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Invariant Interpolation Space for Generalized Multivariate Birkhoff Interpolation

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Abstract: In this paper, we provide a detailed exposition of a generalized multivariate Birkhoff interpolation scheme (Z, S, E) and introduce the notions of invariant interpolation space and singular interpolation space. We prove that the space $\mathcal{P}_S$, which is spanned by the monomial sequence S, is invariant or singular if the incidence matrix E satisfies some conditions. The advantage of our results lie in the fact that we can deduce whether $\mathcal{P}_S$ is always proper or not for all choices of the given node set Z, just from the property of the incidence matrix E, with very low computational complexity.

Keywords: multivariate Birkhoff interpolation; invariant interpolation space; singular interpolation space

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1. Introduction

In the past two decades, there has been tremendous interest in developing the Birkhoff interpolation problems [1,2], originated by Birkhoff in 1906. Different from the well-known Lagrange and Hermite interpolations, Birkhoff interpolation is a more complicated polynomial interpolation because the orders of derivatives at some nodes are noncontinuous. There are many areas about this subject such as discussing the approximation error estimation of Birkhoff interpolation [3], applying the Birkhoff interpolation method to solve some numerical differential Equations [4,5], trying to determine whether an interpolation scheme is regular or almost regular [6–9], and finding a proper interpolation space for the given interpolation conditions and nodes [10,11]. In this paper, we focus on the last area and propose new notions of invariant interpolation space and singular interpolation space.

Given a node set, an interpolation space is said to be proper if there exists a unique solution to the interpolation problem in the space for any given data values. Due to the good property, it is of interest to find a proper interpolation space. A Newton basis was constructed in a more general setting for Birkhoff interpolation by Wang [10]. Jiang et al. [12] proposed the representation of D-invariant polynomial subspace based on symmetric Cartesian tensors. Lei et al. [13] constructed a minimal monomial basis algorithm to compute the proper interpolation space for a multivariate Birkhoff interpolation problem, which was the development of the classic MB algorithm [14]. Considering the perturbation of the interpolation nodes, Cui et al. [15] proposed a stable basis algorithm for a generalized Birkhoff interpolation scheme by modifying the SOI algorithm [16]. In fact, if a space is spanned by a stable monomial basis, then we can call it a stable interpolation space since it is always proper when the node set is perturbed within limits. In this paper, we investigate that for some multivariate Birkhoff interpolation problems there exists an interpolation space which is always proper or not proper for all perturbations of the given node set. We call the former the invariant interpolation space and the latter the singular interpolation space. Furthermore, we study the conditions an interpolation space is invariant or singular and obtain two main results.

2. Preliminaries

In this section, we will recall some basic definitions in symbolic computation and provide a generalized multivariate Birkhoff interpolation problem.

Let $\mathbb{K}[X] = \mathbb{K}[x_1, x_2, \cdots, x_n]$ be the polynomial ring in n variables over the field $\mathbb{K}$. We will denote by $X^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ a monomial in $\mathbb{K}[X]$, where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, i.e., $\alpha \in \mathbb{Z}^n_{\geq 0}$.

Definition 1 ([17])**.** *A monomial order on $\mathbb{K}[X]$ is any relation $\succ$ on $\mathbb{Z}^n_{\geq 0}$, or, equivalently, any relation on the set of monomials $X^\alpha, \alpha \in \mathbb{Z}^n_{\geq 0}$, satisfying:*

(i) $\succ$ *is a total (or linear) order on* $\mathbb{Z}^n_{\geq 0}$.
(ii) *If* $\alpha \succ \beta$ *and* $\gamma \in \mathbb{Z}^n_{\geq 0}$, *then* $\alpha + \gamma \succ \beta + \gamma$.
(iii) $\succ$ *is a well-order on* $\mathbb{Z}^n_{\geq 0}$.

According to the above definition, if $\succ$ is a monomial order, then $X^\alpha \succ X^\beta$ means that $\alpha \succ \beta$.

