symmetry

MDPI

Article

Properties of q-Starlike Functions Associated with the q-Cosine Function

Mohammad Faisal Khan

Department of Basic Science, College of Science and Theoretical Studies, Saudi Electronic University, Riyadh 11673, Saudi Arabia; f.khan@seu.edu.sa

Abstract: In this paper, our main focus is to define a new subfamily of q-analogue of analytic functions associated with the q-cosine function. Furthermore, we investigate some useful results such as the necessary and sufficient condition based on the convolution idea, growth and distortion bounds, closure theorem, convex combination, radii of starlikeness, extreme point theorem and partial sums results for the newly-defined functions class.

Keywords: q-calculus; q-cosine function; analytic function; subordination

MSC: Primary 30C45; 30C50; 30C80; Secondary 11B65; 47B38

check for updates

Citation: Khan, M.F. Properties of q-Starlike Functions Associated with the q-Cosine Function. *Symmetry* **2022**, *14*, 1117. https://doi.org/10.3390/sym14061117

Academic Editors: Dumitru Baleanu and Şahsene Altınkaya

Received: 11 April 2022
Accepted: 26 May 2022
Published: 29 May 2022

1. Introduction and Motivation

In quantum calculus, instead of the limit, we use the parameter q with properties that $0 < q < 1$, the study of such notions is called q-calculus or quantum calculus. Sometimes, we call it calculus without limits. q-derivatives have vast applications in mathematics as well as in other areas of sciences. In mathematics q-calculus has been used in machine learning for designing stochastic activation functions. Jackson [1,2] was the first to provide an application of quantum calculus and introduced q-analogue of the derivatives and integrals.

Aral and Gupta [3–5] used the concepts of q-beta functions and have introduced the new Baskakov–Durrmeyar-type operator. Moreover, Aral and Anastassiu [6,7] discussed a generalization of the complex operator, known as the q-Picard and q-Gauss–Weierstrass singulat operator. Recently, Ahmad et al. [8], introduced a class of meromorphic Janowski-type multivalent q-starlike functions involving the q-differential operator and discussed some of its important geometric properties. Moreover, Ahmad et al. [9] introduced a new q-differential operator and described some applications to the class of convex functions. Furthermore, the q-trigonometric functions are given in [10,11]. In [12,13], the geometric properties of certain classes of analytic functions associated with a q-integral operator have been studied. Moreover, in [14] (see also [15,16]), the authors have used certain higher-order q-derivatives and have defined a number of subclasses of q-starlike functions. For each of their defined functions classes, they have discussed some remarkable results. For some recent study about the q-operator, we refer the readers to [17–19].

In this paper, we first studied some basic concepts and definitions. We then defined a certain new subclass of q-starlike functions, which involved the q-cosine function. We then obtain a number of useful results, including, for example, the necessary and sufficient conditions based on the convolution idea, the growth and distortion bounds, closure theorem, convex combination, radii of starlikeness, extreme point theorem and partial sums results. Some new consequences of our main results are also given in the Remarks and Corollaries.

In the present work, we use the following basic definitions and notations.

Definition 1 ([10]). *Let $q \in (0,1)$. The q-number $[\lambda]_q$ is defined as follows:*

$$[\lambda]_q = \begin{cases} \frac{1-q^\lambda}{1-q} & (\lambda \in \mathbb{C}) \\ \sum\limits_{k=0}^{n-1} q^k = 1 + q + q^2 + ... + q^{n-1} & (\lambda = n \in \mathbb{N}). \end{cases}$$

Definition 2 ([10]). *For $q \in (0,1)$ The q-factorial $[n]_q!$ is defined by*

$$[n]_q! = \begin{cases} 1 & (n = 0) \\ \prod\limits_{k=1}^{n} [k]_q & (n \in \mathbb{N}). \end{cases}$$

The q-difference operator for a function χ is defined as:

$$D_q\chi(\xi) = \frac{\chi(\xi) - \chi(q\xi)}{\xi(1-q)} \qquad (z \neq 0), \tag{1}$$

where $0 < q < 1$. One can easily see that for $n \in \mathbb{N}$ and $\xi \in \mathbf{U}$ (where by $\mathbf{U}$ we mean the open unit disk as defined in (3))

$$D_q\left\{\sum_{n=1}^{\infty} a_n z^n\right\} = \sum_{n=1}^{\infty} [n]_q a_n z^{n-1}, \tag{2}$$

where

$$[n]_q = \frac{1-q^n}{1-q} = 1 + \sum_{l=1}^{n} q^l \quad \text{and} \quad [0]_q = 0.$$

The q-number shift factorial (for any non-negative integer n) is defined by

$$[n]_q! = \begin{cases} 1 & (n = 0) \\ [1]_q[2]_q[3]_q \cdots [n]_q & (n \in \mathbb{N}) \end{cases}$$

Let $\mathcal{A}$ denote the class of all analytic (holomorphic) functions $\chi(\xi)$ defined in the open unit disk

$$\mathbf{U} = \{\xi : \xi \in \mathbb{C} \quad \text{and} \quad |\xi| < 1\}, \tag{3}$$

which are normalized by the conditions

$$\chi(0) = 0 \quad \text{and} \quad \chi'(0) = 1.$$

Thus, each $\chi(\xi) \in \mathcal{A}$ has the following Taylor series expansion:

$$\chi(\xi) = \xi + \sum_{n=2}^{\infty} a_n \xi^n \;\; \xi \in \mathbf{U}. \tag{4}$$

The analytic functions play a vital role in different areas of mathematics. In particular, a study about analytic extension and analytic singularity can be found in [20].

