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Some New Quantum Hermite-Hadamard Type Inequalities for *s*-Convex Functions

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Abstract: In this investigation, we first establish new quantum Hermite–Hadamard type integral inequalities for s-convex functions by utilizing newly defined T_q-integrals. Then, by using obtained inequality, we establish a new Hermite–Hadamard inequality for coordinated (s_1, s_2)-convex functions. The results obtained in this paper provide significant extensions of other related results given in the literature. Finally, some examples are given to illustrate the result obtained in this paper. These types of analytical inequalities, as well as solutions, apply to different areas where the concept of symmetry is important.

Keywords: Hermite–Hadamard inequality; quantum integrals; s-convex functions; ${}^{\rho}T_q$-integral; ${}_{\sigma}T_q$-integral

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1. Introduction

The Hermite–Hadamard inequality discovered by C. Hermite and J. Hadamard (see, e.g., [1,2], p. 137) is one of the most well established inequalities in the theory of convex functions with a geometrical interpretation and many applications. These inequalities state that if $\mathcal{F} : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $\sigma, \rho \in I$ with $\sigma < \rho$, then

$$\mathcal{F}\left(\frac{\sigma+\rho}{2}\right) \leq \frac{1}{\rho-\sigma}\int_{\sigma}^{\rho} \mathcal{F}(\tau)d\tau \leq \frac{\mathcal{F}(\sigma)+\mathcal{F}(\rho)}{2}. \tag{1}$$

Over the years, a large number of studies have focused on finding trapezoid and midpoint type inequalities that provide boundaries to the right and left sides of inequality (1), respectively. Dragomir and Agarwal first considered the trapezoid inequalities for differentiable convex mappings in [3] whereas Kirmacı first proved the midpoint inequalities for differentiable convex mappings in the paper [4]. In [5], Sarikaya et al. extended the inequalities (1) to the case of Riemann–Liouville fractional integrals and the authors also established some corresponding fractional trapezoid type inequalities. What's more, Sarikaya obtained fractional Hermite–Hadamard inequalities and fractional trapezoids in the case of the functions with two variables in [6]. In [7], Dragomir proved Hermite–Hadamard type inequalities for coordinated convex functions $\mathbb{R}^2$. For some other related papers on Hermite–Hadamard type inequalities for convex functions and other kinds of convex classes, please refer to [8–27].

The concept of s-convexity is defined as follows:

Definition 1 ([28]). *Let I be a s-convex set. A function $\mathcal{F} : I \to \mathbb{R}$ is said to be a s-convex function, if*

$$\mathcal{F}(\xi\sigma + (1-\xi)\rho) \leq \xi^s\mathcal{F}(\sigma) + (1-\xi)^s\mathcal{F}(\rho) \tag{2}$$

for all $\sigma, \rho \in I$, and for $\xi \in [0,1]$, $s \in (0,1]$. If the inequality in (2) is reversed, then $\mathcal{F}$ is said to be s-concave.

Dragomir and Fitzpatrick [29] used this class of functions and proved the following Hermite–Hadamard inequality:

$$2^{s-1}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right) \leq \frac{1}{\rho-\sigma}\int_{\sigma}^{\rho} \mathcal{F}(\tau)d\tau \leq \frac{\mathcal{F}(\sigma)+\mathcal{F}(\rho)}{s+1}. \tag{3}$$

Definition 2. *A function $\mathcal{F} : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ will be called coordinated (s_1, s_2)-convex functions on Δ, if the following inequality*

$$\begin{aligned} \mathcal{F}(\xi\tau + (1-\xi)\varsigma, \lambda u + (1-\lambda)v)) \quad &\leq \quad \xi^{s_1}\lambda^{s_2}\mathcal{F}(\xi, u) + \xi^{s_1}(1-\lambda)^{s_2}\mathcal{F}(\tau, v) \\ &+(1-\xi)^{s_1}\lambda^{s_2}\mathcal{F}(\varsigma, u) + (1-\xi)^{s_1}(1-\lambda)^{s_2}\mathcal{F}(\varsigma, v) \end{aligned}$$

holds for all $\xi, \lambda \in [0,1]$, $s_1, s_2 \in (0,1]$ and $(\xi, u), (\tau, w), (\varsigma, u), (\varsigma, w) \in \Delta$, where Δ is bi-dimensional real interval.

Here, if we put $s_1 = 1$ and $s_2 = 1$, then coordinated (s_1, s_2)-convexity reduces to coordinated convexity.

In recent years, by using the concept of s-convexity, several papers have been devoted to Hermite–Hadamard inequalities for functions of one and two variables. For some of them, please refer to [29–36].

2. Quantum Calculus

In this section, we summarize some required definitions of quantum calculus and important quantum integral inequalities. For for information about the related results, one can refer to the papers [37–44]

2.1. q-Integrals and Related Inequalities

Set the following notation (see [38]):

The $[n]_q$ is set of integers and expressed as

$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \ldots\ldots q^{n-1}$ with $q \in (0,1)$.

Jackson derived the q-Jackson integral in [37] from 0 to ρ for $q \in (0,1)$ as follows:

$$\int_0^{\rho} \mathcal{F}(\tau)d_q\tau = (1-q)\rho\sum_{n=0}^{\infty} q^n\mathcal{F}(\rho q^n)$$

provided the sum converges absolutely. The q-Jackson integral in a generic interval $[\sigma, \rho]$ was given by in [37] and defined as follows:

$$\int_{\sigma}^{\rho} \mathcal{F}(\tau)d_q\tau = \int_0^{\rho} \mathcal{F}(\tau)d_q\tau - \int_0^{\sigma} \mathcal{F}(\tau)d_q\tau$$

The quantum integrals on the interval $[\sigma, \rho]$ is defined as follows:

Definition 3 ([39])**.** *Let $\mathcal{F} : [\sigma, \rho] \to \mathbb{R}$ be a continuous function. Then, the q_σ-definite integral on $[\sigma, \rho]$ is defined as*

$$\int_{\sigma}^{\tau} \mathcal{F}(\xi)\ {}_{\sigma}d_q\xi \ = (1-q)(\tau-\sigma)\sum_{n=0}^{\infty} q^n\mathcal{F}(q^n\tau + (1-q^n)\sigma) \tag{4}$$

for $\tau \in [\sigma, \rho]$.

In [45], Alp et al. proved the corresponding Hermite–Hadamard inequalities for convex functions by using q_σ-integrals, as follows:

Theorem 1. *If $\mathcal{F}:[\sigma,\rho]\to\mathbb{R}$ be a convex differentiable function on $[\sigma,\rho]$ and $0<q<1$. Then, q-Hermite–Hadamard inequalities*

$$\mathcal{F}\left(\frac{q\sigma+\rho}{1+q}\right)\leq\frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}\mathcal{F}(\tau)\ {}_{\sigma}d_q\tau\ \leq\frac{q\mathcal{F}(\sigma)+\mathcal{F}(\rho)}{1+q}. \tag{5}$$

On the other hand, Bermudo et al. gave the following new definition of quantum integral on the interval $[\sigma,\rho]$.

