

A Review of Hermite–Hadamard Inequality for α -Type Real-Valued Convex Functions

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Abstract: Inequalities play important roles not only in mathematics but also in other fields, such as economics and engineering. Even though many results are published as Hermite–Hadamard (H-H)-type inequalities, new researchers to these fields often find it difficult to understand them. Thus, some important discoverers, such as the formulations of H-H-type inequalities of α -type real-valued convex functions, along with various classes of convexity through differentiable mappings and for fractional integrals, are presented. Some well-known examples from the previous literature are used as illustrations. In the many above-mentioned inequalities, the symmetrical behavior arises spontaneously.

Keywords: convex functions; generalized convex functions; fractional integrals; Hermite–Hadamard inequality

MSC: primary 26A33; 26D07; 26D15 secondary 26D10; 42B25; 34K37



Citation: Almutairi, O.; Kılıçman, A. A Review of Hermite–Hadamard Inequality for α -Type Real-Valued Convex Functions. *Symmetry* **2022**, *14*, 840. <https://doi.org/10.3390/sym14050840>

Academic Editors: Nicusor Minculete and Shigeru Furuichi

Received: 16 March 2022

Accepted: 14 April 2022

Published: 19 April 2022

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1. Introduction

Mathematical inequalities play a key role in understanding a range of problems in various fields of mathematics. Among the most celebrated ones is the Hermite–Hadamard (H-H) inequality, which made a great impact not only in mathematics but also in other related disciplines. As mentioned by Mitrinović and Lacković [1], this inequality first appeared in the literature through the effort of Hadamard [2]. However, the result was first discovered by Hermite [3]. Following this fact, many researchers referred to the result as the H-H inequality. This inequality was stated in the monograph of [4] to be the first fundamental result for convex functions defined in the interval of real numbers with a natural geometrical interpretation that can be applied to investigate a variety of problems. Inequalities play important roles in understanding many mathematical concepts, such as probability theory, numerical integration and integral operator theory. Throughout the last century, H-H type inequalities have been considered to be among the fastest growing fields in mathematical analysis, through which vast problems in engineering, economics and physics have been studied [4–6]. Due to the enormous importance of these inequalities, many extensions, refinements and generalizations of their related types have been equally investigated [7–10]. One vital problem associated with the H-H inequality is the estimation of the midpoint- and trapezoid-type inequalities. When the difference between the left part of the H-H inequality and the integral of the function under study is observed, the quantity obtained is simply called the midpoint-type inequality. Meanwhile, when such a difference is determined with the right-hand side of the H-H inequality, here, the quantity involved is called the trapezoid-type inequality [11,12].

Therefore, the H-H type inequalities, by which many results are studied, play important roles in the theory of convex functions. The convexities, along with many types of their

generalizations, including the (α, m) -convex function, $(h - m)$ -convex function, $(\alpha, h - m)$ -convex function, refined $(\alpha, h - m)$ -convex function and strongly $(\alpha, h - m)$ -convex function, can be applied in different fields of sciences [13,14], through which many generalizations of the H-H inequality for varying types of convexities have been studied. Other extensions of the H-H inequality include the formulation of problems related to fractional calculus, a branch of calculus dealing with derivatives and integrals of a non-integer order [15–17].

This paper is aimed at introducing the H-H inequality to a new researcher in the field. Thus, we present basic facts on some integral inequalities, fractional inequalities of the H-H type and their constructions via various convexity classes. Some important theorems associated with these inequalities are also discussed, along with some well-known examples to ease the beginner's understanding of the basic concepts of these inequalities. Even though the information presented in this review article can be found in separate studies on inequalities, obtaining a single work combining these results remains elusive. Thus, the sections of this review are chosen to simplify problems related to H-H inequalities.

Therefore, this review is organized in the following order. In Section 2, we present the preliminaries, comprising some basic definitions and theorems on fractional calculus and convex functions. The proof and example of an H-H inequality along with its geometrical representation are described in Section 3. Section 4 describes the generalizations of an H-H inequality involving different types of convex functions. In Section 5, we present several integral inequalities for differentiable convexity, including midpoint-type and trapezoid-type inequalities. Section 6 is devoted to the generalization of the inequalities presented in Section 5 using fractional integrals. In Section 7, we describe the application to special means using the results presented in the previous sections. Meanwhile, Section 8 presents the applications to a quadrature formula. Section 9 is devoted to the conclusion.

2. Preliminaries

In the following, we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. For more details, one can consult [18,19]. The concept of a convex function was first introduced to elementary calculus when discussing the necessary conditions for a minimum or maximum value of a differentiable function. The convex function was later recognized as an active area of study by [20]. In modern studies, a convex function is considered a link between analysis and geometry, which makes it a powerful tool for solving many practical problems:

Definition 1 ([21]). Let V be an interval in \mathbb{R} . A function $\mathcal{G} : V \rightarrow \mathbb{R}$ is said to be convex if

$$\mathcal{G}(\zeta m_1 + (1 - \zeta)m_2) \leq \zeta \mathcal{G}(m_1) + (1 - \zeta)\mathcal{G}(m_2) \quad (1)$$

holds for all $m_1, m_2 \in V$ and $\zeta \in [0, 1]$.

If the inequality in Equation (1) strictly holds for any distinct points m_1 and m_2 , where $\zeta \in (0, 1)$, then the function is said to be a strictly convex. Meanwhile, if a function $-\mathcal{G}$ is convex (strictly convex), then \mathcal{G} is concave (strictly concave).

Geometrically, a function \mathcal{G} is convex, given that the line segment joining any two points on the graph lies above (or on) the graph. Meanwhile, if the line segment connecting the two points is below (or on) the graph, the function is concave.

Example 1. Given a function $\mathcal{G} : V \subseteq \mathbb{R} \rightarrow \mathbb{R}$ for any $m \in \mathbb{R}$, we have the following examples:

- i. $\mathcal{G}(m) = c_1 m + c_2$, where $c_1, c_2 \in \mathbb{R}$. The function $\mathcal{G}(m)$ is both concave and convex on $(-\infty, \infty)$. Thus, it is referred to as an affine.
- ii. The functions $\mathcal{G}(m) = m^2$ and $\mathcal{G}(m) = e^m$ are both convex functions on \mathbb{R} .
- iii. $\mathcal{G}(m) = \ln m$ is a concave function on $(0, \infty)$.

The theory of convexity deals with large classes, such as generalized convex functions on fractal sets and Godunova–Levin, s -convex and preinvex functions. Termed as

the generalization of convexity, these play important roles in optimization theory and mathematical programming. Therefore, we give basic definitions of the different classes of convex functions.

The definition of generalized convex functions on fractal sets \mathbb{R}^α ($0 < \alpha \leq 1$) is given in [22] as follows:

Definition 2. Let $\mathcal{G} : V \subseteq \mathbb{R} \rightarrow \mathbb{R}^\alpha$. For any $m_1, m_2 \in V$ and $\zeta \in [0, 1]$, if the following inequality

$$\mathcal{G}(\zeta m_1 + (1 - \zeta)m_2) \leq \zeta^\alpha \mathcal{G}(m_1) + (1 - \zeta)^\alpha \mathcal{G}(m_2)$$

holds, then \mathcal{G} is called a generalized convex function on V .

The Godunova–Levin space function, denoted by $Q(V)$, was introduced in [23]. They noted that both the positive monotone and positive convex functions belong to $Q(V)$. Due to the importance of this function, we present it as follows:

Definition 3 ([24]). A non-negative function $\mathcal{G} : V \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is called a Godunova–Levin function (denoted by $\mathcal{G} \in Q(V)$) if

$$\mathcal{G}(\zeta m_1 + (1 - \zeta)m_2) \leq \frac{1}{\zeta} \mathcal{G}(m_1) + \frac{1}{1 - \zeta} \mathcal{G}(m_2) \quad (2)$$

holds for all $m_1, m_2 \in V$ and $\zeta \in (0, 1)$.

Example 2 ([25]). For $x \in [m_1, m_2]$, the function

$$\mathcal{G}(x) = \begin{cases} 1, & m_1 \leq x < \frac{m_1 + m_2}{2} \\ 4, & x = \frac{m_1 + m_2}{2} \\ 1, & \frac{m_1 + m_2}{2} < x \leq m_2 \end{cases}$$

is in the class $Q(V)$.

The Godunova–Levin function was restricted to a space called $P(V)$ contained in $Q(V)$. This class was defined in [25] as follows:

Definition 4. A non-negative function $\mathcal{G} : V \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is called a P -function (denoted by $\mathcal{G} \in P(V)$) if

$$\mathcal{G}(\zeta m_1 + (1 - \zeta)m_2) \leq \mathcal{G}(m_1) + \mathcal{G}(m_2)$$

holds for all $m_1, m_2 \in V$ and $\zeta \in [0, 1]$.

Therefore, all non-negative monotone and convex functions are contained in $P(V)$.

For other Godunova–Levin results and P -functions, see [26–28].

The definition of an s -convex function in the second sense or s -Breckner convex function is given as follows:

Definition 5 ([29]). A function $\mathcal{G} : [0, \infty) \rightarrow \mathbb{R}$ is said to be s -convex in the second sense (denoted by $\mathcal{G} \in K_s^2$) if

$$\mathcal{G}(\zeta_1 m_1 + \zeta_2 m_2) \leq \zeta_1^s \mathcal{G}(m_1) + \zeta_2^s \mathcal{G}(m_2) \quad (3)$$

holds for all $m_1, m_2 \in [0, \infty)$, $\zeta_1, \zeta_2 \geq 0$, $\zeta_1 + \zeta_2 = 1$ and $0 < s \leq 1$.

Choosing $s = 1$ reduces the s -convexity in the second sense to the classical convex function on $[0, \infty)$.

The following property that is connected to s -convex function in the second sense is given below:

Theorem 1 ([30]). If $\mathcal{G} \in K_s^2$, then \mathcal{G} is non-negative on $[0, \infty)$.

For some properties of s -convexity in the second sense, see [31–34].

Hudzik and Maligranda [30] presented the example of an s -convex function in the second sense as follows.

