1}-L_2^{\dagger}L_2)y,\ \forall y\in\mathbb{R}^{n^2+n+1}, & \text{if } n \text{ is odd}\\ L_2^{\dagger}\overrightarrow{B}+(I_{n^2+n}-L_2^{\dagger}L_2)y\ \forall y\in\mathbb{R}^{n^2+n}, & \text{if } n \text{ is even}\end{cases}\right.\right\},$$

where $\widetilde{X_{1q}}$ *is the independent elements of* X_{1q}, $q=1,2,3,4$.

Then, the minimal norm skew bisymmetric solution $\widehat{X}_{sb}$ *satisfies*

$$\begin{bmatrix}\widetilde{X_{11}}\\\widetilde{X_{12}}\\\widetilde{X_{13}}\\\widetilde{X_{14}}\end{bmatrix}=L_2^{\dagger}\overrightarrow{B}. \tag{9}$$

5. Algorithms and Numerical Examples

This section provides algorithms and examples of **H**-representation methods for solving bisymmetric and skew bisymmetric solutions of quaternion generalized Lyapunov equation. For convenience, we take the case of $i=1$.

Next, numerical experiments are used to verify the effectiveness of the above algorithms.

Example 3. *Suppose* $A,\ C\in\mathbb{Q}^{n\times n}$ *be generated randomly for* $n=3:50$. *Denote*

$$\begin{aligned}X_b&=X_b^1+X_b^2\mathbf{i}+X_b^3\mathbf{j}+X_b^4\mathbf{k}\in\mathbb{BQ}^{n\times n},\\X_{sb}&=X_{sb}^1+X_{sb}^2\mathbf{i}+X_{sb}^3\mathbf{j}+X_{sb}^4\mathbf{k}\in\mathbb{SBQ}^{n\times n}.\end{aligned}$$

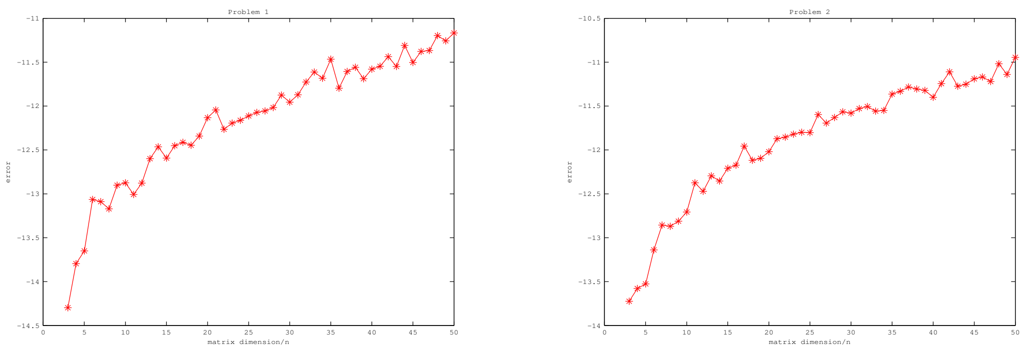
Compute $AX_b+X_bA^{\mathrm{T}}+CX_bC^{\mathrm{T}}=B_b$ *and* $AX_{sb}+X_{sb}A^{\mathrm{T}}+CX_{sb}C^{\mathrm{T}}=B_{sb}$. *For* $AX+XA^{\mathrm{T}}+CXC^{\mathrm{T}}=B_b$ *and* $AX+XA^{\mathrm{T}}+CXC^{\mathrm{T}}=B_{sb}$, *using Algorithms 1 and 2, we can obtain the calculation solutions* $\widehat{X}_b$ *and* $\widehat{X}_{sb}$, *respectively. Denote* $\varepsilon_1=log_{10}\|X_b-\widehat{X}_b\|$, $\varepsilon_2=log_{10}\|X_{sb}-\widehat{X}_{sb}\|$. *As the dimension changes,* $\varepsilon_i,\ (i=1,2)$ *are shown in Figure 1.*

Algorithm 1: (Problem 1)

Step 1: Input $A,\ B,\ C\in\mathbb{Q}^{n\times n}$, *output* $M_1, M_2,\ \overrightarrow{B}$,
Step 2: Input H_b, H_{sb}, *output* H_1, L_1,
Step 3: According to (7), output the minimal norm bisymmetric solution $\widehat{X}_b$ *of (1).*

Algorithm 2: (Problem 2)

Step 1: Input $A,\ B,\ C\in\mathbb{Q}^{n\times n}$, output $M_1, M_2,\ \overrightarrow{B}$,
Step 2: Input H_b, H_{sb}, output H_2, L_2,
Step 3: According to (9), output the minimal norm skew bisymmetric solution $\widehat{X}_{sb}$ of (1).



(**a**) (**b**)

Figure 1. Errors in different dimensions.