Definition 4 ([46]). *Let $\mathcal{F}:[\sigma,\rho]\to\mathbb{R}$ be a continuous function. Then, the q^ρ-definite integral on $[\sigma,\rho]$ is defined as*

$$\int_{\tau}^{\rho}\mathcal{F}(\xi)\ {}^{\rho}d_q\xi\ =(1-q)(\rho-\tau)\sum_{n=0}^{\infty}q^n\mathcal{F}(q^n\tau+(1-q^n)\rho)$$

for $\tau\in[\sigma,\rho]$.

Bermudo et al. proved the corresponding Hermite–Hadamard inequalities for convex functions by using q^ρ-integrals, as follows:

Theorem 2 ([46]). *If $\mathcal{F}:[\sigma,\rho]\to\mathbb{R}$ is a convex differentiable function on $[\sigma,\rho]$ and $0<q<1$. Then, q-Hermite–Hadamard inequalities*

$$\mathcal{F}\left(\frac{\sigma+q\rho}{1+q}\right)\leq\frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}\mathcal{F}(\tau)\ {}^{\rho}d_q\tau\ \leq\frac{\mathcal{F}(\sigma)+q\mathcal{F}(\rho)}{1+q}. \tag{6}$$

From Theorems 1 and 2, one can write the following inequalities:

Corollary 1 ([46]). *For any convex function $\mathcal{F}:[\sigma,\rho]\to\mathbb{R}$ and $0<q<1$, we have*

$$\mathcal{F}\left(\frac{q\sigma+\rho}{1+q}\right)+\mathcal{F}\left(\frac{\sigma+q\rho}{1+q}\right)\leq\frac{1}{\rho-\sigma}\left\{\int_{\sigma}^{\rho}\mathcal{F}(\tau)\ {}_{\sigma}d_q\tau\ +\int_{\sigma}^{\rho}\mathcal{F}(\tau)\ {}^{\rho}d_q\tau\ \right\}\leq\mathcal{F}(\sigma)+\mathcal{F}(\rho) \tag{7}$$

and

$$\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\leq\frac{1}{2(\rho-\sigma)}\left\{\int_{\sigma}^{\rho}\mathcal{F}(\tau)\ {}_{\sigma}d_q\tau\ +\int_{\sigma}^{\rho}\mathcal{F}(\tau)\ {}^{\rho}d_q\tau\ \right\}\leq\frac{\mathcal{F}(\sigma)+\mathcal{F}(\rho)}{2}. \tag{8}$$

In [47], Latif defined $q_{\sigma\varrho}$-integral for functions of two variables and presented important properties of this integral. In [48], Alp and Sarikaya proved quantum Hermite–Hadamard inequalities for co-ordinated convex functions. On the other hand, Budak et al. [49] defined the q_σ^d, q_ϱ^ρ and $q^{\rho d}$-integrals for functions of two variables and they also gave the corresponding Hermite–Hadamard inequalities for these newly defined integrals.

2.2. T_q-Integrals and T_q-Hermite-Hadamard Inequalities

In this subsection, we summarize the definitions and some properties of the T_q-Integrals.

Alp and Sarikaya defined the following new version of quantum integral, which is called ${}_{\sigma}T_q$-integral.

Definition 5 ([42]). *Let $\mathcal{F} : [\sigma, \rho] \to \mathbb{R}$ be function. For $\tau \in [\sigma, \rho]$*

$$\int_{\sigma}^{\rho} \mathcal{F}(s) \ {}_{\sigma}d_q^T s \ = \frac{(1-q)(\rho-\sigma)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n \mathcal{F}(q^n\rho + (1-q^n)\sigma) - \mathcal{F}(\rho)\right] \tag{9}$$

where $0 < q < 1$.

Theorem 3 (${}_{\sigma}T_q$-Hermite–Hadamard [42]). *Let $\mathcal{F} : [\sigma, \rho] \to \mathbb{R}$ be a convex function on $[\sigma, \rho]$ and $0 < q < 1$. Then, we have*

$$\mathcal{F}\left(\frac{\sigma+\rho}{2}\right) \le \frac{1}{\rho-\sigma}\int_{\sigma}^{\rho} \mathcal{F}(\tau) \ {}_{\sigma}d_q^T \tau \ \le \frac{\mathcal{F}(\sigma)+\mathcal{F}(\rho)}{2} \tag{10}$$

In [43], Kara et al. introduced the following generalized quantum integral, which is called ${}^{\rho}T_q$-integral.

Definition 6 ([43]). *Let $\mathcal{F} : [\sigma, \rho] \to \mathbb{R}$ be a function. For $\tau \in [\sigma, \rho]$,*

$$\int_{\sigma}^{\rho} \mathcal{F}(s) \ {}^{\rho}d_q^T s \ = \frac{(1-q)(\rho-\sigma)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n \mathcal{F}(q^n\sigma + (1-q^n)\rho) - \mathcal{F}(\sigma)\right] \tag{11}$$

where $0 < q < 1$.

Theorem 4 (${}^{\rho}T_q$-Hermite-Hadamard). *Let $\mathcal{F} : [\sigma, \rho] \to \mathbb{R}$ be a convex function on $[\sigma, \rho]$ and $0 < q < 1$. Then, we have*

$$\mathcal{F}\left(\frac{\sigma+\rho}{2}\right) \le \frac{1}{\rho-\sigma}\int_{\sigma}^{\rho} \mathcal{F}(\tau) \ {}^{\rho}d_q^T \tau \ \le \frac{\mathcal{F}(\sigma)+\mathcal{F}(\rho)}{2}. \tag{12}$$

Kara and Budak defined T_q-integrals for two-variables functions, as follows

Definition 7 ([44]). *Suppose that $\mathcal{F} : [\sigma, \rho] \times [\varrho, d] \subset \mathbb{R}^2 \to \mathbb{R}$ is a function. Then, the following ${}_{\sigma\varrho}T_q$ ${}_{\sigma}^{d}T_q$, ${}_{\varrho}^{\rho}T_q$ and ${}^{\rho d}T_q$-integrals on $[\sigma, \rho] \times [\varrho, d]$ are defined by*

$$\begin{aligned}
&\int_{\sigma}^{\rho}\int_{\varrho}^{d} \mathcal{F}(\xi, s)\,{}_{\varrho}d_{q_2}^T s\,{}_{\sigma}d_{q_1}^T \xi \\
= \ & \frac{(1-q_1)(1-q_2)(\rho-\sigma)(d-\varrho)}{4q_1q_2} \\
& \times\left[(1+q_1)(1+q_2)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m \mathcal{F}\left(q_1^n\rho + (1-q_1^n)\sigma, q_2^m d + (1-q_2^m)\varrho\right)\right. \\
& -(1+q_2)\sum_{m=0}^{\infty} q_2^m \mathcal{F}(\rho, q_2^m d + (1-q_2^m)\varrho) \\
& \left.-(1+q_1)\sum_{n=0}^{\infty} q_1^n \mathcal{F}\left(q_1^n\rho + (1-q_1^n)\sigma, d\right) + \mathcal{F}(\rho, d)\right],
\end{aligned} \tag{13}$$

