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Geometry of Developable Surfaces of Frenet Type Framed Base Curves from the Singularity Theory Viewpoint

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Abstract: In this paper, we consider Frenet type framed base curves that may have singular points and define one-parameter developable surfaces associated with such curves. By using the singularity theory, we classify the generic singularities of the developable surfaces, which are cuspidal edges and swallowtails. In order to characterize these singularities, two geometric invariants are discovered. At last, an example is given to demonstrate the main results.

Keywords: one-parameter developable surfaces; Frenet type framed base curves; singularities

MSC: 32S25; 57R45

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1. Introduction

The research concerning developable surfaces has many applications. For instance, in industrial design and modelling, P. Bo et al. proposed a method to create seamless, smooth, multi-strip developable surfaces from some design curves in [1]. By curve modifications, their work achieved high accuracy of developability with the controllable error. The superior performance of their method was demonstrated by the fabrication of some paper industrial products. In [2], P. Bo et al. also provided a method to compute design curves to define surfaces with large developability. The surfaces generated by this method do not need any explicit curve interpolation procedure. In mathematics, developable surfaces can be developed into planes without distorting the surface metric. There are many literature studies about developable surfaces (see e.g., [3–7]). A tangent developable surface is a special developable surface, which is formed by a spatial curve and its tangent lines. In classical algebraic geometry, the tangent developable surface plays an important role in the spatial curve duality theory [8]. In [4], the rectifying developable surface of a regular curve was introduced by S. Izumiya et al.; they proved that a regular curve is always the geodesic of its rectifying developable surface. The geodesic properties of a regular curve on its rectifying developable surface were also studied in [9]. For a given geodesic curve, P. Bo and W. Wang showed the expression of the rectifying developable surface of the geodesic curve. In addition, Frenet type framed base curves are smooth curves with moving frames, which may have singular points. S. Honda gave the existence and uniqueness for Frenet type framed base curves in Euclidean space [10]. Here, we study one-parameter developable surfaces, which are generated by Frenet type framed base curves, as a primary case for the study of singular manifolds in Euclidean 3-space.

Singularity theory is a direct descendant of differential calculus, with interest in equations, geometry, astronomy, physics, and other disciplines. There are some research studies about hypersurfaces immersed in different spaces from the viewpoint of singularity theory [11–14]. Furthermore, some of the latest research about singularity theory and the submanifold theory can be seen in [3,4,15–27]. For instance, J. Sun and D. Pei considered the Lorentzian hypersurface on the pseudo n-sphere and classified the singularities of this hypersurface in [11]. They also studied the geometric properties of lightlike hypersurfaces, which are degenerate submanifolds in the Minkowski 4-space [12]. In addition, one-parameter developable surfaces are hypersurface families. They can also be seen as bundles along a Frenet type framed base curve, whose fibers are developable surfaces, and the 'normals' of these developable surfaces are located in the normal planes of the Frenet type framed base curve. Rectifying developable surfaces and tangent developable surfaces of the Frenet type framed base curve are two sections of one-parameter developable surfaces. We prove that Frenet type framed base curves have similar properties to regular curves in their rectifying developable and tangent developable surfaces. We also define one-parameter support functions on the Frenet type framed base curves to investigate the singularities of one-parameter developable surfaces. By using the unfolding theory of functions, the one-parameter support functions can be used to analyze the geometric properties of one-parameter developable surfaces. One-parameter developable surfaces are the discriminant sets of one-parameter support functions. The singularities of developable surfaces are A_k-singularities $(k = 2, 3)$ of these functions. Since Frenet type framed base curves may have singular points, the situation presents some differences when compared with the regular case in [28]. For instance, three cases for developable surfaces have cuspidal edge singularities in the present paper. Theorems 1 and 2 are the main results of this paper.

The organization of this paper is as follows. In Section 2, we review the concepts of Euclidean space that we used in this paper. We define one-parameter developable surfaces of a Frenet type framed base curve and obtain two geometric invariants of the base curve in Section 3. The geometric meaning of these two invariants (Theorem 1) and the classification of singularities of one-parameter developable surfaces (Theorem 2) are also shown in this section. The preparations for the proof of Theorem 2 are in Sections 4 and 5. In Section 6, we provide an example to illustrate the main results in this paper.

All manifolds and maps considered in this paper are differentiable of class C^∞.

2. Basic Notions

In this paper, we suppose that a curve $\gamma : I \to \mathbb{R}^3$ may have singular points. We cannot define the Frenet frame along γ if γ has singular points. Fortunately, S. Honda defined a *Frenet type framed base curve* under a certain condition, as follows [10].

Definition 1. *We say that $\gamma : I \to \mathbb{R}^3$ is a Frenet type framed base curve if there exists a regular spherical curve $\mathcal{T} : I \to S^2$ and a smooth function $\alpha : I \to \mathbb{R}$, such that $\dot{\gamma}(t) = \alpha(t)\mathcal{T}(t)$ for all $t \in I$. Then we call $\mathcal{T}(t)$ a unit tangent vector and $\alpha(t)$ a speed function of $\gamma(t)$.*

Obviously, γ has singular point at t_0 if and only if $\alpha(t_0) = 0$. We define a *unit principal normal vector* by $\mathcal{N}(t) = \frac{\dot{\mathcal{T}}(t)}{\|\dot{\mathcal{T}}(t)\|}$ and a *unit binormal vector* by $\mathcal{B}(t) = \mathcal{T}(t) \times \mathcal{N}(t)$, respectively. Then an orthonormal frame $\{\mathcal{T}(t), \mathcal{N}(t), \mathcal{B}(t)\}$ along $\gamma(t)$ is obtained and we call it the *Frenet type frame* along $\gamma(t)$. We have the following formula:

$$\begin{cases} \dot{\mathcal{T}}(t) = \kappa(t)\mathcal{N}(t) \\ \dot{\mathcal{N}}(t) = -\kappa(t)\mathcal{T}(t) + \tau(t)\mathcal{B}(t) \\ \dot{\mathcal{B}}(t) = -\tau(t)\mathcal{N}(t), \end{cases}$$

where

$$\kappa(t) = \|\dot{\mathcal{T}}(t)\|, \;\; \tau(t) = det(\mathcal{T}(t), \dot{\mathcal{T}}(t), \ddot{\mathcal{T}}(t))/\|\dot{\mathcal{T}}(t)\|^2.$$

The functions $\kappa(t)$ and $\tau(t)$ are called the curvature function and the torsion function of the Frenet type framed base curve γ, respectively.

Now we briefly review the basic notions of developable surfaces and ruled surfaces. Suppose $\gamma : I \to \mathbb{R}^3$ and $\boldsymbol{\xi} : I \to \mathbb{R}^3 \setminus \{\mathbf{0}\}$ are C^∞-mappings. Consider a mapping $\boldsymbol{F}_{(\gamma,\xi)} : I \times \mathbb{R} \to \mathbb{R}^3$, which is defined by

$$\boldsymbol{F}_{(\gamma,\xi)}(t, u) = \gamma(t) + u\boldsymbol{\xi}(t).$$

We call $\boldsymbol{\xi}$ a *director curve*, γ a *base curve*, and $\boldsymbol{F}_{(\gamma,\xi)}$ a *ruled surface*, respectively. For a fixed $t \in I$, $\gamma(t) + u\boldsymbol{\xi}(t)$ is a *ruling* when u varies. A *developable surface* is a ruled surface with the vanishing Gaussian curvature. We know that a ruled surface $\boldsymbol{F}_{(\gamma,\xi)}$ is a developable surface if

$$\det(\boldsymbol{\xi}(t), \dot{\gamma}(t), \dot{\boldsymbol{\xi}}(t)) = 0.$$

Specifically, $\boldsymbol{F}_{(\gamma,\xi)}$ is called a *cylinder* if the direction of $\boldsymbol{\xi}$ is fixed. We denote $\widetilde{\boldsymbol{\xi}}(t) = \boldsymbol{\xi}(t)/\|\boldsymbol{\xi}(t)\|$ and it follows that $\boldsymbol{F}_{(\gamma,\xi)}(I \times \mathbb{R}) = \boldsymbol{F}_{(\gamma,\widetilde{\xi})}(I \times \mathbb{R})$. Then $\boldsymbol{F}_{(\gamma,\xi)}$ is a cylinder if and only if $\dot{\widetilde{\boldsymbol{\xi}}}(t) \equiv 0$. If $\dot{\widetilde{\boldsymbol{\xi}}}(t) \neq 0$, we say that $\boldsymbol{F}_{(\gamma,\xi)}$ is *non-cylindrical*. In this case, a *striction curve* of $\boldsymbol{F}_{(\gamma,\xi)}$ is defined by

$$\boldsymbol{c}(t) = \gamma(t) - \frac{\gamma(t) \cdot \dot{\widetilde{\boldsymbol{\xi}}}(t)}{\|\dot{\widetilde{\boldsymbol{\xi}}}(t)\|^2}\widetilde{\boldsymbol{\xi}}(t).$$

We know that the singular points of a non-cylindrical ruled surface are located on the striction curve. A *cone* is a non-cylindrical ruled surface whose striction curve $\boldsymbol{c}$ is constant.