Example 3. Let $0 < s < 1$ and $c_1, c_2, c_3 \in \mathbb{R}$. When defining

$$\mathcal{G}(m) = \begin{cases} c_1, & m = 0 \\ c_2 m^s + c_3, & m > 0 \end{cases}$$

for $m \in \mathbb{R}_+$, we have

- i. If $c_2 \geq 0$ and $0 \leq c_3 \leq c_1$, then $\mathcal{G} \in K_s^2$;
- ii. If $c_2 > 0$ and $c_3 < 0$, then $\mathcal{G} \notin K_s^2$.

As Hudzik and Maligranda mentioned that the condition $\zeta_1 + \zeta_2 = 1$ in Definition 5 can be replaced by $\zeta_1 + \zeta_2 \leq 1$, then equivalently, the following is true:

Theorem 2 ([30]). Suppose that $\mathcal{G} \in K_s^2$. The inequality in Equation (3) holds for all $c_1, c_2 \in \mathbb{R}_+$ and $\zeta_1, \zeta_2 \geq 0$ with $\zeta_1 + \zeta_2 \leq 1$ if $\mathcal{G}(0) = 0$.

The geometric description of the s -convex curve, given in the definition below, was clearly explained in [35]:

Definition 6. A function $\mathcal{G} : V \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is called s -convex in the second sense for $0 < s < 1$ if the graph of the function is below a bent chord L that is between any two points. This means that for every compact interval $W \subset V$, the following inequality

$$\sup_W (L - \mathcal{G}) \geq \sup_{\partial W} (L - \mathcal{G})$$

holds with a boundary ∂W .

The s -convex function of the second sense can be referred to as the limiting curve. This differentiates the curves that are s -convex in the second sense from others which are not. Following this, Pinheiro [35] determined the effects of the choice of s on the limiting curve. For further results on the s -convex function in the second sense, we refer the reader to [36–38].

The definition of the generalized s -convex function on the fractal sets is given as follows:

Definition 7 ([22]). A function $\mathcal{G} : V \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}^\alpha$ is a generalized s -convex function in the second sense on the fractal sets if

$$\mathcal{G}(\zeta_1 m_1 + \zeta_2 m_2) \leq \zeta_1^{\alpha s} \mathcal{G}(m_1) + \zeta_2^{\alpha s} \mathcal{G}(m_2) \quad (4)$$

holds for all $m_1, m_2 \in V$, $0 < s \leq 1$, $\zeta_1, \zeta_2 \geq 0$ and $\zeta_1 + \zeta_2 = 1$. This class of function is denoted by GK_s^2 .

The generalized s -convex function in the second sense becomes an s -convex function when $\alpha = 1$.

One should note that the following theorems along with the example can be found in [22]:

Theorem 3. Let $\mathcal{G} \in GK_s^2$. The inequality in Equation (4) holds for all $m_1, m_2 \in \mathbb{R}_+$ and $\zeta_1, \zeta_2 \geq 0$ with $\zeta_1 + \zeta_2 < 1$ if $\mathcal{G}(0) = 0^\alpha$.

Theorem 4. Let $0 < s < 1$. If $\mathcal{G} \in GK_s^2$, then \mathcal{G} is non-negative on $[0, +\infty)$.

Theorem 5. Let $0 < s_1 \leq s_2 \leq 1$. If $\mathcal{G} \in GK_{s_2}^2$ and $\mathcal{G}(0) = 0^\alpha$, then $\mathcal{G} \in GK_{s_1}^2$.

Considering the properties of the generalized s -convex function in the second sense, we present the following example.

Example 4. Let $0 < s < 1$ and $a_1^\alpha, a_2^\alpha, a_3^\alpha \in \mathbb{R}^\alpha$. For $m \in \mathbb{R}_+$, we define

$$\mathcal{G}(m) = \begin{cases} a_1^\alpha, & m = 0 \\ a_2^\alpha m^{s\alpha} + a_3^\alpha, & m > 0. \end{cases}$$

Thus, we have the following:

- i. If $a_2^\alpha \geq 0^\alpha$ and $0^\alpha \leq a_3^\alpha \leq a_1^\alpha$, then $\mathcal{G} \in GK_s^2$;
- ii. If $a_2^\alpha > 0^\alpha$ and $a_3^\alpha < 0^\alpha$, then $\mathcal{G} \notin GK_s^2$.

For more results related to the generalized s -convex function in the second sense on the fractal sets, the interested reader is directed to [39,40].

In order to unify the concepts of the Godunova–Levin and P -functions, the authors of [38] introduced the s -Godunova–Levin function as follows:

Definition 8. A function $\mathcal{G} : V \subseteq \mathbb{R} \rightarrow [0, \infty)$ is said to be s -Godunova–Levin (denoted by $\mathcal{G} \in Q_{s_2}(V)$) if

$$\mathcal{G}(\zeta m_1 + (1 - \zeta)m_2) \leq \frac{1}{\zeta^s} \mathcal{G}(m_1) + \frac{1}{(1 - \zeta)^s} \mathcal{G}(m_2) \quad (5)$$

holds for all $m_1, m_2 \in V$, $\zeta \in (0, 1)$ and $0 \leq s \leq 1$.

Choosing $s = 1$ reduces the s -Godunova–Levin function to the class of Godunova–Levin. In addition, when $s = 0$, we have the P -function class. Thus, we have the following: $P(V) = Q_0(V) \subseteq Q_{s_2}(V) \subseteq Q_1(V) = Q(V)$. For more results on s -Godunova–Levin functions of convexity, we refer the reader to [41,42].

Preinvex functions are among the most important classes of generalized convex functions. This concept, playing important roles in many disciplines, was proposed in [43]. Since then, preinvex functions has become an active area of study:

Definition 9 ([44]). A set $V \subseteq \mathbb{R}$ is called invex if there exists a function $\eta : V \times V \rightarrow \mathbb{R}$ such that

$$m_1 + \zeta \eta(m_2, m_1) \in V$$

holds for all $m_1, m_2 \in V$ and $\zeta \in [0, 1]$.

The invex set V can also be referred to as an η -connected set:

Definition 10 ([43]). Suppose that $V \subseteq \mathbb{R}$ is an invex set with respect to $\eta : V \times V \rightarrow \mathbb{R}$. A function $\mathcal{G} : V \rightarrow \mathbb{R}$ is called preinvex with respect to η if

$$\mathcal{G}(m_1 + \zeta \eta(m_2, m_1)) \leq (1 - \zeta) \mathcal{G}(m_1) + \zeta \mathcal{G}(m_2) \quad (6)$$

holds for all $m_1, m_2 \in V$ and $\zeta \in [0, 1]$.

Further generalizations can be found in [45–48].

Fractional calculus, whose application can be found in many disciplines including economics, life and physical sciences as well as engineering, can be considered one of the modern branches of mathematics [49–52]. Many problems of interest from these fields can be analyzed through fractional integrals, which can also be regarded as an interesting sub-discipline of fractional calculus. Some of the applications of integral calculus can be

seen in the following papers [5–10], through which problems in physics, chemistry and population dynamics were studied. The fractional integrals were extended to include the H–H inequality [53–59]. Now, we recall some basic definitions of fractional integrals as follows:

Definition 11. Let $\mathcal{G} \in L_1[m_1, m_2]$. The left- and right-hand Riemann–Liouville integrals denoted by $J_{m_1^+}^\lambda \mathcal{G}$ and $J_{m_2^-}^\lambda \mathcal{G}$ of order $\lambda \in \mathbb{R}_+$ are defined by

$$J_{m_1^+}^\lambda \mathcal{G}(x) = \frac{1}{\Gamma(\lambda)} \int_{m_1}^x (x - \gamma)^{\lambda-1} \mathcal{G}(\gamma) d\gamma, \quad x > m_1$$

and

$$J_{m_2^-}^\lambda \mathcal{G}(x) = \frac{1}{\Gamma(\lambda)} \int_x^{m_2} (\gamma - x)^{\lambda-1} \mathcal{G}(\gamma) d\gamma, \quad x < m_2,$$

respectively.

If $\lambda = 1$ in the above equalities, we obtain the classic integral.

One should note that the Hadamard fractional integrals differ the Riemann–Liouville ones, since in the former, the logarithmic functions of arbitrary exponents are included in the kernels of the integrals. Therefore, the Hadamard fractional integrals are defined as follows:

Definition 12 ([60]). Let $\lambda > 0$ with $m - 1 < \lambda \leq m, m \in \mathbb{N}$ and $m_1 < x < m_2$. The left and right sides of the Hadamard fractional integrals denoted by $H_{m_1^+}^\lambda \mathcal{G}(x)$ and $H_{m_2^-}^\lambda \mathcal{G}(x)$ of order λ of a function \mathcal{G} are given as

$$H_{m_1^+}^\lambda \mathcal{G}(x) = \frac{1}{\Gamma(\lambda)} \int_{m_1}^x \left(\ln \frac{x}{\gamma} \right)^{\lambda-1} \frac{\mathcal{G}(\gamma)}{\gamma} d\gamma$$

and

$$H_{m_2^-}^\lambda \mathcal{G}(x) = \frac{1}{\Gamma(\lambda)} \int_x^{m_2} \left(\ln \frac{\gamma}{x} \right)^{\lambda-1} \frac{\mathcal{G}(\gamma)}{\gamma} d\gamma,$$

respectively.

The research in [55,61–63] provides useful background and the properties of Hadamard fractional integrals.

The following proposition is related to the Hadamard integrals:

Proposition 1 ([55]). If $\lambda > 0$ and $0 < m_1 < m_2 < \infty$, the following relations hold:

$$\left(H_{m_1^+}^\lambda \left(\log \frac{\gamma}{m_1} \right)^{\beta-1} \right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta + \lambda)} \left(\log \frac{x}{m_1} \right)^{\beta+\lambda-1}$$

and

$$\left(H_{m_2^-}^\lambda \left(\log \frac{m_2}{\gamma} \right)^{\beta-1} \right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta + \lambda)} \left(\log \frac{m_2}{x} \right)^{\beta+\lambda-1}.$$

The Riemann–Liouville fractional integrals, along with the Hadamard’s fractional integrals, are generalized through the recent work of [64]. These two integrals were combined and given in a single form. The following definition [64] modifies the old version [65] for Katugampola fractional integrals:

Definition 13. Let $[m_1, m_2] \subset \mathbb{R}$ be a finite interval. The left- and right-hand Katugampola fractional integrals of order $\lambda > 0$ for $\mathcal{G} \in X_c^p(m_1, m_2)$ are defined by

$${}^\rho I_{m_1^+}^\lambda \mathcal{G}(x) = \frac{\rho^{1-\lambda}}{\Gamma(\lambda)} \int_{m_1}^x \frac{\gamma^{\rho-1}}{(x^\rho - \gamma^\rho)^{1-\lambda}} \mathcal{G}(\gamma) d\gamma$$

and

$${}^\rho I_{m_2^-}^\lambda \mathcal{G}(x) = \frac{\rho^{1-\lambda}}{\Gamma(\lambda)} \int_x^{m_2} \frac{\gamma^{\rho-1}}{(\gamma^\rho - x^\rho)^{1-\lambda}} \mathcal{G}(\gamma) d\gamma,$$

with $m_1 < x < m_2$ and $\rho > 0$.