$$\begin{aligned}
&\int_{\sigma}^{\rho}\int_{\varrho}^{d} \mathcal{F}(\xi,s)^{d} d_{q_2}^{T} s_{\sigma} d_{q_1}^{T}\xi \\
= &\frac{(1-q_1)(1-q_2)(\rho-\sigma)(d-\varrho)}{4q_1q_2} \\
&\times\Bigg[(1+q_1)(1+q_2)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m \mathcal{F}\left(q_1^n\rho+(1-q_1^n)\sigma, q_2^m\varrho+(1-q_2^m)d\right) \\
&-(1+q_2)\sum_{m=0}^{\infty} q_2^m \mathcal{F}(\rho, q_2^m\varrho+(1-q_2^m)d) \\
&-(1+q_1)\sum_{n=0}^{\infty} q_1^n \mathcal{F}\left(q_1^n\rho+(1-q_1^n)\sigma,\varrho\right)+\mathcal{F}(\rho,\varrho)\Bigg],
\end{aligned} \tag{14}$$

$$\begin{aligned}
&\int_{\sigma}^{\rho}\int_{\varrho}^{d} \mathcal{F}(\xi,s)_{\varrho} d_{q_2}^{T} s^{\rho} d_{q_1}^{T}\xi \\
= &\frac{(1-q_1)(1-q_2)(\rho-\sigma)(d-\varrho)}{4q_1q_2} \\
&\times\Bigg[(1+q_1)(1+q_2)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m \mathcal{F}\left(q_1^n\sigma+(1-q_1^n)\rho, q_2^m d+(1-q_2^m)\varrho\right) \\
&-(1+q_2)\sum_{m=0}^{\infty} q_2^m \mathcal{F}(\sigma, q_2^m d+(1-q_2^m)\varrho) \\
&-(1+q_1)\sum_{n=0}^{\infty} q_1^n \mathcal{F}\left(q_1^n\sigma+(1-q_1^n)\rho, d\right)+\mathcal{F}(\sigma,d)\Bigg]
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
&\int_{\sigma}^{\rho}\int_{\varrho}^{d} \mathcal{F}(\xi,s)^{d} d_{q_2}^{T} s^{\rho} d_{q_1}^{T}\xi \\
= &\frac{(1-q_1)(1-q_2)(\rho-\sigma)(d-\varrho)}{4q_1q_2} \\
&\times\Bigg[(1+q_1)(1+q_2)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m \mathcal{F}\left(q_1^n\sigma+(1-q_1^n)\rho, q_2^m\varrho+(1-q_2^m)d\right) \\
&-(1+q_2)\sum_{m=0}^{\infty} q_2^m \mathcal{F}(\sigma, q_2^m\varrho+(1-q_2^m)d) \\
&-(1+q_1)\sum_{n=0}^{\infty} q_1^n \mathcal{F}\left(q_1^n\sigma+(1-q_1^n)\rho,\varrho\right)+\mathcal{F}(\sigma,\varrho)\Bigg],
\end{aligned} \tag{16}$$

respectively.

Kara and Budak proved the corresponding Hermite–Hadamard type inequalities for these T_q-integrals. In this paper, we generalize the results proved in the papers [42–44] for s-convex functions.

3. Generalized Quantum Hermite Hadamard Inequalities

In this section, we establish new T_q Hermite–Hadamard type integral inequalities for s-convex and coordinated (s_1, s_2)-convex functions.

Theorem 5. *Let $\mathcal{F} : [\sigma, \rho] \to \mathbb{R}$ be a s-convex functions with $s \in (0, 1]$. Then, we have the quantum following Hermite–Hadamard inequality*

$$\begin{aligned}
2^s\mathcal{F}\left(\frac{\sigma+\rho}{2}\right) &\leq \frac{1}{\rho-\sigma}\left[\int_{\sigma}^{\rho}\mathcal{F}(\tau)\,{}_{\sigma}d_q^T\tau+\int_{\sigma}^{\rho}\mathcal{F}(\tau)\,{}^{\rho}d_q^T\tau\right] \\
&\leq [\mathcal{F}(\sigma)+\mathcal{F}(\rho)]\left[\frac{1}{2q}\left(\frac{1+q}{[s+1]_q}-1+q\right)+\theta_1\right]
\end{aligned} \tag{17}$$

where $q \in (0, 1)$ and

$$\theta_1 = \int_0^1 (1-\xi)^s\,{}_0d_q^T\xi.$$

Proof. Since $\mathcal{F}$ is s-convex, we have

$$\mathcal{F}(\xi\tau + (1-\xi)\varsigma) \leq \xi^s \mathcal{F}(\tau) + (1-\xi)^s \mathcal{F}(\varsigma).$$

For $\xi = \frac{1}{2}$, we can write

$$\mathcal{F}\left(\frac{\tau+\varsigma}{2}\right) \leq \frac{\mathcal{F}(\tau)+\mathcal{F}(\varsigma)}{2^s}. \tag{18}$$

Considering $\tau = \xi\rho + (1-\xi)\sigma$ and $\varsigma = \xi\sigma + (1-\xi)\rho$, in (18), we get

$$2^s \mathcal{F}\left(\frac{\sigma+\rho}{2}\right) \leq \mathcal{F}(\xi\rho + (1-\xi)\sigma) + \mathcal{F}(\xi\sigma + (1-\xi)\rho).$$

By ${}_{\sigma}T_q$- integrating with respect to ξ over $[0,1]$, we have

$$2^s \mathcal{F}\left(\frac{\sigma+\rho}{2}\right) \int_0^1 {}_0d_q^T\xi \leq \int_0^1 \mathcal{F}(\xi\rho + (1-\xi)\sigma)\, {}_0d_q^T\xi + \int_0^1 \mathcal{F}(\xi\sigma + (1-\xi)\rho)\, {}_0d_q^T\xi.$$

From Definitions 5 and 6, we have

$$\begin{aligned}
\int_0^1 \mathcal{F}(\xi\sigma + (1-\xi)\rho)\, {}_0d_q^T\xi &= \frac{1-q}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n \mathcal{F}(q^n\sigma + (1-q^n)\rho) - \mathcal{F}(\sigma)\right] \\
&= \frac{1}{\rho-\sigma}\int_{\sigma}^{\rho} \mathcal{F}(\tau)\, {}^{\rho}d_q^T\tau
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 \mathcal{F}(\xi\rho + (1-\xi)\sigma)\, {}_0d_q^T\xi &= \frac{1-q}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n \mathcal{F}(q^n\rho + (1-q^n)\sigma) - \mathcal{F}(\rho)\right] \\
&= \frac{1}{\rho-\sigma}\int_{\sigma}^{\rho} \mathcal{F}(\tau)\, {}_{\sigma}d_q^T\tau.
\end{aligned}$$

Thus, we can write

$$2^s \mathcal{F}\left(\frac{\sigma+\rho}{2}\right) \leq \frac{1}{\rho-\sigma}\left[\int_{\sigma}^{\rho} \mathcal{F}(\tau)\, {}_{\sigma}d_q^T\tau + \int_{\sigma}^{\rho} \mathcal{F}(\tau)\, {}^{\rho}d_q^T\tau\right]$$

and the first inequality (17) is proved.