Furthermore, let $\mathcal{S}$ denote a subfamily of $\mathcal{A}$, which are univalent in $\mathbf{U}$. For two functions $h_1, h_2 \in \mathcal{A}$ and a holomorphic function w with the property

$$|w(\xi)| \leq |\xi| \quad \text{and} \quad w(0) = 0$$

the subordination can be defined by the following relation

$$h_1(\xi) = h_2(w(\xi)) \qquad (\xi \in \mathbf{U}).$$

Moreover, one can also rewrite the above condition as follows:

$$h_1 \prec h_2 \Leftrightarrow h_1(0) = h_2(0) \quad \text{and} \quad h_1(\mathbf{U}) \subset h_2(\mathbf{U}).$$

Ma and Minda [21] were the first to introduce the class $\mathcal{S}^*(\Phi)$ in 1992, as follows:

$$\mathcal{S}^*(\Phi) = \left\{\chi \in \mathcal{A} : \frac{\xi\chi'(\xi)}{\chi(\xi)} \prec \Phi(\xi)\right\}, \tag{5}$$

where Φ is an holomorphic function with a positive real part in $\mathbf{U}$ and image of function Φ under an open unit disc is a starlike shape with respect to $\Phi(0) = 1$ and $\Phi'(0) > 0$. Moreover, a number of useful geometric properties such as distortion, growth and covering results were studied by them. If we pick $\Phi(\xi) = (1+\xi)/(1-\xi)$ specifically, then the class $\mathcal{S}^*(\Phi)$ is reduced to the familiar class of starlike functions. For the various choices of the function Φ on the right hand side of (5), a number of known and new subclasses of $\mathcal{S}$, whose image domains have some interesting geometrical configurations, can be obtained. Some of them are listed as follows:

1. If we pick
$$\Phi(\xi) = 1 + \sin\xi,$$
then we obtain the class
$$\mathcal{S}^*_{\sin} = \mathcal{S}^*(1 + \sin\xi),$$
which is the class of starlike functions whose image under an open unit is eight-shaped and was introduced and studied by Cho et al. [22].
2. For the choice
$$\Phi(\xi) = 1 + \xi - \frac{1}{3}\xi^3,$$
we obtain the class
$$\mathcal{S}^*_{nep} = \mathcal{S}^*\left(1 + \xi - \frac{1}{3}\xi^3\right),$$
whose image is bounded by a nephroid-shaped region and was introduced and investigated by Wani and Swaminathan [23].
3. If we put $\mathcal{S}^*(\Phi)$ with
$$\Phi(\xi) = \sqrt{1 + \xi},$$
then the functions class leads to the class
$$\mathcal{S}^*_{\mathcal{L}} = \mathcal{S}^*\left(\sqrt{1 + \xi}\right),$$
which is described as the functions of starlike functions, bounded by lemniscate of Bernoulli in right half plan, and was developed by Sokól and Stankiewicz [7].
4. Moreover, if we take
$$\Phi(\xi) = 1 + \frac{4}{3}\xi + \frac{2}{3}\xi^2,$$
we obtain the class
$$\mathcal{S}^*_{car} = \mathcal{S}^*\left(1 + \frac{4}{3}\xi + \frac{2}{3}\xi^2\right),$$
which is a cardioid shape starlike functions class and is studied by Sharma et al. [24].
5. Moreover, if we take $\Phi(\xi) = e^{\xi}$, we obtain the class $\mathcal{S}^*_{\exp} = \mathcal{S}^*(e^{\xi})$, which was introduced and studied by Mendiratta et al. [25]. On the other side, if we take

$\Phi(\xi) = \xi + \sqrt{1+\xi^2}$, we obtain the class $\mathcal{S}_1^* = \mathcal{S}^*\left(\xi + \sqrt{1+\xi^2}\right)$, which maps $\mathbf{U}$ to a crescent-shaped region and was introduced by Raina and Sokól [26].

6. Moreover, if we take

$$\Phi(\xi) = \frac{1+(1-2\alpha)\xi}{1-\xi},$$

we obtain the well-known class of starlike functions of order α.

Furthermore, numerous subclasses of the class of starlike functions were introduced see [27–29] by taking some specific functions such as functions connected with Bell numbers, shell-like curves connected with Fibonacci numbers and functions associated with conic domains, alternatively, of Φ in (5).

The class of q-starlike functions can be defined as follows.

Definition 3 (see [30])**.** *Any function ξ, $\xi \in \mathcal{A}$ is placed in the functional class $\mathcal{S}_q^*$ if*

$$\chi(0) = \chi'(0) - 1 = 0 \tag{6}$$

and

$$\left|\frac{\xi}{\chi(\xi)} D_q\chi(\xi) - \frac{1}{1-q}\right| \leq \frac{1}{1-q}, \tag{7}$$

Equivalently, by using the principle of subordination between analytic functions, we can rewrite the conditions in (6) and (7) as follows:

$$\frac{\xi}{\chi(\xi)} D_q\chi(\xi) \prec \widehat{p}(z) \qquad \left(\widehat{p}(\xi) = \frac{1+\xi}{1-q\xi}\right).$$

One way to generalize the class $\mathcal{S}_q^*$ of q-starlike functions, is by replacing the function $\widehat{p}(\xi)$ with the q-cosine functions; the appropriate definition turns out to be the following.

Definition 4. *A function $\xi \in \mathcal{A}$ is said to be in the functional class $\mathcal{S}_{\cos_q}^*$ if*

$$\frac{\xi D_q\chi(\xi)}{\chi(\xi)} \prec \cos_q(z) \quad (\xi \in \mathbf{U}).$$

Remark 1. *When $q \to 1-$, then we obtain the class*

$$\mathcal{S}_{\cos}^* = \left\{\chi \in \mathcal{A} : \frac{\xi\chi'(\xi)}{\chi(\xi)} \prec \cos(z)\right\} \quad (\xi \in \mathbf{U}),$$

which was defined by Bano and Raza [31].