3. One-Parameter Developable Surfaces

We study one-parameter developable surfaces of a Frenet type framed base curve in this section. Let γ be a Frenet type framed base curve. A spherical vector field $\boldsymbol{L} : I \times [0, \frac{\pi}{2}] \to S^2$ is defined by

$$\boldsymbol{L}(t,\theta) = \frac{\tau(t)\mathcal{T}(t) - \kappa(t)\sin\theta\cos\theta\mathcal{N}(t) + \kappa(t)\cos^2\theta\mathcal{B}(t)}{\sqrt{\tau^2(t) + \kappa^2(t)\cos^2\theta}},$$

where $\tau^2(t) + \kappa^2(t)\cos^2\theta \neq 0$. We assume throughout the whole paper that $\tau^2(t)$ $+\kappa^2(t)\cos^2\theta \neq 0$ for any $(t,\theta) \in I \times [0, \frac{\pi}{2}]$. We denote $\boldsymbol{L}(t,\theta) = \boldsymbol{L}_\theta(t)$ and consider a surface $\boldsymbol{D}_\theta : I \times \mathbb{R} \to \mathbb{R}^3$, which is defined by

$$\boldsymbol{D}_\theta(t,u) = \gamma(t) + u\boldsymbol{L}_\theta(t) = \gamma(t) + u\frac{\tau(t)\mathcal{T}(t) - \kappa(t)\sin\theta\cos\theta\mathcal{N}(t) + \kappa(t)\cos^2\theta\mathcal{B}(t)}{\sqrt{\tau^2(t) + \kappa^2(t)\cos^2\theta}}.$$

We call $\boldsymbol{D}_\theta$ *one-parameter developable surfaces* of γ. For any $\theta_0 \in [0, \frac{\pi}{2}]$, we have

$$\dot{\boldsymbol{L}}_{\theta_0}(t) = \frac{(\kappa\tau^2\sin\theta_0 + \kappa^3\sin\theta_0\cos^2\theta_0 + \kappa\dot{\tau}\cos\theta_0 - \dot{\kappa}\tau\cos\theta_0)(\kappa\cos\theta_0\mathcal{T} + \tau\sin\theta_0\mathcal{N} - \tau\cos\theta_0\mathcal{B})}{(\tau^2 + \kappa^2\cos^2\theta_0)^{\frac{3}{2}}}.$$

Then it follows

$$\begin{aligned}
&\det\big(\dot{\gamma}(t), \boldsymbol{L}_{\theta_0}(t), \dot{\boldsymbol{L}}_{\theta_0}(t)\big) \\
=&\big(\kappa\tau^2\sin\theta_0 + \kappa\dot{\tau}\cos\theta_0 + \kappa^3\sin\theta_0\cos^2\theta_0 - \dot{\kappa}\tau\cos\theta_0\big)\det A \\
=&0
\end{aligned}$$

for all $t \in I$, where

$$A = \left(\alpha\mathcal{T}, \frac{\tau\mathcal{T} - \kappa\sin\theta_0\cos\theta_0\mathcal{N} + \kappa\cos^2\theta_0\mathcal{B}}{\sqrt{\tau^2 + \kappa^2\cos^2\theta_0}}, \frac{\kappa\cos\theta_0\mathcal{T} + \tau\sin\theta_0\mathcal{N} - \tau\cos\theta_0\mathcal{B}}{(\tau^2 + \kappa^2\cos^2\theta_0)^{\frac{3}{2}}}\right).$$

It means $\boldsymbol{D}_{\theta_0}$ is a developable surface. So we call $\boldsymbol{D}_\theta$ the one-parameter developable surfaces of γ. If $\theta_0 = 0, \frac{\pi}{2}$, we have the following proposition.

Proposition 1. *Let $\gamma : I \to \mathbb{R}^3$ be a Frenet type framed base curve. Then we have the following:*

(1) *The normal vector field of $\boldsymbol{D}_0$ along γ is parallel to the unit principal normal vector field of γ.*
(2) *If γ is not a plane curve, then γ is the singularities of $\boldsymbol{D}_{\frac{\pi}{2}}$.*

Proof. By the definition of $\boldsymbol{D}_{\theta_0}$, we have

$$\boldsymbol{D}_0(t,u) = \gamma(t) + u\frac{\kappa(t)\mathcal{B}(t) + \tau(t)\mathcal{T}(t)}{\sqrt{\tau^2(t)+\kappa^2(t)}}, \quad \boldsymbol{D}_{\frac{\pi}{2}}(t,u) = \gamma(t) + u\mathcal{T}(t).$$

We can calculate that

$$\begin{aligned}
&\frac{\partial \boldsymbol{D}_0}{\partial t}(t,0) = \alpha(t)\mathcal{T}(t), \quad \frac{\partial \boldsymbol{D}_0}{\partial u}(t,0) = \frac{\kappa(t)\mathcal{B}(t) + \tau(t)\mathcal{T}(t)}{\sqrt{\tau^2(t)+\kappa^2(t)}},\\
&\frac{\partial \boldsymbol{D}_{\frac{\pi}{2}}}{\partial t}(t,0) = \alpha(t)\mathcal{T}(t), \quad \frac{\partial \boldsymbol{D}_{\frac{\pi}{2}}}{\partial u}(t,0) = \mathcal{T}(t).
\end{aligned}$$

And

$$\begin{aligned}
&\frac{\partial \boldsymbol{D}_0}{\partial t}(t,0) \times \frac{\partial \boldsymbol{D}_0}{\partial u}(t,0) = -\frac{\alpha(t)\kappa(t)}{\sqrt{\tau^2(t)+\kappa^2(t)}}\mathcal{N}(t),\\
&\frac{\partial \boldsymbol{D}_{\frac{\pi}{2}}}{\partial t}(t,0) \times \frac{\partial \boldsymbol{D}_{\frac{\pi}{2}}}{\partial u}(t,0) = 0.
\end{aligned}$$

Therefore, statements (1) and (2) hold. □

If γ is a regular curve, we can easily check that $\boldsymbol{D}_0(t,u)$ and $\boldsymbol{D}_{\frac{\pi}{2}}(t,u)$ are the rectifying developable surface and tangent developable surface of γ, respectively. The regular curve is always the geodesic of its rectifying developable surface (see [4,9]), and it is also always the singularities of its tangent developable surface (see [3]). Here, we call $\boldsymbol{D}_0(t,u)$ and $\boldsymbol{D}_{\frac{\pi}{2}}(t,u)$ the rectifying developable surface and tangent developable surface of the Frenet type framed base curve γ, respectively. By Proposition 1, the Frenet type framed base curve has similar properties to a regular curve in its rectifying developable and tangent developable surfaces.