To prove the second inequality, we use the s-convexity, we have

$$\mathcal{F}(\xi\rho + (1-\xi)\sigma \leq \xi^s \mathcal{F}(\rho) + (1-\xi)^s \mathcal{F}(\sigma)$$

$$\mathcal{F}(\xi\sigma + (1-\xi)\rho) \leq \xi^s \mathcal{F}(\sigma) + (1-\xi)^s \mathcal{F}(\rho).$$

Thus,

$$\mathcal{F}(\xi\rho + (1-\xi)\sigma) + \mathcal{F}(\xi\sigma + (1-\xi)\rho) \leq [\mathcal{F}(\sigma) + \mathcal{F}(\rho)][\xi^s + (1-\xi)^s] \tag{19}$$

By taking ${}_{\sigma}T_q$-integral of (19) on $[0,1]$ and by using Definitions 5 and 6, we have

$$\begin{aligned}
&\frac{1}{\rho-\sigma}\left[\int_{\sigma}^{\rho} \mathcal{F}(\tau)\, {}_{\sigma}d_q^T\tau + \int_{\sigma}^{\rho} \mathcal{F}(\tau)\, {}^{\rho}d_q^T\tau\right] \\
\leq\ & [\mathcal{F}(\sigma) + \mathcal{F}(\rho)]\left[\int_0^1 \xi^s\, {}_0d_q^T\xi + \int_0^1 (1-\xi)^s\, {}_0d_q^T\xi\right] \\
\leq\ & [\mathcal{F}(\sigma) + \mathcal{F}(\rho)]\left[\frac{1}{2q}\left(\frac{1+q}{[s+1]_q} - 1 + q\right) + \theta_1\right].
\end{aligned}$$

Thus, the proof is accomplished. □

Remark 1. *In Theorem 5, if we take the limit as $q \to 1$, then the inequality (17) becomes the inequality (3).*

Theorem 6. *If $\mathcal{F} : [\sigma, \rho] \times [\varrho, d] \to \mathbb{R}$ is coordinated (s_1, s_2)-convex functions on Δ with $s_1, s_2 \in (0, 1]$, then we have the following inequalities:*

$$\begin{aligned}
& 2^{s_1+s_2}\,\mathcal{F}\left(\frac{\sigma+\rho}{2}, \frac{\varrho+d}{2}\right) \qquad (20)\\
\leq\; & \frac{2^{s_2}}{2(\rho-\sigma)}\left(\int_\sigma^\rho \mathcal{F}\left(\tau, \frac{\varrho+d}{2}\right) {}_\sigma d_{q_1}^T\tau + \int_\sigma^\rho \mathcal{F}\left(\tau, \frac{\varrho+d}{2}\right) {}^\rho d_{q_1}^T\tau\right)\\
& + \frac{2^{s_1}}{2(d-\varrho)}\left(\int_\varrho^d \mathcal{F}\left(\frac{\sigma+\rho}{2}, \varsigma\right) {}_\varrho d_{q_2}^T\varsigma + \int_\varrho^d \mathcal{F}\left(\frac{\sigma+\rho}{2}, \varsigma\right) {}^d d_{q_2}^T\varsigma\right)\\
\leq\; & \frac{1}{(\rho-\sigma)(d-\varrho)}\left[\int_\sigma^\rho \int_\varrho^d \mathcal{F}(\tau, \varsigma)\, {}_\varrho d_{q_2}^T\varsigma\, {}_\sigma d_{q_1}^T\tau\right.\\
& + \int_\sigma^\rho \int_\varrho^d \mathcal{F}(\tau, \varsigma)\, {}^d d_{q_2}^T\varsigma\, {}_\sigma d_{q_1}^T\tau\\
& \left. + \int_\sigma^\rho \int_\varrho^d \mathcal{F}(\tau, \varsigma)\, {}_\varrho d_{q_2}^T\varsigma {}^\rho d_{q_1}^T\tau + \int_\sigma^\rho \int_\varrho^d \mathcal{F}(\tau, \varsigma)\, {}^d d_{q_2}^T\varsigma {}^\rho d_{q_1}^T\tau\right]\\
\leq\; & \frac{1}{2(\rho-\sigma)}\left[\frac{1}{2q_2}\left(\frac{1+q_2}{[s_2+1]_{q_2}} - 1 + q_2\right) + \Theta_2\right]\\
& \times\left[\int_\sigma^\rho [\mathcal{F}(\tau, \varrho) + \mathcal{F}(\tau, d)]\, {}_\sigma d_{q_1}^T\tau + \int_\sigma^\rho [\mathcal{F}(\tau, \varrho) + \mathcal{F}(\tau, d)]\, {}^\rho d_{q_1}^T\tau\right]\\
& + \frac{1}{2(d-\varrho)}\left[\frac{1}{2q_1}\left(\frac{1+q_1}{[s_1+1]_{q_1}} - 1 + q_1\right) + \Theta_1\right]\\
& \times\left[\int_\varrho^d [\mathcal{F}(\sigma, \varsigma) + \mathcal{F}(\rho, \varsigma)]\, {}_\varrho d_{q_2}^T\varsigma + \int_\varrho^d [\mathcal{F}(\sigma, \varsigma) + \mathcal{F}(\rho, \varsigma)]\, {}^d d_{q_2}^T\varsigma\right]\\
\leq\; & [F(\sigma, \varrho) + F(\sigma, d) + F(\rho, \varrho) + F(\rho, d)]\\
& \times\left[\frac{1}{2q_2}\left(\frac{1+q_2}{[s_2+1]_{q_2}} - 1 + q_2\right) + \Theta_2\right]\\
& \left[\frac{1}{2q_1}\left(\frac{1+q_1}{[s_1+1]_{q_1}} - 1 + q_1\right) + \Theta_1\right],
\end{aligned}$$

where $q_1, q_2 \in (0, 1)$, $\Theta_1 = \int_0^1 (1-\xi)^{s_1}\, {}_0 d_{q_1}^T\xi$ and $\Theta_2 = \int_0^1 (1-\xi)^{s_2}\, {}_0 d_{q_2}^T\xi$.

