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Kinematic Geometry of Timelike Ruled Surfaces in Minkowski 3-Space $\mathbb{E}_1^3$

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Abstract: Symmetry is a frequently recurring theme in mathematics, nature, science, etc. In mathematics, its most familiar manifestation appears in geometry, most notably line geometry, and in other closely related areas. In this study, we take advantage of the symmetry properties of both dual space and original space in order to transfer problems in original space to dual space. We use E. Study Mappingas a direct method for analyzing the kinematic geometry of timelike ruled and developable surfaces. Then, the invariants for a spacelike line trajectory are studied and the well-known formulae of Hamilton and Mannheim on the theory of surfaces are provenfor the line space. Meanwhile, a timelike Plücker conoid generated by the Disteli-axis is derived and its kinematic geometry is discussed. Finally, some equations for particular timelike ruled surfaces, such as the general timelike helicoid, the Lorentzian sphere, and the timelike cone, are derived and plotted.

Keywords: Hamilton and Mannheim formulae; timelike Plücker conoid

MSC: 53A04; 53A05; 53A17

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1. Introduction

Differential line geometry mainly studies line families in three-dimensional space. The ambient space can be Euclidean or non-Euclidean. Because it is directly related to spatial motion (kinematics), it has been extensively implemented in robot kinematics and mechanism design, which is an interesting subdivision of differential geometry [1–3]. On the other hand, nature organizes itself using the language of symmetry. Symmetry is one of the most basic and important notions in all fields of science, technology, and art. Geometry and symmetry have always been basic tools of scientific investigations, as they are two of the main ingredients in modern mathematical theories. Our methods in this paper rely on symmetry and geometry. We take advantage of symmetry properties between dual space and original space to transfer problems in original space to dual space. In spatial kinematics, it is significant to investigate the inherent properties of linear trajectories in accordance with the concept of straight surfaces in differential geometry. It is well known that it a very effective method of detecting the motion of a straight line trajectory is to use dual numbers. Hence, the E. Study Mapping processleads to the conclusion that the set of all oriented lines in Euclidean 3-space $\mathbb{E}^3$ is immediately linked to the set of points on the dual unit sphere in the dual 3-space $\mathbb{D}^3$. This means that a regular curve on a dual unit sphere represents a ruled surface at $\mathbb{E}^3$ (see, for instance, [4–9]).

In the Minkowski 3-space $\mathbb{E}_1^3$, the differential geometry of ruled surfaces is much more complicated than in the Euclidean case, since the Lorentzian metric is not a positive definite metric. Rather, the distance function $\langle , \rangle$ can be positive, negative, or zero, whereas the distance function in the Euclidean space can only be positive. Hence, if we take the Minkowski 3-space $\mathbb{E}_1^3$ as an alternative of $\mathbb{E}^3$ the E. Study map can be described as follows. The timelike and spacelike dual unit vectors of hyperbolic and Lorentzian dual unit spheres $\mathbb{H}_+^2$ and $\mathbb{S}_1^2$ in the Lorentzian 3-space $\mathbb{D}_1^3$ are in one-to-one correspondence with the oriented timelike and spacelike lines of the space of Lorentzian lines $\mathbb{E}_1^3$, respectively. Then, a differentiable curve on $\mathbb{H}_+^2$ corresponds to a timelike ruled surface at $\mathbb{E}_1^3$. Similarly, the timelike (resp. spacelike) curve on $\mathbb{S}_1^2$ corresponds to any spacelike (resp. timelike) ruled surface at $\mathbb{E}_1^3$ [10–12]. In Euclidean 3-space$\mathbb{E}^3$ the E. Study map can be given as follows. The timelike (resp. spacelike) oriented lines are represented with the timelike (resp. spacelike) dual points on a hyperbolic (resp. Lorentzian) dual unit sphere in the Lorentzian dual 3-Space $\mathbb{D}_1^3$. Hence, a regular curve on $\mathbb{H}_+^2$ represents a timelike ruled surface at $\mathbb{E}_1^3$. Similarly, the spacelike (resp. timelike) curve on $\mathbb{S}_1^2$ represents timelike (resp. spacelike) ruled surface at $\mathbb{E}_1^3$. In consideration of its relationship to engineering and the physical science of Minkowski space, many geometers and engineers have studied straight surfaces and other surfaces and curves and have observed many different properties(see [10–19]).

This work presents an approach to the kinematic geometry of timelike ruled surfaces with a constant Disteli-axis based on E. Study Mapping. Using this method, we obtain and investigate several characterizations and equations of special timelike ruled surfaces undergoing one-parameter screw motion. Additionally, we have obtained necessary and sufficient conditions for constant Disteli-axis timelike ruled surfaces. Consequently, we have also considered some special cases which lead to some timelike ruled surfaces. Moreover, in recent years, many papers have focused on singularity theory, submanifold theory, harmonic quasiconformal mappings, etc. [20–31]. In our future research, we will conduct intersecting studies with singularity theory, submanifolds theory, etc., to obtain further results.

2. Basic Concepts

We begin with requisite concepts relating to dual numbers, dual Lorentzian vectors, and E. Study Mapping (see [1–3,32–36]): An oriented (non-null) line in Minkowski 3-space $\mathbb{E}_1^3$ can be defined by a point $\mathbf{p} \in L$ and a normalized direction vector $\mathbf{x}$ of L, that is, $\langle \mathbf{x}, \mathbf{x} \rangle = \pm 1$. To have coefficients of L, one forms the moment vector $\mathbf{x}^* = \mathbf{p} \times \mathbf{x}$ with respect to the origin point in $\mathbb{E}_1^3$. If $\mathbf{p}$ is substituted by any point $\mathbf{q} = \mathbf{p} + t\mathbf{x}$, $t \in \mathbb{R}$ on L, this impliesthat x^* is independent of $\mathbf{p}$ on L. The two vectors $\mathbf{x}$ and $\mathbf{x}^*$ are dependent on one other; they fulfil the following:

$$\langle \mathbf{x}, \mathbf{x} \rangle = \pm 1, \quad \langle \mathbf{x}^*, \mathbf{x} \rangle = 0.$$

The six coordinates x_i, $x_i^* (i = 1, 2, 3)$ of $\mathbf{x}$ and $\mathbf{x}^*$are named the normalized Plűcker coordinates of the line L.

Thus, the two vectors $\mathbf{x}$ and $\mathbf{x}^*$ locate the oriented line L.