Definition 2 ([17])**.** *(Lexicographic Order). We say $X^\alpha \succ_{lex} X^\beta$, if $\alpha - \beta \in \mathbb{Z}^n$ in the vector difference, the left-most nonzero entry is positive, in which $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \cdots, \beta_n) \in \mathbb{Z}^n_{\geq 0}$.*

Definition 3 ([17])**.** *(Graded Lex Order). Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \cdots, \beta_n) \in \mathbb{Z}^n_{\geq 0}$. We say $X^\alpha \succ_{grlex} X^\beta$ if*

$$\mid \alpha \mid = \sum_{i=1}^{n} \alpha_i > \mid \beta \mid = \sum_{i=1}^{n} \beta_i,$$

or

$$\mid \alpha \mid = \mid \beta \mid \quad \text{and} \quad \alpha \succ_{lex} \beta.$$

In this paper, when dealing with polynomials in two variables, we always assume that $x \succ_{lex} y$.

Definition 4. *Let $S = [X^{\alpha^{(1)}}, X^{\alpha^{(2)}}, \cdots, X^{\alpha^{(l)}}]$ be a monomial sequence ordered by the graded lex order $\prec_{grlex}$, we denote by $D = [D_1, D_2, \cdots, D_l]$ the symmetric differential operator sequence associated with the monomial sequence S, where*

$$X^{\alpha^{(i)}} = x_1^{\alpha_1{}^{(i)}} x_2^{\alpha_2{}^{(i)}} \cdots x_n^{\alpha_n{}^{(i)}}$$

and

$$D_i = \frac{\partial^{\alpha_1{}^{(i)} + \alpha_2{}^{(i)} + \cdots + \alpha_n{}^{(i)}}}{\partial x_1^{\alpha_1{}^{(i)}} \cdots \partial x_n^{\alpha_n{}^{(i)}}}.$$

We next provide the multivariate Birkhoff interpolation problem in a more general setting.

Definition 5. *A generalized multivariate Birkhoff interpolation scheme, (Z, S, E), consists of three components*

- *A set of nodes Z,*
$$Z = \{z_i\}_{i=1}^m = \{(x_{i1}, \cdots, x_{in})\}_{i=1}^m \subset \mathbb{R}^n.$$
- *A monomial sequence ordered by the graded lex order $\prec_{grlex}$,*
$$S = [X^{\alpha^{(1)}}, X^{\alpha^{(2)}}, \cdots, X^{\alpha^{(l)}}].$$

- *An incidence matrix E, which consists of m sub-matrices,*

$$E = \begin{pmatrix} E_1 \\ E_2 \\ \vdots \\ E_m \end{pmatrix},$$

where $E_i = (e_{j,h}^{(i)})$, $1 \le i \le m$, $j = 1, \cdots, j_i, h = 1, \cdots, l, e_{j,h}^{(i)} \in \mathbb{R}$. Every row of E_i corresponds to an interpolation condition on node z_i and the i-th column of E corresponds to the i-th monomial $X^{\alpha^{(i)}}$. Any row of E is not a zero row.

For given data values $c_{ij} \in \mathbb{R}$, the generalized Birkhoff interpolation problem associated with the scheme (Z, S, E) is to find an interpolation space $\mathcal{P} \in \mathbb{K}[X]$ and a polynomial $f \in \mathcal{P}$ satisfying the interpolation conditions

$$\sum_{h=1}^{l} e_{j,h}^{(i)} D_h f(z_i) = \sum_{h=1}^{l} e_{j,h}^{(i)} \frac{\partial^{\alpha_1{}^{(h)}+\alpha_2{}^{(h)}+\cdots+\alpha_n{}^{(h)}}}{\partial x_1^{\alpha_1{}^{(h)}} \cdots \partial x_n^{\alpha_n{}^{(h)}}} f(z_i) = c_{ij}, 1 \le i \le m, 1 \le j \le j_i, \tag{1}$$

where $D = [D_1, D_2, \cdots, D_l]$ is the differential operator sequence associated with the monomial sequence S.