Proof. Let $g_\tau : [\varrho, d] \to \mathbb{R}$, $g_\tau(\varsigma) = \mathcal{F}(\tau, \varsigma)$ is s_2-convex function on $[\varrho, d]$. By using the inequality (17) for the interval $[\varrho, d]$ and $q_2 \in (0, 1)$, we have

$$\begin{aligned}
2^{s_2} g_\tau\left(\frac{\varrho+d}{2}\right) &\leq \frac{1}{d-\varrho}\left[\int_\varrho^d g_\tau(\varsigma)\, {}_\varrho d_{q_2}^T\varsigma + \int_\varrho^d g_\tau(\varsigma)\, {}^d d_{q_2}^T\varsigma\right]\\
&\leq [\mathcal{F}(\varrho) + \mathcal{F}(d)]\left[\frac{1}{2q_2}\left(\frac{1+q_2}{[s_2+1]_{q_2}} - 1 + q_2\right) + \Theta_2\right],
\end{aligned}$$

i.e.,

$$\begin{aligned}
2^{s_2}\mathcal{F}\left(\tau, \tfrac{\varrho+d}{2}\right) &\leq \tfrac{1}{d-\varrho}\left[\int_\varrho^d \mathcal{F}(\tau, \varsigma)\, {}_\varrho d_{q_2}^T\varsigma + \int_\varrho^d \mathcal{F}(\tau, \varsigma)\, {}^d d_{q_2}^T\varsigma\right]\\
&\leq [\mathcal{F}(\tau, \varrho) + \mathcal{F}(\tau, d)]\left[\tfrac{1}{2q_2}\left(\tfrac{1+q_2}{[s_2+1]_{q_2}} - 1 + q_2\right) + \Theta_2\right]
\end{aligned} \qquad (21)$$

for all $\tau \in [\sigma, \rho]$. By ${}_{\sigma}T_{q_1}$ integration of inequality (21) on $[\sigma, \rho]$ for $q_1 \in (0,1)$, we get

$$\begin{aligned}
&\tfrac{2^{s_2}}{\rho-\sigma} \int_{\sigma}^{\rho} F\left(\tau, \tfrac{\varrho+d}{2}\right) {}_{\sigma}d_{q_1}^{T}\tau \\
\leq\ & \tfrac{1}{(\rho-\sigma)(d-\varrho)} \left[\int_{\sigma}^{\rho}\int_{\varrho}^{d} \mathcal{F}(\tau,\varsigma)\, {}_{\varrho}d_{q_2}^{T}\varsigma\, {}_{\sigma}d_{q_1}^{T}\tau + \int_{\sigma}^{\rho}\int_{\varrho}^{d} \mathcal{F}(\tau,\varsigma)\, {}^{d}d_{q_2}^{T}\varsigma\, {}_{\sigma}d_{q_1}^{T}\tau\right] \\
\leq\ & \tfrac{1}{(\rho-\sigma)} \left[\tfrac{1}{2q_2}\left(\tfrac{1+q_2}{[s_2+1]_{q_2}} - 1 + q_2\right) + \Theta_2\right] \\
& \int_{\sigma}^{\rho} [\mathcal{F}(\tau,\varrho) + \mathcal{F}(\tau,d)]\, {}_{\sigma}d_{q_1}^{T}\tau.
\end{aligned} \tag{22}$$

Similarly, by ${}^{\rho}T_{q_1}$ integration of inequality (21) on $[\sigma, \rho]$ for $q_1 \in (0,1)$, we get

$$\begin{aligned}
&\tfrac{2^{s_2}}{\rho-\sigma} \int_{\sigma}^{\rho} \mathcal{F}\left(\tau, \tfrac{\varrho+d}{2}\right) {}^{\rho}d_{q_1}^{T}\tau \\
\leq\ & \tfrac{1}{(\rho-\sigma)(d-\varrho)} \left[\int_{\sigma}^{\rho}\int_{\varrho}^{d} F(\tau,\varsigma)\, {}_{\varrho}d_{q_2}^{T}\varsigma\, {}^{\rho}d_{q_1}^{T}\tau + \int_{\sigma}^{\rho}\int_{\varrho}^{d} \mathcal{F}(\tau,\varsigma)\, {}^{d}d_{q_2}^{T}\varsigma\, {}^{\rho}d_{q_1}^{T}\tau\right] \\
& \tfrac{1}{(\rho-\sigma)} \left[\tfrac{1}{2q_2}\left(\tfrac{1+q_2}{[s_2+1]_{q_2}} - 1 + q_2\right) + \Theta_2\right] \\
& \int_{\sigma}^{\rho} [\mathcal{F}(\tau,\varrho) + \mathcal{F}(\tau,d)]\, {}^{\rho}d_{q_1}^{T}\tau.
\end{aligned} \tag{23}$$

On the other hand, the function $g_{\varsigma} : [\varrho, d] \to \mathbb{R}$, $g_{\varsigma}(\tau) = \mathcal{F}(\tau, \varsigma)$ is s_1-convex function on $[\sigma, \rho]$. By using the inequality (17) for the interval $[\sigma, \rho]$ and $q_1 \in (0,1)$, we have

$$\begin{aligned}
2^{s_1}\, g_{\varsigma}\left(\frac{\sigma+\rho}{2}\right) &\leq \frac{1}{\rho-\sigma}\left[\int_{\sigma}^{\rho} g_{\varsigma}(\tau)\, {}_{\sigma}d_{q_1}^{T}\tau + \int_{\sigma}^{\rho} g_{\varsigma}(\tau)\, {}^{\rho}d_{q_1}^{T}\tau\right] \\
&\leq [\mathcal{F}(\sigma) + \mathcal{F}(\rho)]\left[\frac{1}{2q_1}\left(\frac{1+q_1}{[s_1+1]_{q_1}} - 1 + q_1\right) + \Theta_1\right],
\end{aligned}$$

i.e.,

$$\begin{aligned}
2^{s_1}\, F\left(\tfrac{\sigma+\rho}{2}, \varsigma\right) &\leq \tfrac{1}{\rho-\sigma}\left[\int_{\sigma}^{\rho} F(\tau,\varsigma)\, {}_{\sigma}d_{q_1}^{T}\tau + \int_{\sigma}^{\rho} F(\tau,\varsigma)\, {}^{\rho}d_{q_1}^{T}\tau\right] \\
&\leq [\mathcal{F}(\sigma,\varsigma) + \mathcal{F}(\rho,\varsigma)]\left[\tfrac{1}{2q_1}\left(\tfrac{1+q_1}{[s_1+1]_{q_1}} - 1 + q_1\right) + \Theta_1\right]
\end{aligned} \tag{24}$$

for all $\varsigma \in [\varrho, d]$. By ${}_{\varrho}T_{q_2}$ integration of inequality (24) on $[\varrho, d]$ for $q_2 \in (0,1)$, we get