A dual number $\widehat{x}$ is a number $x + \varepsilon x^*$, where $x, x^* \in \mathbb{R}$, and ε is a dual unit with the property that $\varepsilon^2 = 0$. Then the set:

$$\mathbb{D}^3 = \{\widehat{\mathbf{x}} := \mathbf{x} + \varepsilon \mathbf{x}^* = (\widehat{x}_1, \widehat{x}_2, \widehat{x}_3)\},$$

jointly with the Lorentzian inner product

$$\langle \widehat{\mathbf{x}}, \widehat{\mathbf{x}} \rangle = \widehat{x}_1^2 + \widehat{x}_2^2 - \widehat{x}_3^2,$$

forms the so-called dual Lorentzian 3-space $\mathbb{D}_1^3$. This yields:

$$\left.\begin{array}{l} \widehat{\mathbf{f}}_1 \times \widehat{\mathbf{f}}_2 = \widehat{\mathbf{f}}_3, \widehat{\mathbf{f}}_1 \times \widehat{\mathbf{f}}_3 = \widehat{\mathbf{f}}_2, \widehat{\mathbf{f}}_3 \times \widehat{\mathbf{f}}_2 = \widehat{\mathbf{f}}_1, \\ \langle \widehat{\mathbf{f}}_1, \widehat{\mathbf{f}}_1 \rangle = \langle \widehat{\mathbf{f}}_2, \widehat{\mathbf{f}}_2 \rangle = -\langle \widehat{\mathbf{f}}_3, \widehat{\mathbf{f}}_3 \rangle = 1, \end{array}\right\}$$

where $\widehat{\mathbf{f}}_1$, $\widehat{\mathbf{f}}_2$, and $\widehat{\mathbf{f}}_3$, are the dual base at the origin point $\mathbf{O}(0,0,0)$ of the dual Lorentzian 3-space $\mathbb{D}_1^3$. Thus, a point $\widehat{x} = (\widehat{x}_1, \widehat{x}_2, \widehat{x}_3)^t$ has dual coordinates $\widehat{x}_i = (x_i + \varepsilon x_i^*) \in \mathbb{D}$. If $\mathbf{x} \neq \mathbf{0}$ the norm $\|\widehat{\mathbf{x}}\|$ of $\widehat{\mathbf{x}} = \mathbf{x} + \varepsilon \mathbf{x}^*$ is defined by

$$\|\widehat{\mathbf{x}}\| = \sqrt{|\langle \widehat{\mathbf{x}}, \widehat{\mathbf{x}} \rangle|} = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}(1 + \varepsilon \frac{\langle \mathbf{x}, \mathbf{x}^* \rangle}{\langle \mathbf{x}, \mathbf{x} \rangle}).$$

A dual vector $\widehat{\mathbf{x}}$ with norm 1 is called a dual unit vector, and the vector $\widehat{\mathbf{x}}$ is called a spacelike (resp. timelike) dual unit vector if $\langle \mathbf{x}, \mathbf{x} \rangle = 1$(resp. $\langle \mathbf{x}, \mathbf{x} \rangle = -1$). It is understandable that

$$\langle \widehat{\mathbf{x}}, \widehat{\mathbf{x}} \rangle = \pm 1 \Longleftrightarrow \langle \mathbf{x}, \mathbf{x} \rangle = \pm 1, \ \langle \mathbf{x}, \mathbf{x}^* \rangle = 0.$$

The hyperbolic and Lorentzian (de Sitter space) dual unit spheres with the center $\widehat{\mathbf{o}}$, respectively, are

$$\mathbb{H}_+^2 = \left\{ \widehat{\mathbf{x}} \in \mathbb{D}_1^3 \mid \widehat{x}_1^2 + \widehat{x}_2^2 - \widehat{x}_3^2 = -1 \right\},$$

and

$$\mathbb{S}_1^2 = \left\{ \widehat{\mathbf{x}} \in \mathbb{D}_1^3 \mid \widehat{x}_1^2 + \widehat{x}_2^2 - \widehat{x}_3^2 = 1 \right\}.$$

Another important concept is E. Study Mapping [36]:The main idea underlying E. Study Mapping is the reduction of the dimensions of the objects which we are studying. This map is related with symmetry. An E. study map connects dual space and original space; using this map, we can transfer problems in original space to dual space with a reduction in its dimensions. This make problems easier to solve. Due to the symmetry properties of dual space and original space,the results obtained in dual space can explain and reflect the properties of the objects which we study in original space. We mainly make use of dual unit spheres, which have the shape of a pair of conjugate hyperboloids. In Minkowski 3-space, the common asymptotic cone represents the set of null (lightlike) lines, the oval shaped hyperboloid forms the set of timelike lines, the ring shaped hyperboloid represents the set of spacelike lines, and opposite points of each hyperboloid represent the pair of opposite vectors on a line (see Figure 1).

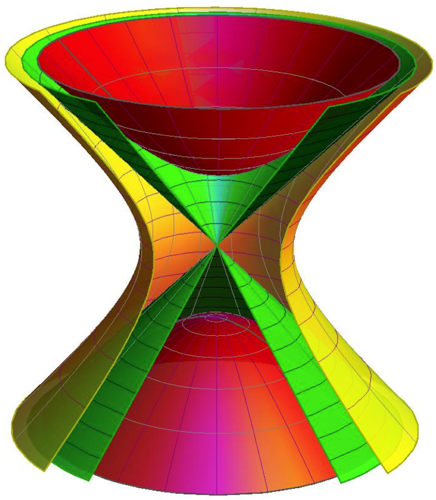

Figure 1. Dual hyperbolic and dual Lorentzian unit spheres.

3. Timelike Ruled Surfaces

According to the concept of E. Study Mapping, a spacelike or timelike ruled surface can be represented by a differentiable curve on $\mathbb{S}_1^2$, that is,

$$t \in \mathbb{R} \mapsto \widehat{\mathbf{x}}(t) \in \mathbb{S}_1^2.$$

where $\widehat{\mathbf{x}}(t)$ are specified with the rulings of the surface and henceforth we do not distinguishbetween the ruled surface and the image of its dual curve. We assume a timelike ruled surface in our study, and denote this surface by (X). The spacelike dual unit vector

$$\widehat{\mathbf{t}}(t)=\mathbf{t}+\varepsilon\mathbf{t}^{*}=\frac{d\widehat{\mathbf{x}}(t)}{dt}\left\|\frac{d\widehat{\mathbf{x}}(t)}{dt}\right\|^{-1}$$