2. Main Results

The convolution of two functions χ and h, where $\chi \in \mathcal{A}$ and given by (4) and h is given by

$$h(z) = \xi + \sum_{n=2}^{\infty} b_n\xi^n,$$

is denoted by $(\chi * h)(\xi)$ and is defined as:

$$(\chi * h)(z) = \xi + \sum_{n=2}^{\infty} a_n b_n \xi^n \qquad (z \in \mathbf{U}).$$

Theorem 1. *Let $\chi(\xi) \in \mathcal{S}_{\cos_q}^*$ then*

$$\frac{1}{\xi}\left(\chi(\xi) * \frac{\xi - \Lambda\xi^2}{(1-\xi)(1-q\xi)}\right) \neq 0, \tag{8}$$

where

$$\Lambda = \frac{\cos_q\left(e^{i\theta}\right)}{\cos_q\left(e^{i\theta}\right) - 1}.$$

Proof. Let $\chi(\xi) \in \mathcal{S}^*_{\cos_q}$ then

$$\frac{\xi D_q \chi(\xi)}{\chi(\xi)} = \cos_q(w(\xi)), \tag{9}$$

where $w(\xi)$ is a Schwarz function, having the properties that $w(0) = 0$ and $|w(\xi)| < 1$. If we take $w(\xi) = e^{i\theta}$, $-\pi \le \theta \le \pi$, then the following holds true

$$\frac{\xi D_q \chi(\xi)}{\chi(\xi)} \ne \cos_q\left(e^{i\theta}\right) \tag{10}$$

$$\Leftrightarrow \xi D_q \chi(\xi) - \left(\cos_q\left(e^{i\theta}\right)\right)\chi(\xi) \ne 0. \tag{11}$$

Using the relation

$$\xi D_q \chi(\xi) = \chi(\xi) * \frac{\xi}{(1-\xi)(1-q\xi)} \text{ and } \chi(\xi) = \chi(\xi) * \frac{\xi}{(1-\xi)}.$$

And Equation (11), becomes

$$\frac{1}{\xi}\left(\chi(\xi) * \frac{\xi - \Lambda\xi^2}{(1-\xi)(1-q\xi)}\right) \ne 0,$$

where Λ is given above and hence the result is completed. □

Corollary 1. *If $\chi(\xi) \in \mathcal{S}^*_{\cos}$ and it has the form given in (4), then*

$$\frac{1}{\xi}\left(\chi(\xi) * \frac{\xi - \Lambda\xi^2}{(1-\xi)^2}\right) \ne 0,$$

where

$$\Lambda = \frac{\cos\left(e^{i\theta}\right)}{\cos\left(e^{i\theta}\right) - 1}.$$

Theorem 2. *If a function $\chi \in \mathcal{S}^*_{\cos_q}$, then*

$$1 - \sum_{n=2}^{\infty} \frac{[n]_q - \cos_q\left(e^{i\theta}\right)}{\cos_q\left(e^{i\theta}\right) - 1} a_n \xi^{n-1} \ne 0. \tag{12}$$

Proof. In Theorem 1, we have proved that if $\chi(\xi) \in \mathcal{S}^*_{\cos_q}$ then the condition in (8) holds true. We can rewrite (8) as follows

$$\begin{aligned}
\frac{1}{\xi}\left(\chi(\xi) * \frac{\xi - \Lambda\xi^2}{(1-\xi)(1-q\xi)}\right) &\ne 0 \\
\frac{1}{\xi}\left(\chi(\xi) * \frac{\xi}{(1-\xi)(1-q\xi)} - \chi(\xi) * \frac{\Lambda\xi^2}{(1-\xi)(1-q\xi)}\right) &\ne 0 \\
\frac{1}{\xi}\left(\xi D_q \chi(\xi) - \Lambda\left(\xi D_q \chi(\xi) - \chi(\xi)\right)\right) &\ne 0 \\
D_q \chi(\xi) - \Lambda\left(D_q \chi(\xi) - \frac{\chi(\xi)}{\xi}\right) &\ne 0.
\end{aligned}$$

Using series of $D_q\chi(\xi)$ and $\chi(\xi)$ in above, we obtain

$$1-\sum_{n=2}^{\infty}[(\Lambda-1)-\Lambda]a_n\xi^{n-1}\neq 0,$$

after putting the value of Λ, we obtain the desired result. □

Corollary 2. *The sufficient criteria for holomorphic function $\chi(\xi)\in\mathcal{S}^*_{\cos}$ is*

$$1-\sum_{n=2}^{\infty}\frac{n-\cos\left(e^{i\theta}\right)}{\cos\left(e^{i\theta}\right)-1}a_n\xi^{n-1}\neq 0. \tag{13}$$

The following Theorem is based on the idea given in [32].

Theorem 3. *If an analytic function $\chi\in\mathcal{A}$ satisfy*

$$\sum_{n=2}^{\infty}\left|[n]_q-\cos_q\left(e^{i\theta}\right)\right||a_n|<\left|\cos_q\left(e^{i\theta}\right)-1\right|, \tag{14}$$

then $\chi(\xi)\in\mathcal{S}^*_{\cos_q}$.

Proof. To prove the desire result, we use the relation (12); we have

$$\begin{aligned}\left|1-\sum_{n=2}^{\infty}\frac{[n]_q-\cos_q\left(e^{i\theta}\right)}{\cos_q\left(e^{i\theta}\right)-1}a_n\xi^{n-1}\right| &> 1-\sum_{n=2}^{\infty}\left|\frac{[n]_q-\cos_q\left(e^{i\theta}\right)}{\cos_q\left(e^{i\theta}\right)-1}a_n\xi^{n-1}\right|\\ &=1-\sum_{n=2}^{\infty}\left|\frac{[n]_q-\cos_q\left(e^{i\theta}\right)}{\cos_q\left(e^{i\theta}\right)-1}\right||a_n|\left|\xi^{n-1}\right|,\end{aligned}$$

from relation (14), we establish

$$1-\sum_{n=2}^{\infty}\left|\frac{[n]_q-\cos_q\left(e^{i\theta}\right)}{\cos_q\left(e^{i\theta}\right)-1}\right||a_n|>0.$$

Hence due to Theorem 1 we conclude that $\chi\in\mathcal{S}^*_{\cos_q}$. □

Corollary 3. *A function $\chi(\xi)\in\mathcal{A}$ be of the form (4), and satisfy*

$$\sum_{n=2}^{\infty}\left|n-\cos\left(e^{i\theta}\right)\right||a_n|<\left|\cos\left(e^{i\theta}\right)-1\right|, \tag{15}$$

then $\chi(\xi)\in\mathcal{S}^*_{\cos}$.