Example 1. *Given a generalized multivariate Birkhoff interpolation scheme (Z, S, E), where $S = [1, y, x, y^2, xy]$, $Z = \{z_1, z_2\} = \{(x_1, y_1), (x_2, y_2)\} \subset \mathbb{R}^2$ and $E = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$, the symmetric differential operator sequence associated with the monomial sequence $S = [1, y, x, y^2, xy]$ is*

$$D = [1, \frac{\partial}{\partial y}, \frac{\partial}{\partial x}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial x \partial y}].$$

E_1 and E_2 are sub-matrices of E, and

$$E_1 = \begin{pmatrix} 1 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & 3 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 2 & 0 & 1 & 3 & 0 \\ 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 5 \end{pmatrix}.$$

The Table 1 expresses the interpolation conditions associated with the given interpolation scheme (Z, S, E).

Table 1. Interpolation conditions table.

S	**1**	y	x	y^2	xy	**Node**
D	**1**	$\frac{\partial}{\partial y}$	$\frac{\partial}{\partial x}$	$\frac{\partial^2}{\partial y^2}$	$\frac{\partial^2}{\partial x \partial y}$	
E_1	1	0	2	1	1	z_1
	0	1	0	3	0	
E_2	2	0	1	3	0	z_2
	1	2	0	0	1	
	0	0	0	2	5	

Let $\{c_{11}, c_{12}, c_{21}, c_{22}, c_{23}\}$ be a set of data values. Given these components, the Birkhoff interpolation problem associated with the scheme (Z, S, E) is to find an interpolation space $\mathcal{P} \in \mathbb{K}[X]$ and a polynomial $f \in \mathcal{P}$ satisfying the interpolation conditions

$$
\begin{aligned}
&f(z_1) + 2\frac{\partial}{\partial x}f(z_1) + \frac{\partial^2}{\partial y^2}f(z_1) + \frac{\partial^2}{\partial x \partial y}f(z_1) = c_{11}, \\
&\frac{\partial}{\partial y}f(z_1) + 3\frac{\partial^2}{\partial y^2}f(z_1) = c_{12}, \\
&2f(z_2) + \frac{\partial}{\partial x}f(z_2) + 3\frac{\partial^2}{\partial y^2}f(z_2) = c_{21}, \\
&f(z_2) + 2\frac{\partial}{\partial y}f(z_2) + \frac{\partial^2}{\partial x \partial y}f(z_2) = c_{22}, \\
&2\frac{\partial^2}{\partial y^2}f(z_2) + 5\frac{\partial^2}{\partial x \partial y}f(z_2) = c_{23}.
\end{aligned}
$$

3. Invariant Interpolation Space

In this section, we will propose the notion of invariant interpolation space and prove that the space spanned by the monomial sequence S is the invariant interpolation space for the given interpolation scheme (Z, S, E) if E satisfies some conditions. First of all, we will introduce the notion of a proper interpolation space for a given node set.

Definition 6. *Given a generalized multivariate Birkhoff interpolation scheme (Z, S, E), Z_1 is one of choice of Z. We say $\mathcal{P} \in \mathbb{K}[X]$ is a proper interpolation space for the node set Z_1, if there exists a unique polynomial $f \in \mathcal{P}$ satisfying the interpolation conditions deduced by the given interpolation scheme (Z, S, E) for any given data values.*

Definition 7. *We say $\mathcal{P} \in \mathbb{K}[X]$ is the invariant interpolation space for a generalized multivariate Birkhoff interpolation scheme (Z, S, E) if $\mathcal{P}$ is a proper interpolation space for any choice of Z.*