is the tangent vector on $\widehat{\mathbf{x}}(t)$. Introducing the timelike dual unit vector $\widehat{\mathbf{g}}(t)=\mathbf{g}(t)+\varepsilon\mathbf{g}^{*}(t)=\widehat{\mathbf{x}}\times\widehat{\mathbf{t}}$, we have the moving frame $\{\widehat{\mathbf{x}}(t),\widehat{\mathbf{t}}(t),\widehat{\mathbf{g}}(t)\}$ on $\widehat{\mathbf{x}}(t)$ called the Blaschke frame. Then

$$\left.\begin{array}{c}\langle\widehat{\mathbf{x}},\widehat{\mathbf{x}}\rangle=\langle\widehat{\mathbf{t}},\widehat{\mathbf{t}}\rangle=-\langle\widehat{\mathbf{g}},\widehat{\mathbf{g}}\rangle=1,\\ \widehat{\mathbf{g}}=\widehat{\mathbf{x}}\times\widehat{\mathbf{t}},\widehat{\mathbf{t}}=\widehat{\mathbf{x}}\times\widehat{\mathbf{g}},\widehat{\mathbf{x}}=-\widehat{\mathbf{t}}\times\widehat{\mathbf{g}}.\end{array}\right\}$$

In terms of the principals of spherical kinematics, the motion of the Blaschke frame at any instant is a rotation around the Darboux vector $\widehat{\omega}$ of this frame, that is,

$$\frac{d}{dt}\begin{pmatrix}\widehat{\mathbf{x}}\\ \widehat{\mathbf{t}}\\ \widehat{\mathbf{g}}\end{pmatrix}=\begin{pmatrix}0&\widehat{p}&0\\ -\widehat{p}&0&\widehat{q}\\ 0&\widehat{q}&0\end{pmatrix}\begin{pmatrix}\widehat{\mathbf{x}}\\ \widehat{\mathbf{t}}\\ \widehat{\mathbf{g}}\end{pmatrix}=\widehat{\omega}\times\begin{pmatrix}\widehat{\mathbf{x}}\\ \widehat{\mathbf{t}}\\ \widehat{\mathbf{g}}\end{pmatrix},\tag{1}$$

where $\widehat{\omega}=\boldsymbol{\omega}+\varepsilon\boldsymbol{\omega}^{*}=\widehat{q}\widehat{\mathbf{x}}-\widehat{p}\widehat{\mathbf{g}}$. Here, $\widehat{p}(t)=p(t)+\varepsilon p^{*}(t)=\left\|\frac{d\widehat{\mathbf{x}}(t)}{dt}\right\|$, and $\widehat{q}(t)=q(t)+\varepsilon q^{*}(t)=\det\left(\widehat{\mathbf{x}},\frac{d\widehat{\mathbf{x}}(t)}{dt},\frac{d^{2}\widehat{\mathbf{x}}(t)}{dt^{2}}\right)$ are the Blaschke invariants of $\widehat{\mathbf{x}}(t)\in\mathbb{S}_{1}^{2}$. The dual unit vectors $\widehat{\mathbf{x}}$, $\widehat{\mathbf{t}}$, and $\widehat{\mathbf{g}}$ are identical to three orthogonally intersected oriented lines at a point $\mathbf{c}$, named the central point. The locus of the central points is the striction curve on (X). The tangent of the striction curve $\mathbf{c}(t)$ is given by [12]:

$$\mathbf{c}^{'}(t)=q^{*}\mathbf{x}(t)-p^{*}\mathbf{g}(t).\tag{2}$$

The distribution parameters of the timelike ruled surface (X), and the spacelike ruled surface (G), respectively, are:

$$\mu(t)=\frac{p^{*}(t)}{p(t)},\ \text{and}\ \lambda(t)=\frac{q^{*}(t)}{q(t)}.\tag{3}$$

3.1. Kinematic Geometry

The Blaschke invariants $\widehat{p}(t)$ and $\widehat{q}(t)$ furnish a kinematic geometry of the moving Blaschke frame. Under the position $|\widehat{q}|\langle|\widehat{p}|$, we locate the timelike dual unit vector

$$\widehat{\mathbf{b}}(t):=\mathbf{b}+\varepsilon\mathbf{b}^{*}=\frac{\widehat{\boldsymbol{\omega}}}{\|\widehat{\boldsymbol{\omega}}\|}=\frac{\widehat{q}}{\sqrt{\widehat{p}^{2}-\widehat{q}^{2}}}\widehat{\mathbf{x}}-\frac{\widehat{p}}{\sqrt{\widehat{p}^{2}-\widehat{q}^{2}}}\widehat{\mathbf{g}}.\tag{4}$$

It is obvious that $\widehat{\mathbf{b}}$ is the Disteli-axis (curvature-axis or striction-axis) of (X). Eventually, we have the following:

(i) The timelike Disteli-axis $\widehat{\mathbf{b}}$ is given by Equation (4).
(ii) The dual angular speed is $\|\widehat{\omega}\|=\omega(1+\varepsilon h)$.
(iii) If $\mathbf{y}(x,y,z)$ is a point on the timelike Disteli-axis $\widehat{\mathbf{b}}$, then

$$\mathbf{y}(t,v)=\mathbf{b}\times\mathbf{b}^{*}+v\mathbf{b},\ v\in\mathbb{R},\tag{5}$$

is a non-developable timelike ruled surface (B).
(iv) If the Blaschke motion is pure rotation, that is, $h(t)=0$, then

$$\widehat{\mathbf{b}}(t)=\mathbf{b}(t)+\varepsilon\mathbf{b}^{*}(t)=\frac{1}{\|\boldsymbol{\omega}\|}(\boldsymbol{\omega}+\varepsilon\boldsymbol{\omega}^{*}).$$

Note that if $h(t) = 0$, and $\|\boldsymbol{\omega}\|^2 = 1$, then $\widehat{\omega}$ is a timelike oriented line. However, in the case that the motion is purely translational, that is, $\widehat{\boldsymbol{\omega}}=0+\varepsilon\boldsymbol{\omega}^*$, we write $\omega^* = \|\boldsymbol{\omega}^*\|$, $\omega^*\mathbf{b} = \boldsymbol{\omega}^*$ and choose an arbitrary $\mathbf{b}^*$under $\omega^* \neq 0$; otherwise the timelike unit vector $\mathbf{b}$ can be taken arbitrarily as well.