In addition, we define two invariants, as follows:

$$\begin{aligned}
\delta(t) =& \frac{\kappa(t)\sin\theta_0(\tau^2(t)+\kappa^2(t)\cos^2\theta_0) + \cos\theta_0(\kappa(t)\dot{\tau}(t) - \dot{\kappa}(t)\tau(t))}{\tau^2(t)+\kappa^2(t)\cos^2\theta_0},\\
\sigma(t) =& \frac{\alpha(t)\tau(t)}{\sqrt{\tau^2(t)+\kappa^2(t)\cos^2\theta_0}} - \frac{d}{dt}\left(\frac{\alpha(t)\kappa(t)\cos\theta_0}{\delta(t)\sqrt{\tau^2(t)+\kappa^2(t)\cos^2\theta_0}}\right), \ \text{(when } \delta(t)\neq 0\text{)}.
\end{aligned}$$

Since

$$\begin{aligned}
\dot{\boldsymbol{L}}_{\theta_0}(t) =& \frac{(\kappa\tau^2\sin\theta_0 + \kappa^3\sin\theta_0\cos^2\theta_0 + \kappa\dot{\tau}\cos\theta_0 - \dot{\kappa}\tau\cos\theta_0)(\kappa\cos\theta_0\mathcal{T} + \tau\sin\theta_0\mathcal{N} - \tau\cos\theta_0\mathcal{B})}{(\tau^2+\kappa^2\cos^2\theta_0)^{\frac{3}{2}}}(t)\\
=&\delta(t)\frac{\kappa\cos\theta_0\mathcal{T} + \tau\sin\theta_0\mathcal{N} - \tau\cos\theta_0\mathcal{B}}{\sqrt{\tau^2+\kappa^2\cos^2\theta_0}}(t),
\end{aligned}$$

then $\delta(t) = 0$ if and only if $\dot{\boldsymbol{L}}_{\theta_0}(t) = \boldsymbol{0}$. By calculation, we have that

$$\frac{\partial \boldsymbol{D}_{\theta_0}}{\partial t}(t,u) \times \frac{\partial \boldsymbol{D}_{\theta_0}}{\partial u}(t,u) = -\left(\frac{\alpha(t)\kappa(t)\cos\theta_0}{\sqrt{\tau^2(t)+\kappa^2(t)\cos^2\theta_0}} + u\delta(t)\right)(\cos\theta_0\mathcal{N}(t) + \sin\theta_0\mathcal{B}(t)).$$

Therefore, (t_0,u_0) is a singular point of $\boldsymbol{D}_{\theta_0}$ if and only if

$$\frac{\alpha(t_0)\kappa(t_0)\cos\theta_0}{\sqrt{\tau^2(t_0)+\kappa^2(t_0)\cos^2\theta_0}} + u_0\frac{\kappa(t_0)\sin\theta_0(\tau^2(t_0)+\kappa^2(t_0)\cos^2\theta_0) + \cos\theta_0(\kappa(t_0)\dot{\tau}(t_0) - \dot{\kappa}(t_0)\tau(t_0))}{\tau^2(t_0)+\kappa^2(t_0)\cos^2\theta_0} = 0.$$

If $\theta_0 \neq \frac{\pi}{2}$ and $\boldsymbol{D}_{\theta_0}$ has singular point at (t_0,u_0), then we have $u_0 \neq 0$. It means that $\boldsymbol{D}_{\theta_0}$ has no singularities on the base curve $\gamma(t)$. For the geometric meaning of the above two invariants, we have the following theorem.

Theorem 1. *Let $\gamma : I \to \mathbb{R}^3$ be a Frenet type framed base curve. Then we have the following:*

(A) For any $\theta_0 \in [0, \frac{\pi}{2}]$, the following statements are equivalent:

(1) $\delta(t) = 0$ for all $t \in I$.

(2) $\boldsymbol{D}_{\theta_0}$ is a cylinder.

(B) If $\delta(t) \neq 0$ for all $t \in I$, then the following statements are equivalent:

(3) $\sigma(t) = 0$ for all $t \in I$.

(4) $\boldsymbol{D}_{\theta_0}$ is a conical surface.

(C) The singular points of one-parameter developable surfaces $\boldsymbol{D}_\theta$ are $\gamma(t)$, $\boldsymbol{D}_{\frac{\pi}{2}}$ and

$$\{\boldsymbol{D}_\theta(t,u) | \alpha(t)\kappa(t)\cos\theta\sqrt{\tau^2(t)+\kappa^2(t)\cos^2\theta} + u[\kappa(t)\sin\theta(\tau^2(t)+\kappa^2(t)\cos^2\theta) + \cos\theta(\kappa(t)\dot{\tau}(t) - \dot{\kappa}(t)\tau(t))] = 0\}.$$

Proof. (A) By the definition, $\boldsymbol{D}_{\theta_0}$ is a cylinder if and only if $\boldsymbol{L}_{\theta_0}(t)$ is a constant vector. Since

$$\dot{\boldsymbol{L}}_{\theta_0}(t) = \delta(t)\frac{\kappa\cos\theta_0\mathcal{T} + \tau\sin\theta_0\mathcal{N} - \tau\cos\theta_0\mathcal{B}}{\sqrt{\tau^2+\kappa^2\cos^2\theta_0}}(t),$$

$\boldsymbol{L}_{\theta_0}(t)$ is a constant vector if and only if $\delta(t) = 0$ for all $t \in I$.

(B) The striction curve $\boldsymbol{c}(t)$ of $\boldsymbol{D}_{\theta_0}$ is expressed by

$$\boldsymbol{c}(t) = \gamma(t) - \frac{\gamma(t)\cdot\dot{\boldsymbol{L}}_{\theta_0}(t)}{\dot{\boldsymbol{L}}_{\theta_0}(t)\cdot\dot{\boldsymbol{L}}_{\theta_0}(t)}\boldsymbol{L}_{\theta_0}(t) = \gamma(t) - \frac{\alpha(t)\kappa(t)\cos\theta_0}{\delta(t)\sqrt{\tau^2(t)+\kappa^2(t)\cos^2\theta_0}}\boldsymbol{L}_{\theta_0}(t).$$

Then (B)-(4) is equivalent to $\dot{\boldsymbol{c}}(t) = 0$ for all $t \in I$. We can calculate that

$$\begin{aligned}
\dot{\boldsymbol{c}} =& \dot{\gamma} - \frac{d}{dt}\left(\frac{\alpha\kappa\cos\theta_0}{\delta\sqrt{\tau^2+\kappa^2\cos^2\theta_0}}\right)\boldsymbol{L}_{\theta_0} - \frac{\alpha\kappa\cos\theta_0}{\delta\sqrt{\tau^2+\kappa^2\cos^2\theta_0}}\dot{\boldsymbol{L}}_{\theta_0} \\
=& \alpha\mathcal{T} - \frac{\alpha\kappa\cos\theta_0}{\delta\sqrt{\tau^2+\kappa^2\cos^2\theta_0}}\frac{\delta(\kappa\cos\theta_0\mathcal{T}+\tau\sin\theta_0\mathcal{N}-\tau\cos\theta_0\mathcal{B})}{\sqrt{\tau^2+\kappa^2\cos^2\theta_0}} - \frac{d}{dt}\left(\frac{\alpha\kappa\cos\theta_0}{\delta\sqrt{\tau^2+\kappa^2\cos^2\theta_0}}\right)\boldsymbol{L}_{\theta_0} \\
=& \frac{\alpha\tau(\tau\mathcal{T} - \kappa\sin\theta_0\cos\theta_0\mathcal{N} + \kappa\cos^2\theta_0\mathcal{B})}{\tau^2+\kappa^2\cos^2\theta_0} - \frac{d}{dt}\left(\frac{\alpha\kappa\cos\theta_0}{\delta\sqrt{\tau^2+\kappa^2\cos^2\theta_0}}\right)\boldsymbol{L}_{\theta_0} \\
=& [\frac{\alpha\tau}{\sqrt{\tau^2+\kappa^2\cos^2\theta_0}} - \frac{d}{dt}\left(\frac{\alpha\kappa\cos\theta_0}{\delta\sqrt{\tau^2+\kappa^2\cos^2\theta_0}}\right)]\boldsymbol{L}_{\theta_0} \\
=& \sigma(t)\boldsymbol{L}_{\theta_0}.
\end{aligned}$$

This means (B)-(4) and (B)-(3) are equivalent.

(C) By straightforward calculations, we have

$$\begin{aligned}
\frac{\partial \boldsymbol{D}_\theta}{\partial t} =& \alpha\mathcal{T} + u\frac{[\kappa\sin\theta(\tau^2+\kappa^2\cos^2\theta) + \cos\theta(\kappa\dot{\tau} - \dot{\kappa}\tau)](\kappa\cos\theta\mathcal{T} + \tau\sin\theta\mathcal{N} - \tau\cos\theta\mathcal{B})}{(\tau^2+\kappa^2\cos^2\theta)^{\frac{3}{2}}}, \\
\frac{\partial \boldsymbol{D}_\theta}{\partial u} =& \frac{\tau\mathcal{T} - \kappa\sin\theta\cos\theta\mathcal{N} + \kappa\cos^2\theta\mathcal{B}}{\sqrt{\tau^2+\kappa^2\cos^2\theta}}, \\
\frac{\partial \boldsymbol{D}_\theta}{\partial \theta} =& u\frac{\kappa^2\tau\sin\theta\cos\theta\mathcal{T} + (\kappa\tau^2\sin^2\theta - \kappa\tau^2\cos^2\theta - \kappa^3\cos^4\theta)\mathcal{N} - (\kappa^3\sin\theta\cos^3\theta + 2\kappa\tau^2\sin\theta\cos\theta)\mathcal{B}}{(\tau^2+\kappa^2\cos^2\theta)^{\frac{3}{2}}}.
\end{aligned}$$