Equation (1) is a set of linear equations with unknown coefficients of the interpolation polynomial f and we denote by $M(E, Z, \mathcal{P})$ the coefficient matrix of these equations. If the number of rows in E equals to $dim\mathcal{P}$, $M(E, Z, \mathcal{P})$ is a square matrix and its determinant is a so-called Vandermonde determinant, denoted by $D(E, Z, \mathcal{P})$. If Z is fixed, there exists a unique solution of Equation (1) for any given data values if, and only if, $D(E, Z, \mathcal{P}) \neq 0$. If Z is not fixed, whether the interpolation problem has a unique solution depends on the choice of the set of nodes Z. Viewing the points in Z as variables, $D(E, Z, \mathcal{P})$ is a polynomial function on the mn coordinates of these points. For an incidence matrix E, one denotes by $|E|$ the number of rows in E and $|S|$ the number of elements in S. Thus, we are led to the following theorem.

Theorem 1. *For a given generalized multivariate Birkhoff interpolation scheme (Z, S, E), $\mathcal{P} \in \mathbb{K}[X]$ is an invariant interpolation space if $dim\mathcal{P} = |E|$ and $D(E, Z, \mathcal{P}) \neq 0$ for all choices of sets of nodes Z.*

For a given interpolation scheme (Z, S, E), maybe there is no invariant interpolation space and it is also difficult to find it when the space exists. If the interpolation conditions are of some good properties, we can easily obtain an invariant interpolation space. Above all let us introduce the permitted elementary row operations of an incidence matrix.

Definition 8. *The permitted elementary row operations of the incidence matrix* $E = \begin{pmatrix} E_1 \\ E_2 \\ \vdots \\ E_m \end{pmatrix}$ *include the following:*

- *if $r_l, r_k \in E_i, i = 1, 2, \cdots, m$, then*
    - *(1) a times the l-th row of E and add ar_l to the k-th row, i.e., $ar_l + r_k$;*

(2) *exchange the l-th row and the k-th row, i.e.,* $r_l \leftrightarrow r_k$*;*

- *if* $r_l \in E_i, r_k \in E_j, i \neq j$*, then* $r_l \leftrightarrow r_k$*.*

Theorem 2. *For a given generalized multivariate Birkhoff interpolation scheme* (Z, S, E)*, if the incidence matrix E can be reduced to an upper triangular matrix* $\hat{E}$ *by performing the permitted elementary row operations described in Definition 7, of which the diagonal elements are nonzero constants, and* $|E| = |S|$*, then* $\mathcal{P}_S = span_{\mathbb{K}}\{X^\alpha | X^\alpha \in S\}$ *is an invariant interpolation space.*