Following this, the space $X_c^p(m_1, m_2)$ ($c \in \mathbb{R}, 1 \leq p \leq \infty$) is introduced as follows:

Definition 14 ([61]). Let the space $X_c^p(m_1, m_2)$ ($c \in \mathbb{R}, 1 \leq p \leq \infty$) of those complex-valued Lebesgue measurable functions \mathcal{G} on $[m_1, m_2]$, for which $\|\mathcal{G}\|_{X_c^p} < \infty$, have the norm defined by

$$\|\mathcal{G}\|_{X_c^p} = \left(\int_{m_1}^{m_2} |\zeta^c \mathcal{G}(\zeta)|^p \frac{d\zeta}{\zeta} \right)^{1/p} < \infty \quad (1 \leq p < \infty, c \in \mathbb{R})$$

and, for the case $p = \infty$, be defined by

$$\|\mathcal{G}\|_{X_c^\infty} = \text{ess sup}_{m_1 \leq \zeta \leq m_2} (\zeta^c |\mathcal{G}(\zeta)|) \quad (c \in \mathbb{R}),$$

where the essential supremum $|\mathcal{G}(\zeta)|$ stands for the essential maximum of $|\mathcal{G}(\zeta)|$.

If $c = 1/p$, $X_c^p(m_1, m_2)$ reduces to $L_p(m_1, m_2)$, the p -integrable function.

Important references on Katugampola fractional integrals and their applications are suggested for further reading [66–69].

The relations among Katugampola fractional integrals, Riemann–Liouville integrals and Hadamard integrals are given in the next theorem. The left-hand version of the relation is considered here for its simplicity, since similar results also exist for the right-hand operators:

Theorem 6 ([69]). Let $\lambda > 0$ and $\rho > 0$. Then, for $x > m_1$, we have

- i. $\lim_{\rho \rightarrow 1} {}^\rho I_{m_1^+}^\lambda \mathcal{G}(x) = J_{m_1^+}^\lambda \mathcal{G}(x);$
- ii. $\lim_{\rho \rightarrow 0^+} {}^\rho I_{m_1^+}^\lambda \mathcal{G}(x) = H_{m_1^+}^\lambda \mathcal{G}(x).$

Remark 1. One should note that while (i) is concerned with the Riemann–Liouville operators, (ii) is related to the Hadamard operators.

The definitions of the conformable fractional derivative and integral were given in [70], and we present them as follows:

Definition 15. Let $\mathcal{G} : [0, \infty) \rightarrow \mathbb{R}$. Then, the conformable fractional derivative of \mathcal{G} of order α is defined as

$$D_\alpha(\mathcal{G})(r) = \lim_{b \rightarrow 0} \frac{\mathcal{G}(r + br^{1-\alpha}) - \mathcal{G}(r)}{b} \quad (7)$$

where \mathcal{G} is said to be α -differentiable at r if $D_\alpha(\mathcal{G})(r)$ exists. In particular, $D_\alpha(\mathcal{G})(0)$ is defined as follows:

$$D_\alpha(\mathcal{G})(0) = \lim_{r \rightarrow 0^+} D_\alpha(\mathcal{G})(r)$$

and we use $\mathcal{G}^\alpha(r)$ or $(d_\alpha/d_\alpha r)(\mathcal{G})$ to denote $D_\alpha(\mathcal{G})(r)$.

Definition 16. Let $\alpha \in (0, 1]$ and $0 \leq m_1 < m_2$. A function $\mathcal{G} : [m_1, m_2] \rightarrow \mathbb{R}$ is α -fractional integrable on $[m_1, m_2]$ if the integral

$$\int_{m_1}^{m_2} \mathcal{G}(x) d_{\alpha} x := \int_{m_1}^{m_2} \mathcal{G}(x) x^{\alpha-1} dx$$

exists and is finite. All α -fractionals integrable on $[m_1, m_2]$ are indicated by $L_{\alpha}^1([m_1, m_2])$.

Remark 2.

$$I_{\alpha}^{m_1}(\mathcal{G})(\gamma) = I_1^{m_1}(\gamma^{\alpha-1}\mathcal{G}) = \int_{m_1}^{\gamma} \frac{\mathcal{G}(x)}{x^{1-\alpha}} dx$$

where the integral is the usual Riemann improper integral and $\alpha \in (0, 1]$.

Theorem 7. Let $\alpha \in (0, 1]$ and $\mathcal{G} : [m_1, m_2] \rightarrow \mathbb{R}$ be continuous on $[m_1, m_2]$ with $0 \leq m_1 < m_2$. Then, the following is true:

$$|I_{\alpha}^{m_1}(\mathcal{G})(x)| \leq I_{\alpha}^{m_1}|\mathcal{G}|(x)$$

For more results on conformable integral operators, we refer the interested reader to [71,72].

The Hölder integral inequality plays an important role in both pure and applied sciences. Other areas applying this inequality include the theory of convexity, which can be considered one of the active and fast-growing fields of study in mathematical science. Thus, the Hölder's integral inequality is described in the following theorem:

Theorem 8 ([73]). Suppose that $p > 1$ and $1/p + 1/q = 1$. If \mathcal{G} and \mathcal{K} are real functions on $[m_1, m_2]$ such that $|\mathcal{G}|^p$ and $|\mathcal{K}|^q$ are integrable functions on $[m_1, m_2]$, then the following holds:

$$\int_{m_1}^{m_2} |\mathcal{G}(x)\mathcal{K}(x)| dx \leq \left(\int_{m_1}^{m_2} |\mathcal{G}(x)|^p dx \right)^{\frac{1}{p}} \left(\int_{m_1}^{m_2} |\mathcal{K}(x)|^q dx \right)^{\frac{1}{q}}$$

The other version of the Hölder integral inequality is called the power-mean integral, which is given in the following theorem.

Theorem 9 ([24]). Suppose that $q \geq 1$. Let \mathcal{G} and \mathcal{K} be real mappings on $[m_1, m_2]$. If $|\mathcal{G}|$ and $|\mathcal{G}||\mathcal{K}|^q$ are integrable functions in the given interval, then the following holds:

$$\int_{m_1}^{m_2} |\mathcal{G}(x)\mathcal{K}(x)| dx \leq \left(\int_{m_1}^{m_2} |\mathcal{G}(x)| dx \right)^{1-\frac{1}{q}} \left(\int_{m_1}^{m_2} |\mathcal{G}(x)||\mathcal{K}(x)|^q dx \right)^{\frac{1}{q}}$$

3. Hermite–Hadamard Inequality

The H-H inequality plays a vital role in the theory of convexity. This inequality estimates the integral average of any convex functions through the midpoint and trapezoidal formula of a given domain. While the midpoint formula estimates the integral from the left, the trapezoidal formula estimates it from the right. More precisely, the classical H-H inequality is considered as follows:

Theorem 10 ([4]). If we let $\mathcal{G} : [m_1, m_2] \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function on $[m_1, m_2]$ with $m_1 < m_2$, then the following holds:

$$(m_2 - m_1)\mathcal{G}\left(\frac{m_1 + m_2}{2}\right) \leq \int_{m_1}^{m_2} \mathcal{G}(x) dx \leq (m_2 - m_1) \frac{\mathcal{G}(m_1) + \mathcal{G}(m_2)}{2} \quad (8)$$

The proof of the inequality in Equation (8) is provided here for simplicity. Though the proof of the theorem exists, the first time Equation (8) was proven was in [7] using a similar technique reported in [74].

Proof. Let \mathcal{G} be a convex function on the interval $[m_1, m_2]$. By taking $\zeta = \frac{1}{2}$ in the inequality in Equation (1) for $x, y \in [m_1, m_2]$, we have

$$\mathcal{G}\left(\frac{x+y}{2}\right) \leq \frac{\mathcal{G}(x) + \mathcal{G}(y)}{2}. \quad (9)$$

By substituting $x = \zeta m_1 + (1 - \zeta)m_2$ and $y = (1 - \zeta)m_1 + \zeta m_2$ in (9), we obtain

$$2\mathcal{G}\left(\frac{m_1 + m_2}{2}\right) \leq \mathcal{G}(\zeta m_1 + (1 - \zeta)m_2) + \mathcal{G}((1 - \zeta)m_1 + \zeta m_2). \quad (10)$$

When integrating the inequality in Equation (10) with respect to ζ over $[0, 1]$, we have

$$\begin{aligned} 2\mathcal{G}\left(\frac{m_1 + m_2}{2}\right) &\leq \int_0^1 \mathcal{G}(\zeta m_1 + (1 - \zeta)m_2) d\zeta + \int_0^1 \mathcal{G}((1 - \zeta)m_1 + \zeta m_2) d\zeta \\ &= \frac{2}{m_2 - m_1} \int_{m_1}^{m_2} \mathcal{G}(x) dx. \end{aligned} \quad (11)$$

In order to prove the second part of the inequality in Equation (8), we used Definition 1 for $\zeta \in [0, 1]$ to arrive at

$$\mathcal{G}(\zeta m_1 + (1 - \zeta)m_2) \leq \zeta \mathcal{G}(m_1) + (1 - \zeta) \mathcal{G}(m_2)$$

and

$$\mathcal{G}((1 - \zeta)m_1 + \zeta m_2) \leq (1 - \zeta) \mathcal{G}(m_1) + \zeta \mathcal{G}(m_2).$$

When the above inequalities are added, we obtain the following:

$$\begin{aligned} &\mathcal{G}(\zeta m_1 + (1 - \zeta)m_2) + \mathcal{G}((1 - \zeta)m_1 + \zeta m_2) \\ &\leq \zeta \mathcal{G}(m_1) + (1 - \zeta) \mathcal{G}(m_2) + (1 - \zeta) \mathcal{G}(m_1) + \zeta \mathcal{G}(m_2). \end{aligned} \quad (12)$$

By integrating the inequality in Equation (12) with respect to ζ over $[0, 1]$, we have

$$\begin{aligned} &\int_0^1 \mathcal{G}(\zeta m_1 + (1 - \zeta)m_2) d\zeta + \int_0^1 \mathcal{G}((1 - \zeta)m_1 + \zeta m_2) d\zeta \\ &\leq [\mathcal{G}(m_1) + \mathcal{G}(m_2)] \int_0^1 d\zeta. \end{aligned}$$

Thus, the following equation completes the proof:

$$\frac{2}{m_2 - m_1} \int_{m_1}^{m_2} \mathcal{G}(x) dx \leq \mathcal{G}(m_1) + \mathcal{G}(m_2)$$

□

The H-H inequality is geometrically described in [21], and we have summarized it as follows.