Furthermore, the timelike Disteli-axis vector allows us to recast the Blaschke formula by:

$$\frac{d\widehat{\mathbf{x}}}{dt} = \|\widehat{\boldsymbol{\omega}}\|\widehat{\mathbf{b}} \times \widehat{\mathbf{x}}, \frac{d\widehat{\mathbf{t}}}{dt} = \|\widehat{\boldsymbol{\omega}}\|\widehat{\mathbf{b}} \times \widehat{\mathbf{t}}, \frac{d\widehat{\mathbf{g}}}{dt} = \|\widehat{\boldsymbol{\omega}}\|\widehat{\mathbf{b}} \times \widehat{\mathbf{g}}.$$

Hence, at any instant $t \in \mathbb{R}$, $\|\widehat{\boldsymbol{\omega}}\| := \widehat{\omega}(t) = \omega(t) + \varepsilon\omega^*(t)$ is the dual angular speed and

$$\omega(t) = \sqrt{p^2 - q^2} \text{ and } \omega^*(t) = \frac{qq^* - pp^*}{\sqrt{p^2 - q^2}}$$

are the rotational angular speed and translational angular speed of the moving Blaschke frame along (X), respectively. Hence, the pitch of the Blaschke frame throughout the length of $\widehat{\mathbf{b}}$ is

$$h(t) = \frac{\langle \boldsymbol{\omega}^*, \boldsymbol{\omega} \rangle}{\|\boldsymbol{\omega}\|} = \frac{pp^* - qq^*}{p^2 - q^2}. \tag{6}$$

So, the timelike Disteli-axis $\widehat{\mathbf{b}}$ is the instantaneous screw axis of the Blaschke frame. According to Equation (4), the timelike Disteli-axis is orthogonal to the spacelike central normal $\widehat{\mathbf{t}}$ and is parallel to the tangent plane of the ruled surface (X). Let $\widehat{\psi}(t) = \psi + \varepsilon\psi^*$ be the spacelike dual angle (dual radius of curvature) between $\widehat{\mathbf{b}}$ and $\widehat{\mathbf{x}}$; then, we have

$$\widehat{\mathbf{b}}(t) := \frac{\widehat{q}}{\sqrt{\widehat{p}^2 - \widehat{q}^2}}\widehat{\mathbf{x}} - \frac{\widehat{p}}{\sqrt{\widehat{p}^2 - \widehat{q}^2}}\widehat{\mathbf{g}} = \sinh\widehat{\psi}\widehat{\mathbf{x}} - \cosh\widehat{\psi}\widehat{\mathbf{g}}. \tag{7}$$

It is understandable that

$$\frac{\widehat{q}}{\widehat{p}} := \tanh\widehat{\psi} = \tanh\psi + \varepsilon\psi^*(1 - \tanh^2\psi).$$

From this equation, we obtain:

$$\psi^*(t) = \frac{pq^* - qp^*}{p^2 - q^2} \tag{8}$$

which is the short distance between the dual unit vectors $\widehat{\mathbf{b}}$ and $\widehat{\mathbf{x}}$. This distance is measured along the spacelike central normal $\widehat{\mathbf{t}}$, and is seen to be the collection of the Blaschke invariants.

From Equations (3), (6) and (8), we obtain

$$\left.\begin{aligned} h(t) &= \mu\cosh^2\psi - \lambda\sinh^2\psi, \\ \psi^*(t) &= (\mu - \lambda)\sinh\psi\cosh\psi. \end{aligned}\right\} \tag{9}$$

These formulas are Lorentzian versions of the Hamilton and Mannhiem formulae in Euclidean 3-space $\mathbb{E}^3$, respectively [1–4]. The surface defined by ψ^* is the Lorentzian version of the Plücker conoid in Euclidean 3-space $\mathbb{E}^3$. The parametric representation can also be given in terms of point coordinates. We may select $\widehat{\mathbf{t}}$, which is coincident with the $y-$axis of a fixed Lorentzian frame (oxyz), whereas the position of the timelike dual unit vector $\widehat{\mathbf{b}}$ is given by the angle ψ and distance ψ^* along the positive direction of the $y-$axis. The edges $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{g}}$ can be selected in the sense of the x- and z-axes, respectively (Figure 2). In view of Equations (5) and (7), it is possible to have the following point coordinates

$$(B) : \mathbf{y}(t, v) = (v\sinh\psi, -\psi^*, -v\cosh\psi), \quad v \in \mathbb{R}. \tag{10}$$

Using this paramerization approach, the timelike dual unit vectors $\widehat{\mathbf{b}}$ are obviously apparent, crossing through the y-axis. It is easily seen based on the latter equation and Equation (9) that

$$(B) : x = v \sinh \psi,\ \psi^* := y = \frac{1}{2}(\lambda - \mu) \sinh 2\psi,\ z = -v \cosh \psi,$$

which is a timelike Plücker conoid; $\lambda - \mu = 1$, $0 \leq \psi \leq 2\pi$, $-2 \leq v \leq 2$ (see Figure 3). Using straightforward calculations,

$$\left(x^2 - z^2\right)y + (\mu - \lambda)xz = 0.$$

It is interesting to note that this is a third-order polynomial in the coordinates x, y, and z. However, the geometric interpretations can be analyzed as follows:

$$\frac{x}{z} = \frac{1}{2y}\left[(\mu - \lambda) \pm \sqrt{(\mu - \lambda)^2 + 4y^2}\right]. \tag{11}$$

The limits of (B) can be obtained by equating the discriminant of Equation (11) to zero, that is,

$$y = \pm\frac{i}{2}(\lambda - \mu);\ i = \sqrt{-1},$$

which are the locations of the two isotropic torsal planes π_1, π_2, and each one of them contains one isotropic line (null line) L. Furthermore, the pitch $h(t)$ is not a periodic function and has at most two extreme values, the distribution parameters μ and λ. Thus, the two edges $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{g}}$ are principal axes of the timelike Plücker conoid. However, the geometric properties are discussed as follows:

(1) If $h(t) \neq 0$, then there are two isotropic lines L_1, and L_2 passing through the isotropic point $(0, y, 0)$ only if $y\langle(\lambda - \mu)$; for the two isotropic limit points $y = \pm\frac{i}{2}(\lambda - \mu)$, they synchronize with the edges $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{g}}$.

(2) If $h(t) = 0$, then the two isotropic torsal lines L_1, and L_2 are obtained by

$$\frac{x}{z} = -\tanh \psi = \mp\sqrt{\frac{\mu}{\lambda}},\ \text{and } y = \pm i\sqrt{\lambda\mu}. \tag{12}$$

These equations mean that the two isotropic torsal lines L_1 and L_2 are orthogonal each other. Hence, in this special case, we have:

(a) In the case of $\lambda = \mu$ the timelike Plücker conoid degenerates into the pencil of lines through "$\mathbf{o}$" in the timelike plane $y = 0$.