Proof. Science $\hat{E}$ is reduced from E by performing the permitted elementary row operations, the interpolation conditions defined by the matrices E and $\hat{E}$ are equivalent. So, we only need to show that $\mathcal{P}_S$ is an invariant interpolation space for the scheme $(Z, S, \hat{E})$. $|E| = |S|$ and $|\hat{E}| = |E|$ imply $|\hat{E}| = dim\mathcal{P}_S$, i.e., the number of interpolation conditions is equal to the dimension of the interpolation space $\mathcal{P}_S$. From this we see the coefficient matrix $M(\hat{E}, Z, \mathcal{P}_{\mathcal{S}})$ is a square matrix. According to the Theorem 1, If we prove that $D(\hat{E}, Z, \mathcal{P}_{\mathcal{S}}) \neq 0$ for all choices of sets of nodes Z, the assertion follows. Let $S = [X^{\alpha^{(1)}}, X^{\alpha^{(2)}}, \cdots, X^{\alpha^{(n)}}]$ and $\hat{E} = (e_{ij})$, $1 \leq i \leq n$,$1 \leq j \leq n$. We recall that $D = [D_1, D_2, \cdots, D_n]$ is the symmetric differential operator sequence associated with the monomial sequence S. Let $f = \sum_{i=1}^{n} a_i X^{\alpha^{(i)}} \in \mathcal{P}_S$ be an interpolation polynomial. The i-th interpolation condition is $\sum_{j=1}^{n} e_{ij} D_j(f) = \sum_{j=i}^{n} e_{ij} D_j(f)$, which is due to the fact that $\hat{E}$ is an upper triangular matrix, i.e., $e_{ij} = 0, i > j$. Suppose that $\sum_{j=i}^{n} e_{ij} D_j(f) = \begin{pmatrix} c_{i1} & c_{i2} & \cdots & c_{in} \end{pmatrix}$ $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$, where $\begin{pmatrix} c_{i1} & c_{i2} & \cdots & c_{in} \end{pmatrix}$ is just the i-th row of $M(\hat{E}, Z, \mathcal{P}_{\mathcal{S}})$. Since S is the monomial sequence ordered by the graded lex order $\prec_{grlex}$, $D_j(X^{\alpha^{(j)}}) = 1$ and $D_j(X^{\alpha^{(k)}}) = 0$ if $k < j$. So, we can conclude that $c_{ii} = e_{ii}$ and $c_{ij} = 0$, where $i > j$. This implies $M(\hat{E}, Z, \mathcal{P}_{\mathcal{S}})$ is also an upper triangular squire matrix, of which the diagonal elements are nonzero constants. Therefore, it is clear that $D(\hat{E}, Z, \mathcal{P}_{\mathcal{S}})$ is also a nonzero constant, i.e., $D(\hat{E}, Z, \mathcal{P}_{\mathcal{S}})$ never vanish for any choice of the set of nodes Z. Thus, the proof is completed. □

The principal significance of the theorem is that it allows one to deduce $\mathcal{P}_{\mathcal{S}}$ is an invariant interpolation space or not, by reducing the incidence matrix E instead of computing complex determinant $D(E, Z, \mathcal{P}_{\mathcal{S}})$.

Example 2. *Given a generalized multivariate Birkhoff interpolation scheme* (Z, S, E)*, where* $S = [1, y, x, y^2, xy, x^2]$, $Z = \{z_1, z_2\} = \{(x_1, y_1), (x_2, y_2)\} \subset \mathbb{R}^2$ *and*

$$E = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

it is clear that $|S| = |E| = 6$*.* E_1 *and* E_2 *are sub-matrices of E, and* $E_1 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$*. By performing the permitted elementary row operations, we can obtain that*

$$\hat{E} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Obviously, $\hat{E}$ is an upper triangular matrix and the diagonal elements are nonzero. According to Theorem 2, we can deduce that $\mathcal{P}_S = span_{\mathbb{K}}[1, y, x, y^2, xy, x^2]$ is an invariant interpolation space. In fact, we can compute that the coefficient matrix is

$$M(E, Z, \mathcal{P}_S) = \begin{pmatrix} 1 & y_1 & x_1 & y_1^2+2 & x_1y_1 & x_1^2+2 \\ 1 & y_1 & x_1 & y_1^2+4 & x_1y_1+1 & x_1^2 \\ 0 & 0 & 1 & 0 & y_1+1 & 2x_1+4 \\ 0 & 1 & 0 & 2y_2 & x_2+1 & 2 \\ 0 & 1 & 0 & 2y_2 & x_2+3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

and it is not difficult to obtain the Vandermonde determinant $D(E, Z, \mathcal{P}_S) \equiv 8$. This shows that $\mathcal{P}_S = span_{\mathbb{K}}[1, y, x, y^2, xy, x^2]$ is indeed an invariant interpolation space.