$$\begin{aligned}
&\tfrac{2^{s_1}}{d-\varrho} \int_{\varrho}^{d} F\left(\tfrac{\sigma+\rho}{2}, \varsigma\right) {}_{\varrho}d_{q_2}^{T}\varsigma \\
\leq\ & \tfrac{1}{(\rho-\sigma)(d-\varrho)} \left[\int_{\varrho}^{d}\int_{\sigma}^{\rho} F(\tau,\varsigma)\, {}_{\sigma}d_{q_1}^{T}\tau\, {}_{\varrho}d_{q_2}^{T}\varsigma + \int_{\varrho}^{d}\int_{\sigma}^{\rho} F(\tau,\varsigma)\, {}^{\rho}d_{q_1}^{T}\tau\, {}_{\varrho}d_{q_2}^{T}\varsigma\right] \\
\leq\ & \tfrac{1}{(d-\varrho)} \left[\tfrac{1}{2q_1}\left(\tfrac{1+q_1}{[s_1+1]_{q_1}} - 1 + q_1\right) + \Theta_1\right] \\
& \int_{\varrho}^{d} [\mathcal{F}(\sigma,\varsigma) + \mathcal{F}(\rho,\varsigma)]\, {}_{\varrho}d_{q_2}^{T}\varsigma.
\end{aligned} \tag{25}$$

Similarly, by ${}^{d}T_{q_2}$ integration of inequality (24) on $[\varrho, d]$ for $q_2 \in (0,1)$, we get

$$\begin{aligned}
&\tfrac{2^{s_1}}{d-\varrho} \int_{\varrho}^{d} \mathcal{F}\left(\tfrac{\sigma+\rho}{2}, \varsigma\right) {}^{d}d_{q_2}^{T}\varsigma \\
\leq\ & \tfrac{1}{(\rho-\sigma)(d-\varrho)} \left[\int_{\varrho}^{d}\int_{\sigma}^{\rho} \mathcal{F}(\tau,\varsigma)\, {}_{\sigma}d_{q_1}^{T}\tau\, {}^{d}d_{q_2}^{T}\varsigma + \int_{\varrho}^{d}\int_{\sigma}^{\rho} F(\tau,\varsigma)\, {}^{\rho}d_{q_1}^{T}\tau\, {}^{d}d_{q_2}^{T}\varsigma\right] \\
\leq\ & \tfrac{1}{(d-\varrho)} \left[\tfrac{1}{2q_1}\left(\tfrac{1+q_1}{[s_1+1]_{q_1}} - 1 + q_1\right) + \Theta_1\right] \\
& \int_{\varrho}^{d} [\mathcal{F}(\sigma,\varsigma) + \mathcal{F}(\rho,\varsigma)]\, {}^{d}d_{q_2}^{T}\varsigma.
\end{aligned} \tag{26}$$

By adding (22), (23), (25) and (26), we get

$$\begin{aligned}
& \frac{2^{s_2}}{(\rho-\sigma)}\left[\int_{\sigma}^{\rho} \mathcal{F}\left(\tau,\frac{\varrho+d}{2}\right) {}_{\sigma}d_{q_1}^{T}\tau + \int_{\sigma}^{\rho} \mathcal{F}\left(\tau,\frac{\varrho+d}{2}\right) {}^{\rho}d_{q_1}^{T}\tau\right] \\
& + \frac{2^{s_1}}{(d-\varrho)}\left[\int_{\varrho}^{d} \mathcal{F}\left(\frac{\sigma+\rho}{2},\varsigma\right) {}_{\varrho}d_{q_2}^{T}\varsigma + \int_{\varrho}^{d} \mathcal{F}\left(\frac{\sigma+\rho}{2},\varsigma\right) {}^{d}d_{q_2}^{T}\varsigma\right] \\
\leq\; & \frac{2}{(\rho-\sigma)(d-\varrho)}\left[\int_{\sigma}^{\rho}\int_{\varrho}^{d} \mathcal{F}(\tau,\varsigma)\, {}_{\varrho}d_{q_2}^{T}\varsigma\, {}_{\sigma}d_{q_1}^{T}\tau + \int_{\sigma}^{\rho}\int_{\varrho}^{d} \mathcal{F}(\tau,\varsigma)\, {}^{d}d_{q_2}^{T}\varsigma\, {}_{\sigma}d_{q_1}^{T}\tau\right. \\
& \left. + \int_{\sigma}^{\rho}\int_{\varrho}^{d} \mathcal{F}(\tau,\varsigma)\, {}_{\varrho}d_{q_2}^{T}\varsigma\, {}^{\rho}d_{q_1}^{T}\tau + \int_{\sigma}^{\rho}\int_{\varrho}^{d} \mathcal{F}(\tau,\varsigma)\, {}^{d}d_{q_2}^{T}\varsigma\, {}^{\rho}d_{q_1}^{T}\tau\right] \\
\leq\; & \frac{1}{(\rho-\sigma)}\left[\frac{1}{2q_2}\left(\frac{1+q_2}{[s_2+1]_{q_2}} - 1 + q_2\right) + \Theta_2\right] \\
& \times\left[\int_{\sigma}^{\rho} [\mathcal{F}(\tau,\varrho)+\mathcal{F}(\tau,d)]\, {}_{\sigma}d_{q_1}^{T}\tau + \int_{\sigma}^{\rho} [\mathcal{F}(\tau,\varrho)+\mathcal{F}(\tau,d)]\, {}^{\rho}d_{q_1}^{T}\tau\right] \\
& + \frac{1}{(d-\varrho)}\left[\frac{1}{2q_1}\left(\frac{1+q_1}{[s_1+1]_{q_1}} - 1 + q_1\right) + \Theta_1\right] \\
& \times\left[\int_{\varrho}^{d} [\mathcal{F}(\sigma,\varsigma)+\mathcal{F}(\rho,\varsigma)]\, {}_{\varrho}d_{q_2}^{T}\varsigma + \int_{\varrho}^{d} [\mathcal{F}(\sigma,\varsigma)+\mathcal{F}(\rho,\varsigma)]\, {}^{d}d_{q_2}^{T}\varsigma\right].
\end{aligned} \tag{27}$$

This completes the proof second and third inequality in (20). From left side of inequality (17), we have

$$2^{s_2}\mathcal{F}\left(\frac{\sigma+\rho}{2},\frac{\varrho+d}{2}\right) \leq \frac{1}{d-\varrho}\left[\int_{\varrho}^{d} F(\frac{\sigma+\rho}{2},\varsigma)\, {}_{\varrho}d_{q_2}^{T}\varsigma + \int_{\varrho}^{d} F(\frac{\sigma+\rho}{2},\varsigma)\, {}^{d}d_{q_2}^{T}\varsigma\right] \tag{28}$$

and

$$2^{s_1}\mathcal{F}\left(\frac{\sigma+\rho}{2},\frac{\varrho+d}{2}\right) \leq \frac{1}{\rho-\sigma}\left[\int_{\sigma}^{\rho} \mathcal{F}(\tau,\frac{\varrho+d}{2})\, {}_{\sigma}d_{q_1}^{T}\tau + \int_{\sigma}^{\rho} \mathcal{F}(\tau,\frac{\varrho+d}{2})\, {}^{\rho}d_{q_1}^{T}\tau\right]. \tag{29}$$