The area under the graph of \mathcal{G} on $[m_1, m_2]$ is between the areas of two trapeziums. While the area of the first trapezium is formed by the points of coordinates $(m_1, \mathcal{G}(m_1))$, $(m_2, \mathcal{G}(m_2))$ with the x -axis, that of the second trapezium is formed by the tangent to the graph of \mathcal{G} at $\left(\frac{m_1 + m_2}{2}, \mathcal{G}\left(\frac{m_1 + m_2}{2}\right)\right)$ with the x -axis.

An example of the H-H inequality is given as follows.

Example 5 ([75]). If we choose $\mathcal{G} = e^x$ with $x \in \mathbb{R}$, the H-H inequality yields

$$e^{(m_1 + m_2)/2} < \frac{e^{m_2} - e^{m_1}}{m_2 - m_1} < \frac{e^{m_1} + e^{m_2}}{2},$$

for $m_1 < m_2$ in \mathbb{R} .

For more examples of the H-H inequality, see [4,76].

The importance of the H-H inequality is that each of its two sides is characterized by a convex function. The necessary and sufficient condition for a continuous function \mathcal{G} to be convex on (m_1, m_2) is given in the following theorem:

Theorem 11 ([77]). Let \mathcal{G} be a continuous function on (m_1, m_2) . Then, \mathcal{G} is convex if

$$\mathcal{G}(x) \leq \frac{1}{2z} \int_{x-z}^{x+z} \mathcal{G}(\zeta) d\zeta, \quad (13)$$

for $m_1 \leq x - z \leq x \leq z + k \leq m_2$.

It can be shown that the inequality in Equation (13) is equivalent to the first part of Equation (8) when \mathcal{G} is continuous on $[m_1, m_2]$ [4].

The second part of the inequality in Equation (8) can be applied as a convexity criterion in the following theorem:

Theorem 12 ([78]). Let \mathcal{G} be a continuous function on $[m_1, m_2]$. Then, \mathcal{G} is convex if

$$\frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \mathcal{G}(x) dx \leq \frac{\mathcal{G}(a_1) + \mathcal{G}(a_2)}{2},$$

for all $m_1 < a_1 < a_2 < m_2$.

4. H-H-Type Inequalities for Various Classes of Convexities

Since different classes of convexity exist, many authors are committed to the improvements and generalizations of H-H inequalities for various types of convex functions. Thus, in this section, we review some generalizations of H-H inequalities involving different convex functions whose definitions were already given in Section 2.

Dragomir et al. [25] established the two inequalities from Equation (8), which hold for classes $Q(V)$ and $P(V)$ as the Godunova–Levin and P-functions, respectively:

Theorem 13 ([25]). Let $m_1, m_2 \in V$ with $m_1 < m_2$ and $\mathcal{G} \in L_1[m_1, m_2]$. If $\mathcal{G} \in Q(V)$, then

$$\mathcal{G}\left(\frac{m_1 + m_2}{2}\right) \leq \frac{4}{m_2 - m_1} \int_{m_1}^{m_2} \mathcal{G}(x) dx \quad (14)$$

and

$$\frac{1}{m_2 - m_1} \int_{m_1}^{m_2} \phi(x) \mathcal{G}(x) dx \leq \frac{\mathcal{G}(m_1) + \mathcal{G}(m_2)}{2}$$

hold, where $\phi(x) = \frac{(m_2 - x)(x - m_1)}{(m_2 - m_1)^2}$ and $x \in V$.

In this sense, since the constant 4 is the best possible choice in Equation (14), it cannot be changed with any smaller constants:

Theorem 14 ([25]). Let $m_1, m_2 \in V$ with $m_1 < m_2$ and $\mathcal{G}(x) \in L_1[m_1, m_2]$. If $\mathcal{G} \in P(V)$, then

$$\mathcal{G}\left(\frac{m_1 + m_2}{2}\right) \leq \frac{2}{m_2 - m_1} \int_{m_1}^{m_2} \mathcal{G}(x) dx \leq 2[\mathcal{G}(m_1) + \mathcal{G}(m_2)] \quad (15)$$

holds.

For more H-H-type inequalities via classes $Q(V)$ and $P(V)$, see [79–81].

A variant of H-H-type inequalities via an s -convex function in second sense was proposed by Dragomir and Fitzpatrick [31]:

Theorem 15. Suppose that $\mathcal{G} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an s -convex function in the second sense, where $0 < s \leq 1$, $m_1, m_2 \in \mathbb{R}_+$ and $m_1 < m_2$. If $\mathcal{G}(x) \in L_1[m_1, m_2]$, then the following holds:

$$2^{s-1} \mathcal{G}\left(\frac{m_1 + m_2}{2}\right) \leq \frac{1}{m_2 - m_1} \int_{m_1}^{m_2} \mathcal{G}(x) dx \leq \frac{\mathcal{G}(m_1) + \mathcal{G}(m_2)}{s + 1} \quad (16)$$

The constant $\frac{1}{s+1}$ is most possible in the second part of the inequality in Equation (16). We refer the reader to [82,83] for more results connected to H-H-type inequalities via an s -convex function in the second sense.

Moreover, Dragomir and Fitzpatrick [31] also defined the following mapping that is closely related to Equation (16):

Theorem 16. If we let $\mathcal{G}(x) : [m_1, m_2] \rightarrow \mathbb{R}$ be an s -convex function in the second sense on $[m_1, m_2]$ such that $\mathcal{G}(x) \in L_1[m_1, m_2]$, and $H : [0, 1] \rightarrow \mathbb{R}$, then

$$H(\zeta) = \frac{1}{m_2 - m_1} \int_{m_1}^{m_2} \mathcal{G}\left(\zeta x + (1 - \zeta) \frac{m_1 + m_2}{2}\right) dx$$

holds for $\zeta \in [0, 1]$.

The properties of the mapping H are given as follows:

- i. $H \in K_s^2$ on $[0, 1]$;
- ii. $H \geq 2^{s-1} \mathcal{G}\left(\frac{m_1 + m_2}{2}\right)$.

These properties are the generalization of some results from [84]. Additionally, for more properties of mappings associated with H-H inequalities, see [85–90].

Another new H-H-type inequality for the preinvex function was given by Noor [91] as follows:

Theorem 17. If we let $\mathcal{G} : V = [m_1, m_1 + \eta(m_2, m_1)] \rightarrow (0, \infty)$ be a preinvex function on V° with $m_1 < m_1 + \eta(m_2, m_1)$, $m_1, m_2 \in V^\circ$ and $\mathcal{G} \in L_1[m_1, m_2]$, then

$$\mathcal{G}\left(\frac{2m_1 + \eta(m_2, m_1)}{2}\right) \leq \frac{1}{\eta(m_2, m_1)} \int_{m_1}^{m_1 + \eta(m_2, m_1)} \mathcal{G}(x) dx \leq \frac{\mathcal{G}(m_1) + \mathcal{G}(m_2)}{2}. \quad (17)$$

Remark 3. In Theorem 17, if we take $\eta(m_2, m_1) = m_2 - m_1$, then the inequality in Equation (17) reduces to Equation (8).

5. H-H-Type Inequalities for Differentiable Functions

An interesting problem in Equation (8) that attracts many researchers is the determination of two bounds of quantities in Equations (18) and (19), given as follows:

$$\left| \frac{1}{m_2 - m_1} \int_{m_1}^{m_2} \mathcal{G}(x) dx - \mathcal{G}\left(\frac{m_1 + m_2}{2}\right) \right| \quad (18)$$

$$\left| \frac{\mathcal{G}(m_1) + \mathcal{G}(m_2)}{2} - \frac{1}{m_2 - m_1} \int_{m_1}^{m_2} \mathcal{G}(x) dx \right|. \quad (19)$$

While Equation (18) estimates the difference between the left and middle parts of Equation (8), the quantity in Equation (19) estimates the difference between the middle and right parts of Equation (8). The quantity in Equation (18) is called the midpoint-type inequality. Meanwhile, the quantity in Equation (19) is named the trapezoid-type inequality. Recently, different integral inequalities were obtained through differentiable convexity.