The above three vectors are linearly dependent at the singularities of $\boldsymbol{D}_\theta$. So that we can obtain these singularities if

$$\frac{u\kappa\cos\theta[u(\kappa\sin\theta(\tau^2+\kappa^2\cos^2\theta) + \cos\theta(\kappa\dot{\tau} - \dot{\kappa}\tau)) + \alpha\kappa\cos\theta\sqrt{\tau^2+\kappa^2\cos^2\theta}]}{(\tau^2+\kappa^2\cos^2\theta)^{\frac{3}{2}}} = 0.$$

It follows that $\cos\theta = 0$ or $u = 0$ or

$$u[\kappa(t)\sin\theta(\tau^2(t)+\kappa^2(t)\cos^2\theta)+\cos\theta(\kappa(t)\dot{\tau}(t)-\dot{\kappa}(t)\tau(t))]+\alpha(t)\kappa(t)\cos\theta\sqrt{\tau^2(t)+\kappa^2(t)\cos^2\theta}=0.$$

Therefore (C) holds. □

Next, we show the relationships between the above two invariants and the singularities of one-parameter developable surfaces, as follows.

Theorem 2. *Let $\gamma : I \to \mathbb{R}^3$ be a Frenet type framed base curve. Then we have the following:*

(1) *(t_0, u_0) is a regular point of $\mathbf{D}_{\theta_0}$ if and only if*

$$\frac{\alpha(t_0)\kappa(t_0)\cos\theta_0}{\sqrt{\tau^2(t_0)+\kappa^2(t_0)\cos^2\theta_0}}+u_0\delta(t_0)\neq 0.$$

(2) *Suppose (t_0, u_0) is a singular point of $\mathbf{D}_{\theta_0}$, then $\mathbf{D}_{\theta_0}$ is locally diffeomorphic to the cuspidal edge $C \times \mathbb{R}$ at (t_0, u_0) if*

(i) *$\delta(t_0)\neq 0, \sigma(t_0)\neq 0$ and*

$$u_0=-\frac{\alpha(t_0)\kappa(t_0)\cos\theta_0}{\delta(t_0)\sqrt{\tau^2(t_0)+\kappa^2(t_0)\cos^2\theta_0}},$$

or

(ii) *$\delta(t_0)=\alpha(t_0)\cos\theta_0=0, \dot{\delta}(t_0)\neq 0$ and*

$$u_0\neq\frac{-((\dot{\alpha}\kappa)(t_0)\cos\theta_0-(\alpha\kappa\tau)(t_0)\sin\theta_0)\sqrt{\tau^2(t_0)+\kappa^2(t_0)\cos\theta_0}}{(2\kappa\tau\dot{\tau})(t_0)\sin\theta_0+(\dot{\kappa}\tau^2)(t_0)\sin\theta_0+(3\kappa^2\dot{\kappa})(t_0)\sin\theta_0\cos^2\theta_0+(\kappa\ddot{\tau})(t_0)\cos\theta_0-(\tau\ddot{\kappa})(t_0)\cos\theta_0},$$

or

(iii) *$\delta(t_0)=\alpha(t_0)\cos\theta_0=\dot{\delta}(t_0)=0$ and*

$$\dot{\alpha}(t_0)\cos\theta_0-\alpha(t_0)\tau(t_0)\sin\theta_0=0.$$

(3) *Suppose (t_0, u_0) is a singular point of $\mathbf{D}_{\theta_0}$, then $\mathbf{D}_{\theta_0}$ is locally diffeomorphic to the swallowtail SW at (t_0, u_0) if $\delta(t_0)\neq 0, \sigma(t_0)=0, \dot{\sigma}(t_0)\neq 0$ and*

$$u_0=-\frac{\alpha(t_0)\kappa(t_0)\cos\theta_0}{\delta(t_0)\sqrt{\tau^2(t_0)+\kappa^2(t_0)\cos^2\theta_0}}.$$

Here $C=\{(x_1,x_2,x_3)|x_1^2=x_2^3\}$ is the cusp (see Figure 1), $C\times\mathbb{R}$ is the cuspidal edge (see Figure 2), $SW=\{(x_1,x_2,x_3)|x_1=3u^4+u^2v, x_2=4u^3+2uv, x_3=v\}$ is the swallowtail (see Figure 3).

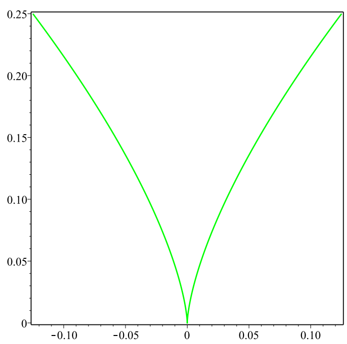

Figure 1. Cusp.

Figure 2. Cuspidal edge.

Figure 3. Swallowtail.

4. One-Parameter Support Functions

For a Frenet type framed base curve $\gamma : I \to \mathbb{R}^3$, we define a function

$$G : I \times [0, \frac{\pi}{2}] \times \mathbb{R}^3 \to \mathbb{R}$$

by $G(t, \theta, \boldsymbol{x}) = \big(\boldsymbol{x} - \gamma(t)\big) \cdot \big(\cos\theta\mathcal{N}(t) + \sin\theta\mathcal{B}(t)\big)$. We call G a *one-parameter support function* of γ with respect to the unit normal vector $\cos\theta\mathcal{N}(t) + \sin\theta\mathcal{B}(t)$. We write $g_{\theta_0,x_0}(t) = G(t, \theta_0, \boldsymbol{x}_0)$ for any $(\theta_0, \boldsymbol{x}_0) \in [0, \frac{\pi}{2}] \times \mathbb{R}^3$. Then we have the following proposition.

Proposition 2. *Let $\gamma : I \to \mathbb{R}^3$ be a Frenet type framed base curve and $g_{\theta_0,x_0}(t) = G(t, \theta_0, \boldsymbol{x}_0)$ the one-parameter support functions. Then the following assertions hold.*

(1) $g_{\theta_0,x_0}(t_0) = 0$ *if and only if there exist $u, v \in \mathbb{R}$ such that*

$$\boldsymbol{x}_0 - \gamma(t_0) = u\mathcal{T}(t_0) + v\big(\sin\theta_0\mathcal{N}(t_0) - \cos\theta_0\mathcal{B}(t_0)\big).$$

(2) $g_{\theta_0,x_0}(t_0) = \dot{g}_{\theta_0,x_0}(t_0) = 0$ *if and only if there exists $u \in \mathbb{R}$ such that*

$$\boldsymbol{x}_0 - \gamma(t_0) = u\frac{\tau(t_0)\mathcal{T}(t_0) - \kappa(t_0)\sin\theta_0\cos\theta_0\mathcal{N}(t_0) + \kappa(t_0)\cos^2\theta_0\mathcal{B}(t_0)}{\sqrt{\tau^2(t_0) + \kappa^2(t_0)\cos^2\theta_0}}.$$

(A) *Suppose $\delta(t_0) \neq 0$. Then the following assertions hold.*

(3) $g_{\theta_0,x_0}(t_0) = \dot{g}_{\theta_0,x_0}(t_0) = \ddot{g}_{\theta_0,x_0}(t_0) = 0$ *if and only if*

$$\boldsymbol{x}_0 - \gamma(t_0) = -\frac{\alpha(t_0)\kappa(t_0)\cos\theta_0}{\delta(t_0)\sqrt{\tau^2(t_0) + \kappa^2(t_0)\cos^2\theta_0}}\frac{\tau(t_0)\mathcal{T}(t_0) - \kappa(t_0)\sin\theta_0\cos\theta_0\mathcal{N}(t_0) + \kappa(t_0)\cos^2\theta_0\mathcal{B}(t_0)}{\sqrt{\tau^2(t_0) + \kappa^2(t_0)\cos^2\theta_0}}.$$

(4) $g_{\theta_0,x_0}(t_0) = \dot{g}_{\theta_0,x_0}(t_0) = \ddot{g}_{\theta_0,x_0}(t_0) = g^{(3)}_{\theta_0,x_0}(t_0) = 0$ *if and only if $\sigma(t_0) = 0$ and*

$$\boldsymbol{x}_0 - \gamma(t_0) = -\frac{\alpha(t_0)\kappa(t_0)\cos\theta_0}{\delta(t_0)\sqrt{\tau^2(t_0) + \kappa^2(t_0)\cos^2\theta_0}}\frac{\tau(t_0)\mathcal{T}(t_0) - \kappa(t_0)\sin\theta_0\cos\theta_0\mathcal{N}(t_0) + \kappa(t_0)\cos^2\theta_0\mathcal{B}(t_0)}{\sqrt{\tau^2(t_0) + \kappa^2(t_0)\cos^2\theta_0}}.$$