To state and prove the following results, we used the idea presented in [8,9].

Theorem 4. *Let $\chi\in\mathcal{S}^*_{\cos_q}$ and $|\xi|=r$. Then*

$$r-\left(\frac{\left|\cos_q\left(e^{i\theta}\right)-1\right|}{\left|[2]_q-\cos_q\left(e^{i\theta}\right)\right|}\right)r^2\leq|\chi(\xi)|\leq r+\left(\frac{\left|\cos_q\left(e^{i\theta}\right)-1\right|}{\left|[2]_q-\cos_q\left(e^{i\theta}\right)\right|}\right)r^2. \tag{16}$$

Proof. Consider

$$\begin{aligned} |\chi(\xi)| &= \left|\xi + \sum_{n=2}^{\infty} a_n \xi^n\right| \\ &\leq r + \sum_{n=2}^{\infty} |a_n| r^n. \end{aligned}$$

Since $r^n \leq r^2$ for $n \geq 2$ and $r < 1$, we have

$$|\chi(\xi)| \leq r + r^2 \sum_{n=2}^{\infty} |a_n|. \tag{17}$$

Similarly,

$$|\chi(\xi)| \geq r - r^2 \sum_{n=2}^{\infty} |a_n|. \tag{18}$$

Now, (14) implies that

$$\sum_{n=2}^{\infty} \frac{\left|[n]_q - \cos_q\left(e^{i\theta}\right)\right|}{\left|\cos_q\left(e^{i\theta}\right) - 1\right|} |a_n| < 1.$$

Since

$$\frac{\left|[2]_q - \cos_q\left(e^{i\theta}\right)\right|}{\left|\cos_q\left(e^{i\theta}\right) - 1\right|} \sum_{n=2}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} \frac{\left|[n]_q - \cos_q\left(e^{i\theta}\right)\right|}{\left|\cos_q\left(e^{i\theta}\right) - 1\right|} |a_n|,$$

from this we obtain

$$\frac{\left|[2]_q - \cos_q\left(e^{i\theta}\right)\right|}{\left|\cos_q\left(e^{i\theta}\right) - 1\right|} \sum_{n=2}^{\infty} |a_n| < 1,$$

one can easily write this as

$$\sum_{n=2}^{\infty} |a_n| < \frac{\left|[2]_q - \cos_q\left(e^{i\theta}\right)\right|}{\left|\cos_q\left(e^{i\theta}\right) - 1\right|},$$

now putting this value in (17) and (18), we obtain the desired inequality. □

Corollary 4. *Let $\chi \in \mathcal{S}^*_{\cos}$ and $|\xi| = r$. Then*

$$r - \left(\frac{\left|\cos\left(e^{i\theta}\right) - 1\right|}{\left|2 - \cos\left(e^{i\theta}\right)\right|}\right) r^2 \leq |\chi(\xi)| \leq r + \left(\frac{\left|\cos\left(e^{i\theta}\right) - 1\right|}{\left|2 - \cos\left(e^{i\theta}\right)\right|}\right) r^2. \tag{19}$$

Theorem 5. *Let $\chi \in \mathcal{S}^*_{\cos_q}$ and $|\xi| = r$. Then*

$$1 - \left(\frac{\left|\cos_q\left(e^{i\theta}\right) - 1\right|}{\left|[2]_q - \cos_q\left(e^{i\theta}\right)\right|}\right) r \leq \left|D_q \chi(\xi)\right| \leq 1 + \left(\frac{\left|\cos_q\left(e^{i\theta}\right) - 1\right|}{\left|[2]_q - \cos_q\left(e^{i\theta}\right)\right|}\right) r. \tag{20}$$

Proof. Consider

$$\begin{aligned} \left|D_q \chi(\xi)\right| &= \left|1 + \sum_{n=2}^{\infty} [n]_q a_n \xi^n\right| \\ &\leq 1 + \sum_{n=2}^{\infty} [n]_q |a_n| r^{n-1}. \end{aligned}$$

Since $r^{n-1} \leq r$ for $n \geq 2$ and $r < 1$, we have

$$\left|D_q\chi(\xi)\right| \leq 1 + r\sum_{n=2}^{\infty}|a_n|. \tag{21}$$

Similarly,

$$\left|D_q\chi(\xi)\right| \geq 1 - r\sum_{n=2}^{\infty}[n]_q|a_n|. \tag{22}$$

Now, (14) implies that

$$\sum_{n=2}^{\infty}\frac{\left|[n]_q - \cos_q\left(e^{i\theta}\right)\right|}{\left|\cos_q\left(e^{i\theta}\right) - 1\right|}|a_n| < 1.$$

Since

$$\frac{\left|[2]_q - \cos_q\left(e^{i\theta}\right)\right|}{\left|\cos_q\left(e^{i\theta}\right) - 1\right|}\sum_{n=2}^{\infty}|a_n| \leq \sum_{n=2}^{\infty}\frac{\left|[n]_q - \cos_q\left(e^{i\theta}\right)\right|}{\left|\cos_q\left(e^{i\theta}\right) - 1\right|}|a_n|,$$

from this we obtain

$$\frac{\left|[2]_q - \cos_q\left(e^{i\theta}\right)\right|}{\left|\cos_q\left(e^{i\theta}\right) - 1\right|}\sum_{n=2}^{\infty}|a_n| < 1,$$

one can easily write this as

$$\sum_{n=2}^{\infty}|a_n| < \frac{\left|\cos_q\left(e^{i\theta}\right) - 1\right|}{\left|[2]_q - \cos_q\left(e^{i\theta}\right)\right|},$$

Now, putting this value in (21) and (22), we obtain the desired inequality. □

Corollary 5. *Let $\chi \in \mathcal{S}^*_{\cos}$ and $|\xi| = r$. Then*

$$1 - \left(\frac{\left|\cos\left(e^{i\theta}\right) - 1\right|}{\left|2 - \cos_q\left(e^{i\theta}\right)\right|}\right)r \leq \left|\chi'(\xi)\right| \leq 1 + \left(\frac{\left|\cos\left(e^{i\theta}\right) - 1\right|}{\left|2 - \cos_q\left(e^{i\theta}\right)\right|}\right)r. \tag{23}$$

Theorem 6. *Let $\chi_k \in \mathcal{S}^*_{\cos_q}$, $k = 1, 2, \cdots$, such that*

$$\chi_k(\xi) = \xi + \sum_{n=2}^{\infty}a_{n,k}\xi^n.$$

Then $H(\xi) = \sum_{k=1}^{\infty}\eta_k\chi_k(\xi)$, where $\sum_{n=1}^{\infty}\eta_k = 1$ is in class $\mathcal{S}^*_{\cos_q}$.