The following result, given by Dragomir and Agarwal [11], can be used to estimate a new bound in Equation (19):

Lemma 1. If we let $\mathcal{G} : V \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on V° , $m_1, m_2 \in V^\circ$ with $m_1 < m_2$ and $\mathcal{G}' \in L_1[m_1, m_2]$, then the following identity holds:

$$\frac{\mathcal{G}(m_1) + \mathcal{G}(m_2)}{2} - \frac{1}{m_2 - m_1} \int_{m_1}^{m_2} \mathcal{G}(x) dx = \frac{m_2 - m_1}{2} \int_0^1 (1 - 2\zeta) \mathcal{G}'(\zeta m_1 + (1 - \zeta)m_2) d\zeta.$$

Therefore, using Lemma 1, the following theorems connected with the second part of Equation (8) for differentiable convex functions hold:

Theorem 18. When letting $\mathcal{G} : V \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on V° , $m_1, m_2 \in V^\circ$ with $m_1 < m_2$ and $\mathcal{G}' \in L_1[m_1, m_2]$, if $|\mathcal{G}'|$ is convex on $[m_1, m_2]$, then

$$\left| \frac{\mathcal{G}(m_1) + \mathcal{G}(m_2)}{2} - \frac{1}{m_2 - m_1} \int_{m_1}^{m_2} \mathcal{G}(x) dx \right| \leq \frac{m_2 - m_1}{8} [|\mathcal{G}'(m_1)| + |\mathcal{G}'(m_2)|]. \quad (20)$$

Theorem 19. When letting $\mathcal{G} : V \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on V° , $m_1, m_2 \in V^\circ$ with $m_1 < m_2$ and $\mathcal{G}' \in L_1[m_1, m_2]$. If $|\mathcal{G}'|^q$ is convex on $[m_1, m_2]$ for $q > 1$ with $q(p - 1) = p$, then

$$\left| \frac{\mathcal{G}(m_1) + \mathcal{G}(m_2)}{2} - \frac{1}{m_2 - m_1} \int_{m_1}^{m_2} \mathcal{G}(x) dx \right| \leq \frac{m_2 - m_1}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left[\frac{|\mathcal{G}'(m_1)|^q + |\mathcal{G}'(m_2)|^q}{2} \right]^{\frac{1}{q}}. \quad (21)$$

The improvement and simplification of the aforementioned result presented in Theorem 19 was provided by Pearce and Pečarić [92]:

Theorem 20. Let $\mathcal{G} : V \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on V° with $m_1, m_2 \in V^\circ$, $m_1 < m_2$ and $\mathcal{G}' \in L_1[m_1, m_2]$. If $|\mathcal{G}'|^q$ for $q > 1$, where $q(p - 1) = p$ is convex on $[m_1, m_2]$, then

$$\left| \frac{\mathcal{G}(m_1) + \mathcal{G}(m_2)}{2} - \frac{1}{m_2 - m_1} \int_{m_1}^{m_2} \mathcal{G}(x) dx \right| \leq \frac{m_2 - m_1}{4} \left[\frac{|\mathcal{G}'(m_1)|^q + |\mathcal{G}'(m_2)|^q}{2} \right]^{\frac{1}{q}}. \quad (22)$$

Remark 4. Choosing $q = 1$ reduces Theorem 20 to Theorem 18. In Theorem 20, taking $q = \frac{p}{p-1}$ improves the constant given in Theorem 19 since $\frac{1}{4} < \frac{1}{2(p+1)^{\frac{1}{p}}}$, where $p > 1$.

Kirmaci [93] proved the following results that give the bounds on Equation (18) by using the assumptions of convexity:

Lemma 2. Let $\mathcal{G} : V \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on V° , $m_1, m_2 \in V^\circ$ with $m_1 < m_2$. If $\mathcal{G}' \in L_1[m_1, m_2]$, then we have

$$\frac{1}{m_2 - m_1} \int_{m_1}^{m_2} \mathcal{G}(x) dx - \mathcal{G}\left(\frac{m_1 + m_2}{2}\right) = (m_2 - m_1) \int_0^1 Q(\zeta) \mathcal{G}'(\zeta m_1 + (1 - \zeta)m_2) d\zeta,$$

where

$$Q(\zeta) = \begin{cases} \zeta, & \zeta \in \left[0, \frac{1}{2}\right) \\ \zeta - 1, & \zeta \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Theorem 21. Let $\mathcal{G} : V \subseteq \mathbb{R}$ be a differentiable mapping on V° , $m_1, m_2 \in V^\circ$ with $m_1 < m_2$. If $|\mathcal{G}'|$ is convex on $[m_1, m_2]$, then we have

$$\left| \frac{1}{m_2 - m_1} \int_{m_1}^{m_2} \mathcal{G}(x) dx - \mathcal{G}\left(\frac{m_1 + m_2}{2}\right) \right| \leq \frac{m_2 - m_1}{8} (|\mathcal{G}'(m_1)| + |\mathcal{G}'(m_2)|). \quad (23)$$

Some new inequalities for twice-differentiable functions connected to the inequality in Equation (8) were given by Dragomir and Pearce [4] through the following lemma:

Lemma 3. If we let $\mathcal{G} : V \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a twice-differentiable function on V° , $m_1, m_2 \in V^\circ$ with $m_1 < m_2$ and $\mathcal{G}'' \in L_1[m_1, m_2]$, then the following holds:

$$\frac{\mathcal{G}(m_1) + \mathcal{G}(m_2)}{2} - \frac{1}{m_2 - m_1} \int_{m_1}^{m_2} \mathcal{G}(x) dx = \frac{(m_2 - m_1)^2}{2} \int_0^1 \zeta(1 - \zeta) \mathcal{G}''(\zeta m_1 + (1 - \zeta)m_2) d\zeta$$

Kirmaci et al. [12] studied a new inequality of the H-H type for differentiable mappings involving s -convexity:

Theorem 22. Let $\mathcal{G} : V \subset [0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on V° such that $\mathcal{G}' \in L_1[m_1, m_2]$, where $m_1, m_2 \in V$ with $m_1 < m_2$. If $|\mathcal{G}'|^q$ is s -convex on $[m_1, m_2]$, where $q \geq 1$ and $s \in (0, 1]$, we have

$$\left| \frac{\mathcal{G}(m_1) + \mathcal{G}(m_2)}{2} - \frac{1}{m_2 - m_1} \int_{m_1}^{m_2} \mathcal{G}(x) dx \right| \leq \frac{m_2 - m_1}{2} \left(\frac{1}{2} \right)^{\frac{q-1}{q}} \left[\frac{s + \left(\frac{1}{2} \right)^s}{(s+1)(s+2)} \right]^{\frac{1}{q}} \left(|\mathcal{G}'(m_1)|^q + |\mathcal{G}'(m_2)|^q \right)^{\frac{1}{q}}. \quad (24)$$

Barani et al. [94] generalized Lemma 1 to estimate the trapezoid type inequalities connected with Equation (8) for a preinvex function.

Lemma 4. Suppose that $\mathcal{G} : V = [m_1, m_1 + \eta(m_2, m_1)] \rightarrow (0, \infty)$ is a differentiable function, where $m_1, m_1 + \eta(m_2, m_1) \in V$ with $m_1 < m_1 + \eta(m_2, m_1)$. If $\mathcal{G}' \in L_1[m_1, m_1 + \eta(m_2, m_1)]$, we have

$$\begin{aligned} & \frac{1}{\eta(m_2, m_1)} \int_{m_1}^{m_1 + \eta(m_2, m_1)} \mathcal{G}(x) dx - \frac{\mathcal{G}(m_1) + \mathcal{G}(m_1 + \eta(m_2, m_1))}{2} \\ &= \frac{\eta(m_2, m_1)}{2} \left[\int_0^1 (1 - 2\zeta) \mathcal{G}'(m_1 + \zeta \eta(m_2, m_1)) d\zeta \right]. \end{aligned}$$

Recently, presumably new H-H-type inequalities were established by Mehrez and Agarwal [95], whose findings are reported in the next theorem:

Theorem 23. Suppose that $\mathcal{G} : V \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a differentiable mapping on V° , $m_1, m_2 \in V^\circ$ with $m_1 < m_2$. Let the derivative of \mathcal{G} be $\mathcal{G}' : [\frac{3m_1 - m_2}{2}, \frac{3m_2 - m_1}{2}] \rightarrow \mathbb{R}$, a continuous function on $[\frac{3m_1 - m_2}{2}, \frac{3m_2 - m_1}{2}]$. When letting $q \geq 1$, if $|\mathcal{G}'|$ is convex on $[\frac{3m_1 - m_2}{2}, \frac{3m_2 - m_1}{2}]$, then the following holds:

$$\left| \frac{1}{m_2 - m_1} \int_{m_1}^{m_2} \mathcal{G}(x) dx - \mathcal{G}\left(\frac{m_1 + m_2}{2}\right) \right| \leq \frac{m_2 - m_1}{8} \left(\left| \mathcal{G}'\left(\frac{3m_1 - m_2}{2}\right) \right|^q + \left| \mathcal{G}'\left(\frac{3m_2 - m_1}{2}\right) \right|^q \right)^{\frac{1}{q}} \quad (25)$$

Almutairi and Kiliçman [8] extended Theorem 23 to an s -convex function in the second sense as follows:

Theorem 24. Suppose that $\mathcal{G} : V \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}$ is a differentiable mapping on V° , $m_1, m_2 \in V^\circ$ with $m_1 < m_2$. Let the derivative of \mathcal{G} be $\mathcal{G}' : [\frac{3m_1 - m_2}{2}, \frac{3m_2 - m_1}{2}] \rightarrow \mathbb{R}$, a continuous function on $[\frac{3m_1 - m_2}{2}, \frac{3m_2 - m_1}{2}]$. When letting $q \geq 1$, if $|\mathcal{G}'|^q$ is an s -convex function on $[\frac{3m_1 - m_2}{2}, \frac{3m_2 - m_1}{2}]$ for some fixed $s \in (0, 1)$, then we have the following:

$$\left| \frac{1}{m_2 - m_1} \int_{m_1}^{m_2} \mathcal{G}(x) dx - \mathcal{G}\left(\frac{m_1 + m_2}{2}\right) \right| \leq (m_2 - m_1) \left(\frac{1}{8}\right)^{\frac{q-1}{q}} \left[\frac{2 - 2^{-s}}{2(s+1)(s+2)} \right]^{\frac{1}{q}} \left[\left| \mathcal{G}'\left(\frac{3m_1 - m_2}{2}\right) \right|^q + \left| \mathcal{G}'\left(\frac{3m_2 - m_1}{2}\right) \right|^q \right]^{\frac{1}{q}}. \quad (26)$$

6. Generalized H-H-Type Inequalities Involving Different Fractional Integrals

This section presents some results on the generalization of inequalities introduced in Section 5. Therefore, many generalizations of H-H-type inequalities established using fractional integrals for different classes of convexities are discussed here, since they can be frequently used in other parts of the article. For example, the work of Sarikaya et al. [74] was the first to present inequalities of the H-H type involving Riemann–Liouville fractional integrals. This is given below:

Theorem 25. Suppose that $\mathcal{G} : [m_1, m_2] \rightarrow \mathbb{R}$ is a non-negative function with $0 \leq m_1 < m_2$ and $\mathcal{G} \in L_1[m_1, m_2]$. If \mathcal{G} is a convex function on $[m_1, m_2]$, we have

$$\mathcal{G}\left(\frac{m_1 + m_2}{2}\right) \leq \frac{\Gamma(\lambda + 1)}{2(m_2 - m_1)^\lambda} \left[J_{m_1^+}^\lambda \mathcal{G}(m_2) + J_{m_2^-}^\lambda \mathcal{G}(m_1) \right] \leq \frac{\mathcal{G}(m_1) + \mathcal{G}(m_2)}{2}, \quad (27)$$

where $\lambda > 0$.