(5) $g_{\theta_0,x_0}(t_0) = \dot{g}_{\theta_0,x_0}(t_0) = \ddot{g}_{\theta_0,x_0}(t_0) = g^{(3)}_{\theta_0,x_0}(t_0) = g^{(4)}_{\theta_0,x_0}(t_0) = 0$ *if and only if $\sigma(t_0) = 0$, $\dot{\sigma}(t_0) = 0$*
and

$$\boldsymbol{x}_0 - \gamma(t_0) = -\frac{\alpha(t_0)\kappa(t_0)\cos\theta_0}{\delta(t_0)\sqrt{\tau^2(t_0) + \kappa^2(t_0)\cos^2\theta_0}}\frac{\tau(t_0)\mathcal{T}(t_0) - \kappa(t_0)\sin\theta_0\cos\theta_0\mathcal{N}(t_0) + \kappa(t_0)\cos^2\theta_0\mathcal{B}(t_0)}{\sqrt{\tau^2(t_0) + \kappa^2(t_0)\cos^2\theta_0}}.$$

(B) *Suppose $\delta(t_0) = 0$. Then the following assertions hold.*

(6) $g_{\theta_0,x_0}(t_0) = \dot{g}_{\theta_0,x_0}(t_0) = \ddot{g}_{\theta_0,x_0}(t_0) = 0$ *if and only if* $\alpha(t_0)\cos\theta_0 = 0$ *and there exists* $u \in \mathbb{R}$, *such that*

$$\boldsymbol{x}_0 - \gamma(t_0) = u\frac{\tau(t_0)\mathcal{T}(t_0) - \kappa(t_0)\sin\theta_0\cos\theta_0\mathcal{N}(t_0) + \kappa(t_0)\cos^2\theta_0\mathcal{B}(t_0)}{\sqrt{\tau^2(t_0) + \kappa^2(t_0)\cos^2\theta_0}}.$$

(7) $g_{\theta_0,x_0}(t_0) = \dot{g}_{\theta_0,x_0}(t_0) = \ddot{g}_{\theta_0,x_0}(t_0) = g^{(3)}_{\theta_0,x_0}(t_0) = 0$ *if and only if one of the following holds:*

(a) $\alpha(t_0)\cos\theta_0 = 0, \dot{\delta}(t_0) \neq 0$ *and*

$$\boldsymbol{x}_0 - \gamma(t_0) = \frac{(\dot{\alpha}\kappa\cos\theta_0 - \alpha\kappa\tau\sin\theta_0)(\tau\mathcal{T} - \kappa\sin\theta_0\cos\theta_0\mathcal{N} + \kappa\cos^2\theta_0\mathcal{B})(t_0)}{\dot{\kappa}\tau^2\sin\theta_0 + 3\kappa^2\dot{\kappa}\sin\theta_0\cos^2\theta_0 + 2\kappa\tau\dot{\tau}\sin\theta_0 - \tau\ddot{\kappa}\cos\theta_0 + \kappa\ddot{\tau}\cos\theta_0}.$$

(b) $\alpha(t_0)\cos\theta_0 = \dot{\delta}(t_0) = \dot{\alpha}(t_0)\kappa(t_0)\cos\theta_0 - \alpha(t_0)\kappa(t_0)\tau(t_0)\sin\theta_0 = 0$ *and*

$$\boldsymbol{x}_0 - \gamma(t_0) = u\frac{\tau(t_0)\mathcal{T}(t_0) - \kappa(t_0)\sin\theta_0\cos\theta_0\mathcal{N}(t_0) + \kappa(t_0)\cos^2\theta_0\mathcal{B}(t_0)}{\sqrt{\tau^2(t_0) + \kappa^2(t_0)\cos^2\theta_0}}.$$

Proof. Since $g_{\theta_0,x_0}(t) = \big(\boldsymbol{x}_0 - \gamma(t)\big) \cdot \big(\cos\theta_0\mathcal{N}(t) + \sin\theta_0\mathcal{B}(t)\big)$, we have the following:

$$\begin{aligned}
(i)\ \dot{g}_{\theta_0,x_0} =& (\boldsymbol{x}_0 - \gamma) \cdot (-\kappa\cos\theta_0\mathcal{T} - \tau\sin\theta_0\mathcal{N} + \tau\cos\theta_0\mathcal{B}),\\
(ii)\ \ddot{g}_{\theta_0,x_0} =& \alpha\kappa\cos\theta_0 + (\boldsymbol{x}_0 - \gamma) \cdot [(\kappa\tau\sin\theta_0 - \dot{\kappa}\cos\theta_0)\mathcal{T} - (\kappa^2\cos\theta_0 + \dot{\tau}\sin\theta_0 + \tau^2\cos\theta_0)\mathcal{N}\\
&+ (\dot{\tau}\cos\theta_0 - \tau^2\sin\theta_0)\mathcal{B}],\\
(iii)\ g^{(3)}_{\theta_0,x_0} =& 2\alpha\dot{\kappa}\cos\theta_0 + \dot{\alpha}\kappa\cos\theta_0 - \alpha\kappa\tau\sin\theta_0\\
&+ (\boldsymbol{x}_0 - \gamma) \cdot [(\dot{\kappa}\tau\sin\theta_0 + 2\kappa\dot{\tau}\sin\theta_0 - \ddot{\kappa}\cos\theta_0 + \kappa^3\cos\theta_0 + \kappa\tau^2\cos\theta_0)\mathcal{T}\\
&+ (\kappa^2\tau\sin\theta_0 - 3\kappa\dot{\kappa}\cos\theta_0 - 3\tau\dot{\tau}\cos\theta_0 + \tau^3\sin\theta_0 - \ddot{\tau}\sin\theta_0)\mathcal{N}\\
&+ (\ddot{\tau}\cos\theta_0 - 3\tau\dot{\tau}\sin\theta_0 - \kappa^2\tau\cos\theta_0 - \tau^3\cos\theta_0)\mathcal{B}],
\end{aligned}$$

$$\begin{aligned}
(iv)\ g^{(4)}_{\theta_0,x_0} =& 3\alpha\ddot{\kappa}\cos\theta_0 + 3\dot{\alpha}\dot{\kappa}\cos\theta_0 + \ddot{\alpha}\kappa\cos\theta_0 - 2\alpha\dot{\kappa}\tau\sin\theta_0\\
&- 3\alpha\kappa\dot{\tau}\sin\theta_0 - \dot{\alpha}\kappa\tau\sin\theta_0 - \alpha\kappa^3\cos\theta_0 - \alpha\kappa\tau^2\cos\theta_0\\
&+ (\boldsymbol{x}_0 - \gamma) \cdot [(\ddot{\kappa}\tau\sin\theta_0 + 3\dot{\kappa}\dot{\tau}\sin\theta_0 - \dddot{\kappa}\cos\theta_0 + \dot{\kappa}\tau^2\cos\theta_0 + 3\kappa\ddot{\tau}\sin\theta_0\\
&+ 6\kappa^2\dot{\kappa}\cos\theta_0 + 5\kappa\tau\dot{\tau}\cos\theta_0 - \kappa^3\tau\sin\theta_0 - \kappa\tau^3\sin\theta_0)\mathcal{T}\\
&+ (3\kappa\dot{\kappa}\tau\sin\theta_0 + 3\kappa^2\dot{\tau}\sin\theta_0 - 4\kappa\ddot{\kappa}\cos\theta_0 + \kappa^4\cos\theta_0 + 2\kappa^2\tau^2\cos\theta_0 - \dddot{\tau}\sin\theta_0\\
&- 3(\dot{\kappa})^2\cos\theta_0 - 4\tau\ddot{\tau}\cos\theta_0 - 3(\dot{\tau})^2\cos\theta_0 + 6\tau^2\dot{\tau}\sin\theta_0 + \tau^4\cos\theta_0)\mathcal{N}\\
&+ (\dddot{\tau}\cos\theta_0 - 4\tau\ddot{\tau}\sin\theta_0 - 3(\dot{\tau})^2\sin\theta_0 - 5\kappa\dot{\kappa}\tau\cos\theta_0 - \kappa^2\dot{\tau}\cos\theta_0\\
&- 6\tau^2\dot{\tau}\cos\theta_0 + \kappa^2\tau^2\sin\theta_0 + \tau^4\sin\theta_0)\mathcal{B}].
\end{aligned}$$

By the definition, $g_{\theta_0,x_0}(t_0) = 0$ if there exist $u, a, b \in \mathbb{R}$, such that

$$\boldsymbol{x}_0 - \gamma(t_0) = u\mathcal{T}(t_0) + a\mathcal{N}(t_0) + b\mathcal{B}(t_0)$$

and $a\cos\theta_0 + b\sin\theta_0 = 0$. So that there exists $v \in \mathbb{R}$, such that $a = v\sin\theta_0$ and $b = -v\cos\theta_0$, then we have

$$\boldsymbol{x}_0 - \gamma(t_0) = u\mathcal{T}(t_0) + v\big(\sin\theta_0\mathcal{N}(t_0) - \cos\theta_0\mathcal{B}(t_0)\big).$$

Therefore (1) holds.