Remark 1. *The condition "E can be reduced to an upper triangular matrix" is sufficient but not necessary. After some minor modifications with Example 2, we arrive at the following example.*

Example 3. *Let $S = [1, y, x]$, $Z = \{z_1, z_2\} = \{(x_1, y_1), (x_2, y_2)\} \subset \mathbb{R}^2$ and $E = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$, where $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 1 & 2 \end{pmatrix}$. Since $r_1, r_2 \in E_1$, by performing $-2r_1 + r_2$, we obtain*

$$\hat{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

Since r_2 and r_3 belong to different sub-matrices ($r_2 \in E_1, r_3 \in E_2$,), we cannot use r_2 to eliminate r_3. Thus, $\hat{E}$ is reduced and it is not an upper triangular matrix. We can compute the matrix $M(E, Z, \mathcal{P}_S) = \begin{pmatrix} 1 & y_1 & x_1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ and the Vandermonde determinant $D(E, Z, \mathcal{P}_S) \equiv 1$. This implies that $\mathcal{P}_S$ is an invariant interpolation space though the matrix E does not satisfy the condition of Theorem 2.

4. Singular Interpolation Space

In this section, we will propose the definition of a singular interpolation space, which is also a special interpolation space, and provide a criterion for the singularity.

Definition 9. *We say $\mathcal{P} \in \mathbb{K}[X]$ is the singular interpolation space for a generalized multivariate Birkhoff interpolation scheme (Z, S, E) if $\mathcal{P}$ is not a proper interpolation space for any choice of Z.*

Definition 10. *A monomial sequence B is called a lower sequence if $X^\alpha \prec_{lex} X^\beta$ and $X^\beta \in B$ imply that $X^\alpha \in B$.*

Given a Birkhoff interpolation scheme (Z, S, E), let $A \subseteq S$ be a subsequence, also a monomial sequence ordered by the graded lex order. We denote by E_A a sub-matrix of E, whose columns correspond to the elements of A. Thus, $E_S = E$. We define $\mid E_A \mid$ to be the number of nonzero rows in E_A and $\mid A \mid$ to be the number of elements in A.

Example 4. *Let $S = [1, y, x, y^2]$ and $E = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 3 & 0 \end{pmatrix}$. $A = [1, x] \in S$ is a subsequence, whose elements are the first and the third monomial of the sequence S. So, $E_A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 3 \end{pmatrix}$, whose columns are just the first and the third column of E. It is easy to see that $\mid E_A \mid = 2 = \mid A \mid$.*

Theorem 3. *For a given generalized multivariate Birkhoff interpolation scheme (Z, S, E), if there exists a lower subsequence $A \subseteq S$ such that $\mid E_A \mid < \mid A \mid$, then $\mathcal{P}_S = span_{\mathbb{K}}\{X^\alpha | X^\alpha \in S\}$ is a singular interpolation space.*

Proof. Let us consider the interpolation scheme (Z, A, E_A). For any given set of nodes Z, there are less rows than the columns for the matrix $M(E_A, Z, \mathcal{P}_A)$ because the condition $\mid E_A \mid < \mid A \mid$ implies that the number of interpolation conditions is less than the dimension of the interpolation space P_A. Then, the homogeneous interpolation problem associated with the scheme (Z, A, E_A) has a non-trivial solution $g = \sum_{X^{\alpha^{(i)}} \in A} a_i X^{\alpha^{(i)}}$, $X^{\alpha^{(i)}} = x_1^{\alpha_1{}^{(i)}} x_2^{\alpha_2{}^{(i)}} \cdots x_n^{\alpha_n{}^{(i)}}$. For any $X^{\alpha^{(j)}} = x_1^{\alpha_1{}^{(j)}} x_2^{\alpha_2{}^{(j)}} \cdots x_n^{\alpha_n{}^{(j)}} \in S \setminus A$, it follows that

$$\frac{\partial^{\alpha_1{}^{(j)} + \cdots + \alpha_n{}^{(j)}}}{\partial x_1^{\alpha_1{}^{(j)}} \cdots \partial x_n^{\alpha_n{}^{(j)}}} g \equiv 0$$

since A is a lower sequence and there exists some k, such that $\alpha_k{}^{(j)} > \alpha_k{}^{(i)}$. Then, it is easy to see that

$$\sum_{h=1}^{l} e_{j,h}^{(i)} \frac{\partial^{\alpha_1{}^{(h)} + \alpha_2{}^{(h)} + \cdots + \alpha_n{}^{(h)}}}{\partial x_1^{\alpha_1{}^{(h)}} \cdots \partial x_n^{\alpha_n{}^{(h)}}} g \equiv 0, 1 \le i \le m, 1 \le j \le j_i. \tag{2}$$