Remark 5. In Theorem 25, choosing $\lambda = 1$ reduces the inequality in Equation (27) to Equation (8).

Moreover, Sarikaya et al. [74] presented the following fractional integral identity:

Lemma 5. Let $\mathcal{G} : [m_1, m_2] \rightarrow \mathbb{R}$ be a differentiable function on (m_1, m_2) with $m_1 < m_2$. If $\mathcal{G}' \in L_1[m_1, m_2]$, then we have

$$\begin{aligned} \frac{\mathcal{G}(m_1) + \mathcal{G}(m_2)}{2} - \frac{\Gamma(\lambda + 1)}{2(m_2 - m_1)^\lambda} \left[J_{m_1^+}^\lambda \mathcal{G}(m_2) + J_{m_2^-}^\lambda \mathcal{G}(m_1) \right] \\ = \frac{m_2 - m_1}{2} \int_0^1 \left[(1 - \zeta)^\lambda - \zeta^\lambda \right] \mathcal{G}'(\zeta m_1 + (1 - \zeta)m_2) d\zeta. \end{aligned}$$

The identity presented in the above lemma was also used by Sarikaya when determining the trapezoid-type inequalities connected with Equation (8) for Riemann–Liouville fractional integrals.

Theorem 26. Let $\mathcal{G} : [m_1, m_2] \rightarrow \mathbb{R}$ be a differentiable function on (m_1, m_2) with $m_1 < m_2$ and $\mathcal{G}' \in L_1[m_1, m_2]$. If $|\mathcal{G}'|$ is convex on $[m_1, m_2]$, then we have

$$\begin{aligned} \left| \frac{\mathcal{G}(m_1) + \mathcal{G}(m_2)}{2} - \frac{\Gamma(\lambda + 1)}{2(m_2 - m_1)^\lambda} \left[J_{m_1^+}^\lambda \mathcal{G}(m_2) + J_{m_2^-}^\lambda \mathcal{G}(m_1) \right] \right| \\ \leq \frac{m_2 - m_1}{2(\lambda + 1)} \left(1 - \frac{1}{2^\lambda} \right) [\mathcal{G}'(m_1) + \mathcal{G}'(m_2)]. \end{aligned} \quad (28)$$

Remark 6. Taking $\lambda = 1$ in Theorem 26 reduces the inequality in Equation (28) to the inequality in Equation (20) of Theorem 18.

Zhu et al. [96] studied a new fractional integral identity for differentiable convex mappings. The results are presented below:

Lemma 6. Let $\mathcal{G} : [m_1, m_2] \rightarrow \mathbb{R}$ be a differentiable mapping on (m_1, m_2) with $m_1 < m_2$. If $\mathcal{G}' \in L_1[m_1, m_2]$, then the equality for fractional integrals holds as follows:

$$\begin{aligned} & \frac{\Gamma(\lambda+1)}{2(m_2-m_1)^\lambda} \left[J_{m_1^+}^\lambda \mathcal{G}(m_2) + J_{m_2^-}^\lambda \mathcal{G}(m_1) \right] - \mathcal{G}\left(\frac{m_1+m_2}{2}\right) \\ &= \frac{m_2-m_1}{2} \left[\int_0^1 P \mathcal{G}'(\zeta m_1 + (1-\zeta)m_2) d\zeta - \int_0^1 \left[(1-\zeta)^\lambda - \zeta^\lambda \right] \mathcal{G}'(\zeta m_1 + (1-\zeta)m_2) d\zeta \right], \end{aligned} \quad (29)$$

where

$$P = \begin{cases} 1, & 0 \leq \zeta < \frac{1}{2} \\ -1, & \frac{1}{2} \leq \zeta < 1. \end{cases}$$

Using the above identity, the following result estimates the midpoint-type inequalities related to Equation (8), which involves Riemann–Liouville fractional integrals:

Theorem 27. Let $\mathcal{G} : [m_1, m_2] \rightarrow \mathbb{R}$ be a differentiable mapping on (m_1, m_2) with $m_1 < m_2$. If $|\mathcal{G}'|$ is convex on $[m_1, m_2]$, then the following inequality holds:

$$\begin{aligned} & \left| \frac{\Gamma(\lambda+1)}{2(m_2-m_1)^\lambda} \left[J_{m_1^+}^\lambda \mathcal{G}(m_2) + J_{m_2^-}^\lambda \mathcal{G}(m_1) \right] - \mathcal{G}\left(\frac{m_1+m_2}{2}\right) \right| \\ & \leq \frac{m_2-m_1}{4(\lambda+1)} \left(\lambda + 3 - \frac{1}{2^{\lambda-1}} \right) \left[|\mathcal{G}'(m_1)| + |\mathcal{G}'(m_2)| \right]. \end{aligned} \quad (30)$$

Almutair and Kiliçman [14] extended Lemma 6 and Theorem 27 for Katugampola fractional integrals as follows:

Lemma 7. Let $\mathcal{G} : [m_1^\rho, m_2^\rho] \rightarrow \mathbb{R}$ be a differentiable mapping on (m_1^ρ, m_2^ρ) , where $m_1 < m_2$. The following equality holds if the fractional integrals exist:

$$\begin{aligned} & \frac{\lambda \rho^\lambda \Gamma(\lambda+1)}{2(m_2^\rho - m_1^\rho)^\lambda} \left[{}^\rho I_{m_1^+}^\lambda \mathcal{G}(m_2^\rho) + {}^\rho I_{m_2^-}^\lambda \mathcal{G}(m_1^\rho) \right] - \mathcal{G}\left(\frac{m_1^\rho + m_2^\rho}{2}\right) \\ &= \frac{m_2^\rho - m_1^\rho}{2} \left[\int_0^1 M \mathcal{G}'(\zeta^\rho m_1^\rho + (1-\zeta^\rho)m_2^\rho) d\zeta - \int_0^1 \left[(1-\zeta^\rho)^\lambda - \zeta^{\rho\lambda} \right] \zeta^{\rho-1} \mathcal{G}'(\zeta^\rho m_1^\rho + (1-\zeta^\rho)m_2^\rho) d\zeta \right], \end{aligned} \quad (31)$$

where

$$M = \begin{cases} \zeta^{\rho-1}, & 0 \leq \zeta < \frac{1}{\rho\sqrt{2}} \\ -\zeta^{\rho-1}, & \frac{1}{\rho\sqrt{2}} \leq \zeta < 1. \end{cases}$$

Remark 7. If $\rho = 1$, then the identity in Equation (31) in Lemma 7 reduces to the identity in Equation (29) in Lemma 6.

Theorem 28. Let $\mathcal{G} : [m_1^\rho, m_2^\rho] \rightarrow \mathbb{R}$ be a differentiable mapping on (m_1^ρ, m_2^ρ) with $0 \leq m_1 < m_2$. If $|\mathcal{G}'|$ is convex on $[m_1^\rho, m_2^\rho]$, then the following inequality holds:

$$\begin{aligned} & \left| \frac{\lambda \rho^\lambda \Gamma(\lambda+1)}{2(m_2^\rho - m_1^\rho)^\lambda} \left[{}^\rho I_{m_1^+}^\lambda \mathcal{G}(m_2^\rho) + {}^\rho I_{m_2^-}^\lambda \mathcal{G}(m_1^\rho) \right] - \mathcal{G}\left(\frac{m_1^\rho + m_2^\rho}{2}\right) \right| \\ & \leq \frac{m_2^\rho - m_1^\rho}{4\rho(\lambda+1)} \left[3 + \lambda - \frac{1}{2^{\lambda-1}} \right] \left[|\mathcal{G}'(m_1^\rho)| + |\mathcal{G}'(m_2^\rho)| \right] \end{aligned} \quad (32)$$

Remark 8. Considering the inequality in Equation (32) of Theorem 28, we have the following:

- i. Choosing $\rho = 1$ reduces the inequality in Equation (32) to the inequality in Equation (30) of Theorem 27.

- ii. Taking $\rho = 1$ and $\lambda = 1$ reduces the inequality in Equation (32) to the inequality in Equation (16) in [96], which is given as follows:

$$\left| \frac{1}{m_2 - m_1} \int_{m_1}^{m_2} \mathcal{G}(x) dx - \mathcal{G}\left(\frac{m_1 + m_2}{2}\right) \right| \leq \frac{3(m_2 - m_1)}{8} (|\mathcal{G}'(m_1)| + |\mathcal{G}'(m_2)|).$$

Meanwhile, Wang et al. [97] extended Lemma 5 to include a twice-differentiable mapping:

Lemma 8. Let $\mathcal{G} : [m_1, m_2] \rightarrow \mathbb{R}$ be a twice-differentiable function on (m_1, m_2) with $m_1 < m_2$. If $\mathcal{G}'' \in L_1[m_1, m_2]$, then the following holds:

$$\begin{aligned} & \frac{\mathcal{G}(m_1) + \mathcal{G}(m_2)}{2} - \frac{\Gamma(\lambda + 1)}{2(m_2 - m_1)^\lambda} \left[J_{m_1^+}^\lambda \mathcal{G}(m_2) + J_{m_2^-}^\lambda \mathcal{G}(m_1) \right] \\ &= \frac{(m_2 - m_1)^2}{2} \int_0^1 \frac{1 - (1 - \zeta)^{\lambda+1} - \zeta^{\lambda+1}}{\lambda + 1} \mathcal{G}''(\zeta m_1 + (1 - \zeta)m_2) d\zeta \end{aligned}$$

Set et al. [98] generalized Theorem 15 for fractional integrals, and the result is given as follows:

Theorem 29. Suppose that $\mathcal{G} : [m_1, m_2] \rightarrow \mathbb{R}$ is a non-negative function with $0 \leq m_1 < m_2$ and $\mathcal{G} \in L_1[m_1, m_2]$. If \mathcal{G} is an s -convex function in the second sense on $[m_1, m_2]$, we have

$$\begin{aligned} 2^{s-1} \mathcal{G}\left(\frac{m_1 + m_2}{2}\right) &\leq \frac{\Gamma(\lambda + 1)}{2(m_2 - m_1)^\lambda} \left[J_{m_1^+}^\lambda \mathcal{G}(m_2) + J_{m_2^-}^\lambda \mathcal{G}(m_1) \right] \\ &\leq \left[\frac{\lambda}{(\lambda + s)} + \lambda \beta(\lambda, s + 1) \right] \frac{\mathcal{G}(m_1) + \mathcal{G}(m_2)}{2}, \end{aligned} \quad (33)$$

where $\lambda > 0$ and $0 < s < 1$.