Using (28) and (29) in (27), we get first inequality of (20). Now from right side of inequality (17), we have

$$\begin{aligned}
& \frac{1}{(\rho-\sigma)}\left[\int_{\sigma}^{\rho} \mathcal{F}(\tau,\varrho){}_{\sigma}d_{q_1}^{T}\tau + \int_{\sigma}^{\rho} \mathcal{F}(\tau,\varrho){}^{\rho}d_{q_1}^{T}\tau\right. \\
& \left. + \int_{\sigma}^{\rho} \mathcal{F}(\tau,d){}_{\sigma}d_{q_1}^{T}\tau + \int_{\sigma}^{\rho} \mathcal{F}(\tau,d){}^{\rho}d_{q_1}^{T}\tau\right] \\
\leq\; & [\mathcal{F}(\sigma,\varrho)+\mathcal{F}(\rho,\varrho)+\mathcal{F}(\sigma,d)+\mathcal{F}(\rho,d)] \\
& \times\left[\frac{1}{2q_1}\left(\frac{1+q_1}{[s_1+1]_{q_1}} - 1 + q_1\right) + \Theta_1\right]
\end{aligned} \tag{30}$$

and

$$\begin{aligned}
& \frac{1}{(d-\varrho)}\left[\int_{\varrho}^{d} \mathcal{F}(\sigma,\varsigma)\, {}_{\varrho}d_{q_2}^{T}\varsigma + \int_{\varrho}^{d} \mathcal{F}(\sigma,\varsigma)\, {}^{d}d_{q_2}^{T}\varsigma\right. \\
& \left. + \int_{\varrho}^{d} \mathcal{F}(\rho,\varsigma)\, {}_{\varrho}d_{q_2}^{T}\varsigma + \int_{\varrho}^{d} \mathcal{F}(\rho,\varsigma)\, {}^{d}d_{q_2}^{T}\varsigma\right] \\
\leq\; & [\mathcal{F}(\sigma,\varrho)+\mathcal{F}(\rho,\varrho)+\mathcal{F}(\sigma,d)+\mathcal{F}(\rho,d)] \\
& \times\left[\frac{1}{2q_2}\left(\frac{1+q_2}{[s_2+1]_{q_2}} - 1 + q_2\right) + \Theta_2\right].
\end{aligned} \tag{31}$$

Using (30) and (31) in (27), we obtain the last inequality of (20).

Thus, the proof is accomplished. □

Corollary 2. *If we take the limit $q_1, q_2 \to 1^-$ in Theorem 6, then the inequality (20) becomes the following Hermite–Hadamard inequality for coordinated (s_1, s_2)-convex functions*

$$\begin{aligned} 2^{s_1+s_2-2}\,\mathcal{F}\left(\frac{\sigma+\rho}{2}, \frac{\varrho+d}{2}\right) &\leq \left[\frac{2^{s_2}}{4(\rho-\sigma)}\int_\sigma^\rho \mathcal{F}\left(\tau, \frac{\varrho+d}{2}\right)d\tau \right. \\ &\quad \left. + \frac{2^{s_1}}{4(d-\varrho)}\int_\varrho^d \mathcal{F}\left(\frac{\sigma+\rho}{2}, \varsigma\right)d\varsigma\right] \\ &\leq \frac{1}{(\rho-\sigma)(d-\varrho)}\int_\sigma^\rho\int_\varrho^d \mathcal{F}(\tau,\varsigma)\,d\varsigma d\tau \\ &\leq \frac{1}{2(\rho-\sigma)(s_2+1)}\left[\int_\sigma^\rho [\mathcal{F}(\tau,\varrho)+\mathcal{F}(\tau,d)]d\tau\right] \\ &\quad + \frac{1}{2(d-\varrho)(s_1+1)}\left[\int_\varrho^d [\mathcal{F}(\sigma,\varsigma)+\mathcal{F}(\rho,\varsigma)]d\varsigma\right] \\ &\leq \frac{\mathcal{F}(\sigma,\varrho)+\mathcal{F}(\sigma,d)+\mathcal{F}(\rho,\varrho)+\mathcal{F}(\rho,d)}{(s_1+1)(s_2+1)}. \end{aligned} \tag{32}$$

Remark 2. *If we take $s_1 = s_2 = 1$ in Corollary 2, then (32) reduces to ([7], Theorem 1).*

4. Applications

In this section, we give some applications to illustrate our main outcomes.

Example 1. *Define the function $\mathcal{F}(\tau) = \tau^2$ on $[0,1]$. Applying Theorem 5 with $q = s = \frac{1}{2}$, we have*

$$2^s\mathcal{F}\left(\frac{\sigma+\rho}{2}\right) = \frac{\sqrt{2}}{2}\mathcal{F}\left(\frac{1}{2}\right) = \frac{1}{2\sqrt{2}},$$

$$\frac{1}{(\rho-\sigma)}\left[\int_\sigma^\rho \mathcal{F}(\tau)\,{}_\sigma d_q^T\tau + \int_\sigma^\rho \mathcal{F}(\tau)\,{}^\rho d_q^T\tau\right] = \int_0^1 \tau^2\,{}_0 d_{\frac{1}{2}}^T\tau + \int_0^1 \tau^2\,{}^1 d_{\frac{1}{2}}^T\tau = 0.714$$

and

$$\begin{aligned} &[\mathcal{F}(\sigma)+\mathcal{F}(\rho)]\left[\frac{1}{2q}\left(\frac{1+q}{[s+1]_q} - 1 + q\right) + \theta_1\right] \\ =\; & \frac{3}{2[\frac{3}{2}]_{\frac{1}{2}}} - \frac{1}{2} + \theta_1 \approx \frac{3\sqrt{2}}{2(2\sqrt{2}-1)} - \frac{1}{2} + \frac{6\sqrt{2}+3\sqrt{3}}{16} \\ =\; & 1.515 \end{aligned}$$

Hence, the result is verified.