Proof. We have

$$H(\xi) = \xi + \sum_{k=1}^{\infty}\sum_{n=2}^{\infty}\eta_k a_{n,k}\xi^n.$$

Consider

$$\begin{aligned}
&\sum_{n=2}^{\infty}\frac{\left|[n]_q - \cos_q\left(e^{i\theta}\right)\right|}{\left|\cos_q\left(e^{i\theta}\right) - 1\right|}\sum_{k=1}^{\infty}\eta_k|a_{n,k}| \\
&= \sum_{k=1}^{\infty}\eta_k\left(\sum_{n=2}^{\infty}\frac{\left|[n]_q - \cos_q\left(e^{i\theta}\right)\right|}{\left|\cos_q\left(e^{i\theta}\right) - 1\right|}|a_{n,k}|\right) \\
&< \sum_{k=1}^{\infty}\eta_k = 1,
\end{aligned}$$

hence $H(\xi) \in \mathcal{S}^{*}_{\cos_q}$. $\square$

Theorem 7. *The class $\mathcal{S}^{*}_{\cos_q}$ is closed under the convex combination.*

Proof. Let g_1 and g_2 be any functions in set $\mathcal{S}^{*}_{\cos_q}$ with the following series representation

$$g_1(\xi) = \xi + \sum_{n=2}^{\infty} a_n \xi^n \text{ and } g_2(\xi) = \xi + \sum_{n=2}^{\infty} b_n \xi^n.$$

We have to show that $G(\xi) = \lambda g_1(\xi) + (1-\lambda) g_2(\xi)$, with $0 \leq \lambda \leq 1$, is in the class $\mathcal{S}^{*}_{\cos_q}$. Since

$$G(\xi) = \xi + \sum_{n=2}^{\infty} [\lambda a_n + (1-\lambda) b_n] \xi^n.$$

Consider

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{\left|[n]_q - \cos_q\left(e^{i\theta}\right)\right|}{\left|\cos_q\left(e^{i\theta}\right) - 1\right|} |\lambda a_n + (1-\lambda) b_n| \\
&\leq \lambda \sum_{n=2}^{\infty} \frac{\left|[n]_q - \cos_q\left(e^{i\theta}\right)\right|}{\left|\cos_q\left(e^{i\theta}\right) - 1\right|} |a_n| \\
&+ (1-\lambda) \sum_{n=2}^{\infty} \frac{\left|[n]_q - \cos_q\left(e^{i\theta}\right)\right|}{\left|\cos_q\left(e^{i\theta}\right) - 1\right|} |b_n| \\
&< 1.
\end{aligned}$$

Hence $G(\xi) \in \mathcal{S}^{*}_{\cos_q}$. $\square$

Theorem 8. *Let $\chi \in \mathcal{S}^{*}_{\cos_q}$. Then, for $|\xi| = r_1$, the function is starlike of the order α, where*

$$r_1 = \inf\left\{ \frac{(1-\alpha)}{(n-\alpha)} \frac{\left|[n]_q - \cos_q\left(e^{i\theta}\right)\right|}{\left|\cos_q\left(e^{i\theta}\right) - 1\right|} \right\}^{\frac{1}{n-1}}, \text{ for } n \in \mathbb{N}\backslash\{1\}.$$

Proof. We know that a function is starlike of the order α if and only if

$$\left| \frac{\xi \chi'(\xi) - \chi(\xi)}{\xi \chi'(\xi) - (2\alpha - 1)\chi(\xi)} \right| < 1.$$

After simple computation, we obtain

$$\left| \frac{\xi \chi'(\xi) - \chi(\xi)}{\xi \chi'(\xi) - (2\alpha - 1)\chi(\xi)} \right| \leq \sum_{n=2}^{\infty} \left(\frac{n-\alpha}{1-\alpha} \right) |a_n| |\xi|^{n-1}. \tag{24}$$

Moreover, from (14), we have

$$\sum_{n=2}^{\infty} \frac{\left|[n]_q - \cos_q\left(e^{i\theta}\right)\right|}{\left|\cos_q\left(e^{i\theta}\right) - 1\right|} |a_n| < 1. \tag{25}$$

From (24) is bounded by 1 if

$$\sum_{n=2}^{\infty} \left(\frac{n-\alpha}{1-\alpha} \right) |\xi|^{n-1} < \sum_{n=2}^{\infty} \frac{\left|[n]_q - \cos_q\left(e^{i\theta}\right)\right|}{\left|\cos_q\left(e^{i\theta}\right) - 1\right|},$$

implies that

$$|\xi| < \left(\frac{(1-\alpha)}{(n-\alpha)} \frac{\left| [n]_q - \cos_q\left(e^{i\theta}\right) \right|}{\left|\cos_q\left(e^{i\theta}\right) - 1\right|} \right)^{\frac{1}{n-1}} = r_1,$$

hence the proof is completed. □

Corollary 6. *Let $\chi \in \mathcal{S}^*_{\cos}$. Then, for $|\xi| = r_1$, the function is starlike of the order α, where*

$$r_1 = \inf\left\{ \frac{(1-\alpha)}{(n-\alpha)} \frac{\left| n - \cos\left(e^{i\theta}\right) \right|}{\left|\cos\left(e^{i\theta}\right) - 1\right|} \right\}^{\frac{1}{n-1}}, \text{ for } n \in \mathbb{N}\backslash\{1\}.$$