(b) In the case of $\lambda + \mu = 0$ the two torsal isotropic lines L_1, and L_2 are coincident with the edges $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{g}}$.

(c) In the case of $\mu = 0$, $\lambda \neq 0$ the two torsal isotropic lines L_1, and L_2 both coincide with the $x-$axis; for $\mu \neq 0$, $\lambda = 0$ they coincide with the $z-$axis.

(d) In the case of $\lambda = \mu = 0$ the ruled timelike surface (X) and the spacelike surface (G) are developable surfaces (cones); the central point $\mathbf{c}$ is a fixed point.

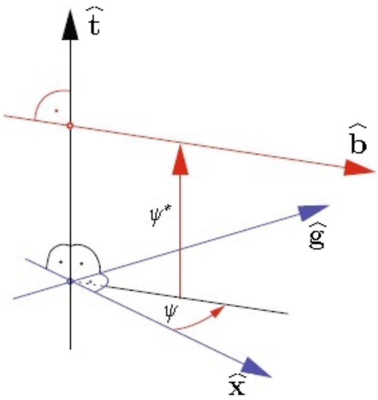

Figure 2. $\widehat{\mathbf{b}}(t) = \sinh\widehat{\psi}\widehat{\mathbf{x}} - \cosh\widehat{\psi}\widehat{\mathbf{g}}$.

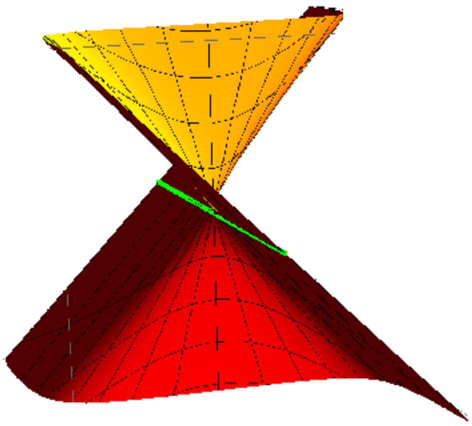

Figure 3. A timelike Plücker conoid.

Serret–Frenet Motion

(a) If $\mu = 0$, then the tangent vector of the striction curve is parallel to $\mathbf{x}$, that is, $\mathbf{c}' \parallel \mathbf{x}$. This means that (X) is a timelike tangential surface. For the curvature κ, and the torsion τ, we can find the following calculations simply:

$$\kappa(t) = \frac{\left\|\mathbf{c}' \times \mathbf{c}''\right\|}{\left\|\mathbf{c}'\right\|^3} = \frac{p}{q^*}, \text{ and } \tau(t) = \frac{\det(\mathbf{c}', \mathbf{c}'', \mathbf{c}''')}{\left\|\mathbf{c}' \times \mathbf{c}''\right\|^2} = \frac{1}{\lambda}, \text{ with } \lambda \neq 0.$$

Thus, the distribution parameter λ is the radius of torsion of the spacelike striction curve. We arrive, therefore, at the conclusion that the spacelike striction curve $\mathbf{c}(t)$ is the edge of regression of (X). Based on [35], we summarize this result in the following.

Theorem 1. *Any timelike ruled surface (X) with the curvature function*

$$q^*(t) = p(a\cosh\theta - b\sinh\theta); \ \theta(t) = \int_0^t \frac{dt}{\lambda}(t) \neq 0,$$

with real constants $(a, b) \neq (0, 0)$ is a timelike tangential surface of a spacelike curve lying on a Lorentzian sphere with radius $\sqrt{a^2 - b^2} > 0$.

Corollary 1. *The curvature function $\kappa(t)$ and torsion function $\tau(t)$ of the Lorentzian spherical curve in Theorem 1, respectively, are:*

$$\kappa(t) = \frac{1}{a\cosh\theta - b\sinh\theta}, \text{ and } \tau(t) = \frac{q}{p(a\cosh\theta - b\sinh\theta)}. \quad (13)$$

Furthermore, based on Equations (7) and (13), we can write:

$$\frac{\tau(t)}{\kappa(t)} = \frac{q(t)}{p(t)} \Rightarrow \cosh\psi = \frac{\kappa}{\sqrt{\kappa^2 - \tau^2}}, \text{ and } \sinh\psi = \frac{\tau}{\sqrt{\kappa^2 - \tau^2}}. \quad (14)$$

On the basis of this and Equation (9), it follows that

$$h(t) = -\frac{1}{\tau}\sinh^2\psi = -\frac{\tau}{\kappa^2 - \tau^2}, \text{ and } \psi^*(t) = -\frac{1}{\tau}\sinh\psi\cosh\psi = -\frac{\kappa}{\kappa^2 - \tau^2}.$$

The corresponding timelike Plücker conoid is

$$\tau\left(x^2 - z^2\right)y - xz = 0.$$

(b) If $\lambda(t) = 0$, then the striction curve is tangent to **g**; it is normal to the ruling through $\mathbf{c}(t)$. In this case (X) is a timelike binormal ruled surface. Similarly, we find

$$\kappa(t) = \frac{q}{p^*}, \ \tau(t) = \frac{1}{\mu}, \text{ with } p^* \neq 0.$$

Therefore, the curvature function $\mu(t)$ is the radius of torsion of the timelike striction curve $\mathbf{c}(t)$. Similarly, we summarize this result in the following.

Theorem 2. *Any timelike ruled surface (X) with the curvature function*

$$\left(\frac{p^*}{q}\right)(t) = a\cos\theta + b\sin\theta; \ \theta(t) = \int_0^t \frac{dt}{\mu} \neq 0,$$

with real constants $(a, b) \neq (0, 0)$ is a timelike binormal surface of a spacelike curve lying on a Lorentzian sphere with radius $\sqrt{a^2 + a^2}$.