This implies that g is also a non-trivial solution of the homogeneous interpolation problem associated with the original scheme (Z, S, E) and the coefficient matrix $M(Z, E, \mathcal{P}_S)$ of the linear system (2) is not invertible for any set of nodes Z. Thus, $D(Z, E, \mathcal{P}_S) \equiv 0$, which means that $\mathcal{P}_S$ is a singular interpolation space. □

Example 5. *Let $S = [1, y, x, y^2]$, $Z = \{z_1, z_2\} = \{(x_1, y_1), (x_2, y_2)\} \subset \mathbb{R}^2$, and $E = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$, where $E_1 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. All the lower sequences of S are $L_1 = [1]$, $L_2 = [1, y]$, $L_3 = [1, x]$, $L_4 = [1, y, x]$, $L_5 = [1, y, y^2]$ and $L_6 = S = [1, y, x, y^2]$. We can see that $E_{L_2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $1 = \mid E_{L_2} \mid < \mid L_2 \mid = 2$. According to Theorem 3, we can deduce $\mathcal{P}_S = span_{\mathbb{K}}[1, y, x, y^2]$ is a singular interpolation space. In fact, we can compute that the coefficient matrix is*

$$M(E, Z, \mathcal{P}_S) = \begin{pmatrix} 1 & y_1 & x_1 + 1 & x_1^2 + 2x_1 \\ 0 & 0 & 1 & 2 + 2x_1 \\ 0 & 0 & 1 & 2x_2 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

A simple calculation gives that $D(E, Z, \mathcal{P}_S) \equiv 0$ and this shows that the interpolation space $\mathcal{P}_S = span_{\mathbb{K}}[1, y, x, y^2]$ is indeed singular.

Remark 2. *It is worth pointing out that even though all the lower sequence $L_i \subseteq S$ satisfy the condition $|E_{L_i}| \geq |L_i|$, we can not ensure that $\mathcal{P}_S = span_{\mathbb{K}}\{X^\alpha | X^\alpha \in S\}$ is not a singular interpolation space.*

Example 6. *Let $S = [1, y, x]$, $Z = \{z_1, z_2\} = \{(x_1, y_1), (x_2, y_2)\} \subset \mathbb{R}^2$ and $E = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$, where $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 2 & 4 \end{pmatrix}$. All the lower sequences of S are $L_1 = [1]$, $L_2 = [1, y]$, $L_3 = [1, x]$ and $L_4 = S = [1, y, x]$. Accordingly,*

$$E_{L_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, E_{L_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{pmatrix}, E_{L_3} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 4 \end{pmatrix}, E_{L_4} = E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix}.$$

It is easy to see that $1 =| E_{L_1} |\geq| L_1 |= 1$, $3 =| E_{L_2} |\geq| L_2 |= 2$, $3 =| E_{L_3} |\geq| L_3 |= 2$, and $3 =| E_{L_4} |\geq| L_4 |= 3$. We can compute $M(E, Z, \mathcal{P}_S) = \begin{pmatrix} 1 & y_1 & x_1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix}$. Obviously, $D(E, Z, \mathcal{P}_S) \equiv 0$ and thus the interpolation space $\mathcal{P}_S$ is singular.

5. Conclusions

We propose a generalized multivariate Birkhoff interpolation scheme, which differs from the usual one in the definition of the incidence matrix. Accordingly, based on the proper interpolation space, we introduce the notions of invariant and singular interpolation space. Theorems 2 and 3 allow us to deduce whether $\mathcal{P}_S$ is invariant or singular just from the properties of their incidence matrices and there is no need for the tedious computations for a determinant.

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