Theorem 9. *Let $\chi \in \mathcal{S}^*_{\cos_q}$. Then, for $|\xi| = r_2$, the function is close-to-convex of the order α, $(0 \leq \alpha \leq 1)$ where*

$$r_1 = \inf\left\{ \frac{(1-\alpha)}{n} \frac{\left| [n]_q - \cos_q\left(e^{i\theta}\right) \right|}{\left|\cos_q\left(e^{i\theta}\right) - 1\right|} \right\}^{\frac{1}{n-1}}, \text{ for } n \in \mathbb{N}\backslash\{1\}.$$

Proof. Let $\chi \in \mathcal{S}^*_{\cos_q}$. To establish that χ is in a class of close-to-convex functions of order α, it is enough to show that

$$\left|\chi'(\xi) - 1\right| < 1 - \alpha.$$

Using simple computation, we obtain

$$\sum_{n=2}^{\infty} \frac{n}{1-\alpha} |a_n| |\xi|^{n-1} < 1. \tag{26}$$

Since $\chi \in \mathcal{S}^*_{\cos_q}$, in light of Theorem 3, we have

$$\sum_{n=2}^{\infty} \frac{\left| [n]_q - \cos_q\left(e^{i\theta}\right) \right|}{\left|\cos_q\left(e^{i\theta}\right) - 1\right|} |a_n| < 1. \tag{27}$$

Inequality (26) holds true if the following relation holds:

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{n}{1-\alpha} |a_n| |\xi|^{n-1} \\ <\; &\sum_{n=2}^{\infty} \frac{\left| [n]_q - \cos_q\left(e^{i\theta}\right) \right|}{\left|\cos_q\left(e^{i\theta}\right) - 1\right|} |a_n|, \end{aligned}$$

after simple calculation, we obtain

$$|\xi| < \left(\frac{(1-\alpha)}{n} \frac{\left| [n]_q - \cos_q\left(e^{i\theta}\right) \right|}{\left|\cos_q\left(e^{i\theta}\right) - 1\right|} \right)^{\frac{1}{n-1}} = r_2.$$

Hence complete the proof. □

Corollary 7. *Let $\chi \in \mathcal{S}^*_{\cos}$. Then, for $|\xi| = r_2$, the function is close-to-convex of order α, $(0 \leq \alpha \leq 1)$ where*

$$r_1 = \inf\left\{ \frac{(1-\alpha)}{n} \frac{\left| n - \cos\left(e^{i\theta}\right) \right|}{\left|\cos\left(e^{i\theta}\right) - 1\right|} \right\}^{\frac{1}{n-1}}, \text{ for } n \in \mathbb{N}\backslash\{1\}.$$

3. Partial Sums

Silverman [33] in the year 1997, studied the partial sums results for the class of starlike and convex functions and developed through

$$\begin{aligned} \chi_1(\xi) &= \xi, \\ \chi_m(\xi) &= \xi + \sum_{n=2}^{m} a_n \xi^n, \quad = 2, 3, 4, ... \end{aligned}$$

Several authors have been motivated by the idea of Silverman and have investigated partial sums for different subclasses. For some recent investigations, we refer authors to view [34–36].

Theorem 10. *If χ of the form (4) satisfies condition (14), then*

$$\Re\left(\frac{\chi(\xi)}{\chi_j(\xi)}\right) \geq 1 - \frac{1}{u_{j+1}} \qquad (\forall \xi \in \mathbf{U}) \tag{28}$$

and

$$\Re\left(\frac{\chi_j(\xi)}{\chi(\xi)}\right) \geq \frac{u_{j+1}}{1 + u_{j+1}} \qquad (\forall \xi \in \mathbf{U}), \tag{29}$$

where

$$u_j = \frac{\left|[n]_q - \cos_q\left(e^{i\theta}\right)\right|}{\left|\cos_q\left(e^{i\theta}\right) - 1\right|}, \tag{30}$$

and

$$u_j \geq \begin{cases} 1 \text{ for } j = 2, 3, \cdots, m \\ u_{j+1} \text{ for } j = m+1, \cdots. \end{cases}$$

Proof. To prove inequality (28), we set:

$$\begin{aligned} \zeta_{j+1}\left[\frac{\chi(\xi)}{\chi_j(\xi)} - \left(1 - \frac{1}{\zeta_{j+1}}\right)\right] &= \frac{1 + \sum_{n=2}^{j} a_n \xi^{n-1} + \zeta_{j+1} \sum_{n=j+1}^{\infty} a_n \xi^{n-1}}{1 + \sum_{n=2}^{j} a_n \xi^{n-1}} \\ &= \frac{1 + \phi_1(\xi)}{1 + \phi_2(\xi)}. \end{aligned}$$

We now set:

$$\frac{1 + \phi_1(\xi)}{1 + \phi_2(\xi)} = \frac{1 + w(\xi)}{1 - w(\xi)}.$$

Then, we find after some suitable simplifications, that:

$$w(\xi) = \frac{\phi_1(\xi) - \phi_2(\xi)}{2 + \phi_1(\xi) + \phi_2(\xi)}.$$

Thus, clearly, we find that:

$$w(\xi) = \frac{\zeta_{j+1} \sum_{n=j+1}^{\infty} a_n \xi^{n-1}}{2 + 2\sum_{n=2}^{j} a_n \xi^{n-1} + \zeta_{j+1} \sum_{n=j+1}^{\infty} a_n \xi^{n-1}}$$

By making use of the trigonometric inequalities with the condition $|\xi| < 1$, we obtain the following inequality:

$$|w(\xi)| \leq \frac{\zeta_{j+1} \sum_{n=j+1}^{\infty} |a_n|}{2 - 2\sum_{n=2}^{j} |a_n| - \zeta_{j+1} \sum_{n=j+1}^{\infty} |a_n|}.$$

We can now see that:

$$|w(\xi)| \leq 1$$

if and only if

$$2\zeta_{j+1} \sum_{n=j+1}^{\infty} |a_n| \leq 2 - 2\sum_{n=2}^{j} |a_n|,$$

which implies that:

$$\sum_{n=2}^{j} |a_n| + \zeta_{j+1} \sum_{n=j+1}^{\infty} |a_n| \leq 1. \tag{31}$$

Finally, to show the inequality in (28), it enough to establish that the left hand side of (31) is bounded above by the following sum:

$$\sum_{n=2}^{\infty} \zeta_n |a_n|,$$

which is equivalent to

$$\sum_{n=2}^{j} (\zeta_n - 1)|a_n| + \sum_{n=j+1}^{\infty} (\zeta_n - \zeta_{j+1})|a_n| \geq 0. \tag{32}$$

In virtue of (32), the proof of approximation in (28) is now completed.