The H-H inequality for Hadamard fractional integrals that was established by Wang et al. [99] also received the attention of many researchers. This refinement is given as follows:

Theorem 30. Suppose that $\mathcal{G} : [m_1, m_2] \rightarrow \mathbb{R}$ is a non-negative function with $0 < m_1 < m_2$ and $\mathcal{G} \in L_1[m_1, m_2]$. If \mathcal{G} is a non-decreasing convex function on $[m_1, m_2]$, then the following inequality holds:

$$\mathcal{G}(\sqrt{m_1 m_2}) \leq \frac{\Gamma(\lambda + 1)}{2(\ln m_2 - \ln m_1)^\lambda} \left[H_{m_1^+}^\lambda \mathcal{G}(m_2) + H_{m_2^-}^\lambda \mathcal{G}(m_1) \right] \leq \mathcal{G}(m_2)$$

Mo et al. [22] provided the generalized H-H-type inequalities involving local fractional integrals for generalized convex functions on fractal sets as follows:

Theorem 31. If we let $\mathcal{G}(x) \in I_x^\alpha[m_1, m_2]$ be a generalized convex function on $[m_1, m_2]$ with $m_1 < m_2$, then the following holds:

$$\mathcal{G}\left(\frac{m_1 + m_2}{2}\right) \leq \frac{\Gamma(1 + \alpha)}{(m_2 - m_1)^\alpha} m_1 I_{m_2}^\alpha \mathcal{G}(x) \leq \frac{\mathcal{G}(m_1) + \mathcal{G}(m_2)}{2^\alpha} \quad (34)$$

Remark 9. By choosing $\alpha = 1$ in the inequality in Equation (34), we obtain the inequality in Equation (8).

Furthermore, the H-H-type inequalities for the generalized s -convex function in the second sense on fractal sets were proposed by Mo and Sui [100]:

Theorem 32. Suppose that $\mathcal{G} : \mathbb{R}_+ \rightarrow \mathbb{R}^\alpha$ is a generalized s -convex function in the second sense for $0 < s < 1$ and $m_1, m_2 \in [0, \infty)$ with $m_1 < m_2$. Then, for $\mathcal{G} \in C_\alpha[m_1, m_2]$, the following inequality holds:

$$\frac{2^{(s-1)\alpha}}{\Gamma(1+\alpha)} \mathcal{G}\left(\frac{m_1+m_2}{2}\right) \leq \frac{m_1 I_{m_2}^\alpha \mathcal{G}(x)}{(m_2-m_1)^\alpha} \leq \frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)} (\mathcal{G}(m_1) + \mathcal{G}(m_2)) \quad (35)$$

Remark 10. When taking $\alpha = 1$ in Equation (35), we obtained Equation (16).

For more results on the generalizations of H-H-type inequalities involving fractal sets via fractional integrals, one should consult the following references [101,102].

The result in Theorem 33 involving Katugampola fractional integrals is the generalization of the result presented earlier in Theorem 25:

Theorem 33 ([103]). Let $\lambda > 0$ and $\rho > 0$. Let $\mathcal{G} : [m_1^\rho, m_2^\rho] \rightarrow \mathbb{R}$ be a positive function with $0 \leq m_1 < m_2$ and $\mathcal{G} \in X_c^\rho(m_1^\rho, m_2^\rho)$. If \mathcal{G} is also a convex function on $[m_1, m_2]$, then the following inequality

$$\mathcal{G}\left(\frac{m_1^\rho + m_2^\rho}{2}\right) \leq \frac{\rho^\lambda \Gamma(\lambda+1)}{2(m_2^\rho - m_1^\rho)^\lambda} \left[{}^\rho I_{m_1^+}^\lambda \mathcal{G}(m_2^\rho) + {}^\rho I_{m_2^-}^\lambda \mathcal{G}(m_1^\rho) \right] \leq \frac{\mathcal{G}(m_1^\rho) + \mathcal{G}(m_2^\rho)}{2} \quad (36)$$

holds, where the fractional integrals are considered for the function $\mathcal{G}(x^\rho)$ and evaluated at m_1 and m_2 , respectively.

The estimate of the difference between the right term and the middle term of the inequality in Equation (36) is obtained using the following lemma:

Lemma 9. Suppose that $\mathcal{G} : [m_1^\rho, m_2^\rho] \subset \mathbb{R}_+ \rightarrow \mathbb{R}$ is a differentiable mapping on (m_1^ρ, m_2^ρ) , where $0 \leq m_1 < m_2$ and $\lambda, \rho > 0$. If the fractional integrals exist, we have

$$\begin{aligned} & \frac{\mathcal{G}(m_1^\rho) + \mathcal{G}(m_2^\rho)}{2} - \frac{\lambda \rho^\lambda \Gamma(\lambda+1)}{2(m_2^\rho - m_1^\rho)^\lambda} \left[{}^\rho I_{m_1^+}^\lambda \mathcal{G}(m_2^\rho) + {}^\rho I_{m_2^-}^\lambda \mathcal{G}(m_1^\rho) \right] \\ &= \frac{m_2^\rho - m_1^\rho}{2} \int_0^1 \left[(1-\zeta^\rho)^\lambda - \zeta^{\rho\lambda} \right] \zeta^{\rho-1} \mathcal{G}'(\zeta^\rho m_1^\rho + (1-\zeta^\rho)m_2^\rho) d\zeta. \end{aligned} \quad (37)$$

Theorem 34. Suppose that $\mathcal{G} : [m_1^\rho, m_2^\rho] \subset \mathbb{R}_+ \rightarrow \mathbb{R}$ is a differentiable mapping on (m_1^ρ, m_2^ρ) with $0 \leq m_1 < m_2$. If $|\mathcal{G}'|$ is convex on $[m_1^\rho, m_2^\rho]$, then the following inequality holds:

$$\begin{aligned} & \left| \frac{\mathcal{G}(m_1^\rho) + \mathcal{G}(m_2^\rho)}{2} - \frac{\lambda \rho^\lambda \Gamma(\lambda+1)}{2(m_2^\rho - m_1^\rho)^\lambda} \left[{}^\rho I_{m_1^+}^\lambda \mathcal{G}(m_2^\rho) + {}^\rho I_{m_2^-}^\lambda \mathcal{G}(m_1^\rho) \right] \right| \\ & \leq \frac{m_2^\rho - m_1^\rho}{2\rho(\lambda+1)} \left(1 - \frac{1}{2^\lambda} \right) \left[|\mathcal{G}'(m_1^\rho)| + |\mathcal{G}'(m_2^\rho)| \right] \end{aligned} \quad (38)$$

Remark 11. Choosing $\rho = 1$ in Theorem 34 reduces the inequality in Equation (38) to the inequality in Equation (28) in Theorem 26.

Other important results involving Katugampola fractional integrals include the work of Mehreen and Anwar [104], who generalized Theorem 29, given as follows:

Theorem 35. Suppose that $\lambda > 0$ and $\rho > 0$. Let $\mathcal{G} : [m_1^\rho, m_2^\rho] \rightarrow \mathbb{R}$ be a positive function with $0 \leq m_1 < m_2$ and $\mathcal{G} \in X_c^p(m_1^\rho, m_2^\rho)$. If \mathcal{G} is also a convex function on $[m_1, m_2]$, then the following inequality holds:

$$\begin{aligned} 2^{s-1} \mathcal{G}\left(\frac{m_1^\rho + m_2^\rho}{2}\right) &\leq \frac{\rho^\lambda \Gamma(\lambda + 1)}{2(m_2^\rho - m_1^\rho)^\lambda} \left[{}^\rho I_{m_1^+}^\lambda \mathcal{G}(m_2^\rho) + {}^\rho I_{m_2^-}^\lambda \mathcal{G}(m_1^\rho) \right] \\ &\leq \left[\frac{\lambda}{(\lambda + s)} + \lambda \beta(\lambda, s + 1) \right] \frac{\mathcal{G}(m_1^\rho) + \mathcal{G}(m_2^\rho)}{2} \end{aligned} \quad (39)$$

Anderson [105] provided generalized H-H-type inequalities involving conformable fractional integrals as follows:

Theorem 36. Suppose that $\alpha \in (0, 1]$, $m_1, m_2 \in \mathbb{R}$ where $m_1 < m_2$, and $\mathcal{G} : [m_1, m_2] \rightarrow \mathbb{R}$ is an α -fractional differentiable function such that $D_\alpha(\mathcal{G})$ is increasing. Then, we have

$$\frac{\alpha}{m_2^\alpha - m_1^\alpha} \int_{m_1}^{m_2} \mathcal{G}(y) d_\alpha y \leq \frac{\mathcal{G}(m_1) + \mathcal{G}(m_2)}{2} \quad (40)$$

In addition, if the mapping \mathcal{G} is decreasing on $[m_1, m_2]$, then

$$\mathcal{G}\left(\frac{m_1 + m_2}{2}\right) \leq \frac{\alpha}{m_2^\alpha - m_1^\alpha} \int_{m_1}^{m_2} \mathcal{G}(y) d_\alpha y \quad (41)$$

Remark 12. If $\alpha = 1$, then we clearly see that the inequalities in Equations (40) and (41) reduce to the inequality in Equation (8).