By (i), $g_{\theta_0,x_0}(t_0) = \dot{g}_{\theta_0,x_0}(t_0) = 0$ if and only if

$$\boldsymbol{x}_0 - \boldsymbol{\gamma}(t_0) = u\mathcal{T}(t_0) + v\big(\sin\theta_0\mathcal{N}(t_0) - \cos\theta_0\mathcal{B}(t_0)\big)$$

and $u\kappa\cos\theta_0 + v\tau = 0$. Since $\kappa \neq 0$ and $\kappa^2\cos\theta_0^2 + \tau^2 \neq 0$, so that there exists $u \in \mathbb{R}$, such that

$$\boldsymbol{x}_0 - \boldsymbol{\gamma}(t_0) = u\frac{\tau(t_0)\mathcal{T}(t_0) - \kappa(t_0)\sin\theta_0\cos\theta_0\mathcal{N}(t_0) + \kappa(t_0)\cos^2\theta_0\mathcal{B}(t_0)}{\sqrt{\tau^2(t_0) + \kappa^2(t_0)\cos^2\theta_0}}.$$

Therefore (2) holds.

By (ii), $g_{\theta_0,x_0}(t_0) = \dot{g}_{\theta_0,x_0}(t_0) = \ddot{g}_{\theta_0,x_0}(t_0) = 0$ if and only if there exists $u \in \mathbb{R}$, such that

$$\boldsymbol{x}_0 - \boldsymbol{\gamma}(t_0) = u\frac{\tau(t_0)\mathcal{T}(t_0) - \kappa(t_0)\sin\theta_0\cos\theta_0\mathcal{N}(t_0) + \kappa(t_0)\cos^2\theta_0\mathcal{B}(t_0)}{\sqrt{\tau^2(t_0) + \kappa^2(t_0)\cos^2\theta_0}}$$

and

$$\alpha(t_0)\kappa(t_0)\cos\theta_0 + u\frac{\kappa(t_0)\sin\theta_0(\tau^2(t_0) + \kappa^2(t_0)\cos^2\theta_0) + \cos\theta_0(\kappa(t_0)\dot{\tau}(t_0) - \dot{\kappa}(t_0)\tau(t_0))}{\sqrt{\tau^2(t_0) + \kappa^2(t_0)\cos^2\theta_0}} = 0.$$

It follows

$$\alpha(t_0)\kappa(t_0)\cos\theta_0 + u\delta(t_0)\sqrt{\tau^2(t_0) + \kappa^2(t_0)\cos^2\theta_0} = 0.$$

Thus

$$\delta(t_0) = \frac{\kappa(t_0)\sin\theta_0(\tau^2(t_0) + \kappa^2(t_0)\cos^2\theta_0) + \cos\theta_0(\kappa(t_0)\dot{\tau}(t_0) - \dot{\kappa}(t_0)\tau(t_0))}{\tau^2(t_0) + \kappa^2(t_0)\cos^2\theta_0} \neq 0$$

and

$$u = -\frac{\alpha(t_0)\kappa(t_0)\cos\theta_0}{\delta(t_0)\sqrt{\tau^2(t_0) + \kappa^2(t_0)\cos^2\theta_0}},$$

or $\delta(t_0) = 0$ and $\alpha(t_0)\cos\theta_0 = 0$. This completes the proof of (A)-(3) and (B)-(6).

If $\delta(t_0) \neq 0$, by (iii), we have

$$g_{\theta_0,x_0}(t_0) = \dot{g}_{\theta_0,x_0}(t_0) = \ddot{g}_{\theta_0,x_0}(t_0) = g^{(3)}_{\theta_0,x_0}(t_0) = 0$$

if and only if

$$\boldsymbol{x}_0 - \boldsymbol{\gamma}(t_0) = -\frac{\alpha(t_0)\kappa(t_0)\cos\theta_0}{\delta(t_0)\sqrt{\tau^2(t_0) + \kappa^2(t_0)\cos^2\theta_0}}\frac{\tau(t_0)\mathcal{T}(t_0) - \kappa(t_0)\sin\theta_0\cos\theta_0\mathcal{N}(t_0) + \kappa(t_0)\cos^2\theta_0\mathcal{B}(t_0)}{\sqrt{\tau^2(s_0) + \kappa^2(s_0)\cos^2\theta_0}}$$

and

$$\begin{aligned}&\Big[2\alpha\dot{\kappa}\cos\theta_0 + \dot{\alpha}\kappa\cos\theta_0 - \alpha\kappa\tau\sin\theta_0 - \frac{\alpha\kappa\cos\theta_0}{\delta(\tau^2 + \kappa^2\cos^2\theta_0)}(\dot{\kappa}\tau^2\sin\theta_0 + 3\kappa^2\dot{\kappa}\sin\theta_0\cos^2\theta_0 + 2\kappa\tau\dot{\tau}\sin\theta_0 \\ &- \tau\ddot{\kappa}\cos\theta_0 + \kappa\ddot{\tau}\cos\theta_0)\Big](t_0) = 0.\end{aligned}$$

We denote $\sigma(t_0)$ as following:

$$\begin{aligned}\sigma(t_0) = &-\sqrt{\tau^2 + \kappa^2\cos^2\theta_0}\Big[2\alpha\dot{\kappa}\cos\theta_0 + \dot{\alpha}\kappa\cos\theta_0 - \alpha\kappa\tau\sin\theta_0 - \frac{\alpha\kappa\cos\theta_0}{\delta(\tau^2 + \kappa^2\cos^2\theta_0)}(\dot{\kappa}\tau^2\sin\theta_0 \\ &+ 3\kappa^2\dot{\kappa}\sin\theta_0\cos^2\theta_0 + 2\kappa\tau\dot{\tau}\sin\theta_0 - \tau\ddot{\kappa}\cos\theta_0 + \kappa\ddot{\tau}\cos\theta_0)\Big](t_0).\end{aligned}$$

Therefore $\sigma(t_0) = 0$, we have (A)-(4). By the similar arguments to the above, we have (A)-(5).

If $\delta(t_0) = 0$, by (iii),

$$g_{\theta_0,x_0}(t_0) = \dot{g}_{\theta_0,x_0}(t_0) = \ddot{g}_{\theta_0,x_0}(t_0) = g^{(3)}_{\theta_0,x_0}(t_0) = 0$$

if and only if $\alpha(t_0)\cos\theta_0 = 0$ and there exists $u \in \mathbb{R}$, such that

$$[(\dot{\alpha}\kappa\cos\theta_0 - \alpha\kappa\tau\sin\theta_0)\sqrt{\tau^2+\kappa^2\cos\theta_0} + u(\dot{\kappa}\tau^2\sin\theta_0 + 3\kappa^2\dot{\kappa}\sin\theta_0\cos^2\theta_0 \\ + 2\kappa\tau\dot{\tau}\sin\theta_0 - \tau\ddot{\kappa}\cos\theta_0 + \kappa\ddot{\tau}\cos\theta_0)](t_0) = 0.$$

It means that

$$(\dot{\kappa}\tau^2\sin\theta_0 + 3\kappa^2\dot{\kappa}\sin\theta_0\cos^2\theta_0 + 2\kappa\tau\dot{\tau}\sin\theta_0 - \tau\ddot{\kappa}\cos\theta_0 + \kappa\ddot{\tau}\cos\theta_0)(t_0) \neq 0$$

and

$$u = \frac{(\dot{\alpha}\kappa\cos\theta_0 - \alpha\kappa\tau\sin\theta_0)\sqrt{\tau^2+\kappa^2\cos\theta_0}}{\dot{\kappa}\tau^2\sin\theta_0 + 3\kappa^2\dot{\kappa}\sin\theta_0\cos^2\theta_0 + 2\kappa\tau\dot{\tau}\sin\theta_0 - \tau\ddot{\kappa}\cos\theta_0 + \kappa\ddot{\tau}\cos\theta_0}(t_0),$$

or

$$(\dot{\kappa}\tau^2\sin\theta_0 + 3\kappa^2\dot{\kappa}\sin\theta_0\cos^2\theta_0 + 2\kappa\tau\dot{\tau}\sin\theta_0 - \tau\ddot{\kappa}\cos\theta_0 + \kappa\ddot{\tau}\cos\theta_0)(t_0) = 0,$$

$\dot{\delta}(t_0) = 0$ and $(\dot{\alpha}\kappa\cos\theta_0 - \alpha\kappa\tau\sin\theta_0)(t_0) = 0$. Therefore, we have (B)-(7). This completes the proof. □

5. Unfolding of Functions

We classify the singularities of one-parameter developable surfaces by using the unfolding theory of functions in this section .