Next, in order to prove the inequality (29), we set:

$$\begin{aligned}(1+\zeta_{j+1})\left(\frac{\chi_j(\xi)}{\chi(\xi)} - \frac{\zeta_{j+1}}{1+\zeta_{j+1}}\right) &= \frac{1 + \sum_{n=2}^{j} a_n \xi^{n-1} - \zeta_{j+1} \sum_{n=j+1}^{\infty} a_n \xi^{n-1}}{1 + \sum_{n=2}^{\infty} a_n \xi^{n-1}} \\ &= \frac{1+w(\xi)}{1-w(\xi)},\end{aligned}$$

where

$$|w(\xi)| \leq \frac{(1+\zeta_{j+1}) \sum_{n=j+1}^{\infty} |a_n|}{2 - 2\sum_{n=2}^{j} |a_n| - (\zeta_{j+1} - 1) \sum_{n=j+1}^{\infty} |a_n|} \leq 1. \tag{33}$$

This last inequality in (33) is equivalent to

$$\sum_{n=2}^{j} |a_n| + \zeta_{j+1} \sum_{n=j+1}^{\infty} |a_n| \leq 1. \tag{34}$$

Finally, it can be observed that the inequality in the left hand side of (34) is bounded above by:

$$\sum_{n=2}^{\infty} \zeta_n |a_n|,$$

and its complete the proof of the assertion (29), which completes the proof of Theorem 10. □

We next prove the result involving the derivatives.

Theorem 11. *If χ of the form (4) satisfies condition (14), then*

$$\Re\left(\frac{\chi'(\xi)}{\chi_j'(\xi)}\right) \geq 1 - \frac{j+1}{u_{j+1}} \quad (\forall \xi \in \mathbf{U}) \tag{35}$$

and

$$\Re\left(\frac{\chi_j'(\xi)}{\chi'(\xi)}\right) \geq \frac{u_{j+1}}{u_{j+1}+j+1} \quad (\forall \xi \in \mathbf{U}), \tag{36}$$

where u_j is given by (30).

Proof. Theorem 11 can be proven similarly as we proved Theorem 10, therefore we omit the analogous details of the proof. □

4. Conclusions

Using the subordinations principle, we have established a new subfamily of q -starlike functions based on the q-analogue of the cosine function. This new subfamily generalized the family of holomorphic functions associated with the cosine, which was introduced by Bano et al. [31]. Moreover, we have investigated different geometric properties such as inclusions, relations and radii problems. For this particular family, we have also investigated a convolutions-type result, and based on this convolution result we have derived necessary and sufficient conditions. Furthermore, we have established some useful results such as closure theorem, growth and distortion approximation, convex combination, radii of starlikness and partial sums results.

Moreover, this idea can be extended to find some other problems, such as Fekete-Szegö inequalities, second and third order Hankel determinant and Toeplitz determinants. Moreover, these types of results can be obtained for other subfamilies whose image domain lies in other different trigonometric functions. The same techniques can be used to define another functions classes of symmetric starlike functions. Interested readers may also use iteration processes to develop an approximate common fixed point of the mapping $\frac{\xi D_q \chi(\xi)}{\chi(\xi)}$ (see for details [37]).

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The author is thankful to the reviewers for their many valuable suggestions and recommendations, which significantly improved the revised version.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Jackson, F.H. On q-functions and certain difference operator. *Trans. R. Soc. Edinb.* **1909**, *46*, 253–281. [CrossRef]
2. Jackson, F.H. On q-definite integrals. *Quart. J. Pure Appl. Math.* **1910**, *41*, 193–203.
3. Aral, A.; Gupta, V. On q-Baskakov type operators. *Demostr. Math.* **2009**, *42*, 109–122.