Corollary 2. *The curvature function $\kappa(t)$ and torsion function $\tau(t)$ of the Lorentzian spherical curve in Theorem 2, respectively, are:*

$$\kappa(t) = \frac{1}{a\cos\theta + b\sin\theta}, \text{ and } \tau(t) = \frac{p}{q(a\cos\theta + b\sin\theta)}.$$

Using similar arguments, we can give the identical equations for case (a);

$$\left.\begin{array}{l} \cosh\psi = \frac{\tau}{\sqrt{\tau^2-\kappa^2}}, \ \sinh\psi = \frac{\kappa}{\sqrt{\tau^2-\kappa^2}}, \ |\kappa|\langle|\tau|, \\ h(t) = \frac{\tau^2}{\tau^2-\kappa^2}, \ \psi^*(t) = -\frac{\kappa}{\tau^2-\kappa^2}, \\ \tau\left(x^2 - z^2\right)y + xz = 0. \end{array}\right\} \quad (15)$$

3.2. Timelike Ruled Surfaces with Constant Disteli-Axis

In this section we examine the propertiesof the Blaschke invariants of (X). The dual arc-length $\widehat{s}$ of $\widehat{\mathbf{x}}(t)$ is given by

$$d\widehat{s} = ds + \varepsilon ds^* = \left\| \frac{d\widehat{\mathbf{x}}(t)}{dt} \right\| dt = \widehat{p}(t)dt. \tag{16}$$

Based on Equations (1) and (16), we obtain

$$\begin{pmatrix} \widehat{\mathbf{x}}' \\ \widehat{\mathbf{t}}' \\ \widehat{\mathbf{g}}' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -\widehat{\gamma} \\ 0 & \widehat{\gamma} & 0 \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{x}} \\ \widehat{\mathbf{t}} \\ \widehat{\mathbf{g}} \end{pmatrix}; \ ('= \frac{d}{d\widehat{s}}), \tag{17}$$

where $\widehat{\gamma}(\widehat{s}) := \frac{\widehat{q}}{\widehat{p}} = \gamma + \varepsilon\gamma^*$ is the dual geodesic curvature of $\widehat{\mathbf{x}}(\widehat{s})$ on $\mathbb{S}^2_1$. Furthermore, we have:

$$\left.\begin{array}{l} \widehat{\gamma}(\widehat{s}) = \gamma[1 + \varepsilon(\lambda - \mu)] = \tanh\widehat{\psi} = \tanh\psi + \varepsilon\psi^*(1 - \tanh^2\psi), \\ \widehat{\kappa}(\widehat{s}) := \kappa + \varepsilon\kappa^* = \sqrt{1 - \widehat{\gamma}^2} = \frac{1}{\cosh\widehat{\psi}}, \\ \widehat{\tau}(\widehat{s}) := \tau + \varepsilon\tau^* = \pm\widehat{\psi}' = \pm\frac{\widehat{\gamma}'}{1-\widehat{\gamma}^2}, \end{array}\right\} \tag{18}$$

where $\widehat{\kappa}(\widehat{s})$ is the dual curvature and $\widehat{\tau}(\widehat{s})$ is the dual torsion of the spacelike dual curve $\widehat{\mathbf{x}}(\widehat{s}) \in \mathbb{S}^2_1$.

Proposition 1. *If the dual geodesic curvature function $\widehat{\gamma}(\widehat{s})$ is constant, $\widehat{\mathbf{x}}(\widehat{s})$ is a spacelike dual circle on $\mathbb{S}^2_1$.*

Proof. From Equation (18), we can find that $\widehat{\gamma}(\widehat{s})$ = constant yields that $\widehat{\tau}(\widehat{s}) = 0$, and $\widehat{\kappa}(\widehat{s})$ is constant, which implies that $\widehat{\mathbf{x}}(\widehat{s})$ is a spacelike dual circle on $\mathbb{S}^2_1$. □

Definition 1. *A non-developable timelike ruled surface (X) is defined as a constant Disteli-axis timelike ruled surface if its dual geodesic curvature $\widehat{\gamma}(\widehat{s})$ is constant.*

According to the E. Study map, the constant Disteli-axis timelike ruled surface (X) is generated by a spacelike line undergoing a Lorentzian helical motion of constant pitch h about the timelike Disteli-axis $\widehat{\mathbf{b}}$ As a special case, if $\widehat{\gamma}(\widehat{s}) = 0$, then $\widehat{\mathbf{x}}(\widehat{s})$ is a great spacelike dual circle on $\mathbb{S}^2_1$, that is,

$$\widehat{c} = \{\widehat{\mathbf{x}} \in \mathbb{S}^2_1 \mid \langle \widehat{\mathbf{x}}, \widehat{\mathbf{b}} \rangle = 0, \text{with } \langle \widehat{\mathbf{b}}, \widehat{\mathbf{b}} \rangle = -1\}.$$

In this case, all the rulings of (X) intersected orthogonally with the timelike Disteli-axis $\widehat{\mathbf{b}}$, that is, $\psi = \psi^* = 0$. Thus, we can observe that $\widehat{\gamma}(\widehat{s}) = 0 \Leftrightarrow (X)$ is a timelike helicoidal surface. The class of the constant-Disteli-axis ruled surfaceis fundamental to the curvature theory of ruled surfaces. We therefore examine its properties in some detail.

Example 1. *In the following, we establish the constant Disteli-axis timelike ruled surface (X). Since $\widehat{\gamma}(\widehat{s})$ is constant, based on Equations (17) and (18) we obtain the ODE $\widehat{\mathbf{t}}'' + \widehat{\kappa}^2\widehat{\mathbf{t}} = \mathbf{0}$. Without the loss of generality, we may assume $\widehat{\mathbf{t}}(0) = (0, 1, 0)$, and the general solution of the ODE becomes:*

$$\widehat{\mathbf{t}}(\widehat{s}) = \left(\widehat{b}_1 \sin(\widehat{\kappa}\widehat{s}), \cos(\widehat{\kappa}\widehat{s}) + \widehat{b}_2 \sin(\widehat{\kappa}\widehat{s}), \widehat{b}_3 \sin(\widehat{\kappa}\widehat{s})\right), \tag{19}$$

where $\widehat{b}_1$, $\widehat{b}_2$, and $\widehat{b}_3$ are some dual constants fulfilling $\widehat{b}_1^2 - \widehat{b}_3^2 = 1$, and $\widehat{b}_2 = 0$. Equation (19) gives us:

$$\widehat{\mathbf{x}}(\widehat{s}) = \left(-\widehat{b}_1 \frac{1}{\widehat{\kappa}} \cos(\widehat{\kappa}\widehat{s}) + \widehat{d}_1, \frac{1}{\widehat{\kappa}} \sin(\widehat{\kappa}\widehat{s}), -\widehat{b}_3 \frac{1}{\widehat{\kappa}} \cos(\widehat{\kappa}\widehat{s}) + \widehat{d}_3\right),$$

where $\widehat{d}_1$ and $\widehat{d}_3$ are some dual constants fulfilling $\widehat{b}_3\widehat{d}_3 - \widehat{b}_1\widehat{d}_1 = 0$ and $\widehat{d}_1^2 - \widehat{d}_3^2 = 1 - \widehat{\rho}^2$, where $\widehat{\rho}\widehat{\kappa} = 1$. We now replace the dual coordinates by $\widehat{\widehat{x}}_1$, $\widehat{\widehat{x}}_2$, $\widehat{\widehat{x}}_3$ as:

$$\begin{pmatrix} \widehat{\widehat{x}}_1 \\ \widehat{\widehat{x}}_2 \\ \widehat{\widehat{x}}_3 \end{pmatrix} = \begin{pmatrix} \widehat{b}_1 & 0 & -\widehat{b}_3 \\ 0 & 1 & 0 \\ -\widehat{b}_3 & 0 & \widehat{b}_1 \end{pmatrix} \begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \\ \widehat{x}_3 \end{pmatrix}.$$