Let $F : (\mathbb{R}\times\mathbb{R}^r, (t_0, \boldsymbol{x}_0)) \to \mathbb{R}$ be a function germ and $f(t) = F_{x_0}(t, \boldsymbol{x}_0)$, then F is called an *r-parameter unfolding* of f. We say that f has A_k-singularity at t_0 if $f^{(p)}(t_0) = 0$ for all $1 \le p \le k$ and $f^{(k+1)}(t_0) \neq 0$. We also say that f has $A_{\ge k}$-singularity at t_0 if $f^{(p)}(t_0) = 0$ for all $1 \le p \le k$. If f has A_k-singularity $(k \ge 1)$ at t_0 and F is an r-parameter unfolding of f, the $(k-1)$-jet of the partial derivative $\partial F/\partial x_i$ at t_0 is defined by

$$j^{(k-1)}\frac{\partial F}{\partial x_i}(t, \boldsymbol{x}_0)(t_0) = \sum_{j=1}^{k+1}\alpha_{ji}(t-t_0)^j, \quad (i = 1, \dots, r).$$

We call F an *R-versal unfolding* of f if the rank of $k\times r$ matrix $(\alpha_{0i}, \alpha_{ji})$ is k $(k \le r)$, where $\alpha_{0i} = \frac{\partial F}{\partial x_i}(t_0, \boldsymbol{x}_0)$. The *discriminant set* of F is defined to be

$$D_F = \{\boldsymbol{x} \in \mathbb{R}^r \mid \exists\, t \in \mathbb{R}, F(t, \boldsymbol{x}) = \frac{\partial F}{\partial t}(t, \boldsymbol{x}) = 0\}.$$

There is the following classification theorem in [29].

Theorem 3. *Let $F : (\mathbb{R}\times\mathbb{R}^r, (t_0, \boldsymbol{x}_0)) \to \mathbb{R}$ be an r-parameter unfolding of f which has A_k-singularity at t_0. Suppose F is an R-versal unfolding of f. Then D_F is locally diffeomorphic to $C\times\mathbb{R}^{r-2}$ if $k = 2$; D_F is locally diffeomorphic to $SW\times\mathbb{R}^{r-3}$ if $k = 3$.*

By Proposition 2, the discriminant set of the one-parameter support functions $G(t, \theta, \boldsymbol{x})$ is

$$D_G = \left\{\gamma(t) + u\frac{\tau(t)\mathcal{T}(t) - \kappa(t)\sin\theta\cos\theta\mathcal{N}(t) + \kappa(t)\cos^2\theta\mathcal{B}(t)}{\sqrt{\tau^2(t)+\kappa^2(t)\cos^2\theta}} | t, u \in \mathbb{R}, \theta \in [0, \frac{\pi}{2}]\right\}.$$

We have the following proposition for the proof of Theorem 2.

Proposition 3. *Let $\gamma : I \to \mathbb{R}^3$ be a Frenet type framed base curve. If g_{θ_0,x_0} has the A_k-singularity $(k = 2,3)$ at t_0, then G is an R-versal unfolding of g_{θ_0,x_0}. Here, we assume $\delta(t_0) \neq 0$ for $k = 3$.*

Proof. We write $\boldsymbol{x} = (x_1, x_2, x_3)$, $\gamma(t) = \big(\gamma_1(t), \gamma_2(t), \gamma_3(t)\big)$ and $\cos\theta_0\mathcal{N}(t) + \sin\theta_0\mathcal{B}(t) = \big(l_1(t), l_2(t), l_3(t)\big)$. Then we have

$$G(t, \theta_0, \boldsymbol{x}) = l_1(t)\big(x_1 - \gamma_1(t)\big) + l_2(t)\big(x_2 - \gamma_2(t)\big) + l_3(t)\big(x_3 - \gamma_3(t)\big)$$

and

$$\frac{\partial G}{\partial x_i}(t, \boldsymbol{x}) = l_i(t), \quad (i = 1, 2, 3).$$

Therefore the 2-jet is as following:

$$j^2\frac{\partial G}{\partial x_i}(t_0, \boldsymbol{x}_0) = l_i(t_0) + \dot{l}_i(t_0)(t - t_0) + \frac{1}{2}\ddot{l}_i(t_0)(t - t_0)^2.$$

We consider the following 3×3 matrix:

$$A = \begin{pmatrix} l_1(t_0) & l_2(t_0) & l_3(t_0) \\ \dot{l}_1(t_0) & \dot{l}_2(t_0) & \dot{l}_3(t_0) \\ \ddot{l}_1(t_0) & \ddot{l}_2(t_0) & \ddot{l}_3(t_0) \end{pmatrix} = \begin{pmatrix} \cos\theta_0\mathcal{N}(t_0) + \sin\theta_0\mathcal{B}(t_0) \\ \cos\theta_0\dot{\mathcal{N}}(t_0) + \sin\theta_0\dot{\mathcal{B}}(t_0) \\ \cos\theta_0\ddot{\mathcal{N}}(t_0) + \sin\theta_0\ddot{\mathcal{B}}(t_0) \end{pmatrix}.$$

By the Frenet type formula, we have

$$\begin{aligned} \cos\theta_0\dot{\mathcal{N}}(t_0) + \sin\theta_0\dot{\mathcal{B}}(t_0) =& -\kappa(t_0)\cos\theta_0\mathcal{T}(t_0) - \tau(t_0)\sin\theta_0\mathcal{N}(t_0) + \tau(t_0)\cos\theta_0\mathcal{B}(t_0), \\ \cos\theta_0\ddot{\mathcal{N}}(t_0) + \sin\theta_0\ddot{\mathcal{B}}(t_0) =& \big(\kappa(t_0)\tau(t_0)\sin\theta_0 - \dot{\kappa}(t_0)\cos\theta_0\big)\mathcal{T}(t_0) - \big[(\kappa^2(t_0) + \tau^2(t_0))\cos\theta_0 \\ &+ \dot{\tau}(t_0)\sin\theta_0\big]\mathcal{N}(t_0) + \big(\dot{\tau}(t_0)\cos\theta_0 - \tau^2(t_0)\sin\theta_0\big)\mathcal{B}(t_0). \end{aligned}$$

Since $\{\mathcal{T}(t_0), \mathcal{N}(t_0), \mathcal{B}(t_0)\}$ is an orthonormal basis of $\mathbb{R}^3$, then the rank of

$$A = \begin{pmatrix} \cos\theta_0\mathcal{N}(t_0) + \sin\theta_0\mathcal{B}(t_0) \\ \cos\theta_0\dot{\mathcal{N}}(t_0) + \sin\theta_0\dot{\mathcal{B}}(t_0) \\ \cos\theta_0\ddot{\mathcal{N}}(t_0) + \sin\theta_0\ddot{\mathcal{B}}(t_0) \end{pmatrix}$$

is equal to the rank of

$$\begin{pmatrix} 0 & \cos\theta_0 & \sin\theta_0 \\ -\kappa(t_0)\cos\theta_0 & -\tau(t_0)\sin\theta_0 & \tau(t_0)\cos\theta_0 \\ \kappa(t_0)\tau(t_0)\sin\theta_0 - \dot{\kappa}(t_0)\cos\theta_0 & -(\kappa^2(t_0) + \tau^2(t_0))\cos\theta_0 - \dot{\tau}(t_0)\sin\theta_0 & \dot{\tau}(t_0)\cos\theta_0 - \tau^2(t_0)\sin\theta_0 \end{pmatrix}.$$