Example 2. *Define a function $\mathcal{F} : \Delta \to \mathbb{R}$ $\mathcal{F}(\tau,\varsigma) = \tau^2\varsigma^2$ if $\Delta \in [0,1]\times[0,1]$. Applying Theorem 6 with $q_1 = q_2 = \frac{1}{2}$, and $s_1 = s_2 = \frac{1}{2}$, we have*

$$2^{s_1+s_2}\,\mathcal{F}\left(\frac{\sigma+\rho}{2}, \frac{\varrho+d}{2}\right) = 2^{\frac{1}{2}+\frac{1}{2}}\mathcal{F}\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{8},$$

$$
\begin{aligned}
& \frac{2^{s_2}}{2(\rho-\sigma)}\left(\int_\sigma^\rho \mathcal{F}\left(\tau,\frac{\varrho+d}{2}\right) {}_\sigma d_{q_1}^T\tau+\int_\sigma^\rho \mathcal{F}\left(\tau,\frac{\varrho+d}{2}\right) {}^\rho d_{q_1}^T\tau\right) \\
& +\frac{2^{s_1}}{2(d-\varrho)}\left(\int_\varrho^d \mathcal{F}\left(\frac{\sigma+\rho}{2},\varsigma\right) {}_\varrho d_{q_2}^T\varsigma+\int_\varrho^d \mathcal{F}\left(\frac{\sigma+\rho}{2},\varsigma\right) {}^d d_{q_2}^T\varsigma\right) \\
= \; & \frac{1}{\sqrt{2}}\left(\int_0^1 \frac{\tau^2}{4} {}_0 d_{\frac{1}{2}}^T\tau+\int_0^1 \frac{\tau^2}{4} {}^1 d_{\frac{1}{2}}^T\tau\right) \\
& +\frac{1}{\sqrt{2}}\left(\int_0^1 \frac{\varsigma^2}{4} {}_0 d_{\frac{1}{2}}^T+\int_0^1 \frac{\varsigma^2}{4} {}^1 d_{\frac{1}{2}}^T\varsigma\right) \\
= \; & \frac{5}{14\sqrt{2}},
\end{aligned}
$$

$$
\begin{aligned}
& \frac{1}{(\rho-\sigma)(d-\varrho)}\left[\int_\sigma^\rho\int_\varrho^d \mathcal{F}(\tau,\varsigma) {}_\varrho d_{q_2}^T\varsigma\, {}_\sigma d_{q_1}^T\tau+\int_\sigma^\rho\int_\varrho^d \mathcal{F}(\tau,\varsigma) {}^d d_{q_2}^T\varsigma\, {}_\sigma d_{q_1}^T\tau\right. \\
& \left.+\int_\sigma^\rho\int_\varrho^d \mathcal{F}(\tau,\varsigma) {}_\varrho d_{q_2}^T\varsigma {}^\rho d_{q_1}^T\tau+\int_\sigma^\rho\int_\varrho^d \mathcal{F}(\tau,\varsigma) {}^d d_{q_2}^T\varsigma {}^\rho d_{q_1}^T\tau\right] \\
= \; & \int_0^1\int_0^1 \tau^2\varsigma^2 {}_0 d_{q_2}^T\varsigma\, {}_0 d_{q_1}^T\tau+\int_0^1\int_0^1 \tau^2\varsigma^2 {}^1 d_{q_2}^T\varsigma\, {}_0 d_{q_1}^T\tau \\
& +\int_0^1\int_0^1 \tau^2\varsigma^2 {}_0 d_{q_2}^T\varsigma {}^1 d_{q_1}^T\tau+\int_0^1\int_0^1 \tau^2\varsigma^2 {}^1 d_{q_2}^T\varsigma\, {}^1 d_{q_1}^T\tau \\
= \; & \frac{25}{196}+\frac{25}{196}+\frac{25}{196}+\frac{25}{196} \\
= \; & \frac{25}{49},
\end{aligned}
$$

$$
\begin{aligned}
& \frac{1}{2(\rho-\sigma)}\left[\frac{1}{2q_2}\left(\frac{1+q_2}{[s_2+1]_{q_2}}-1+q_2\right)+\Theta_2\right] \\
& \times\left[\int_\sigma^\rho [\mathcal{F}(\tau,\varrho)+\mathcal{F}(\tau,d)]_\sigma d_{q_1}^T\tau+\int_\sigma^\rho [\mathcal{F}(\tau,\varrho)+\mathcal{F}(\tau,d)]^\rho d_{q_1}^T\tau\right] \\
& +\frac{1}{2(d-\varrho)}\left[\frac{1}{2q_1}\left(\frac{1+q_1}{[s_1+1]_{q_1}}-1+q_1\right)+\Theta_1\right] \\
& \times\left[\int_\varrho^d [\mathcal{F}(\sigma,\varsigma)+\mathcal{F}(\rho,\varsigma)]_\varrho d_{q_2}^T\varsigma+\int_\varrho^d [\mathcal{F}(\sigma,\varsigma)+\mathcal{F}(\rho,\varsigma)]^d d_{q_2}^T\varsigma\right] \\
= \; & \frac{1}{2}\left[\left(\frac{\frac{3}{2}}{[\frac{3}{2}]_{\frac{1}{2}}}-\frac{1}{2}\right)+\Theta_2\right]\frac{10}{14}+\frac{1}{2}\left[\left(\frac{\frac{3}{2}}{[\frac{3}{2}]_{\frac{1}{2}}}-\frac{1}{2}\right)+\Theta_1\right]\frac{10}{14} \\
\approx \; & \left[\left(\frac{3\sqrt{2}}{2(2\sqrt{2}-1)}-\frac{1}{2}\right)+\frac{6\sqrt{2}+3\sqrt{3}}{16}\right]\frac{5}{7} \\
= \; & \frac{5}{7}\left[\frac{\left(12\sqrt{2}+3\right)\left(2\sqrt{2}+\sqrt{3}\right)}{64\sqrt{2}-32}\right] \\
= \; & 1.11,
\end{aligned}
$$

$$
\begin{aligned}
& [F(\sigma,\varrho)+F(\sigma,d)+F(\rho,\varrho)+F(\rho,d)] \\
& \times\left[\frac{1}{2q_2}\left(\frac{1+q_2}{[s_2+1]_{q_2}}-1+q_2\right)+\Theta_2\right]\left[\frac{1}{2q_1}\left(\frac{1+q_1}{[s_1+1]_{q_1}}-1+q_1\right)+\Theta_1\right] \\
= & \left[\frac{\left(12\sqrt{2}+3\right)\left(2\sqrt{2}+\sqrt{3}\right)}{64\sqrt{2}-32}\right]\left[\frac{\left(12\sqrt{2}+3\right)\left(2\sqrt{2}+\sqrt{3}\right)}{64\sqrt{2}-32}\right] \\
= & \ 2.423.
\end{aligned}
$$

Hence, the result is verified.

5. Conclusions

In current work, some new quantum Hermite–Hadamard-type inequalities for s-convex and coordinated (s_1, s_2)-convex functions by utilizing the T_q-integrals are obtained. We also presented some examples which satisfied our main outcomes. It is also shown that some classical results can be obtained by the results presented in the current study by taking the limit $q \to 1$. It is a novel and fascinating problem that the researcher will be able to obtain similar inequalities in their future work for various types of convexity and coordinated convexity.

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