With respect to the dual coordinates by $\widehat{\widehat{x}}_1$, $\widehat{\widehat{x}}_2$, $\widehat{\widehat{x}}_3$, $\widehat{\mathbf{x}}(\widehat{s})$ becomes

$$\widehat{\mathbf{x}}(\widehat{s}) = \left(-\cosh\widehat{\psi}\cos(\widehat{\kappa}\widehat{s}), \cosh\widehat{\psi}\sin(\widehat{\kappa}\widehat{s}), \widehat{d}\right), \tag{20}$$

for a dual constant $\widehat{d} = \widehat{b}_1\widehat{d}_3 - \widehat{b}_3\widehat{d}_1$, with $\widehat{d} = \pm\sinh\widehat{\psi}$. Note that $\widehat{\mathbf{x}}(\widehat{s})$ is independent of the choice of the lower sign or upper sign of $\pm$. Therefore, using the method described in this paper, we choose upper sign, that is,

$$\widehat{\mathbf{x}}(\widehat{\varphi}) = (-\cosh\widehat{\psi}\cos\widehat{\varphi}, \cosh\widehat{\psi}\sin\widehat{\varphi}, \sinh\widehat{\psi}), \tag{21}$$

where $\widehat{\varphi} = \widehat{\kappa}\widehat{s}$ It is a spacelike spherical curve with the dual curvature $\widehat{\kappa} = \sqrt{1 - \widehat{\gamma}^2}$ on the Lorentzian dual unit sphere. $\mathbb{S}_1^2$. Let $\widehat{\varphi} = \varphi(1 + \varepsilon h)$, h indicate the pitch of the helical motion; then Equation (21) represents a timelike ruled surface. Thus, the Blaschke frame is found as follows:

$$\begin{pmatrix} \widehat{\mathbf{x}} \\ \widehat{\mathbf{t}} \\ \widehat{\mathbf{g}} \end{pmatrix} = \begin{pmatrix} -\cosh\widehat{\psi}\cos\widehat{\varphi} & \cosh\widehat{\psi}\sin\widehat{\varphi} & \sinh\widehat{\psi} \\ \sin\widehat{\varphi} & \cos\widehat{\varphi} & 0 \\ -\sinh\widehat{\psi}\cos\widehat{\varphi} & \sinh\widehat{\psi}\sinh\widehat{\varphi} & \cosh\widehat{\psi} \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{f}}_1 \\ \widehat{\mathbf{f}}_2 \\ \widehat{\mathbf{f}}_3 \end{pmatrix}. \tag{22}$$

It can be readily seen from Equation (22) that

$$\left.\begin{aligned} \widehat{p}(\varphi) = (1 + \varepsilon h)\cosh\widehat{\psi}, \widehat{q}(\varphi) = (1 + \varepsilon h)\sinh\widehat{\psi}, \\ d\widehat{s} = \widehat{p}(\varphi)d\varphi, \widehat{\gamma}(\varphi) =: \tfrac{\widehat{q}(\varphi)}{\widehat{p}(\varphi)} = \tan\widehat{\psi}. \end{aligned}\right\} \tag{23}$$

From the real and dual parts of Equation (23), we find

$$\mu = \psi^* \tanh\psi + h, \text{ and } \lambda = \psi^* \coth\psi + h. \tag{24}$$

Furthermore, we have

$$\widehat{\mathbf{b}} = \sinh\widehat{\psi}\widehat{\mathbf{x}} - \cosh\widehat{\psi}\widehat{\mathbf{g}} = -\widehat{\mathbf{f}}_3. \tag{25}$$

This means that the axis of the Lorentzian helical motion is the constant timelike Disteli-axis $\widehat{\mathbf{b}}$.

Now we derive the equation of the timelike ruled surface (X). If we separate Equation (21) into real and dual parts, we obtain

$$\mathbf{x}(\varphi)=(-\cosh\psi\cos\varphi, \cosh\psi\sin\varphi, \sinh\psi), \tag{26}$$

and

$$\mathbf{x}^*(\varphi) = \begin{pmatrix} x_1^* \\ x_2^* \\ x_3^* \end{pmatrix} = \begin{pmatrix} \varphi^*\sin\varphi\cosh\psi - \psi^*\sinh\psi\cos\varphi \\ \varphi^*\cos\varphi\cosh\psi + \psi^*\sinh\psi\sin\varphi \\ \psi^*\cosh\psi \end{pmatrix}. \tag{27}$$

Let $\boldsymbol{\beta}(\beta_1, \beta_2, \beta_3)$ be a point on $\widehat{\mathbf{x}}$. Since $\boldsymbol{\beta} \times \mathbf{x} = \mathbf{x}^*$ we have the system of linear equations in β_1, β_2, and β_3:

$$\left.\begin{array}{c}\beta_2 \sinh \psi - \beta_3 \cosh \psi \sin \varphi = x_1^*, \\ -\beta_1 \sinh \psi - \beta_3 \cosh \psi \cos \varphi = x_2^*, \\ -(\beta_1 \sin \varphi + \beta_2 \cos \varphi) \cosh \psi = x_3^*.\end{array}\right\}$$