In [106], Set et al. provided the generalized H-H-type inequalities involving conformable fractional integrals as follows:

Theorem 37. Let $\mathcal{G} : [m_1, m_2] \rightarrow \mathbb{R}$ be a function with $0 \leq a < b$ and $f \in L_1[m_1, m_2]$. If \mathcal{G} is a convex function on $[m_1, m_2]$, then the following inequality

$$\mathcal{G}\left(\frac{m_1 + m_2}{2}\right) \leq \frac{\Gamma(\alpha + 1)}{2(m_2 - m_1)^\alpha \Gamma(\alpha - n)} \left[(I_\alpha^{m_1} \mathcal{G})(m_2) + ({}^{m_2} I_\alpha \mathcal{G})(m_1) \right] \leq \frac{\mathcal{G}(m_1) + \mathcal{G}(m_2)}{2} \quad (42)$$

holds, where $\alpha \in (n, n + 1]$.

Sarikaya et al. [107], presented the following H-H inequalities for conformable fractional integrals:

Theorem 38. Let $0 < m_1 < m_2$, $\alpha \in (0, 1)$, $\mathcal{G} : [m_1, m_2] \rightarrow \mathbb{R}$ be a convex function and $J_\alpha f$ exist on $[m_1, m_2]$. Then, one has

$$\mathcal{G}\left(\frac{m_1^\alpha + m_2^\alpha}{2}\right) \leq \frac{\alpha}{m_2^\alpha - m_1^\alpha} \int_{m_1}^{m_2} \mathcal{G}(\gamma^\alpha) d_\alpha \gamma \leq \frac{\mathcal{G}(m_1^\alpha) + \mathcal{G}(m_2^\alpha)}{2} \quad (43)$$

For more results on the generalization of H-H-type inequalities, we refer interested readers to [106,108–110].

7. Applications to Special Means

The following means for positive real numbers $m_1, m_2 \in \mathbb{R}_+$ ($m_1 \neq m_2$) exist in the literature [4,5]:

1. The arithmetic mean:

$$A(m_1, m_2) = \frac{m_1 + m_2}{2}.$$

2. The geometric mean:

$$G(m_1, m_2) = \sqrt{m_1 m_2}.$$

3. The logarithmic mean:

$$L(m_1, m_2) = \frac{m_2 - m_1}{\ln m_2 - \ln m_1}, m_1, m_2 \neq 0.$$

4. The generalized log mean:

$$L_\vartheta(m_1, m_2) = \left[\frac{m_2^{\vartheta+1} - m_1^{\vartheta+1}}{(\vartheta+1)(m_2 - m_1)} \right]^{1/\vartheta}, \vartheta \in \mathbb{Z} \setminus \{-1, 0\}.$$

We note that L_ϑ is monotonically increasing over $\vartheta \in \mathbb{R}$ with $L_{-1} = L$. We in particular obtained the following inequality: $G \leq L \leq A$. These special means can be frequently applied to numerical approximations, as well as other related problems that can be obtained in different fields. Several results that deal with special means have been reported in the literature (see [11,111]).

Dragomir and Agarwal [112] applied the results of Theorem 18 to establish the following new inequalities connecting the above means:

Proposition 2. Let $m_1, m_2 \in \mathbb{R}, m_1 < m_2$ and $\theta \in \mathbb{N}, \theta \geq 2$. Then, the following inequality holds:

$$\left| A(m_1^\theta, m_2^\theta) - L_\theta(m_1, m_2) \right| \leq \frac{\theta(m_2 - m_1)}{4} A(|m_1|^{\theta-1}, |m_2|^{\theta-1})$$

Proposition 3. Let $m_1, m_2 \in \mathbb{R}, m_1 < m_2$, and $0 \notin [m_1, m_2]$. Then, the following inequality holds:

$$\left| A(m_1^{-1}, m_2^{-1}) - L^{-1}(m_1, m_2) \right| \leq \frac{(m_2 - m_1)}{4} A(|m_1|^{-2}, |m_2|^{-2})$$

Furthermore, Kirmaci [93] established an application to special means using the result of Theorem 21 as follows:

Proposition 4. Let $m_1, m_2 \in \mathbb{R}, m_1 < m_2$ and $\theta \in \mathbb{N}, \theta \geq 2$. Then, we obtain

$$\left| L_\theta(m_1, m_2) - A^\theta(m_1, m_2) \right| \leq \frac{\theta(m_2 - m_1)}{4} A(|m_1|^{\theta-1}, |m_2|^{\theta-1})$$

One can consult the following references [113,114] for a comprehensive study on special means.

8. Applications to the Quadrature Formula

Let $\mathcal{G} : [m_1, m_2] \rightarrow \mathbb{R}$ be a twice-differentiable function on (m_1, m_2) , such that $\mathcal{G}''(x)$ is bounded on the given interval. This can be written as

$$\|\mathcal{G}''\|_\infty = \sup_{x \in (m_1, m_2)} |\mathcal{G}''(x)| < \infty.$$

The following results are referred to as the midpoint and trapezoid inequalities, respectively:

$$\left| \int_{m_1}^{m_2} \mathcal{G}(x) dx - (m_2 - m_1) \mathcal{G}\left(\frac{m_1 + m_2}{2}\right) \right| \leq \frac{(m_2 - m_1)^3}{24} \|\mathcal{G}''\|_\infty \quad (44)$$

$$\left| \int_{m_1}^{m_2} \mathcal{G}(x) dx - (m_2 - m_1) \frac{\mathcal{G}(m_1) + \mathcal{G}(m_2)}{2} \right| \leq \frac{(m_2 - m_1)^3}{12} \|\mathcal{G}''\|_{\infty}, \quad (45)$$

Therefore, the integral $\int_{m_1}^{m_2} \mathcal{G}(x) dx$ can be approximated in terms of the midpoint formula and the trapezoid formula, respectively:

$$\begin{aligned} \int_{m_1}^{m_2} \mathcal{G}(x) dx &\cong (m_2 - m_1) \mathcal{G}\left(\frac{m_1 + m_2}{2}\right), \\ \int_{m_1}^{m_2} \mathcal{G}(x) dx &\cong (m_2 - m_1) \frac{\mathcal{G}(m_1) + \mathcal{G}(m_2)}{2}, \end{aligned}$$

The midpoint and trapezoid inequalities can be grouped in the most important relationship: the H-H inequality (8).

Suppose that d is a partition of the interval $[m_1, m_2]$ such that $m_1 = z_0 < z_1 < \dots < z_{n-1} < z_n = m_2$. Therefore, we write the following quadrature formula:

$$\int_{m_1}^{m_2} \mathcal{G}(x) dx = T_i(\mathcal{G}, d) + E_i(\mathcal{G}, d), i = 1, 2,$$

whereby

$$T_1(\mathcal{G}, d) = \sum_{i=0}^{n-1} \frac{\mathcal{G}(z_i) + \mathcal{G}(z_{i+1})}{2} (z_{i+1} - z_i)$$

is the trapezoidal version, and

$$T_2(\mathcal{G}, d) = \sum_{i=0}^{n-1} \mathcal{G}\left(\frac{z_i + z_{i+1}}{2}\right) (z_{i+1} - z_i)$$

stands for the midpoint version.

The remainder term $E_1(\mathcal{G}, d)$ for the integral $\int_{m_1}^{m_2} \mathcal{G}(x) dx$ estimated by the trapezoidal formula $T_1(\mathcal{G}, d)$ satisfies

$$|E_1(\mathcal{G}, d)| \leq \frac{M}{12} \sum_{i=0}^{n-1} (z_{i+1} - z_i)^3. \quad (46)$$

Meanwhile, that of the midpoint formula $T_2(\mathcal{G}, d)$ satisfies

$$|E_2(\mathcal{G}, d)| \leq \frac{M}{24} \sum_{i=0}^{n-1} (z_{i+1} - z_i)^3. \quad (47)$$

These remainder terms in Equations (46) and (47) can be used to estimate the error bounds of many numerical integrations. Furthermore, the inequalities in Equations (44) and (45) can only hold if the second derivative is bounded on the interval (m_1, m_2) , and \mathcal{G} is a twice-differentiable function. This encourages many researchers to determine inequalities with a less than or equal to one derivative.

For example, Dragomir and Agarwal [112] estimated the remainder term through one derivative as follows:

Proposition 5. Let \mathcal{G} be a differentiable function on K^o , $m_1, m_2 \in K^o$ with $m_1 < m_2$. If $|\mathcal{G}'|$ is convex on $[m_1, m_2]$, then the following holds:

$$\begin{aligned} |E(\mathcal{G}, d)| &\leq \frac{1}{8} \sum_{j=0}^{\theta-1} (z_{j+1} - z_j)^2 (|\mathcal{G}'(z_j)| + |\mathcal{G}'(z_{j+1})|) \\ &\leq \frac{\max\{|\mathcal{G}'(m_1)|, |\mathcal{G}'(m_2)|\}}{4} \sum_{j=0}^{n-1} (z_{j+1} - z_j)^2 \end{aligned}$$

Another important result was established by Kirmaci [93], who estimated the remainder term through one derivative as follows:

Proposition 6. Let \mathcal{G} be a differentiable function on K^0 , $m_1, m_2 \in K^0$ with $m_1 < m_2$. If $|\mathcal{G}'|$ is convex on $[m_1, m_2]$, then the following holds:

$$|E(\mathcal{G}, d)| \leq \frac{1}{8} \sum_{j=0}^{\theta-1} (x_{j+1} - x_j)^2 (|\mathcal{G}'(x_j)| + |\mathcal{G}'(x_{j+1})|)$$

Thus, these estimates remain as open-ended problems when considering their wider areas of application [11,93].

9. Conclusions

H-H-type inequalities are introduced in this article to ease the concepts for beginners in the field of the theory of inequality. We described some basic facts including integral inequalities and fractional inequalities of the H-H type through various classes of convexity so as to encourage more new research in this field of study. In order to achieve our goal, we provided and discussed some important definitions, examples and theorems related to the H-H inequality. For example, the formulations of H-H-type inequalities of α -type real-valued convex functions, together various classes of convexity, were discussed in detail in this review. Using the concept presented in this study, more results can be produced as extensions of some basic information discussed in the review.

Author Contributions: Both authors contributed equally. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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