It means rank $A = 3$ if and only if

$$\kappa(t_0)\sin\theta_0(\tau^2(t_0) + \kappa^2(t_0)\cos^2\theta_0) + \cos\theta_0(\kappa(t_0)\dot{\tau}(t_0) - \dot{\kappa}(t_0)\tau(t_0)) \neq 0.$$

The above inequality is equivalent to the condition $\delta(t_0) \neq 0$. Moreover, the rank of

$$\begin{pmatrix} \cos\theta_0\mathcal{N}(t_0) + \sin\theta_0\mathcal{B}(t_0) \\ \cos\theta_0\dot{\mathcal{N}}(t_0) + \sin\theta_0\dot{\mathcal{B}}(t_0) \end{pmatrix} = \begin{pmatrix} \cos\theta_0\mathcal{N}(t_0) + \sin\theta_0\mathcal{B}(t_0) \\ -\kappa(t_0)\cos\theta_0\mathcal{T}(t_0) - \tau(t_0)\sin\theta_0\mathcal{N}(t_0) + \tau(t_0)\cos\theta_0\mathcal{B}(t_0) \end{pmatrix}$$

is always two under the condition $\kappa^2(t_0)\cos^2\theta_0 + \tau^2(t_0) \neq 0$.

Then G is an R-versal unfolding of g_{θ_0,x_0} if g_{θ_0,x_0} has A_k-singularity $(k = 2,3)$ at t_0. This completes the proof. $\square$

Proof of Theorem 2. By straightforward calculations, we have

$$\frac{\partial \boldsymbol{D}_{\theta_0}}{\partial t}(t,u) \times \frac{\partial \boldsymbol{D}_{\theta_0}}{\partial u}(t,u) = -\left(\frac{\alpha(t)\kappa(t)\cos\theta_0}{\sqrt{\tau^2(t)+\kappa^2(t)\cos^2\theta_0}} + u\delta(t)\right)(\cos\theta_0 \mathcal{N}(t) + \sin\theta_0 \mathcal{B}(t)).$$

Then (t_0, u_0) is a regular point of $\boldsymbol{D}_{\theta_0}$ if and only if

$$\frac{\alpha(t_0)\kappa(t_0)\cos\theta_0}{\sqrt{\tau^2(t_0)+\kappa^2(t_0)\cos^2\theta_0}} + u_0\delta(t_0) \neq 0.$$

The assertion (1) holds.

By Proposition 2-(2), the image of one-parameter developable surfaces of γ is the discriminant set D_G of the one-parameter support functions G.

If $\delta(t_0) \neq 0$, by Proposition 2-(A)-(3), (4) and (5), g_{θ_0,x_0} has the A_2-type singularity (respectively, the A_3-type singularity) at $t = t_0$ if and only if

$$u_0 = -\frac{\alpha(t_0)\kappa(t_0)\cos\theta_0}{\delta(t_0)\sqrt{\tau^2(t_0)+\kappa^2(t_0)\cos^2\theta_0}} \quad (*)$$

and $\sigma(t_0) \neq 0$ (respectively, $(*)$, $\sigma(t_0) = 0$ and $\dot{\sigma}(t_0) \neq 0$). It follows from Theorem 3 and Proposition 3 that assertions (2)-(i) and (3) hold.

If $\delta(t_0) = 0$, by Proposition 2-(B)-(6) and (7), g_{θ_0,x_0} has the A_2-type singularity if and only if $\alpha(t_0)\cos\theta_0 = 0$ and

$$u_0 \neq \frac{(\dot{\alpha}\kappa\cos\theta_0 - \alpha\kappa\tau\sin\theta_0)\sqrt{\tau^2+\kappa^2\cos\theta_0}}{\dot{\kappa}\tau^2\sin\theta_0 + 3\kappa^2\dot{\kappa}\sin\theta_0\cos^2\theta_0 + 2\kappa\tau\dot{\tau}\sin\theta_0 - \tau\ddot{\kappa}\cos\theta_0 + \kappa\ddot{\tau}\cos\theta_0}(t_0).$$

It follows from Theorem 3 and Proposition 3 that assertion (2)-(ii) holds. This completes the proof. □

6. Example

In this section, we define a Frenet type framed base curve that has a singular point, and consider the one-parameter developable surfaces associated with this curve. We study two sections of the one-parameter developable surfaces of the base curve. They are the rectifying developable surface and tangent developable surface. These two developable surfaces can also be seen as the wavefronts of the base curve.

Let $\gamma(t) = (\frac{1}{2}t^2, \frac{1}{3}t^3, \frac{1}{5}t^5)$ be a Frenet type framed base curve with a singular point. Then we have $\alpha(t) = t\sqrt{1+t^2+t^6}$ and

$$\mathcal{T}(t) = \frac{1}{\sqrt{1+t^2+t^6}}(1, t, t^3),$$
$$\mathcal{N}(t) = \frac{1}{\sqrt{(1+t^2+t^6)(1+9t^4+4t^6)}}(-t-3t^5, 1-2t^6, t^2(3+2t^2)),$$
$$\mathcal{B}(t) = \frac{1}{\sqrt{1+9t^4+4t^6}}(2t^3, -3t^2, 1).$$

We can calculate that

$$\kappa(t) = \frac{\sqrt{1+9t^4+4t^6}}{1+t^2+t^6}, \quad \tau(t) = \frac{6t\sqrt{1+t^2+t^6}}{1+9t^4+4t^6}.$$

Since $\alpha(0) = 0$, so that $t = 0$ is a singular point of γ. We also have $\delta(0) = 6$ and $\sigma(0) = \frac{1}{6}$. The tangent developable surface of γ is as follows:

$$\boldsymbol{D}_{\frac{\pi}{2}}(t,u) = (\frac{t^2}{2} + \frac{u}{\sqrt{1+t^2+t^6}}, \frac{t^3}{3} + \frac{ut}{\sqrt{1+t^2+t^6}}, \frac{t^5}{5} + \frac{ut^3}{\sqrt{1+t^2+t^6}}).$$

By Theorem 2, the tangent developable surface $\boldsymbol{D}_{\frac{\pi}{2}}(t,u)$ is locally diffeomorphic to the cuspidal edge at $u=0$ (Figure 4).

The expression of the rectifying developable surface is as follows:

$$\begin{aligned}\boldsymbol{D}_0(t,u) = (\frac{1}{2}t^2, \frac{1}{3}t^3, \frac{1}{5}t^5) + \frac{u}{\sqrt{36t^2(1+t^2+t^6)^3+(1+9t^4+4t^6)^3}}&((1+9t^4+4t^6)(2t^3,-3t^2,1)\\ &+(1+t^2+t^6)(1,t,t^3)).\end{aligned}$$

By Theorem 2, the rectifying developable surface $\boldsymbol{D}_0(t,u)$ has cuspidal edge singularities at $(0,0)$ (Figure 5).

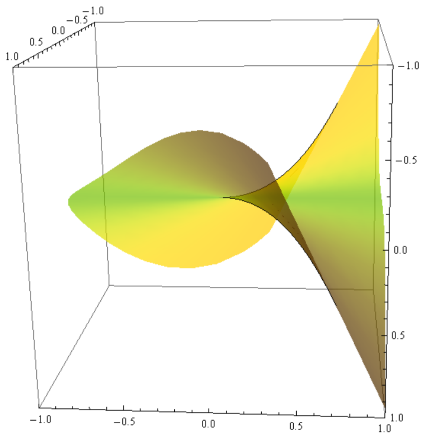

Figure 4. γ and the tangent developable surface $\boldsymbol{D}_{\frac{\pi}{2}}(t,u)$.

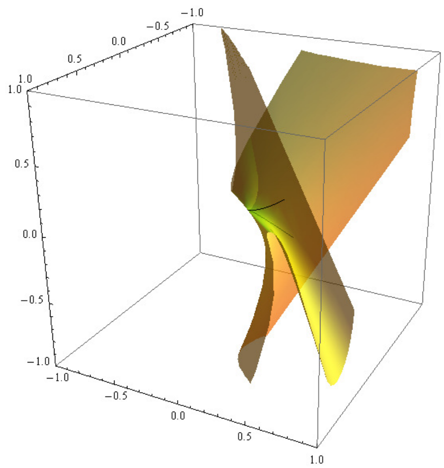

Figure 5. γ and the rectifying developable surface $\boldsymbol{D}_0(t,u)$.

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