4. Aral, A.; Gupta, V. On the Durrmeyer type modification of the q-Baskakov type operators. *Nonlinear Anal. Theory Methods Appl.* **2010**, *72*, 1171–1181. [CrossRef]
5. Aral, A.; Gupta, V. Generalized q-Baskakov type operators. *Math. Slovaca* **2011**, *61*, 619–634. [CrossRef]
6. Anastassiu, G.A.; Gal, S.G. Geometric and approximation properties of some singular integrals in the unit disc. *J. Comput. Anal. Appl.* **2006**, *8*, 249–261.
7. Sokól, J.; Stankiewicz, J. Radius of convexity of some subclasses of strongly starlike functions. *Zesz. Nauk. Politech. Rzeszowskiej Mat.* **1996**, *19*, 101–105.
8. Ahmad, B.; Mashwani, W.K.; Araci, S.; Mustafa, S.; Khan, M.G.; Khan, B. A subclass of meromorphic Janowski-type multivalent q -starlike functions involving a q-differential operator. *Adv. Continous Discret. Model.* **2022**, *2*, 2022. [CrossRef]
9. Ahmad, B.; Khan, M.G.; Aouf, M.K.; Salleh, Z.; Tang, H. Applications of a new difference operator in the Janowski-type meromorphic convex functions. *J. Funct. Spaces* **2021**, *2021*, 9967640. [CrossRef]
10. Gasper, G.; Rahman, M. *Basic Hypergeometric Series*; Cambridge University Press: Cambridge, UK, 2004; Volume 96.
11. Ashish, V.; Sarasvati, Y. Recursion formulas for Srivastava's general triple q-hypergeometric series. *Afr. Mat.* **2020**, *31*, 869–885.
12. Ullah, K.; Zainab, S.; Arif, M.; Darus, M.; Shutaywi, M. Radius problems for starlike functions associated with the tan hyperbolic function. *J. Funct. Spaces* **2021**, *2021*, 9967640. [CrossRef]
13. Alotaibi, A.; Arif, M.; Alghamdi, M.A.; Hussain, S. Starlikness associated with cosine hyperbolic function. *Mathematics* **2020**, *8*, 1118. [CrossRef]
14. Khan, B.; Liu, Z.-G.; Srivastava, H.M.; Khan, N.; Darus, M.; Tahir, M. A study of some families of multivalent q-starlike functions involving higher-order q-derivatives. *Mathematics* **2020**, *8*, 1470. [CrossRef]
15. Hu, Q.; Srivastava, H.M.; Ahmad, B.; Khan, N.; Khan, M.G.; Mashwani, W.K.; Khan, B. A subclass of multivalent Janowski type q-starlike functions and its consequences. *Symmetry* **2021**, *13*, 1275. [CrossRef]
16. Rehman, M.S.U.; Ahmad, Q.Z.; Srivastava, H.M.; Khan, N.; Darus, M.; Khan, B. Applications of higher-order q-derivatives to the subclass of q-starlike functions associated with the Janowski functions. *AIMS Math.* **2021**, *6*, 1110–1125. [CrossRef]
17. Khan, B.; Liu, Z.-G.; Shaba, T.G.; Araci, S.; Khan, N.; Khan, M.G. Applications of q-Derivative Operator to the Subclass of Bi-Univalent Functions Involving q-Chebyshev Polynomials. *J. Math.* **2022**, *2022*, 8162182. [CrossRef]
18. Shi, L.; Ahmad, B.; Khan, N.; Khan, M.G.; Araci, S.; Mashwani, W.K.; Khan, B. Coefficient Estimates for a Subclass of Meromorphic Multivalent q-Close-to-Convex Functions. *Symmetry* **2021**, *13*, 1840. [CrossRef]
19. Ahmad, Q.Z.; Khan, N.; Raza, M.; Tahir, M.; Khan, B. Certain q-difference operators and their applications to the subclass of meromorphic q-starlike functions. *Filomat* **2019**, *33*, 3385–3397. [CrossRef]
20. Diao, H.; Liu, H.; Wang, L. On generalized Holmgren's principle to the Lamé operator with applications to inverse elastic problems. *Calc. Var. Partial. Differ. Equ.* **2020**, *59*, 179. [CrossRef]
21. Ma, W.C.; Minda, D. A unified treatment of some special classes of univalent functions. In Proceedings of the Conference on Complex Analysis, Tianjin, China, 19–23 June 1992; Int. Press: Cambridge, UK, 1992; pp. 157–169.
22. Cho, N.E.; Kumar, V.; Kumar, S.S.; Ravichandran, V. Radius problems for starlike functions associated with the sine function. *Bull. Iran. Math. Soc.* **2019**, *45*, 213–232. [CrossRef]
23. Wani, L.A.; Swaminathan, A. Starlike and convex functions associated with a Nephroid domain. *Bull. Malays. Math. Sci. Soc.* **2021**, *44*, 79–104. [CrossRef]
24. Sharma, K.; Jain, N.K.; Ravichandran, V. Starlike functions associated with cardioid. *Afrika Math.* **2016**, *27*, 923–939. [CrossRef]
25. Mendiratta, R.; Nagpal, S.; Ravichandran, V. On a subclass of strongly starlike functions associated with exponential function. *Bull. Malays. Math. Sci. Soc.* **2015**, *38*, 365–386. [CrossRef]
26. Raina, R.K.; Sokól, J. On Coefficient estimates for a certain class of starlike functions. *Hacet. J. Math. Statist.* **2015**, *44*, 1427–1433. [CrossRef]
27. Cho, N.E.; Kumar, S.; Kumar, V.; Ravichandran, V.; Srivastava, H.M. Starlike functions related to the Bell numbers. *Symmetry* **2019**, *11*, 219. [CrossRef]
28. Dziok, J.; Raina, R.K.; Sokól, J. On certain subclasses of starlike functions related to a shell-like curve connected with Fibonacci numbers. *Math. Comput. Model.* **2013**, *57*, 1203–1211. [CrossRef]
29. Kanas, S.; Răducanu, D. Some classes of analytic functions related to conic domains. *Math. Slovaca* **2014**, *64*, 1183–1196. [CrossRef]
30. Ismail, M.E.-H.; Merkes, E.; Styer, D. A generalization of starlike functions. *Complex Var. Theory Appl.* **1990**, *14*, 77–84. [CrossRef]
31. Bano, K.; Raza, M. Starlike functions associated with cosine functions. *Bull. Iran. Math. Soc.* **2021**, *47*, 1513–1532. [CrossRef]
32. Noor, K.I.; Malik, S.N. On coefficient inequalities of functions associated with conic domains. *Comput. Math. Appl.* **2011**, *62*, 2209–2217. [CrossRef]
33. Silverman, H. Partial sums of starlike and convex functions. *J. Math. Anal. Appl.* **1997**, *209*, 221–227. [CrossRef]
34. Khan, M.G.; Ahmad, B.; Khan, N.; Mashwani, W.K.; Arjika, S.; Khan, B.; Chinram, R. Applications of Mittag-Leffler type Poisson distribution to a subclass of analytic functions involving conic-type regions. *J. Funct. Spaces* **2021**, *2021*, 9.

35. Khan, M.G.; Darus, M.; Ahmad, B.; Murugusundaramoorty, G.; Khan, R.; Khan, N. Meromorphic starlike functions with respect to symmetric points. *Int. J. Anal. Appl.* **2020**, *18*, 1037–1047.
36. Khan, M.G.; Ahmad, B.; Salleh, Z.; Lukman, I. On subclasses of harmonic mappings involving Frasin operator. *Afr. Mat.* **2021**, *32*, 1159–1171. [CrossRef]
37. Zada, M.B.; Sarwar, M.; Tunc, C. Fixed point theorems in b-metric spaces and their applications to non-linear fractional differential and integral equations. *J. Fixed Point Theory Appl.* **2018**, *20*, 25. [CrossRef]