The matrix of coefficients of unknowns β_1, β_2, and β_3 is:

$$\begin{pmatrix} 0 & \sinh \psi & -\cosh \psi \sin \varphi \\ -\sinh \psi & 0 & -\cosh \psi \cos \varphi \\ -\cosh \psi \sin \varphi & -\cosh \psi \cos \varphi & 0 \end{pmatrix},$$

and therefore its rank is 2 with $\varphi \neq p\pi$ (p is an integer), and $\psi \neq 0$. In addendum, the rank of the augmented matrix

$$\begin{pmatrix} 0 & \sinh \psi & -\cosh \psi \sin \varphi & x_1^* \\ -\sinh \psi & 0 & -\cosh \psi \cos \varphi & x_2^* \\ -\cosh \psi \sin \varphi & -\cosh \psi \cos \varphi & 0 & x_3^* \end{pmatrix},$$

is 2. Hence, this system has infinitely many solutions, represented as

$$\begin{array}{c}\beta_1 = -\psi^* \sin \varphi - (\varphi^* + \beta_3) \coth \psi \cos \varphi, \\ \beta_2 = -\psi^* \cos \varphi + (\varphi^* + \beta_3) \coth \psi \sin \varphi, \\ -\beta_1 \sin \varphi - \beta_2 \cos \varphi = \psi^*.\end{array} \tag{28}$$

Since β_3 is taken at random, then we may take $\varphi^* + \beta_3 = 0$. In this case, Equation (28) reduces to

$$\beta_1 = -\psi^* \sin \varphi, \ \beta_2 = -\psi^* \cos \varphi, \ \beta_3 = -\varphi^*.$$

We now straightforwardly find the base curve:

$$\boldsymbol{\beta}(\varphi) = (-\psi^* \sin \varphi, -\psi^* \cos \varphi, -h\varphi).$$

It can be show that $\langle \boldsymbol{\beta}', \mathbf{x}' \rangle = 0$; $(' = \frac{d}{d\varphi})$, so the base curve of (X) is its striction curve. Furthermore, it can be shown that $\boldsymbol{\beta}(\varphi)$ is a spacelike (resp., a timelike) if and only if $|\psi^*| > h$ (resp., $|\psi^*| < h$). Regarding the curvature κ, and the torsion τ, we can find these using the following calculations.

$$\kappa(\varphi) = \frac{\psi^*}{\psi^{*2} - h^2}, \text{ and } \quad \tau(\varphi) = \frac{h}{\psi^{*2} - h^2}.$$

So, $\boldsymbol{\beta}(\varphi)$ is a spacelike helix (resp., timelike) if and only if $|\psi^*| > h$ (resp., $|\psi^*| < h$). In addition, if $\mathbf{p}(x, y, z)$ is a point on (X), then we have

$$(X) : \mathbf{p}(\varphi, v)) = \begin{pmatrix} -\psi^* \sin \varphi - v \cosh \psi \cos \varphi \\ -\psi^* \cos \varphi + v \cosh \psi \sin \varphi \\ -h\varphi + v \sinh \psi \end{pmatrix}, \tag{29}$$

from which we attain,

$$(X) : \frac{x^2}{\psi^{*2}} + \frac{y^2}{\psi^{*2}} - \frac{Z^2}{\chi^2} = 1,$$

where $\chi = \psi^* \coth \psi$, and $Z = z + h\varphi$. The parameters h, ψ, and ψ^* can control the shape of the timelike surface (X). Thus, (X) is a 3-parameter family of Lorentzian unit spheres. The intersection of each Lorentzian unit sphere and the corresponding spacelike plane $z = -h\varphi$ is a one-parameter set of Euclidean circles $x^2 + y^2 = \psi^{*2}$ Therefore, the envelope of (X) is a one-parameter set of Lorentzian cylinders. According to Equation (29), we have the following:

(1) A general timelike helicoid: for $h = -0.7$, $\psi^* = -2$, $\psi = 1$, $-1.5 \leq v \leq 1.5$, and $0 \leq \varphi \leq 2\pi$ (see Figure 4).
(2) Lorentzian sphere: for $h = 0$, $\psi^* = -2$, $\psi = 1$, $-1.5 \leq v \leq 1.5$, and $0 \leq \varphi \leq 2\pi$ (see Figure 5).
(3) A timelike helicoid: for $h = -1$, $\psi^* = \psi = 0$, $-2.5 \leq v \leq 2.5$, and $0 \leq \varphi \leq 2\pi$ (see Figure 6).
(4) A timelike cone: for $h = \psi^* = 0$, $\psi = 1$, $-2.5 \leq v \leq 2.5$, and $0 \leq \varphi \leq 2\pi$ (see Figure 7).

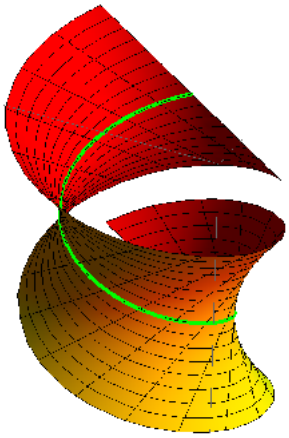

Figure 4. A general timelike helicoid.

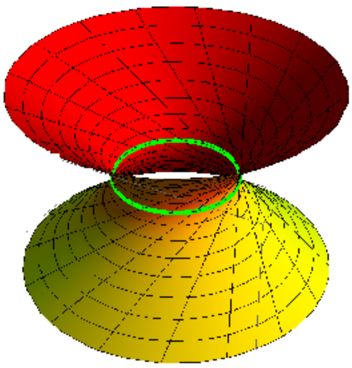

Figure 5. Lorentzian sphere.

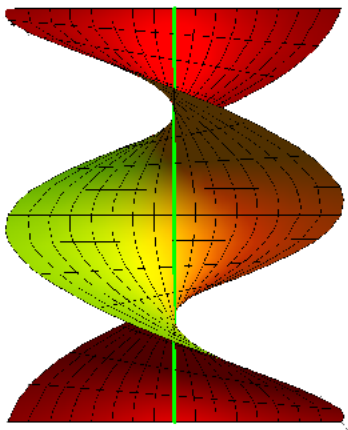

Figure 6. A timelike helicoid.

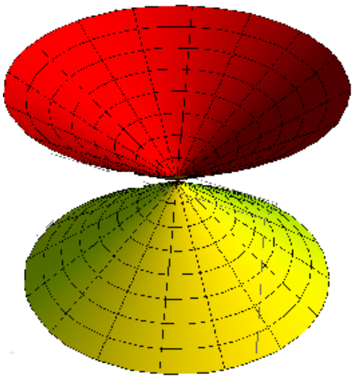

Figure 7. Timelike cone.

4. Conclusions

The main result of this study was to provide the necessary and sufficient conditions to analyze constant dual angles with respect to a constant spacelike Disteli-axis.We presented and examined some characterizations of special timelike ruled surfaces undergoing one-parameter helical motion. Furthermore, we provided an approach for studying the kinematic geometry of a timelike ruled surface with constant Disteli-axis based on E. Study Mapping. As a result, we have obtained and examined the timelike Plücker conoid associated with the Blaschke frame associated with the surface.

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