Next, the proposed method is compared with the real vector representation method in Reference [16]. Next, we briefly introduce the real vector representation of quaternion.

Let $q = q_1 + q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k} \in \mathbb{Q}$, denote

$$\overleftarrow{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix},$$

$\overleftarrow{q}$ is called the real vector representation of q.

Let $x = [x_1, x_2, \cdots, x_n]$, $y = [y_1, y_2, \cdots, y_n]^{\mathrm{T}}$ be quaternion vectors. Denote

$$\overleftarrow{x} = \begin{bmatrix} \overleftarrow{x_1} \\ \vdots \\ \overleftarrow{x_n} \end{bmatrix}, \overleftarrow{y} = \begin{bmatrix} \overleftarrow{y_1} \\ \vdots \\ \overleftarrow{y_n} \end{bmatrix},$$

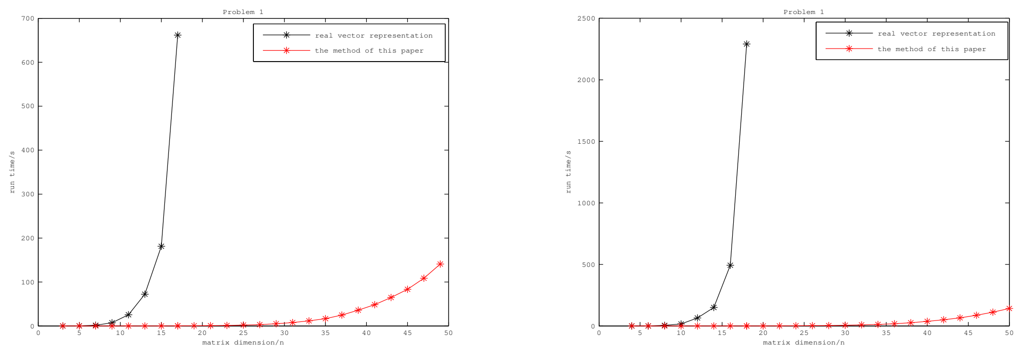
$\overleftarrow{x}$, $\overleftarrow{y}$ are called as the real vector representation of quaternion vector x and y, respectively.

For $A = (A_{ij}) \in \mathbb{Q}^{m \times n}$, denote

$$\overleftarrow{A}^c = \begin{pmatrix} \overleftarrow{Col_1(A)} \\ \overleftarrow{Col_2(A)} \\ \vdots \\ \overleftarrow{Col_n(A)} \end{pmatrix} = \begin{pmatrix} \overleftarrow{A_{11}} \\ \vdots \\ \overleftarrow{A_{m1}} \\ \vdots \\ \overleftarrow{A_{1n}} \\ \vdots \\ \overleftarrow{A_{mn}} \end{pmatrix}, \overleftarrow{A}^r = \begin{pmatrix} \overleftarrow{Row_1(A)} \\ \overleftarrow{Row_2(A)} \\ \vdots \\ \overleftarrow{Row_m(A)} \end{pmatrix} = \begin{pmatrix} \overleftarrow{A_{11}} \\ \vdots \\ \overleftarrow{A_{1n}} \\ \vdots \\ \overleftarrow{A_{m1}} \\ \vdots \\ \overleftarrow{A_{mn}} \end{pmatrix},$$

where $Col_k(A)$ is the kth column of $A(k = 1, \cdots, n)$, $Row_l(A)$ is the lth row of $A(l = 1, \cdots, m)$. $\overleftarrow{A}^c$, $\overleftarrow{A}^r$ are called the real column stacking form and the real row stacking form of A, respectively. Real column stacking form and real row stacking form of A are collectively called real vector representation of A.

By comparing the computer running time and the computable dimension, the superiority of the proposed method is illustrated. Taking the bisymmetric matrix as an example, when n is odd or even, the figures are shown in Figure 2.



(**a**) (**b**)

Figure 2. Comparison of running time of different methods.

Since the real vector representation method takes too long in calculating large dimensions, the real vector representation method only calculates $n = 18$. By comparison, in terms of time, under the premise of calculating the same dimension, the time required by this method is shorter, and with the increase of dimension, the advantages of this method are more obvious. Secondly, in terms of computable dimension, due to the smaller matrix size and less data to be stored in this method, the computable dimension is larger.

6. Conclusions

With the help of semi-tensor product of matrices, this paper puts forward some new conclusions about the product expansion rules of quaternion matrices and applies them to the calculation of matrix equation. For solving quaternion matrix equation with special structural solutions, **H**-representation is used to extract the independent elements of the matrix to participate in the operation, by comparing with the real vector representation of quaternion, we illustrate the effectiveness and superiority of the method. In addition, the event-triggered $L_2 - L_\infty$ filtering for network-based neutral systems with time-varying delay via $T - S$ fuzzy approach based on proposed method is a good direction worthy of studying.

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